KŌDAI MATH. SEM. REP.
23 (1971), 402—472

CASCADE SEMIGROUPS AND THEIR CHARACTERIZATION

BY TETSUO FUJIMAGARI AND MINORU MOTOO

Introduction.

A mathematical theory of cascade processes with infinite cross section has been developed by Harris in his book ([3], Chap. VII).[1] By a cascade process with infinite cross section, we mean a process in which each particle splits infinitely often in any finite time interval. In our paper, we will treat a model which satisfies "Approximation A" in Harris' book. Our model is less general than Harris' one in the sense that it consists of only one type of particles such as electrons. On the other hand, it includes the case where the particle may split into infinitely many new particles simultaneously and may lose its energy continuously.[2] Inspired by recent developments of the theory of continuous state branching processes ([7], [8], [9], [13], [14], [15]), we shall define a cascade process as a branching Markov process satisfying a condition of homogeneity on a certain space of discrete measures. Each measure in the space represents a configuration of a system of countably many particles. Moreover we shall specify the process by its characteristic quantities.

In §1, we investigate fundamental properties of a space $\boldsymbol{M}_p$ of measures. Any element μ in $\boldsymbol{M}_p$ $(0<p<\infty)$ has a form $\sum_{i=1}^{\infty} x_i\delta_{x_i}$ $(0<x_i\leqq 1,\ \sum_{i=1}^{\infty} x_i\leqq p)$ or 0, where δ_{x_i} is a unit measure concentrated at x_i. A measure $\mu=\sum_i x_i\delta_{x_i}$ corresponds to a configuration of a system of particles, with energy x_i $(i=1, 2, \cdots)$. The total mass $\|\mu\|=\sum_i x_i$ of μ represents the total energy of the system. Endowed with weak*-topology, the space $\boldsymbol{M}_p$ is considered as a compact metrizable space. In §2, we first define a cascade semigroup on the space of continuous functions on $\boldsymbol{M}_1$. It has the branching property and certain property of homogeneity in addition to usual ones of conservative Markov semigroups. Then, we define a cascade process (μ_t) corresponding to the cascade semigroup, where μ_t is considered to specify the state of the system at time t. In §3, we derive an underlying process (x_t) on $[0, 1]$ where x_t may be considered as the energy of each specific particle at time t. It is shown that $(-\log x_t)$ is an increasing additive process. In §4, a branching measure Π on $\boldsymbol{M}_1-\{\delta_1\}$ is introduced. The measure represents the law of splitting of each particle. It has a close connection with the Levy measure of

Received November 12, 1970.

1) Historical notes and physical meanings of the theory are also seen in the book (Chap. VII, §1 in [3]).

2) Even in this case it is different from "Approximation B" in §2 of [3].

the underlying process (x_t) (see (4.15)).[3] The underlying process is uniquely determined by a branching measure Π and a nonnegative constant m. The constant m represents a rate of continuous loss of energy of each particle. In § 5, using the underlying process and the branching measure, we derive a system of (S_d)-equations. They are fundamental integral equations, and their unique bounded solution is given by the cascade semigroup. The system of (S_d)-equations is an analogue of the equation given by Skorohod [12] (see, also, [5]). In § 6, we show that for a given cascade semigroup the underlying process and the branching measure are uniquely determined through (S_d)-equations. The result is used in § 8. In § 7, we have the expression of the generator of the cascade semigroup. A non-linear evolution equation for the cascade semigroup is derived by using the branching measure Π and the nonnegative constant m. The equation corresponds to that given by Harris (Theorem 11.1 of Chap. VII in [3]). In § 8, we start with a process (x_t) on $[0, 1]$ and a measure Π on $\boldsymbol{M}_1-\{\delta_1\}$ satisfying centain conditions which are known to be necessary for an underlying process and a branching measure. By solving (S_d)-equations constructed by (x_t) and Π, we obtain a cascade semigroup. Moreover, it is shown that (x_t) and Π are the underlying process and the branching measure of the cascade semigroup. Finally, we have the following result: There is a one-to-one correspondence between cascade processes and pairs (m, Π), where m is a nonnegative constant and Π is a measure on $\boldsymbol{M}_1-\{\delta_1\}$ such that $\int_{\boldsymbol{M}_1-\{\delta_1\}}(1-M(\mu))\Pi(d\mu)<+\infty$ where $M(\mu)=\max_i x_i$ for $\mu=\sum_i x_i\delta_{x_i}\in\boldsymbol{M}_1$. Π is the branching measure and m is the constant mentioned in § 4.

Main results of the paper were published in [2] without detailed proofs. We would like to express our thanks to Professores N. Ikeda, M. Nagasawa, and S. Watanabe for their valuable opinions and encouragement.

§ 1. Preliminaries.

In this section we shall present several notions which will be necessary to formulate cascade processes.

First of all, to define the state space, let S be an interval $(0, 1]$, and set

$$\boldsymbol{M}_p=\{\mu;\ \mu \text{ is a measure on } S \text{ such that } \mu=0 \text{ or}$$
$$\mu=\textstyle\sum_i x_i\delta_{x_i}\ (x_i\in S) \text{ and } \|\mu\|=\sum_i x_i\leqq p\}$$

for each p $(0<p<\infty)$, where $x_i\delta_{x_i}$ is a measure which is concentrated at a point x_i and has a mass x_i at the point, $\sum_i$ denotes a finite or countably infinite sum, and $\|\mu\|$ is the total mass of a measure μ. Setting $p=1$, $\boldsymbol{M}_1$ will be the state space of cascade processes.

For introducing a topology on the space $\boldsymbol{M}_p$, let C_0 be the set of all con-

3) Π is σ-finite, but not necessarily finite. If Π is finite, the process has a finite cross section.

tinuous functions on S vanishing in a neighborhood of 0, then there exists a countable family $\{f_n\}$ of functions in C_0 such that $0 \leqq f_n \leqq 1$ for all $n \geqq 1$ and the linear hull $\mathcal{L}\{f_n\}$ is dense in C_0 with the uniform topology.

Given such an $\{f_n\}$, set

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} |(f_n, \mu) - (f_n, \nu)|$$

for each $\mu, \nu \in \boldsymbol{M}_p$, where (f, μ) is defined as

$$(f, \mu) = \int_S f(x)\mu(dx)$$

for any Borel function f on S and any measure μ on S. By the definition, it is clear that $\rho(\mu, \nu) \leqq 2p$ for all $\mu, \nu \in \boldsymbol{M}_p$.

Then, we have the following

PROPOSITION 1. 1. *$(\boldsymbol{M}_p, \rho)$ is a compact metric space and the convergence with respect to ρ is equivalent to the weak*-convergence, i.e. for $\{\mu_n\} \subset \boldsymbol{M}_p$, $\mu \in \boldsymbol{M}_p$, $\mu_n \to \mu$ in ρ if and only if $(f, \mu_n) \to (f, \mu)$ for all $f \in C_0$.*

Proof. $(\boldsymbol{M}_p, \rho)$ is obviously a metric space and the equivalence of convergence in ρ and weak*-convergence is also easily shown. Hence, we have only to show the compactness of $(\boldsymbol{M}_p, \rho)$. To begin with, it should be noticed that a bounded closed set in the dual space C_0' of C_0 is compact with respect to the weak*-topology and the space $\boldsymbol{M}_p$ can be considered as a set in C_0' in the usual way. Therefore, as the boundedness of $\boldsymbol{M}_p \subset C_0'$ is clear, it is sufficient to verify the closedness of $\boldsymbol{M}_p$ in C_0'.

Let $\{\mu_n\}$ be a sequence in $\boldsymbol{M}_p$ converging to some continuous linear functional $l \in C_0'$, then l may be identified with a measure μ on S which satisfies $\|\mu\| \leqq p$. If infinitely many μ_n are equal to 0, then obviously $\mu = 0 \in \boldsymbol{M}_p$. Hence it is sufficient to consider the case $\mu_n = \sum_i x_i^n \delta_{x_i^n}$ for all n. If we denote the restricted measure on $[\varepsilon, 1]$ $(0 < \varepsilon < 1)$ of a measure μ on S by $\mu|_\varepsilon$, then $\mu_n|_\varepsilon$ converges weakly to $\mu|_\varepsilon$ as $n \to \infty$, if $\mu(\{\varepsilon\}) = 0$. Let $\mu_n|_\varepsilon = \sum_{i=1}^{N_n} x_i^n \delta_{x_i^n}$, then $N_n = (f_0, \mu_n|_\varepsilon)$ for a continuous function $f_0(x) = x^{-1}$ on $[\varepsilon, 1]$ and this converges to $(f_0, \mu|_\varepsilon) = N$ if $\mu(\{\varepsilon\}) = 0$. From this, $N_n = N$ for all sufficiently large n and so we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $N_{n_k} = N$ and $x_i^{n_k}$ converges to some x_i as $k \to \infty$ for all i $(1 \leqq i \leqq N)$. Thus $\mu_{n_k}|_\varepsilon$ converges to $\sum_{i=1}^N x_i \delta_{x_i}$ and we have $\mu|_\varepsilon = \sum_{i=1}^N x_i \delta_{x_i}$ if $\mu(\{\varepsilon\}) = 0$. Therefore, taking a sequence $\varepsilon_n \downarrow 0$ such as $\mu(\{\varepsilon_n\}) = 0$, we can conclude $\mu = \sum_i x_i \delta_{x_i}$, i.e. $\mu \in \boldsymbol{M}_p$.

We remark that the topology of $\boldsymbol{M}_p$ does not depend on the choice of $\{f_n\}$ because it is equivalent to the weak*-topology by proposition 1. 1.

Next we define a function $\hat{f}$ which will play a fundamental role in formulating the branching Markov process. For this, set

$$B_0^*=\{f;\ f \text{ is a Borel function on } S \text{ such that } 0\leqq f\leqq 1 \text{ and } f=1 \text{ in some neighborhood of } 0\}$$

and

$$C_0^*=\{e^{-\varphi};\ \varphi\in C_0 \text{ and } \varphi\geqq 0\}\subset B_0^*.$$

Then we define $\hat{f}(\mu)$ for any $f\in B_0^*$ and any $\mu\in \boldsymbol{M}_p$ by

$$\hat{f}(\mu)=\exp\left(\int_S \frac{1}{x}\log f(x)\mu(dx)\right), \tag{1.1}$$

where $\log 0=-\infty$ and $e^{-\infty}=0$. It follows by the definition that $\hat{1}=1$, $\hat{f}$ is a Borel function on $\boldsymbol{M}_p$, $0\leqq \hat{f}\leqq 1$ and $\hat{f}(0)=1$, and if $f\in C_0^*$, then $\hat{f}>0$.

We will consider $\hat{f}$ for $f\in C_0^*$ almost all time, partly because of the following proposition.

PROPOSITION 1. 2. *The linear hull* $\mathcal{L}\{\hat{f};\ f\in C_0^*\}$ *is dense in the space* $C(\boldsymbol{M}_p)$ *of all continuous functions on* $\boldsymbol{M}_p$ *with the uniform topology.*

Proof. If $f\in C_0^*$, then $x^{-1}\log f(x)\in C_0$ and so $\hat{f}\in C(\boldsymbol{M}_p)$ by the definition. Thus Proposition 1. 2 follows from the theorem of Stone and Weierstrass.

We shall state some of properties of $\hat{f}$.

LEMMA 1. 1. (i) *If* $f, g\in B_0^*$, $\hat{f}(\mu)\hat{g}(\mu)=\widehat{f\cdot g}(\mu)$ *for any* $\mu\in\boldsymbol{M}_p$. (ii) *If* $f\in B_0^*$, $\hat{f}(\mu+\nu)=\hat{f}(\mu)\hat{f}(\nu)$ *for any* $\mu, \nu\in\boldsymbol{M}_p$ *such as* $\mu+\nu\in\boldsymbol{M}_p$ *also.* (iii) *If* $f\in B_0^*$, $\hat{f}(\mu)=\prod_i f(x_i)$ *for any* $\mu=\sum_i x_i\delta_{x_i}\in\boldsymbol{M}_p$.

Proof. (i)
$$\hat{f}(\mu)\hat{g}(\mu)=\exp\left(\int\frac{1}{x}\log f(x)\mu(dx)\right)\exp\left(\int\frac{1}{x}\log g(x)\mu(dx)\right)$$
$$=\exp\left(\int\frac{1}{x}\log f(x)g(x)\mu(dx)\right)=\widehat{f\cdot g}(\mu).$$

(ii)
$$\hat{f}(\mu+\nu)=\exp\left(\int\frac{1}{x}\log f(x)(\mu+\nu)(dx)\right)$$
$$=\exp\left(\int\frac{1}{x}\log f(x)\mu(dx)\right)\exp\left(\int\frac{1}{x}\log f(x)\nu(dx)\right)$$
$$=\hat{f}(\mu)\hat{f}(\nu).$$

(iii) Since $f\in B_0^*$, $f(x)=1$ on $(0, \varepsilon)$ for some $\varepsilon>0$ and so

$$\hat{f}(\mu)=\exp\left(\int_S\frac{1}{x}\log f(x)\mu(dx)\right)=\exp\left(\int_{[\varepsilon,1]}\frac{1}{x}\log f(x)\mu(dx)\right)$$
$$=\prod_{x_i\geqq\varepsilon} f(x_i)=\prod_i f(x_i)$$

when $\mu=\sum_i x_i\delta_{x_i}$.

We now define a multiplication of $a\in S$ and $\mu\in \boldsymbol{M}_p$ by

$$a\cdot\mu=\begin{cases}\sum_i ax_i\delta_{ax_i} & \text{if } \mu=\sum_i x_i\delta_{x_i},\\ 0 & \text{if } \mu=0,\end{cases} \tag{1.2}$$

and set

$$\theta_a f(x)=f(ax) \tag{1.3}$$

for any Borel function f on S. Clearly $a\cdot\mu\in\boldsymbol{M}_p$, $\theta_a f\in B_0^*$ for $f\in B_0^*$ and $\theta_a f\in C_0^*$ for $f\in C_0^*$.

LEMMA 1.2. (i) *For any $a\in S$, $a\cdot\mu$ is a continuous mapping of μ on $\boldsymbol{M}_p$.* (ii) *If $a\in S$ and $f\in B_0^*$, $\hat{f}(a\cdot\mu)=\widehat{\theta_a f}(\mu)$ for every $\mu\in\boldsymbol{M}_p$.*

Proof. (i) Let $\{\mu_n\}$ be a sequence in $\boldsymbol{M}_p$ which converges to μ. Since

$$(\varphi, a\cdot\mu_n)=\int\varphi(x)a\cdot\mu_n(dx)=\int a\varphi(ax)\mu_n(dx)$$

and $a\varphi(ax)\in C_0$ for any $\varphi\in C_0$, $(\varphi, a\cdot\mu_n)$ converges to

$$\int a\varphi(ax)\mu(dx)=(\varphi, a\cdot\mu).$$

Thus $a\cdot\mu_n\to a\cdot\mu$ as $n\to\infty$.

(ii) When $\mu=\sum x_i\delta_{x_i}$, it follows from Lemma 1.1 (iii) that

$$\hat{f}(a\cdot\mu)=\hat{f}(\sum ax_i\delta_{ax_i})=\prod_i f(ax_i)$$

and

$$\widehat{\theta_a f}(\mu)=\prod_i\theta_a f(x_i)=\prod_i f(ax_i),$$

so that we have the lemma.

We define a function $M(\mu)$ on $\boldsymbol{M}_p$ by

$$M(\mu)=\begin{cases}\max_i x_i & \text{if } \mu=\sum_i x_i\delta_{x_i},\\ 0 & \text{if } \mu=0,\end{cases} \tag{1.4}$$

which will play an important role to characterize a cascade process.

LEMMA 1.3. *$M(\mu)$ is a continuous function of μ on $\boldsymbol{M}_p$.*

Proof. Let $\{\mu_n\}$ be a sequence which converges to μ. When $\mu=0$, choose a function $f\in C_0$ such that $f\geqq 0$ and $f=1$ on $[\varepsilon, 1]$ for $0<\varepsilon<1$. Since (f, μ_n) converges to $(f, \mu)=0$, $(f, \mu_n)<\varepsilon$ and so $M(\mu_n)<\varepsilon$ for all sufficiently large n. This implies $M(\mu_n)\to 0=M(\mu)$ as $n\to\infty$.

As for the case $\mu\neq 0$, put $a=M(\mu)>0$ and choose a function $f_1\in C_0$ such that $f_1\geqq 0$ and

$$f_1(x)=\begin{cases}1 & \text{if } a\leqq x\leqq 1,\\ 0 & \text{if } 0<x\leqq a-\varepsilon\end{cases}$$

for each sufficiently small $\varepsilon>0$. Since the limit of (f_1, μ_n) is $(f_1, \mu)\geqq a>0$, $(f_1, \mu_n)>0$ for all sufficiently large n and so $M(\mu_n)>a-\varepsilon$. Similarly, choosing a nonnegative function $f_2\in C_0$ which satisfies

$$f_2(x)=\begin{cases}1 & \text{if } a+\varepsilon\leqq x\leqq 1,\\ 0 & \text{if } 0<x\leqq a,\end{cases}$$

we have $M(\mu_n)<a+\varepsilon$ for all sufficiently large n. Therefore we have $M(\mu_n)\to M(\mu)$ as $n\to\infty$.

Finally we shall introduce some more notations which are of technical use in sections 4, 6, and 8.

For any d $(2/3<d<1)$, set $S_d=((1-d)/d, 1]$ and

$$\boldsymbol{M}_p^d=\{\mu;\ \mu \text{ is a measure on } S_d \text{ such that } \mu=0$$
$$\text{or } \mu=\textstyle\sum_i x_i\delta_{x_i}(x_i\in S_d) \text{ and } \|\mu\|\leqq p\}.$$

Then, introducing weak*-topology on $\boldsymbol{M}_p^d$ as in the case of $\boldsymbol{M}_p$, $\boldsymbol{M}_p^d$ is also a compact metrizable space. For $\mu\in\boldsymbol{M}_p$, denote the restriction of μ on S_d by $\varphi_d(\mu)=\mu|_{S_d}$, then φ_d maps $\boldsymbol{M}_p$ onto $\boldsymbol{M}_p^d$. Set

$$B_d^*=\{f;\ f \text{ is a Borel function on } S \text{ such that}$$
$$0\leqq f\leqq 1 \text{ and } f=1 \text{ on } S-S_d\}$$

and denote the Borel fields on $\boldsymbol{M}_p$ and $\boldsymbol{M}_p^d$ by $\mathcal{B}_p$ and $\mathcal{B}_p^d$, respectively. Then we have the following lemma.

LEMMA 1.4. (i) *φ_d is a continuous mapping from $\boldsymbol{M}_p$ onto $\boldsymbol{M}_p^d$.* (ii) *$\varphi_d^{-1}(\mathcal{B}_p^d)\subset\mathcal{B}_p$ and if $2/3<d<d'<1$, $\varphi_d^{-1}(\mathcal{B}_p^d)\subset\varphi_{d'}^{-1}(\mathcal{B}_p^{d'})$.* (iii) *$\bigvee_{2/3<d<1}\varphi_d^{-1}(\mathcal{B}_p^d)=\mathcal{B}_p$.* (iv) *If $f\in B_d^*$, $\hat{f}$ is a $\varphi_d^{-1}(\mathcal{B}_p^d)$-measurable function.*

Proof. (i) Denote $C_0(S_d)$ the set of all continuous functions with compact support on S_d and define $\bar{f}\in C_0(S)$ for $f\in C_0(S_d)$ by

$$\bar{f}=\begin{cases}f & \text{on } S_d,\\ 0 & \text{on } S-S_d.\end{cases}$$

Let $\{\mu_n\}$ be a sequence which converges to μ in $\boldsymbol{M}_p$, then for each $f\in C_0(S_d)$,

$$(f, \varphi_d(\mu_n))=\int_{S_d} f d\varphi_d(\mu_n)=\int_S \bar{f} d\mu_n=(f, \mu_n).$$

Thus letting $n\to\infty$, we have $\varphi_d(\mu_n)\to\varphi_d(\mu)$ in $\boldsymbol{M}_p^d$. (ii) It is clear from (i) that $\varphi_d^{-1}(\mathscr{B}_p^d)\subset\mathscr{B}_p$. If $B\in\mathscr{B}_p^d$, $\varphi_{d'}^{-1}(\varphi_{d'}(\varphi_d^{-1}(B)))=\varphi_d^{-1}(B)$ and in addition $\varphi_{d'}(\varphi_d^{-1}(B))\in\mathscr{B}_p^{d'}$ since it is shown in the same way as (i) that a mapping $\boldsymbol{M}_p^{d'}\ni\mu\to\mu|_{S_d}\in\boldsymbol{M}_p^d$ is continuous, and $\mu|_{S_d}=\varphi_d(\varphi_{d'}^{-1}(\mu))$. Thus we have (ii). (iii) $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)\subset\mathscr{B}_p$ is obvious from (ii) above, and so, noting the fact that $\mathscr{B}_p$ is equal to the minimal σ-field on $\boldsymbol{M}_p$ with respect to which all continuous functions on $\boldsymbol{M}_p$ are measurable, in order to prove (iv) it is sufficient to show that all continuous functions on $\boldsymbol{M}_p$ are $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)$-measurable.

For each nonnegative function $g\in C_0(S)$, there exists such a d $(2/3<d<1)$ that $g=0$ on $S-S_d$. Then, if $d<d'<1$, $(g, \mu)=(g|_{S_{d'}}, \varphi_{d'}(\mu))$ for any $\mu\in\boldsymbol{M}_p$. Since $g|_{S_{d'}}\in C_0(S_{d'})$, $(g|_{S_{d'}}, \nu)$ is a continuous function of ν on $\boldsymbol{M}_p^{d'}$ and so (g, μ) is $\varphi_{d'}^{-1}(\mathscr{B}_p^{d'})$-measurable. It follows from this that $\hat{f}$ is $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)$-measurable for any $f\in C_0^*$. Therefore, from Proposition 1. 2, all continuous functions on $\boldsymbol{M}_p$ are $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)$-measurable. (iv) If $f\in B_d^*$,

$$\hat{f}(\mu)=\exp\left(\int_S \frac{1}{x}\log f(x)\mu(dx)\right)$$

$$=\exp\left(\int_{S_d} \frac{1}{x}\log f(x)\varphi_d(\mu)(dx)\right)$$

for all $\mu\in\boldsymbol{M}_p$, and the right hand side of this equality is a $\mathscr{B}_p^d$-measurable function of $\varphi_d(\mu)$, so that $\hat{f}$ is $\varphi_d^{-1}(\mathscr{B}_p^d)$-measurable.

Furthermore, we shall state the following extension theorem.

LEMMA 1. 5. *Let* $\{\Lambda_d: 2/3<d<1\}$ *be a family of measures such that each* Λ_d *is a finite measure on* $(\boldsymbol{M}_p, \varphi_d^{-1}(\mathscr{B}_p^d))$ *and if* $2/3<d<d'<1$, Λ_d *is the restriction of* $\Lambda_{d'}$ *on* $\varphi_d^{-1}(\mathscr{B}_p^d)$. *Then, there exists a unique measure* Λ *on* $(\boldsymbol{M}_p, \mathscr{B}_p)$ *such that* Λ_d *is the restriction of* Λ *on* $\varphi_d^{-1}(\mathscr{B}_p^d)$.

Proof. Set $\Lambda^*(A)=\Lambda_d(A)$ for $A\in\varphi_d^{-1}(\mathscr{B}_p^d)$. Then Λ^* is well defined on $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)$ and a finitely additive set function on it. Let A be any set in $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)$, then $A=\varphi_d^{-1}(\tilde{A})$ for some d and $\tilde{A}\in\mathscr{B}_p^d$. Since $\tilde{\Lambda}_d(\cdot)\equiv\Lambda_d(\varphi_d^{-1}(\cdot))$ is a finite Borel measure on the compact metrizable space $\boldsymbol{M}_p^d$, there exists for any $\varepsilon>0$ a compact set $\tilde{K}$ such that $\tilde{K}\subset\tilde{A}$ and

$$\Lambda_d(\varphi_d^{-1}(\tilde{K}))=\tilde{\Lambda}_d(\tilde{K})\geqq\tilde{\Lambda}_d(\tilde{A})-\varepsilon=\Lambda_d(A)-\varepsilon.$$

Since φ_d is continuous and $\boldsymbol{M}_p$ is compact, $\varphi_d^{-1}(\tilde{K})$ is a compact subset of A and in $\varphi_d^{-1}(\mathscr{B}_p^d)$. It follows from this fact that Λ^* is countably additive in $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)$. Thus Λ^* can be extended uniquely to a measure Λ on $\vee_d\varphi_d^{-1}(\mathscr{B}_p^d)=\mathscr{B}_p$. This Λ is a required one and the uniqueness of Λ is obvious.

§ 2. Formulation.

We now define a cascade semigroup and a cascade process.

DEFINITION 2.1. $\{T_t;\ t\geqq 0\}$ is said to be a *cascade semigroup* when it satisfies (a) $\{T_t;\ t\geqq 0\}$ is a strongly continuous and contraction semigroup of nonnegative linear operators on $C(\boldsymbol{M}_1)$ and $T_t\,1=1$ for $t\geqq 0$, (b) $T_t f(\mu+\nu)=T_t f(\mu)T_t f(\nu)$ for any $f\in C_0^*$, if μ, ν and $\mu+\nu\in\boldsymbol{M}_1$, and (c) $T_t\hat{f}(a\delta_a)=T_t\widehat{\theta_a f}(\delta_1)$ for any $f\in C_0^*$, if $a\in S$.

By the general theory of Markov processes (see, for example, [1]), there exists a strong Markov process $\{W, \mu_t, \mathcal{N}_t, P_\mu;\ \mu\in\boldsymbol{M}_1\}$ on the state space $\boldsymbol{M}_1$ with right continuous sample paths with left limit at each $t\geqq 0$ such that $T_t F(\mu)=E_\mu[F(\mu_t)]$, where W is the space of sample paths and $\mu_t(w)=w(t)$ for $w\in W$, $\mathcal{N}_t$ is a σ-field of subsets of W generated by the sets $\{w;\ \mu_s(w)\in A\}$ for $A\in\mathcal{B}(\boldsymbol{M}_1)$ and $s\in[0, t]$, and $E_\mu[\cdot]$ denotes the expectation by the probability measure P_μ on $(W, \mathcal{N}_\infty)$ in which $\mathcal{N}_\infty$ is the smallest σ-field including $\mathcal{N}_t$ for all $t\geqq 0$. We shall call the Markov process (μ_t) a *cascade process*. The property (b) in Definition 2.1 will be called a *branching property* of the semigroup (or of the process (μ_t)) which is an abstruction of the independence of each particle of the cascade process (cf. Ikeda, Nagasawa, and Watanabe [5]), where we interprete a state $\mu=\sum_i x_i\delta_{x_i}\in\boldsymbol{M}_1$ as the existence of particles with energy (or mass etc.) x_i, $i=1, 2, \cdots$. The property (c) in the definition is an analogue of the "Approximation A" of Harris' book [3] (p. 167) representing a *homogeneity* of a medium in some sense. This will become clear in later sections.

In the following, we shall study some general properties of a cascade semigroup $\{T_t\}$ (or of a cascade process (μ_t)).

LEMMA 2.1. *Let $(T_t\hat{f})|_S$ be the restriction on S of $T_t\hat{f}$ defined by $(T_t\hat{f})|_S(x)=T_t\hat{f}(x\delta_x)$ for $f\in C_0^*$. Then, $(T_t\hat{f})|_S\in C_0^*$ and*

$$T_t\hat{f}(\mu)=\widehat{(T_t\hat{f})|_S}(\mu) \tag{2.1}$$

for all $\mu\in\boldsymbol{M}_1$.

Proof. Put $\mu|_\varepsilon=\sum_{x_i\geqq\varepsilon}x_i\delta_{x_i}$ for any $\mu=\sum_i x_i\delta_{x_i}\in\boldsymbol{M}_1$, then $\mu|_\varepsilon\to\mu$ as $\varepsilon\downarrow 0$ and

$$T_t\hat{f}(\mu)=\lim_{\varepsilon\downarrow 0} T_t\hat{f}(\mu|_\varepsilon)$$

by $T_t\hat{f}\in C(\boldsymbol{M}_1)$. Since the branching property (b) implies

$$T_t\hat{f}(\mu|_\varepsilon)=\prod_{x_i\geqq\varepsilon} T_t\hat{f}(x_i\delta_{x_i}),$$

we have

$$T_t\hat{f}(\mu)=\prod_i T_t\hat{f}(x_i\delta_{x_i}). \tag{2.2}$$

On the other hand, there exists $0<\varepsilon<1$ for each $f\in C_0^*$ such that $f(x)=1$ for

$0<x<\varepsilon$. Since

$$T_t\hat{f}(x\delta_x)=T_t\widehat{\theta_x f}(\delta_1)$$
$$=E_{\delta_1}[\widehat{\theta_x f}(\mu_t)]$$
$$=E_{\delta_1}\Big[\exp\Big(\int_S \frac{1}{y}\log f(xy)\mu_t(dy)\Big)\Big],$$

it follows $T_t\hat{f}(x\delta_x)=1$ if $0<x<\varepsilon$. Therefore $(T_t\hat{f})|_S\in C_0^*$ follows from the continuity of the mapping $x\to x\delta_x$. Thus, (2. 1) follows from (2. 2) and

$$(\widehat{T_t\hat{f}})|_S(\mu)=\prod_i (T_t\hat{f})|_S(x_i)=\prod_i T_t\hat{f}(x_i\delta_{x_i})$$

by Lemma 1. 1 (iii).

It is shown that the total mass in a cascade process does not exceed the initial one.

PROPOSITION 2. 1. *For any* $\mu\in \boldsymbol{M}_1$,

(2. 3) $$P_\mu(\|\mu_t\|\leqq\|\mu\| \text{ for all } t\geqq 0)=1.$$

Proof. Take a sequence $\{\varphi_n\}\subset C_0$ such as $\varphi_n\geqq 0$ and $\varphi_n\uparrow 1$ for $n\to\infty$, and put

$$f_n(x)=e^{\lambda x\varphi_n(x)}.$$

Then, since $f_n\in C_0^*$ if $\lambda\leqq 0$, it holds

$$T_t\hat{f}_n(\mu+\nu)=T_t\hat{f}_n(\mu)T_t\hat{f}_n(\nu)$$

and

$$T_t\hat{f}_n(a\delta_a)=T_t\widehat{\theta_a f_n}(\delta_1)$$

for any $\lambda\leqq 0$. Moreover, since $T_t\hat{f}_n(\mu)$ turns out to be an analytic function of λ in the whole complex plane from the expression:

$$T_t\hat{f}_n(\mu)=E_\mu\Big[\exp\Big(\int_S \lambda\varphi_n(x)\mu_t(dx)\Big)\Big]=E_\mu[e^{\lambda\langle\varphi_n,\mu_t\rangle}],$$

the above two equations hold for all real λ. Thus we have

$$T_t\hat{f}_n(\mu)=\prod_i T_t\widehat{\theta_{x_i} f_n}(\delta_1)$$

for any $\mu=\sum_i x_i\delta_{x_i}\in\boldsymbol{M}_1$ and any real λ because $\hat{f}_n\in C(\boldsymbol{M}_1)$ implies $T_t\hat{f}_n\in C(\boldsymbol{M}_1)$.

Since

$$\widehat{\theta_a f_n}(\mu)=\exp\Big(a\lambda\int_S \varphi_n(ax)\mu(dx)\Big)\leqq e^{\lambda a\|\mu\|}$$

for $\lambda>0$,

$$\prod_i T_t\widehat{\theta_{x_i}f_n}(\delta_1)=\prod_i E_{\delta_1}[\widehat{\theta_{x_i}f_n}(\mu_t)]\leqq\prod_i E_{\delta_1}[e^{x_i\lambda||\mu_t||}]$$
$$\leqq\prod_i E_{\delta_1}[e^{x_i\lambda}]=e^{\lambda\Sigma_i x_i}=e^{\lambda||\mu||}$$

for $\lambda>0$. On the other hand,

$$T_t\hat{f}_n(\mu)=E_\mu[e^{\lambda\langle\varphi_n,\mu_t\rangle}]\uparrow E_\mu[e^{\lambda||\mu_t||}]$$

as $n\to\infty$ for $\lambda>0$. Therefore we have

$$E_\mu[e^{\lambda||\mu_t||}]\leqq e^{\lambda||\mu||},$$

or

$$E_\mu[e^{\lambda(||\mu_t||-||\mu||)}]\leqq 1$$

for all $\lambda>0$. Thus, letting $\lambda\to\infty$, we have

$$P_\mu(||\mu_t||-||\mu||>0)=0,$$

or

$$P_\mu(||\mu_t||\leqq||\mu||)=1$$

for all $t\geqq 0$. Moreover, by the right continuity of $\mu_t(w)$, we have

$$P_\mu(||\mu_t||\leqq||\mu|| \text{ for all } t\geqq 0)=1.$$

The branching property may be extended in Proposition 2.2 below, but for this it needs the following

LEMMA 2.2. *For any integer* $n\geqq 1$, $f_1, f_2, \cdots, f_n\in C_0^*$, *and* $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n<\infty$,

$$E_{\mu+\nu}\left[\prod_{i=1}^n \hat{f}_i(\mu_{t_i})\right]=E_\mu\left[\prod_{i=1}^n \hat{f}_i(\mu_{t_i})\right]\cdot E_\nu\left[\prod_{i=1}^n \hat{f}_i(\mu_{t_i})\right] \tag{2.4}$$

if μ, ν *and* $\mu+\nu\in \boldsymbol{M}_1$.

Proof. (2.4) is reduced to the branching property for $n=1$. Since, by the Markov property at time t_{n-1},

$$E_{\mu+\nu}\left[\prod_{i=1}^n \hat{f}_i(\mu_{t_i})\right]=E_{\mu+\nu}\left[\prod_{i=1}^{n-1}\hat{f}_i(\mu_{t_i})E_{\mu_{t_{n-1}}}\hat{f}_n(\mu_{t_n-t_{n-1}})\right]$$

where by Lemma 2.1

$$E_{\mu_{t_{n-1}}}[\hat{f}_n(\mu_{t_n-t_{n-1}})]=T_{t_n-t_{n-1}}\hat{f}_n(\mu_{t_{n-1}})=\widehat{(T_{t_n-t_{n-1}}\hat{f}_n)|S}(\mu_{t_{n-1}}),$$

and

$$E_{\mu+\nu}\left[\prod_{i=1}^{n}\hat{f}_i(\mu_{t_i})\right]=E_{\mu+\nu}\left[\prod_{i=1}^{n-2}\hat{f}_i(\mu_{t_i})\hat{f}_{n-1}(\mu_{t_{n-1}})\widehat{(T_{t_n-t_{n-1}}\hat{f}_n)}|s(\mu_{t_{n-1}})\right]$$

$$=E_{\mu+\nu}\left[\prod_{i=1}^{n-2}\hat{f}_i(\mu_{t_i})\widehat{f_{n-1}(T_{t_n-t_{n-1}}\hat{f}_n)}|s(\mu_{t_{n-1}})\right],$$

we have, if we assume (2. 4) for $n-1$,

$$E_{\mu+\nu}\left[\prod_{i=1}^{n}\hat{f}_i(\mu_{t_i})\right]=E_{\mu}\left[\prod_{i=1}^{n-2}\hat{f}_i(\mu_{t_i})\widehat{f_{n-1}(T_{t_n-t_{n-1}}\hat{f}_n)}|s(\mu_{t_{n-1}})\right]$$

$$\times E_{\nu}\left[\prod_{i=1}^{n-2}\hat{f}_i(\mu_{t_i})\widehat{f_{n-1}(T_{t_n-t_{n-1}}\hat{f}_n)}|s(\mu_{t_{n-1}})\right]$$

$$=E_{\mu}\left[\prod_{i=1}^{n}\hat{f}_i(\mu_{t_i})\right]E_{\nu}\left[\prod_{i=1}^{n}\hat{f}_i(\mu_{t_i})\right].$$

Thus the proof is completed by induction.

We now define $w=w_1+w_2\in W$ for $w_1, w_2\in W$ by $\mu_t(w)=\mu_t(w_1)+\mu_t(w_2)$, $t\geqq 0$ if it belongs to $\boldsymbol{M}_1$ for all $t\geqq 0$.

Then we have the following

PROPOSITION 2. 2. *For any bounded $\mathfrak{N}_\infty$-measurable function G,*

$$E_{\mu+\nu}[G(w)]=E_{\mu}^{(1)}\otimes E_{\nu}^{(2)}[G(w_1+w_2)] \tag{2. 5}$$

if μ, ν and $\mu+\nu\in\boldsymbol{M}_1$, where $P_{\mu}^{(1)}=P_{\mu}$, $P_{\nu}^{(2)}=P_{\nu}$, and $E_{\mu}^{(1)}\otimes E_{\nu}^{(2)}$ denotes the expectation by the product measure $P_{\mu}^{(1)}\otimes P_{\nu}^{(2)}$.

Proof. For $f_1, f_2, \cdots, f_n\in C_0^*$ and $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n<\infty$, we have, by Lemma 2. 2,

$$E_{\mu+\nu}[\hat{f}_1(\mu_{t_1}(w))\cdots\hat{f}_n(\mu_{t_n}(w))]$$

$$=E_{\mu}^{(1)}\otimes E_{\nu}^{(2)}[\hat{f}_1(\mu_{t_1}(w_1))\cdots\hat{f}_n(\mu_{t_n}(w_1))\hat{f}_1(\mu_{t_1}(w_2))\cdots\hat{f}_n(\mu_{t_n}(w_2))]$$

$$=E_{\mu}^{(1)}\otimes E_{\nu}^{(2)}[\hat{f}_1(\mu_{t_1}(w_1)+\mu_{t_1}(w_2))\cdots\hat{f}_n(\mu_{t_n}(w_1)+\mu_{t_n}(w_2))]$$

$$=E_{\mu}^{(1)}\otimes E_{\nu}^{(2)}[\hat{f}_1(\mu_{t_1}(w_1+w_2))\cdots\hat{f}_n(\mu_{t_n}(w_1+w_2))],$$

where we note that $w_1+w_2\in W$, $P_{\mu}^{(1)}\otimes P_{\nu}^{(2)}$-a. s. because $||\mu_t(w_1)||\leqq||\mu||$ and $||\mu_t(w_2)||\leqq||\nu||$ by Proposition 2. 1, and hence $||\mu_t(w_1)+\mu_t(w_2)||\leqq||\mu||+||\nu||=||\mu+\nu||\leqq 1$ $P_{\mu}^{(1)}\otimes P_{\nu}^{(2)}$-a. s. for all $t\geqq 0$ since $\mu+\nu\in\boldsymbol{M}_1$.

Therefore, by Proposition 1. 2, (2. 5) holds for the function $G(w)=F_1(\mu_{t_1}(w))F_2(\mu_{t_2}(w))\cdots F_n(\mu_{t_n}(w))$ where $F_1, F_2, \cdots, F_n\in C(\boldsymbol{M}_1)$. Thus we have (2. 5) for any bounded $\mathfrak{N}_\infty$-measurable function G by the standard argument.

From this proposition, we can see the probabilistic meaning of the branching property which is originally defined by means of a semigroup $\{T_t\}$: there are no interactions between particles in a cascade process and they move independently each other and obey the same probability law.

The notion of homogeneity can also be expressed by (2. 7) in the same way as Proposition 2. 2. We first show the following

LEMMA 2. 3. *For any integer* $n\geqq 1$, $f_1, f_2, \cdots, f_n \in C_0^*$, *and* $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n<\infty$.

$$E_{a\cdot\mu}\left[\prod_{i=1}^{n}\hat{f}_i(\mu_{t_i})\right]=E_\mu\left[\prod_{i=1}^{n}\hat{f}_i(a\cdot\mu_{t_i})\right] \tag{2. 6}$$

if $a\in S$ *and* $\mu\in \boldsymbol{M}_1$.

Proof. Since, by Lemma 1. 2 (ii) and Lemma 2. 1,

$$E_{a\cdot\mu}[\hat{f}_1(\mu_{t_1})]=T_{t_1}\hat{f}_1(a\cdot\mu)=(\widehat{T_{t_1}\hat{f}_1)|s}(a\cdot\mu)=\widehat{\theta_a(T_{t_1}\hat{f}_1)|s}(\mu)$$

and

$$\theta_a(T_{t_1}\hat{f}_1)|s(x)=(T_{t_1}\hat{f}_1)s(ax)=T_{t_1}\hat{f}_1(ax\delta_{ax})=T_{t_1}\widehat{\theta_{ax}f_1}(\delta_1)$$

$$=T_{t_1}\widehat{\theta_x\theta_a f_1}(\delta_1)=T_{t_1}\widehat{\theta_a f_1}(x\delta_x)=(T_{t_1}\widehat{\theta_a f_1})|s(x),$$

$$E_{a\cdot\mu}[\hat{f}_1(\mu_{t_1})]=\widehat{(T_{t_1}\widehat{\theta_a f_1})|s}(\mu)=T_{t_1}\widehat{\theta_a f_1}(\mu)$$

$$=E_\mu[\widehat{\theta_a f_1}(\mu_{t_1})]=E_\mu[\hat{f}_1(a\cdot\mu_{t_1})].$$

Thus (2. 6) holds for $n=1$. Then (2. 6) is verified by induction for all $n\geqq 1$ as follows. Assume (2. 6) for $n-1$. Then

$$E_{a\cdot\mu}\left[\prod_{i=1}^{n}\hat{f}_i(\mu_{t_i})\right]=E_{a\cdot\mu}\left[\prod_{i=1}^{n-1}\hat{f}_i(\mu_{t_i})E_{\mu_{t_{n-1}}}\hat{f}_n(\mu_{t_n-t_{n-1}})\right]$$

$$=E_{a\cdot\mu}\left[\prod_{i=1}^{n-2}\hat{f}_i(\mu_{t_i})\widehat{f_{n-1}(T_{t_n-t_{n-1}}\hat{f}_n)|s}(\mu_{t_{n-1}})\right]$$

$$=E_\mu\left[\prod_{i=1}^{n-2}\hat{f}_i(a\cdot\mu_{t_i})\widehat{f_{n-1}(T_{t_n-t_{n-1}}\hat{f}_n)|s}(a\cdot\mu_{t_{n-1}})\right]$$

$$=E_\mu\left[\prod_{i=1}^{n-1}\hat{f}_i(a\cdot\mu_{t_i})T_{t_n-t_{n-1}}\hat{f}_n(a\cdot\mu_{t_{n-1}})\right]$$

$$=E_\mu\left[\prod_{i=1}^{n-1}\hat{f}_i(a\cdot\mu_{t_i})E_{\mu_{t_{n-1}}}[\hat{f}_n(a\cdot\mu_{t_n-t_{n-1}})]\right]$$

$$=E_\mu\left[\prod_{i=1}^{n-1}\hat{f}_i(a\cdot\mu_{t_i})\hat{f}_n(a\cdot\mu_{t_n})\right],$$

and the proof is completed.

Define $a \cdot w \in W$ for $a \in S$ and $w \in W$ by $\mu_t(a \cdot w) = a \cdot \mu_t(w)$, then we have

PROPOSITION 2. 3. *For any bounded $\mathcal{N}_\infty$-measurable function G,*

$$E_{a\cdot\mu}[G(w)] = E_\mu[G(a \cdot w)] \tag{2. 7}$$

if $a \in S$ and $\mu \in \boldsymbol{M}_1$.

Proof of Proposition 2. 3 is completed in the same way as that of Proposition 2. 2 making use of Lemma 2. 3.

This proposition suggests a homogeneity property of a cascade process.

Put $M_t = M(\mu_t)$ where $M(\mu)$ is defined by (1. 4). Then M_t is right continuous and has left limit at each $t \geqq 0$ because of Lemma 1. 3. We shall denote $P_a = P_{a\delta_a}$ and $E_a = E_{a\delta_a}$, $a \in S$, from now on.

PROPOSITION 2. 4. *For all $\mu \in \boldsymbol{M}_1$,*

$$P_\mu(M_t \textit{ is non-increasing for all } t \geqq 0) = 1. \tag{2. 8}$$

Proof. When $\mu = a\delta_a$, $P_a(||\mu_t|| \leqq a) = 1$ by (2. 3) and hence $P_a(M_t \leqq M_0) = 1$. When $\mu = a\delta_a + b\delta_b$, we have, by Proposition 2. 2,

$$\begin{aligned} P_\mu(M_t \leqq M_0) &= P_{a\delta_a + b\delta_b}(M_t \leqq (a \vee b)) = P_a^{(1)} \otimes P_b^{(2)}(M_t(w_1 + w_2) \leqq (a \vee b)) \\ &= P_a^{(1)} \otimes P_b^{(2)}(M_t(w_1) \leqq (a \vee b),\ M_t(w_2) \leqq (a \vee b)) \\ &= P_a(M_t \leqq (a \vee b)) P_b(M_t \leqq (a \vee b)) \geqq P_a(M_t \leqq a) P_b(M_t \leqq b) \end{aligned}$$

and hence

$$P_\mu(M_t \leqq M_0) = 1.$$

In the same way, $P_\mu(M_t \leqq M_0) = 1$ holds for $\mu = \sum_{i=1}^n x_i \delta_{x_i}$. For any $\mu \in \boldsymbol{M}_1$, take $\varepsilon > 0$ such that $\mu((0, \varepsilon]) \leqq M(\mu)$, then we can write as $\mu = \sum_{x_i \geqq \varepsilon} x_i \delta_{x_i} + \mu_0$ where $||\mu_0|| \leqq M(\mu)$. Since, then,

$$\begin{aligned} P_\mu(M_t \leqq M_0) &= P_\mu(M_t \leqq M(\mu)) = P_{\sum_{x_i \geqq \varepsilon} x_i \delta_{x_i}}(M_t \leqq M(\mu)) \cdot P_{\mu_0}(M_t \leqq M(\mu)) \\ &= P_{\mu_0}(M_t \leqq M(\mu)) \geqq P_{\mu_0}(M_t \leqq ||\mu_0||) \geqq P_{\mu_0}(||\mu_t|| \leqq ||\mu_0||) = 1, \end{aligned}$$

we have $P_\mu(M_t \leqq M_0) = 1$ for any $\mu \in \boldsymbol{M}_1$.

Now, by the Markov property,

$$P_\mu(M_s \geqq M_t) = E_\mu[P_{\mu_s}(M_0 \geqq M_{t-s})] = 1, \qquad 0 \leqq s \leqq t < \infty$$

and hence we have

$$P_\mu(M_s \geqq M_t, \quad 0 \leqq s \leqq t < \infty) = 1$$

by the right continuity of M_t.

Put $W'=\{w\in W;\ M_t(w)$ is non-increasing for all $t\geqq 0\}$, then $W'\in\mathcal{N}_\infty$ and hence, by Proposition 2.4, we can restrict the sample space W to W'. Thus we take W' as our sample space from now on, writing it again W.

We now define a Markov time τ_d which will play an important role in studying a cascade process.

Let d be $2/3<d<1$, and define

$$\tau_d(w)=\begin{cases}\inf\left\{s;\ \dfrac{M_s(w)}{M_0(w)}\leqq d\right\},\\ +\infty \quad \text{if } \{\cdots\}=\phi.\end{cases} \tag{2.9}$$

Since

$$\{\tau_d\leqq t\}=\left\{\frac{M_t}{M_0}\leqq d\right\}\in\mathcal{N}_t,$$

τ_d is an $\mathcal{N}_t$-Markov time. Moreover it satisfies $\tau_d(a\cdot w)=\tau_d(w)$ for any $a\in S$. For any $0\leqq\varepsilon\leqq 1$, define

$$\sigma_\varepsilon(w)=\begin{cases}\inf\ \{s;\ M_s(w)\leqq\varepsilon\},\\ +\infty \quad \text{if } \{\cdots\}=\phi,\end{cases} \tag{2.10}$$

then σ_ε is also an $\mathcal{N}_t$-Markov time and the following will be usefull later:

$$\begin{aligned}\tau_d(w)&=t+\tau_{p(w)}(w_t^+), \quad p(w)=\frac{M_0(w)}{M_t(w)}d,\\ &=t+\sigma_{q(w)}(w_t^+), \quad q(w)=M_0(w)d,\end{aligned} \tag{2.11}$$

if $t<\tau_d(w)<\infty$, where $d\leqq p(w)<1$.

We can assume τ_d is finite except a trivial case. To see this, we first show the following

LEMMA 2.4. *If* $P_1(\tau_d=\infty)>0$ *for all* d $(2/3<d<1)$, *then each* $\mu\in\boldsymbol{M}_1$ *is a trap, i.e.* $P_\mu(\mu_t=\mu$ *for all* $t\geqq 0)=1$.

Proof. Put $\alpha=P_1(\tau_d=\infty)$. Then

$$\begin{aligned}\alpha&=P_1(M_s>d \text{ for all } s\geqq 0)\\ &\leqq P_1(M_{s+t}>d \text{ for all } s\geqq 0)\\ &=E_1[M_t>d;\ P_{\mu_t}(M_s>d \text{ for all } s\geqq 0)]\end{aligned}$$

for any $t\geqq 0$. Putting $\mu_t=M_t\delta_{M_t}+\mu'$, we have

$$\begin{aligned}&P_{\mu_t}(M_s>d \text{ for all } s\geqq 0)\\ &=P^{(1)}_{M_t}\otimes P^{(2)}_{\mu'}(M_s(w_1+w_2)>d \text{ for all } s\geqq 0)\end{aligned}$$

$$=P^{(1)}_{M_t}\otimes P^{(2)}_{\mu'}(M_s(w_1)>d \text{ for all } s\geqq 0)$$

$$=P_{M_t}(M_s>d \text{ for all } s\geqq 0)$$

since $M_s(w_2)\leqq M(\mu')<1-d<d$. Moreover,

$$P_a(M_s>d \text{ for all } s\geqq 0)$$

$$=P_1(M_s(a\cdot w)>d \text{ for all } s\geqq 0)$$

$$=P_1(aM_s(w)>d \text{ for all } s\geqq 0)$$

$$\leqq P_1(M_s>d \text{ for all } s\geqq 0)=\alpha$$

for any $a\in S$. Thus we have

$$\alpha\leqq E_1[M_t>d;\ P_{M_t}(M_s>d \text{ for all } s\geqq 0)]<\alpha P_1(M_t>d).$$

Therefore, $\alpha>0$ for all d $(2/3<d<1)$ implies $P_1(M_t>d)=1$ for all d $(2/3<d<1)$ and hence $P_1(M_t=1)=1$. Since

$$P_a(\mu_t=a\delta_a)=P_a(M_t=a)=P_1(M_t=1)=1,$$

we have $T_t\hat{f}(a\delta_a)=f(a)$ for all $f\in C_0^*$, and, by the branching property,

$$T_t\hat{f}(\mu)=(\widehat{T_t\hat{f}})|_S(\mu)=\hat{f}(\mu)$$

for all $\mu\in \boldsymbol{M}_1$. Therefore $T_t=I$ (identity), which concludes the proof.

PROPOSITION 2.5. *It holds* $P_\mu(\tau_d<\infty)=1$ *for any* d $(2/3<d<1)$ *and any* $\mu\in\boldsymbol{M}_1-\{0\}$, *except the case where each* $\mu\in\boldsymbol{M}_1$ *is a trap.*

Proof. If any $\mu\in\boldsymbol{M}_1-\{0\}$ is not a trap, there exists d_0 $(2/3<d_0<1)$ such that $P_1(\tau_{d_0}<\infty)=1$, by Lemma 2.4. Since $\tau_d\leqq\tau_{d_0}$ for $d\geqq d_0$, $P_1(\tau_d<\infty)=1$ for any d $(d_0\leqq d<1)$. Clearly $P_a(\tau_d<\infty)=P_1(\tau_d<\infty)=1$ for $a\in S$, and

$$P_{a\delta_a+b\delta_b}(\tau_d<\infty)=P^{(1)}_a\otimes P^{(2)}_b(\tau_d(w_1+w_2)<\infty)$$

$$\geqq P^{(1)}_a\otimes P^{(2)}_b(\tau_d(w_1)<\infty,\ \tau_d(w_2)<\infty)$$

$$=P_a(\tau_d<\infty)P_b(\tau_d<\infty)=1.$$

Thus, by the same argument it holds $P_\mu(\tau_d<\infty)=1$ for $\mu=\sum_{i=1}^n x_i\delta_{x_i}$. For any $\mu\in\boldsymbol{M}_1-\{0\}$, we write $\mu=\sum x_i\delta_{x_i}=\mu_d+\mu'$ where $\mu_d=\sum_{x_i\geqq M(\mu)d} x_i\delta_{x_i}$ and $M(\mu')<M(\mu)d$. Then

$$P_\mu(\tau_d<\infty)=P_\mu(\sigma_{M(\mu)d}<\infty)=P^{(1)}_{\mu_d}\otimes P^{(2)}_{\mu'}(\sigma_{M(\mu)d}(w_1)\vee\sigma_{M(\mu)d}(w_2)<\infty)$$

$$=P_{\mu_d}(\sigma_{M(\mu)d}<\infty)P_{\mu'}(\sigma_{M(\mu)d}<\infty)=P_{\mu_d}(\tau_d<\infty)=1$$

for any d $(d_0\leqq d<1)$.

Since $\tau_{d^2}(w)\leqq\tau_d(w)+\tau_d(w^+_{\tau_d})$ if $\mu_{\tau_d}\neq 0$, and $\tau_{d^2}(w)=\tau_d(w)$ if $\tau_d(w)<\infty$ and $\mu_{\tau_d(w)}(w)=0$,

$$P(\tau_{d^2}<\infty)\geqq P_\mu(\{w;\ \tau_d(w)+\tau_d(w^+_{\tau_d})<\infty\})+P_\mu(\tau_d<\infty,\ \mu_{\tau_d}=0)$$

$$=E_\mu[\tau_d<\infty,\ \mu_{\tau_d}\neq 0:\ P_{\mu_{\tau_d}}(\tau_d<\infty)]+P_\mu(\tau_d<\infty,\ \mu_{\tau_d}=0).$$

Therefore, if $d_0\leqq d<1$, we have

$$P_\mu(\tau_{d^2}<\infty)\geqq P_\mu(\tau_d<\infty,\ \mu_{\tau_d}\neq 0)+P_\mu(\tau_d<\infty,\ \mu_{\tau_d}=0)$$

$$=P_\mu(\tau_d<\infty)=1$$

for any $\mu\in\boldsymbol{M}_1-\{0\}$. Thus we have $P_\mu(\tau_{d^n}<\infty)=1$ for any d $(d_0\leqq d<1)$ and any $\mu\in\boldsymbol{M}_1-\{0\}$, and hence $P_\mu(\tau_d<\infty)=1$ for any d $(2/3<d<1)$ and $\mu\in\boldsymbol{M}_1-\{0\}$.

By Proposition 2. 5, we assume $P_\mu(\tau_d<\infty)=1$ for any d $(2/3<d<1)$ and any $\mu\in\boldsymbol{M}_1-\{0\}$ from now on. We remark that it follows from the last part of the proof of Proposition 2. 5 that M_t decreases to zero and hence μ_t converges to zero almost surely when t tends to infinity.

Since M_t is right continuous, it is obvious that $P_\mu(\tau_d>0)=1$ and hence it can be made $P_\mu(\tau_d\leqq t_0)<1$, taking $t_0>0$ to be sufficiently small. Moreover we have the following

LEMMA 2. 5. *Set*

$$\begin{cases} q_1(t)=P_x(\tau_d\leqq t), \\ q_{n+1}(t)=E_x[q_n(t-\tau_d);\ t\geqq\tau_d], \qquad n\geqq 1. \end{cases} \tag{2. 12}$$

Then $q_n(t)$ does not depend on $x\in S$ and for any $t>0$ it holds $q_n(t)<1$ for sufficiently large n.

Proof. By Proposition 2. 3 and $\tau_d(a\cdot w)=\tau_d(w)$, it is clear that $q_n(t)$ does not depend on x. By definition $q_n(t)$ is non-decreasing in $t\geqq 0$ and non-increasing in $n\geqq 1$. Take $t_0>0$ such as $q_1(t_0)<1$. Then we show $q_n(n\,t_0)<1$ for all $n\geqq 1$ by induction as follows. Since

$$\begin{aligned} q_{n+1}((n+1)t_0)&=E_x[q_n((n+1)t_0-\tau_d);\ (n+1)t_0\geqq\tau_d] \\ &=E_x[q_n((n+1)t_0-\tau_d);\ t_0<\tau_d\leqq(n+1)t_0] \\ &\quad+E_x[q_n((n+1)t_0-\tau_d);\ \tau_d\leqq t_0] \\ &\leqq q_n(n\,t_0)P_x(t_0<\tau_d\leqq(n+1)t_0)+P_x(\tau_d\leqq t_0), \end{aligned}$$

we have, by assuming $q_n(n\,t_0)<1$,

$$q_{n+1}((n+1)t_0)\leqq P_x(\tau_d\leqq t_0)<1$$

when $P_x(t_0<\tau_d\leqq(n+1)t_0)=0,$

and

$$q_{n+1}((n+1)t_0) < P_x(t_0 < \tau_d \leqq (n+1)t_0) + P_x(\tau_d \leqq t_0)$$
$$= P_x(\tau_d \leqq (n+1)t_0) \leqq 1$$

when $P_x(t_0 < \tau_d \leqq (n+1)t_0) > 0$, and hence $q_{n+1}((n+1)t_0) < 1$.

Therefore, for a given $t > 0$, taking n large enough to satisfy $t \leqq n\, t_0$, we have

$$q_n(t) \leqq q_n(n\, t_0) < 1.$$

LEMMA 2.6. *The nonnegative bounded function $u(t)$ which satisfies:*

$$u(t) \leqq E_x[u(t-\tau_d);\ t \geqq \tau_d], \qquad 0 \leqq t \leqq T \tag{2.13}$$

is necessarily zero for any fixed $T > 0$.

Proof. Put $\|u\| = \sup_{0 \leqq t \leqq T} |u(t)|$. Then

$$u(t) \leqq \|u\| P_x(t \geqq \tau_d) = \|u\| q_1(t), \qquad 0 \leqq t \leqq T.$$

Now, assume $|u(t)| \leqq \|u\| q_n(t)$, $0 \leqq t \leqq T$. Then

$$|u(t)| \leqq \|u\| E_x[q_n(t-\tau_d);\ t \geqq \tau_d]$$
$$= \|u\| q_{n+1}(t), \qquad 0 \leqq t \leqq T.$$

Thus we have $\|u\| \leqq \|u\| q_n(T)$ for all $n \geqq 1$ and hence $u = 0$ since $q_n(T) < 1$ for sufficiently large n by Lemma 2.5.

§ 3. Underlying process.

The process (M_t) which is obtained by tracing out a particle with the maximum energy (or mass, etc.) of a given cascade process (μ_t) is not generally a Markov process. However, we can obtain a nice Markov process (x_t) on $\bar{S} = [0, 1]$ which is equivalent to the process (M_t) till the time τ_d. We shall call the Markov process (x_t) the underlying process of a given cascade process (μ_t), because it can be considered to represent the mode of movement of each particle of which the cascade process consists.

In this section we shall construct the Markov process (x_t) and prove some properties of it, especially the relation between the processes (x_t) and (M_t).

We will fix d $(2/3 < d < 1)$ and put $\tau_d = \tau$ in what follows. Set $B(\bar{S})$ the set of all bounded Borel functions on $\bar{S}$ and $B^+(\bar{S})$ the set of all nonnegative functions in $B(\bar{S})$.

For $f \in B^+(\bar{S})$, we define $u_n(t, x; f)$ $(n = 0, 1, 2, \cdots)$ successively by

$$\begin{cases} u_0(t, x; f) = 0, \\ u_{n+1}(t, x; f) = E_x[f(M_t):\ t < \tau] + E_x[u_n(t-\tau, M_\tau; f):\ t \geqq \tau] \end{cases} \tag{3.1}$$

for $n\geqq 0$. Then it is plain to see

$$0\leqq u_n\leqq u_{n+1}\leqq\|f\| \qquad \text{for} \quad n\geqq 0,$$

where $\|f\|=\sup_{x\in\bar{S}}|f(x)|$, and the limit

$$u(t, x; f)=\lim_{n\to\infty} u_n(t, x; f)$$

is a solution of the following equation:

$$u(t, x; f)=E_x[f(M_t): t<\tau]+E_x[u(t-\tau, M_\tau; f): t\geqq\tau]. \tag{3. 2}$$

We now define an operator T_t^d on $B(\bar{S})$ by

$$T_t^d f(x)=u(t, x; f^+)-u(t, x; f^-),$$

where $f^+(x)=\max(f(x), 0)$ and $f^-(x)=\max(-f(x), 0)$. Then we have the following

LEMMA 3. 1. *$T_t^d f(x)$ is a unique bounded solution of* (3. 2) *for all $f\in B(\bar{S})$, and each of T_t^d is a nonnegative contraction linear operator on $B(\bar{S})$ such that $T_0^d=I$ and $T_t^d 1=1$.*

Proof. The uniqueness of a solution of (3. 2) is as follows. For $f\in B(\bar{S})$, let $u(t, x)$ and $\tilde{u}(t, x)$ be two bounded solutions of (3. 2). Then

$$u(t, x)-\tilde{u}(t, x)=E_x[u(t-\tau, M_\tau)-\tilde{u}(t-\tau, M_\tau): t\geqq\tau].$$

Set $v(t)=\sup_{x\in\bar{S}}|u(t, x)-\tilde{u}(t, x)|$, we have

$$|u(t, x)-\tilde{u}(t, x)|\leqq E_x[v(t-\tau): t\geqq\tau],$$

and, since the right-hand side is independent of x,

$$v(t)\leqq E_x[v(t-\tau): t\geqq\tau].$$

Thus, by Lemma 2. 6, we have $v(t)=0$, and hence $u(t, x)=\tilde{u}(t, x)$.

Therefore $T_t^d f(x)$ is a unique bounded solution of (3. 2), and the rest of the lemma is obvious from the definition of T_t^d.

Now, let C_d be the set of all continuous functions on $\bar{S}$ such that $f(x)=0$ for $0\leqq x\leqq(1-d)/d$. Then $C_d\subset C_{d'}$ if $d\leqq d'$.

LEMMA 3. 2. (i) *For $f\in B(\bar{S})$ and $a\in\bar{S}$,*

$$\theta_a T_t^d f(x)=T_t^d\theta_a f(x), \qquad x\in\bar{S}, \tag{3. 3}$$

(ii) *T_t^d maps $C(\bar{S})$ into $C(\bar{S})$, especially it maps C_d into C_d.*

Proof. (i) Since $T_t^d f(0)=f(0)$, (3. 3) is clear for $a=0$. Let $a>0$, then from the homogeneity of the cascade process,

$$T_t^d f(ax) = E_{ax}[f(M_t): t<\tau] + E_{ax}[T_{t-\tau}^d f(M_\tau): t \geqq \tau]$$
$$= E_x[f(M_t(a\cdot w)): t<\tau(a\cdot w)] + E_x[T_{t-\tau(a\cdot w)}^d f(M_{\tau(a\cdot w)}(a\cdot w)): t \geqq \tau(a\cdot w)]$$
$$= E_x[f(aM_t): t<\tau] + E_x[T_{t-\tau}^d f(aM_\tau): t\geqq \tau].$$

Thus we have

$$\theta_a T_t^d f(x) = E_x[\theta_a f(M_t): t<\tau] + E_x[\theta_a T_{t-\tau}^d f(M_\tau): t \geqq \tau],$$

and hence $\theta_a T_t^d f(x) = T_t^d \theta_a f(x)$, by the uniqueness of a solution of (3. 2).

(ii) If $B(\bar{S}) \ni f_n \downarrow 0$, $T_t^d f_n \downarrow 0$ because we have from (3. 2),

$$\lim_{n\to\infty} T_t^d f_n(x) = E_x[\lim_{n\to\infty} T_{t-\tau}^d f_n(M_\tau): t \geqq \tau]$$

and $\lim_{n\to\infty} T_t^d f_n(x) = 0$ by Lemma 2. 6. Since, moreover, T_t^d is a linear operator on $B(\bar{S})$, there exists a system of probability measures $\{P_t^d(x, dy)\}$ such that $T_t^d f(x)$ can be expressed by

$$T_t^d f(x) = \int_{\bar{S}} f(y) P_t^d(x, dy), \qquad f \in B(\bar{S}).$$

Since, by (3. 3),

$$T_t^d f(x) = T_t^d \theta_x f(1) = \int \theta_x f(y) P_t^d(1, dy) = \int f(xy) P_t^d(1, dy),$$

we have $T_t^d f \in C(\bar{S})$ if $f \in C(\bar{S})$.

For the case $f \in C_d$, we have $T_t^d f \in C_d$ by the definition because M_t is non-increasing.

LEMMA 3. 3. *When* $2/3 < d \leqq d' < 1$,

$$T_t^{d'} f = T_t^d f \tag{3. 4}$$

for all $f \in C_d$.

Proof. Put $\tau_d = \tau$ and $\tau_{d'} = \tau'$. Since a solution of (3. 2) is unique, it is sufficient to show that $T_t^d f(x)$ for $f \geqq 0$ is a solution of the following equation:

$$u(t, x) = E_x[f(M_t): t<\tau'] + E_x[u(t-\tau', M_{\tau'}): t \geqq \tau']. \tag{*}$$

To begin with, we recall that $T_t^d f(x) = \lim_{n\to\infty} u_n(t, x)$, where

$$u_{n+1}(t, x) = E_x[f(M_t): t<\tau] + E_x[u_n(t-\tau, M_\tau): t \geqq \tau].$$

Since $\tau' \leqq \tau$,

$$I \equiv E_x[f(M_t): t<\tau] = E_x[f(M_t): t<\tau'] + E_x[f(M_t): \tau' \leqq t < \tau].$$

Since $\tau(w) = \tau'(w) + \sigma_{xd}(w_{\tau'}^+)$ when $\tau'(w) < \tau(w)$, we have, by a strong Markov property of μ_t,

$$E_x[f(M_t):\ \tau'\leqq t<\tau]=E_x[E_{\mu_{\tau'}}[f(M_{t-s}):\ t-s<\sigma_{xd}]_{s=\tau'}:\ \tau'\leqq t,\ \tau'<\tau],$$

where, if we put $\mu_{\tau'}=M_{\tau'}\delta_{M_{\tau'}}+\mu'$, then $M(\mu')<x(1-d)$, and hence

$$\begin{aligned}&E_{\mu_{\tau'}}[f(M_{t-s}):\ t-s<\sigma_{xd}]\\ &=E^{(1)}_{M_{\tau'}}\otimes E^{(2)}_{\mu'}[f(M_{t-s}(w_1+w_2)):\ t-s<\sigma_{xd}(w_1+w_2)]\\ &=E^{(1)}_{M_{\tau'}}\otimes E^{(2)}_{\mu'}[f(M_{t-s}(w_1)\vee M_{t-s}(w_2)):\ t-s<(\sigma_{xd}(w_1)\vee\sigma_{xd}(w_2))]\\ &=E^{(1)}_{M_{\tau'}}\otimes E^{(2)}_{\mu'}[f(M_{t-s}(w_1)):\ t-s<\sigma_{xd}(w_1)]\\ &=E_{M_{\tau'}}[f(M_{t-s}):\ t-s<\sigma_{xd}].\end{aligned}$$

Thus we have

$$\mathrm{I}=E_x[f(M_t):\ t<\tau']+E_x[E_{M_{\tau'}}[f(M_r):\ r<\tau_p]_{\substack{r=t-\tau'\\ p=xd/M_{\tau'}}}:\ \tau'\leqq t,\quad \tau'<\tau].$$

We next consider the second term.

$$\begin{aligned}\mathrm{II}&\equiv E_x[u_n(t-\tau,M_\tau):\ t\geqq\tau]\\ &=E_x[u_n(t-\tau,M_\tau):\ t\geqq\tau,\ t\geqq\tau',\ \tau>\tau']+E_x[u_n(t-\tau,M_\tau):\ t\geqq\tau,\ \tau=\tau'].\end{aligned}$$

The first term of the above can be written as

$$E_x[E_{\mu_{\tau'}}[u_n(r-\sigma_{xd},M_{\sigma_{xd}}):\ r\geqq\sigma_{xd}]_{r=t-\tau'}:\ t\geqq\tau',\ \tau>\tau'].$$

As in the case of I, the integrand is equal to

$$\begin{aligned}&E^{(1)}_{M_{\tau'}}\otimes E^{(2)}_{\mu'}[u_n(r-\sigma_{xd}(w_1+w_2),\ M_{\sigma_{xd}(w_1+w_2)}(w_1+w_2)):\ r\geqq\sigma_{xd}(w_1+w_2)]\\ &=E^{(1)}_{M_{\tau'}}\otimes E^{(2)}_{\mu'}[u_n(r-\sigma_{xd}(w_1),\ M_{\sigma_{xd}(w_1)}(w_1)):\ M_{\sigma_{xd}(w_1)}(w_1)>x(1-d),\ r\geqq\sigma_{xd}(w_1)]\\ &\quad+E^{(1)}_{M_{\tau'}}\otimes E^{(2)}_{\mu'}[u_n(r-\sigma_{xd}(w_1),\ M_{\sigma_{xd}(w_1)}(w_1)\vee M_{\sigma_{xd}(w_1)}(w_2)):\ M_{\sigma_{xd}(w_1)}(w_1)\\ &\quad\leqq x(1-d),\ r\geqq\sigma_{xd}(w_1)],\end{aligned}$$

where the second term is zero because $u_n(t,x)=0$ for $0\leqq x\leqq(1-d)/d$, and so

$$\begin{aligned}&=E_{M_{\tau'}}[u_n(r-\sigma_{xd},M_{\sigma_{xd}}):\ M_{\sigma_{xd}}>x(1-d),\ r\geqq\sigma_{xd}]\\ &=E_{M_{\tau'}}[u_n(r-\sigma_{xd},M_{\sigma_{xd}}):\ r\geqq\sigma_{xd}].\end{aligned}$$

Thus we have

$$\mathrm{II}=E_x[E_{M_{\tau'}}[u_n(r-\tau_p,M_{\tau_p}):\ r\geqq\tau_p]_{\substack{r=t-\tau'\\ p=xd/M_{\tau'}}}:\ t\geqq\tau',\ \tau>\tau']+E_x[u_n(t-\tau',M_{\tau'}):\ t\geqq\tau=\tau'],$$

and hence

$$(**)\qquad u_{n+1}(t, x)=E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau=\tau']$$
$$+E_x[\{E_{M_{\tau'}}[f(M_r)\colon r<\tau_p]$$
$$+E_{M_{\tau'}}[u_n(r-\tau_p, M_{\tau_p})\colon r\geqq\tau_p]\}_{\substack{r=t-\tau'\\ p=xd/M_{\tau'}}}\colon t\geqq\tau',\ \tau>\tau'].$$

If we assume

$$u_n(t, x)\leqq E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau']$$

for all $d'(\geqq d)$, then, since $p\geqq d$ in the last term of (**), we have

$$u_{n+1}(t, x)\geqq E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau=\tau']$$
$$+E_x[u_n(r, M_{\tau'})_{r=t-\tau'}\colon t\geqq\tau',\ \tau>\tau']$$
$$=E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau'].$$

Going back to (**) and using this inequality,

$$u_{n+1}(t, x)\leqq E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau=\tau']$$
$$+E_x[u_{n+1}(t-\tau', M_{\tau'})\colon t\geqq\tau',\ \tau>\tau']$$
$$\leqq E_x[f(M_t)\colon t<\tau']+E_x[u_{n+1}(t-\tau', M_{\tau'})\colon t\geqq\tau=\tau']$$
$$+E_x[u_{n+1}(t-\tau', M_{\tau'})\colon t\geqq\tau',\ \tau>\tau']$$
$$=E_x[f(M_t)\colon t<\tau']+E_x[u_{n+1}(t-\tau', M_{\tau'})\colon t\geqq\tau'].$$

Therefore we have, by induction,

$$u_n(t, x)\leqq E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau']$$

for all $n\geqq 0$, because it is obvious for $n=0$. We also have

$$u_{n+1}(t, x)\geqq E_x[f(M_t)\colon t<\tau']+E_x[u_n(t-\tau', M_{\tau'})\colon t\geqq\tau']$$

for all $n\geqq 0$. Letting n to infinity in the above two inequalities, we have

$$T_t^d f(x)=E_x[f(M_t)\colon t<\tau']+E_x[T_{t-\tau'}^d f(M_{\tau'})\colon t\geqq\tau'],$$

and complete the proof.

LEMMA 3.4. *For* $f\in C_d$,

$$(3.5)\qquad T_t^d T_s^d f=T_{t+s}^d f,\qquad t, s\geqq 0.$$

Proof. By the definition,

$$T_{t+s}^d f(x)=E_x[f(M_{t+s})\colon t+s<\tau]+E_x[T_{t+s-\tau}^d f(M_\tau)\colon t+s\geqq\tau],$$

where $\tau=\tau_d$. Since $T_t^d T_s^d f(x)=T_{t+s}^d f(x)$ is obvious for $x=0$, we assume $x\in S$.

$$\text{I} \equiv E_x[f(M_{t+s}) \colon t+s<\tau]$$
$$=E_x[E_{\mu_t}[f(M_s) \colon s<\sigma_{xd}] \colon t<\tau],$$

where, if we put $\mu_t = M_t\delta_{M_t}+\mu'$, then $M_t>xd$ and $M(\mu')<x(1-d)$ because $t<\tau$, and hence the integrand is equal to

$$E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}[f(M_s(w_1+w_2)) \colon s<\sigma_{xd}(w_1+w_2)]$$
$$=E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}[f(M_s(w_1)) \colon s<\sigma_{xd}(w_1)]$$
$$=E_{M_t}[f(M_s) \colon s<\sigma_{xd}].$$

Thus we have

$$\text{I}=E_x[E_{M_t}[f(M_s) \colon s<\tau_p]_{p=xd/M_t} \colon t<\tau].$$

$$\text{II} \equiv E_x[T^d_{t+s-\tau}f(M_\tau) \colon t+s \geqq \tau]$$
$$=E_x[T^d_{t+s-\tau}f(M_\tau) \colon t \geqq \tau]+E_x[T^d_{t+s-\tau}f(M_\tau) \colon t<\tau \leqq t+s]$$
$$=E_x[T^d_{t+s-\tau}f(M_\tau) \colon t \geqq \tau]+E_x[E_{\mu_t}[T^d_{s-\sigma_{xd}}f(M_{\sigma_{xd}}) \colon s \geqq \sigma_{xd}] \colon t<\tau].$$

As in the case of I, the integrand of the second term is equal to

$$E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}[T^d_{s-\sigma_{xd}(w_1+w_2)}f(M_{\sigma_{xd}(w_1+w_2)}(w_1+w_2)) \colon s \geqq \sigma_{xd}(w_1+w_2)]$$
$$=E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}[T^d_{s-\sigma_{xd}(w_1)}f(M_{\sigma_{xd}(w_1)}(w_1) \vee M_{\sigma_{xd}(w_1)}(w_2)) \colon s \geqq \sigma_{xd}(w_1)]$$
$$=E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}[T^d_{s-\sigma_{xd}(w_1)}f(M_{\sigma_{xd}(w_1)}) \colon M_{\sigma_{xd}(w_1)}(w_1)>x(1-d),\ s \geqq \sigma_{xd}(w_1)]$$
$$+E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}[T^d_{s-\sigma_{xd}(w_1)}f(M_{\sigma_{xd}(w_1)} \vee M_{\sigma_{xd}(w_1)}(w_2)) \colon M_{\sigma_{xd}(w_1)}(w_1) \leqq x(1-d),\ s \geqq \sigma_{xd}(w_1)]$$

where the second term is equal to zero because $T^d_t f \in C_d$, and hence we can continue as

$$E_{M_t}[T^d_{s-\sigma_{xd}}f(M_{\sigma_{xd}}) \colon M_{\sigma_{xd}}>x(1-d),\ s \geqq \sigma_{xd}]=E_{M_t}[T^d_{s-\sigma_{xd}}f(M_{\sigma_{xd}}) \colon s \geqq \sigma_{xd}].$$

Thus we have

$$\text{II}=E_x[T^d_{t+s-\tau}f(M_\tau) \colon t \geqq \tau]+E_x[E_{M_t}[T^d_{s-\tau_p}f(M_{\tau_p}) \colon s \geqq \tau_p]_{p=xd/M_t} \colon t<\tau],$$

and hence

$$T^d_{t+s}f(x)=E_x[\{E_y[f(M_s) \colon s<\tau_p]+E_y[T^d_{s-\tau_p}f(M_{\tau_p}) \colon s \geqq \tau_p]\}_{\substack{y=M_t \\ p=xd/M_t}} \colon t<\tau]$$
$$+E_x[T^d_{t+s-\tau}f(M_\tau) \colon t \geqq \tau].$$

Making use of Lemma 3. 3 for the first term of the right-hand side, we have

$$T^d_{t+s}f(x)=E_x[T^d_s f(M_t) \colon t<\tau]+E_x[T^d_{t+s-\tau}f(M_\tau) \colon t \geqq \tau].$$

Now, consider the above equality as an equation for a function of a variable (t, x)

where s is fixed, then we have

$$T^d_{t+s}f(x)=T^d_t T^d_s f(x)$$

from the uniqueness of a solution of (3. 2).

Now we shall extend the semigroup $\{T^d_t\}$ on C_d to a semigroup $\{T^0_t\}$ on $\overline{C}_0=\{f\in C(S)\colon \lim_{x\to 0}f(x)=0\}$. By Lemma 3. 3, we can define $T^0_t f$ for all $f\in C_0 =\bigcup_{2/3<d<1}C_d$ as $T^0_t f=T^d_t f$ if $f\in C_d$. Since, then, T^0_t is a bounded linear operator on C_0, T^0_t can be uniquely extended to an operator on $\overline{C}_0$. Then we have the following theorem.

THEOREM 3. 1. *$\{T^0_t\}$ is a strongly continuous contraction semigroup of non-negative linear operators on $\overline{C}_0$. Moreover it satisfies*

(3. 6) $$\theta_a T^0_t f=T^0_t\theta_a f$$

for $f\in\overline{C}_0$ and $a\in S$.

Proof. In the equation (3. 2), we have $\lim_{t\downarrow 0}u(t, x; f)=f(x)$ from the right continuity of M_t and hence

(*) $$\lim_{t\downarrow 0} T^0_t f(x)=f(x)$$

for $f\in C_0$. By the limiting procedure, we have (*) for all $f\in\overline{C}_0$ and hence the strong continuity of $\{T^0_t\}$ follows.[4] (3. 6) follows from (3. 3) and the remainder of the theorem follows by the limiting procedure from the argument on C_d.

By Theorem 3. 1, there exists a strong Markov process $\{W^0, x_t, \zeta^0, \mathcal{N}^0_t, P^0_x\colon x\in S\}$ on S such that $T^0_t f(x)=E^0_x[f(x_t)]$ for $f\in\overline{C}_0$, where W^0 is the set of all right continuous functions w^0: $[0, \zeta^0(w))\to S$ with left limit and $x_t(w^0)=w^0(t)$, $\mathcal{N}^0_t$ is a σ-field generated by the sets $\{x_s\in E\}$, $0\leqq s\leqq t$, $E\in\mathcal{B}(S)$, and P^0_x is a probability measure on $\mathcal{N}^0_\infty=\bigvee_{t>0}\mathcal{N}^0_t$ and E^0_x denotes the expectation by P^0_x. We shall call the Markov process (x_t) the *underlying process* of a given cascade process (μ_t).

In what follows, we shall study some properties of the underlying process, particularly its relation to the process (M_t). We remark that the process (x_t) can be considered as the strong Markov process on $\overline{S}$ where 0 is a trap and $x_t(w^0)=0$ for $t\geqq\zeta^0(w^0)$.

LEMMA 3. 5. *For $x\in S$,*

(3. 7) $$P^0_x\ (x_t \textit{ is non-increasing for all } t\geqq 0)=1.$$

Proof. Take $f\in C_0$ such that $f(y)=0$ for $y\leqq x$. Since f is a function in C_d for such d as $(1-d)/d=x$, $T^0_t f=T^d_t f\in C_d$ and hence $T^0_t f(y)=0$ for $y\leqq(1-d)/d=x$. Thus, $P^0_x(x_t>x+\varepsilon_n)=0$ for any sequence $\varepsilon_n\downarrow 0$, and we have

4) See, e.g., p. 233 of Yosida [17].

$$P_x^0(x_t>x)=\lim_{\varepsilon_n\to 0} P_x^0(x_t>x+\varepsilon_n)=0,$$

or $P_x^0(x_t\leqq x)=1$. From this, it follows

$$P_x^0(x_s\geqq x_t)=E_x^0[P_{x_s}^0(x_0\geqq x_{t-s})]=1$$

for any $0\leqq s<t<\infty$. Since x_t is right continuous, we have

$$P_x^0(x_s\geqq x_t \text{ for all } 0\leqq s<t<\infty)=1.$$

Put $\widetilde{W}^0=\{w^0\in W^0: w^0(t) \text{ is non-increasing for all } t\geqq 0\}$. Then, by Lemma 3. 5, we can take $\widetilde{W}^0$ as a sample space of the underlying process (x_t) and so we shall denote it again by W^0 in the following.

We define τ_d^0 and σ_ε^0 of the underlying process (x_t) in the same way as in the case of the process (M_t):

$$\tau_d^0=\begin{cases}\inf\left\{t: \dfrac{x_t}{x_0}\leqq d\right\}, \\ +\infty, \quad \text{if } \{\cdots\}=\phi\end{cases} \tag{3. 8}$$

$$\sigma_\varepsilon^0=\begin{cases}\inf\{t: x_t\leqq\varepsilon\}, \\ +\infty, \quad \text{if } \{\cdots\}=\phi.\end{cases} \tag{3. 9}$$

Then, τ_d^0 and σ_ε^0 are also $\mathcal{N}_t^0$-Markov times.

For $w^0\in W^0$ and $a\in S$, let $a\cdot w^0$ be a sample path in W^0 such that $x_t(a\cdot w^0)=ax_t(w^0)$, $t\geqq 0$. It should be noted that $\tau_d^0(a\cdot w^0)=\tau_d^0(w^0)$.

LEMMA 3. 6. *For any bounded $\mathcal{N}_\infty^0$-measurable function F and $a\in S$, it holds*

$$E_{ax}^0[F(w^0)]=E_x^0[F(a\cdot w^0)], \qquad x\in S. \tag{3. 10}$$

Proof. When $F(w^0)=f(x_t(w^0))$, $f\in\overline{C}_0$, (3, 10) reduces to (3. 6). (3. 10) is verified as usual by induction for $F(w^0)=f_1(x_{t_1}(w^0))f_2(x_{t_2}(w^0))\cdots f_n(x_{t_n}(w^0))$ where $n\geqq 1$, $f_1, f_2, \cdots, f_n\in\overline{C}_0$, and $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n$. Hence (3. 10) holds for any bounded $\mathcal{N}_\infty^0$-measurable function F by the standard argument.

LEMMA 3. 7. *For any $f\in B(S)$,*

$$E_x[f(M_t): t<\tau_d]=E_x^0[f(x_t): t<\tau_d^0], \qquad x\in S. \tag{3. 11}$$

Proof. It is sufficient to show (3. 11) for $f\geqq 0$, $f\in C_0$, and $x=1$. For, since

$$E_x[f(M_t): t<\tau_d]=E_1[f(M_t(x\cdot w)): t<\tau_d(x\cdot w)]=E_1[\theta_x f(M_t): t<\tau_d],$$

and

$$E_x^0[f(x_t): t<\tau_d^0]=E_1^0[\theta_x f(x_t): t<\tau_d^0],$$

(3. 11) is equivalent to

$$E_1[\theta_x f(M_t)\colon t<\tau_d]=E_1^0[\theta_x f(x_t)\colon t<\tau_d^0].$$

Let $g_n \in C_0$ be a sequence such that $0 \leqq g_n \leqq 1$ and $g_n(x) \uparrow \chi_{(d,1]}(x)$ as $n \to \infty$. Then $f_n = f \cdot g_n \in C_0$ and $T_t^0 f_n(x)=0$ for $x \leqq d$. Hence we have

$$T_t^0 f_n(1)=E_1[f_n(M_t)\colon t<\tau_d]+E_1[T_{t-\tau_d}^0 f_n(M_{\tau_d})\colon t\geqq\tau_d]=E_1[f_n(M_t)\colon t<\tau_d],$$

that is,

$$E_1^0[f_n(x_t)]=E_1^0[f(x_t)g_n(x_t)]=E_1[f(M_t)g_n(M_t)\colon t<\tau_d].$$

Letting n to infinity, we have

$$E_1^0[f(x_t)\colon x_t>d]=E_1[f(M_t)\colon M_t>d,\ t<\tau_d].$$

Since $x_t>d$ is equivalent to $t<\tau_d^0$ and $M_t>d$ to $t<\tau_d$, we have

$$E_1^0[f(x_t)\colon t<\tau_d^0]=E_1[f(M_t)\colon t<\tau_d].$$

We remark that, by putting $f=1$ in (3. 11), we have $P_x(t<\tau_d)=P_x^0(t<\tau_d^0)$, or $P_x(\tau_d \leqq t)=P_x^0(\tau_d^0 \leqq t)$.

LEMMA 3. 8. *For any* $f \in C_d$ *and* $g \in C([0,\infty])$, *it holds*

$$E_x[f(M_{\tau_d})g(\tau_d)\colon \tau_d \leqq t]=E_x^0[f(x_{\tau_d^0})g(\tau_d^0)\colon \tau_d^0 \leqq t], \qquad x \in S. \tag{3. 12}$$

Proof. Put $\tau=\tau_d$ and $\tau^0=\tau_d^0$. As in the proof of Lemma 3. 7, it is sufficient to show (3. 12) for $x=1$ and $t>0$.

Putting $t_k=(k/n)t$, $k=0,1,2,\cdots$, we define τ_n and τ_n^0 by

$$\tau_n=t_k \qquad \text{if} \quad t_{k-1}<\tau \leqq t_k,$$

$$\tau_n^0=t_k \qquad \text{if} \quad t_{k-1}<\tau^0 \leqq t_k.$$

Then it is clear $\tau_n \downarrow \tau$ and $\tau_n^0 \downarrow \tau^0$ as $n \to \infty$.

Now we calculate the following:

$$\begin{aligned}
&E_1^0[f(x_{\tau_n^0})g(\tau_n^0)\colon t\geqq\tau^0]\\
&=\sum_{k=1}^{n} E_1^0[f(x_{t_k})g(t_k)\colon t_{k-1}<\tau^0\leqq t_k]\\
&=\sum_{k=1}^{n} g(t_k)E_1^0\left[E_{x_{t_{k-1}}}^0\left[f(x_{t/n})\colon \tau_p^0\leqq \frac{t}{n}\right]_{p=d/x_{t_{k-1}}}\colon t_{k-1}<\tau^0\right]\\
&=\sum_{k=1}^{n} g(t_k)E_1\left[E_{M_{t_{k-1}}}^0\left[f(x_{t/n})\colon \tau_p^0\leqq \frac{t}{n}\right]_{p=d/M_{t_{k-1}}}\colon t_{k-1}<\tau\right],
\end{aligned}$$

where we made use of Lemma 3. 7 in the last step. On the other hand, we have

$$E_y^0\left[f(x_{t/n}):\ \tau_p^0\leqq\frac{t}{n}\right]=T_{t/n}^0f(y)-E_y^0\left[f(x_{t/n}):\ \tau_p^0>\frac{t}{n}\right]$$

$$=E_y\left[f(M_{t/n}):\ \frac{t}{n}<\tau_p\right]+E_y\left[T_{t/n-\tau_p}^0f(M_{\tau_p}):\ \tau_p\leqq\frac{t}{n}\right]$$

$$-E_y^0\left[f(x_{t/n}):\ \tau_p^0>\frac{t}{n}\right]$$

$$=E_y\left[T_{t/n-\tau_p}^0f(M_{\tau_p}):\ \tau_p\leqq\frac{t}{n}\right],$$

and hence

$$E_1^0[f(x_{\tau_n^0})g(\tau_n^0):\ t\geqq\tau^0]$$

$$=\sum_{k=1}^n g(t_k)E_1\left[E_{M_{t_{k-1}}}\left[T_{t/n-\tau_p}^0f(M_{\tau_p}):\ \tau_p\leqq\frac{t}{n}\right]_{p=d/M_{t_{k-1}}}:\ t_{k-1}<\tau\right]$$

$$=\sum_{k=1}^n g(t_k)E_1\left[E_{\mu_{t_{k-1}}}\left[T_{t/n-\tau_p}^0f(M_{\tau_p}):\ \tau_p\leqq\frac{t}{n}\right]_{p=d/M_{t_{k-1}}}:\ t_{k-1}<\tau\right],$$

where, by putting $\mu_{t_{k-1}}=M_{t_{k-1}}\delta_{M_{t_{k-1}}}+\mu'$, we used the branching property. Applying Markov property of (μ_t) at time t_{k-1}, we have

$$E_1^0[f(x_{\tau_n^0})g(\tau_n^0):\ t\geqq\tau^0]$$

$$=\sum_{k=1}^n E_1[g(t_k)T_{t_k-\tau}^0f(M_\tau):\ t_{k-1}<\tau\leqq t_k]$$

$$=E_1[g(\tau_n)T_{\tau_n-\tau}^0f(M_\tau):\ \tau\leqq t],$$

and hence, by letting $n\to\infty$,

$$E_1^0[f(x_{\tau^0})g(\tau^0):\ \tau^0\leqq t]=E_1[f(M_\tau)g(\tau):\ \tau\leqq t].$$

We remark that it can be shown by Lemma 3. 7 and Lemma 3. 8 that the two processes $(x_t, P_x^0, t\leqq\tau_d^0)$ and $(M_t, P_x, t\leqq\tau_d)$ are equivalent, that is, they obey the same probability law.

It is seen by Theorem 3. 1 and Lemma 3. 5 that the process $(-\log x_t, P_1^0)$ is a nondecreasing additive process on $[0,\infty]$ with ∞ as a trap. Then it is well-known that the Laplace transform of $-\log x_t$ is represented in the form:[5)]

$$(3.13)\qquad\begin{cases}E_1^0[e^{-\alpha(-\log x_t)}]=e^{-t\phi(\alpha)}, & \alpha>0,\\ \phi(\alpha)=m\alpha+\displaystyle\int_{(0,\infty]}(1-e^{-\alpha u})l(du),\end{cases}$$

5) See, e.g., Ito-Mckean [6], pp. 31-32.

where m is a nonnegative constant and $l(du)$ is a measure on $(0, \infty]$ such that

$$\int_{(0,\infty]} \frac{u}{1+u} l(du) < +\infty.$$

In addition, it is known that

$$E_x^0\Big[\sum_{s\leq\rho} f(-\log x_{s-}, -\log x_s)\Big] = E_x^0\Big[\int_0^\rho ds \int_{(0,\infty]} f(-\log x_s, -\log x_s+u) l(du)\Big]^{6)} \tag{3.14}$$

for any Markov time ρ and for any $f \in B([0, \infty] \times [0, \infty])$ such that $f \geqq 0$ and $f(x, x)=0$ for $x \in [0, \infty]$. From the discussion on the additive process $(-\log x_t, P_x^0)$, we have the following statements on the underlying process (x_t, P_x^0).

PROPOSITION 3.1. *There exists uniquely a measure $k(da)$ on $[0, 1)$ and a constant m such that*

$$\begin{cases} \int_{[0,1)} (1-a) k(da) < +\infty, \\ m \geqq 0, \end{cases} \tag{3.15}$$

and

$$A^0 f(x) = -mxf'(x) + \int_{[0,1)} k(da)(f(xa)-f(x)),^{7)} \qquad x \in \bar{S} \tag{3.16}$$

for all $f \in C^1[0, 1]$ where A^0 is an infinitesimal generator of the underlying process. The set $C^1[0, 1]$ is a core of A^0 on $C[0, 1]$. Moreover, it holds

$$E_x^0\Big[\sum_{s\leq\rho} g(x_{s-}, x_s)\Big] = E_x^0\Big[\int_0^\rho ds \int_{[0,1)} g(x_s, ax_s) k(da)\Big] \tag{3.17}$$

for any Markov time ρ and for any $g \in B(\bar{S} \times \bar{S})$ such that $g \geqq 0$ and $g(x, x)=0$ for $x \in \bar{S}$.

REMARK 1. The measures l and k satisfy the following relation:

$$k(A) = \int_{\{u:\, e^{-u} \in A\}} l(du), \qquad A \in \mathcal{B}[0, 1). \tag{3.18}$$

REMARK 2. The measure k and the constant m determine the underlying process.

§ 4. Branching measure.

In the preceding section we constructed a strong Markov process called

---

6) See, Ikeda-Watanabe [4] and Watanabe [16] (cf. Motoo [10]).

7) This is easily obtained from the expression of a generator of an additive process for which it is refered to Sato [11], Chap. 3.

underlying process, describing the behavior of each particle of which a cascade process consists, while the purpose of this section is to construct a measure $\Pi(d\mu)$—this will be called a branching measure—on $\boldsymbol{M}_1-\{\delta_1\}$ which gives the law according to which new particles are born when a parent particle has splitted.

For ε $(0<\varepsilon<1/2)$, set

$$B^{\varepsilon}=\{F\colon F \text{ is a nonnegative bounded Borel function on } \boldsymbol{M}_2 \text{ and satisfies } F(\mu)=0 \text{ if } M(\mu)>1-\varepsilon\}.$$

Taking $F\in B^{\varepsilon}$, we concern with the quantity

$$E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]$$

for $x\in S$, where $\tau=\tau_d$ $(2/3<d<1)$ and $\tau_t=\tau\wedge t$. Then we have

$$\left\|\frac{1}{M_{s-}}\cdot\mu_s\right\|=\frac{1}{M_{s-}}\|\mu_s\|\leqq\frac{x}{xd}=\frac{1}{d}<2, \qquad P_x\text{-a. s.}$$

Moreover, since $F((1/M_{s-})\cdot\mu_s)=0$ if $M((1/M_{s-})\cdot\mu_s)=M_s/M_{s-}>1-\varepsilon$, the number N of s such that $s\leqq\tau$ and $F((1/M_{s-})\cdot\mu_s)\neq 0$ has a bound:

$$N\leqq K_d^{\varepsilon}\equiv\frac{\log d}{\log(1-\varepsilon)}+1.$$

Thus it follows

$$\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\leqq\sum_{s\leqq\tau} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\leqq K_d^{\varepsilon}\|F\|$$

with $\|F\|=\sup_{\mu\in\boldsymbol{M}_2}|F(\mu)|$. In addition, we have by the homogeneity of the cascade process (μ_t)

$$E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]=E_1\left[\sum_{s\leqq\tau_t(x\cdot w)} F\left(\frac{1}{M_{s-}(x\cdot w)}\cdot\mu_s(x\cdot w)\right)\right]$$
$$=E_1\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right],$$

which means the left-hand term does not depend on $x\in S$.

We now define $u_n(t)$, $n=0, 1, 2, \cdots$ successively by

$$(4.1)\qquad \begin{cases} u_0(t)\equiv 0, \\ u_{n+1}(t)=E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]+E_x[u_n(t-\tau);\ t\geqq\tau], \qquad n\geqq 0. \end{cases}$$

It is easy to see that $u_n(t)$ does not depend on $x\in S$ and $0\leqq u_n(t)\leqq u_{n+1}(t)$ for all

$n \geqq 0$. Therefore, $\lim_{n\to\infty} u_n(t) \leqq +\infty$ exists and we denote it by $A_d^{\epsilon}(t; F)$. Then, by the definition, $A_d^{\epsilon}(t; F)$ is a solution of the following equation:

$$u(t)=E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]+E_x[u(t-\tau);\ t\geqq\tau]. \tag{4. 2}$$

Moreover, it is plain to see that $u_n(t)$ is non-decreasing in t:

$$u_n(s)\leqq u_n(t) \qquad \text{for} \quad 0\leqq s<t<\infty \tag{4. 3}$$

and we have as a limit

$$A_d^{\epsilon}(s; F)\leqq A_d^{\epsilon}(t; F) \qquad \text{for} \quad 0\leqq s<t<\infty. \tag{4. 4}$$

LEMMA 4. 1. *For any* $0<T<\infty$, $A_d^{\epsilon}(t; F)$ *is bounded in* $t\in[0, T]$ *and right continuous in* t.

Proof. Each of $u_n(t)$ is obviously bounded, or more precisely $|u_n(t)| \leqq K_d^{\epsilon}\|F\|(1+n)$. Moreover, since $E_x[\sum_{s\leqq\tau_t} F((1/M_{s-})\cdot\mu_s)]$ is right continuous in t, we can see by induction that $u_n(t)$ is right continuous in t for all n.

Since

$$u_{n+1}(t)-u_n(t)=E_x[u_n(t-\tau)-u_{n-1}(t-\tau);\ t\geqq\tau],$$

we have

$$a_n(t)\leqq E_x[a_{n-1}(t-\tau);\ t\geqq\tau] \tag{*}$$

where $a_n(t)=\sup_{0\leqq s\leqq t}|u_{n+1}(s)-u_n(s)|$. Using notations in Lemma 2. 5,

$$a_n(t)\leqq a_{n-1}(t)q_1(t),$$

and hence by the inequality (*), we have

$$\begin{aligned} a_n(t)&\leqq E_x[a_{n-2}(t-\tau)q_1(t-\tau);\ t\geqq\tau] \\ &\leqq a_{n-2}(t)E_x[q_1(t-\tau);\ t\geqq\tau]=a_{n-2}(t)q_2(t). \end{aligned}$$

Thus, we have by induction

$$a_n(t)\leqq a_{n-m}(t)q_m(t)$$

for $1\leqq m\leqq n$.

By Lemma 2. 5, there exists a positive integer n_0 for any fixed $T>0$ such that $q_{n_0}(T)<1$. Since

$$a_{mn_0+k}(T)\leqq a_{(m-1)n_0+k}(T)q_{n_0}(T)\leqq\cdots\leqq a_k(T)q_{n_0}(T)^m$$

for any $m\geqq 1$ and $0\leqq k<n_0$, it follows

$$\sum_{n=n_0}^{\infty} a_n(T) = \sum_{k=0}^{n_0-1} \sum_{m=1}^{\infty} a_{mn_0+k}(T) \leqq \sum_{k=0}^{n_0-1} \sum_{m=1}^{\infty} a_k(T) q_{n_0}(T)^m$$

$$= \sum_{k=0}^{n_0-1} a_k(T) \sum_{m=1}^{\infty} q_{n_0}(T)^m < +\infty.$$

Therefore, $u_n(t)$ converges uniformly in $0 \leqq t \leqq T$, and $A_d^\varepsilon(t; F) = \lim_{n\to\infty} u_n(t)$ is bounded in $t \in [0, T]$ and right continuous in t.

By Lemma 2. 6, it is easily seen that the bounded solution on $[0, T]$ of the equation (4. 2) is unique for any $T > 0$. Thus, $A_d^\varepsilon(t; F)$ is a unique solution of (4. 2) in this sense, and especially we have

$$(4.5) \qquad A_d^\varepsilon(t; F+G) = A_d^\varepsilon(t; F) + A_d^\varepsilon(t; G), \qquad A_d^\varepsilon(t; \alpha F) = \alpha A_d^\varepsilon(t; F)$$

for $F, G \in B^\varepsilon$ and $\alpha \geqq 0$.

LEMMA 4. 2. *If* $\{F_n\} \subset B^\varepsilon$ *satisfies* $F_n(\mu) \downarrow 0$ *as* $n \to \infty$, *then* $A_d^\varepsilon(t; F_n) \downarrow 0$ *as* $n \to \infty$.

Proof. Since $A_d^\varepsilon(t; F) \geqq 0$ for $F \in B^\varepsilon$, (4. 5) implies $A_d^\varepsilon(t; F_n)$ is non-increasing in n, so that $\lim_{n\to\infty} A_d^\varepsilon(t; F_n) = A(t)$ exists and $A(t)$ satisfies the equation:

$$A(t) = E_x[A(t-\tau);\ t \geqq \tau],$$

because $E_x[\sum_{s \leqq \tau_t} F_n((1/M_{s-}) \cdot \mu_s)] \downarrow 0$ as $n \to \infty$. Hence $A(t) = 0$ by Lemma 2. 6.

Now, we set

$$B_d^\varepsilon = \{F \in B^\varepsilon \colon F \text{ is } \varphi_d^{-1}(\mathcal{B}_2^d)\text{-measurable}\},$$

where φ_d is defined in §1. $F \in B_d^\varepsilon$ means that $F(\mu)$ depends only on $\varphi_d(\mu)$, or $F(\mu) = F(\mu')$ if $\varphi_d(\mu) = \varphi_d(\mu')$.

LEMMA 4. 3. *For* $2/3 < d \leqq d' < 1$,

$$A_d^\varepsilon(t; F) = A_{d'}^\varepsilon(t; F)$$

if $F \in B_d^\varepsilon$.

Proof. Put $\tau_d = \tau$ and $\tau_{d'} = \tau'$. Since the equation (4. 2) has a unique solution, $A_d^\varepsilon(t; F) = A_{d'}^\varepsilon(t; F)$ follows if we show that $A_d^\varepsilon(t; F)$ satisfies

$$A_d^\varepsilon(t; F) = E_x\left[\sum_{s \leqq \tau'_t} F\left(\frac{1}{M_{s-}} \cdot \mu_s\right)\right] + E_x[A_d^\varepsilon(t-\tau'; F);\ t \geqq \tau'].$$

For later use in §5, we shall show the above equation for any Markov time ρ which satisfies $\rho \leqq \tau$ (obviously, $\tau' \leqq \tau$).

Let $A_d^\varepsilon(t; F) = \lim_{n\to\infty} u_n(t)$, where $u_n(t)$ is defined by (4. 1). Putting $B(t) = E_x[\sum_{s \leqq \rho_t} F((1/M_{s-}) \cdot \mu_s)]$, we rewrite $u_{n+1}(t)$ as follows;

$$u_{n+1}(t)=E_x\Big[\sum_{s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]+E_x[u_n(t-\tau);\ t\geqq\tau]$$

$$=E_x\Big[\sum_{s\leqq\rho_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]+E_x\Big[\sum_{\rho_t<s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]+E_x[u_n(t-\tau);\ t\geqq\tau]$$

$$=B(t)+E_x\Big[\sum_{\rho<s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big);\ t\geqq\rho,\quad\tau>\rho\Big]$$

$$+E_x[u_n(t-\tau);\ t\geqq\tau,\ \tau>\rho]+E_x[u_n(t-\tau);\ t\geqq\tau=\rho]$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},\ \text{say}.$$

$$\mathrm{II}=E_x\Big[E_{\mu_\rho}\Big[\sum_{r<s\leqq(r+\sigma_{xd})\wedge t}F\Big(\frac{1}{M_{(s-r)-}}\cdot\mu_{s-r}\Big)\Big]_{r=\rho};\ t\geqq\rho,\quad\tau>\rho\Big]$$

$$=E_x\Big[E_{\mu_\rho}\Big[\sum_{s\leqq\sigma_{xd}\wedge(t-r)}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]_{r=\rho};\ t\geqq\rho,\quad\tau>\rho\Big].$$

Putting $\mu_\rho=M_\rho\delta_{M_\rho}+\mu'$ where $M(\mu')<x(1-d)$, and making use of the branching property, the integrand is equal to

$$E^{(1)}_{M_\rho}\otimes E^{(2)}_{\mu'}\Big[\sum_{s\leqq\sigma_{xd}(w_1+w_2)\wedge(t-r)}F\Big(\frac{1}{M_{s-}(w_1+w_2)}\cdot\mu_s(w_1+w_2)\Big)\Big]$$

$$=E^{(1)}_{M_\rho}\otimes E^{(2)}_{\mu'}\Big[\sum_{s\leqq\sigma_{xd}(w_1)\wedge(t-r)}F\Big(\frac{1}{M_{s-}(w_1)}\cdot(\mu_s(w_1)+\mu_s(w_2)\Big)\Big]$$

$$=E^{(1)}_{M_\rho}\otimes E^{(2)}_{\mu'}\Big[\sum_{s\leqq\sigma_{xd}(w_1)\wedge(t-r)}F\Big(\frac{1}{M_{s-}(w_1)}\cdot\mu_s(w_1)\Big)\Big]$$

$$=E_{M_\rho}\Big[\sum_{s\leqq\sigma_{xd}\wedge(t-r)}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]$$

because $F\in B^\epsilon_d$ and

$$M\Big(\frac{1}{M_{s-}(w_1)}\cdot\mu_s(w_2)\Big)=\frac{M_s(w_2)}{M_{s-}(w_1)}<\frac{x(1-d)}{xd}=\frac{1-d}{d}.$$

Thus, we have

$$\mathrm{II}=E_x\Big[E_{M_\rho}\Big[\sum_{s\leqq\tau_p\wedge(t-r)}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]_{\substack{r=\rho\\ p=xd/M_\rho}};\ t\geqq\rho,\quad\tau>\rho\Big],$$

$$\mathrm{III}=E_x[E_{\mu_\rho}[u_n(t-r-\sigma_{xd});\ t-r\geqq\sigma_{xd}]_{r=\rho};\ t\geqq\rho,\quad\tau>\rho],$$

where again putting $\mu_\rho=M_\rho\delta_{M_\rho}+\mu'$ and using the branching property, we have

$$\mathrm{III}=E_x[E_{M_\rho}[u_n(t-r-\tau_p);\ t-r\geqq\tau_p]_{\substack{r=\rho\\ p=xd/M_\rho}};\ t\geqq\rho,\quad\tau>\rho].$$

Therefore $u_{n+1}(t)$ is expressed as

$$(*)\qquad \begin{aligned} u_{n+1}(t) &= B(t)+E_x[u_n(t-\rho);\ t\geqq\tau=\rho] \\ &+E_x\Big[\Big\{E_y\Big[\sum_{s\leqq r\wedge\tau_p} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big] \\ &+E_y[u_n(r-\tau_p);\ r\geqq\tau_p]\Big\}_{\substack{r=t-\rho\\ p=xd/M_\rho\\ y=M_\rho}} :\ t\geqq\rho,\quad \tau>\rho\Big]. \end{aligned}$$

Now, if we assume

$$(**)\qquad u_n(t)\leqq B(t)+E_x[u_n(t-\rho);\ t\geqq\rho]$$

for any Markov time $\rho\leqq\tau$, then it follows from (*)

$$(***)\qquad \begin{aligned} u_{n+1}(t) &\geqq B(t)+E_x[u_n(t-\rho);\ t\geqq\tau=\rho]+E_x[u_n(t-\rho);\ t\geqq\rho,\quad \tau>\rho] \\ &= B(t)+E_x[u_n(t-\rho);\ t\geqq\rho], \end{aligned}$$

and again applying this to (*), we have

$$\begin{aligned} u_{n+1}(t) &\leqq B(t)+E_x[u_n(t-\rho);\ t\geqq\tau=\rho]+E_x[u_{n+1}(t-\rho);\ t\geqq\rho,\quad \tau>\rho] \\ &\leqq B(t)+E_x[u_{n+1}(t-\rho);\ t\geqq\rho]. \end{aligned}$$

Since (**) is obvious for $n=0$, (**) is verified by induction for all $n\geqq 0$ and (***) also. Thus, letting $n\to\infty$, $\lim_{n\to\infty} u_n(t)=A_d^\epsilon(t;\ F)$ satisfies

$$A_d^\epsilon(t;\ F)=B(t)+E_x[A_d^\epsilon(t-\rho;\ F);\ t\geqq\rho].$$

LEMMA 4. 4. *For* $F\in B_d^\epsilon$,

$$A_d^\epsilon(t+s;\ F)=A_d^\epsilon(t;\ F)+A_d^\epsilon(s;\ F),\quad 0\leqq t,\quad s<\infty.$$

Proof. This is shown in a similar way as in the proof of the semigroup property of $\{T_t^0\}$.

Putting $A(t)=A_d^\epsilon(t;\ F)$ and $B(t)=E_x[\sum_{s\leqq\tau_t} F((1/M_{s-})\cdot\mu_s)]$, we have

$$\begin{aligned} A(t+s) &= B(t+s)+E_x[A(t+s-\tau);\ t+s\geqq\tau) \\ &= B(t)+E_x\Big[\sum_{\tau_t<u\leqq\tau_{t+s}} F\Big(\frac{1}{M_{u-}}\cdot\mu_u\Big)\Big]+E_x[A(t+s-\tau);\ t+s\geqq\tau] \\ &= B(t)+E_x\Big[\sum_{t<u\leqq\tau_{t+s}} F\Big(\frac{1}{M_{u-}}\cdot\mu_u\Big);\ t<\tau\Big] \\ &\quad +E_x[A(t+s-\tau);\ t<\tau,\ t+s\geqq\tau]+E_x[A(t+s-\tau);\ t\geqq\tau] \\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},\ \text{say}. \end{aligned}$$

$$\mathrm{II}=E_x\Big[E_{\mu_t}\Big[\sum_{t<u\leqq(t+\sigma_{xd})\wedge(t+s)}F\Big(\frac{1}{M_{(u-t)-}}\cdot\mu_{u-t}\Big)\Big];\ t<\tau\Big]$$

$$=E_x\Big[E_{\mu_t}\Big[\sum_{u\leqq\sigma_{xd}\wedge s}F\Big(\frac{1}{M_{u-}}\cdot\mu_u\Big)\Big];\ t<\tau\Big].$$

If we put $\mu_t=M_t\delta_{M_t}+\mu'$ where $M(\mu')<x(1-d)$, we have by the branching property that the integrand is equal to

$$E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}\Big[\sum_{u\leqq\sigma_{xd}(w_1+w_2)\wedge s}F\Big(\frac{1}{M_{u-}(w_1+w_2)}\cdot\mu_u(w_1+w_2)\Big)\Big]$$

$$=E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}\Big[\sum_{u\leqq\sigma_{xd}(w_1)\wedge s}F\Big(\frac{1}{M_{u-}(w_1)}\cdot(\mu_u(w_1)+\mu_u(w_2))\Big)\Big]$$

$$=E^{(1)}_{M_t}\otimes E^{(2)}_{\mu'}\Big[\sum_{u\leqq\sigma_{xd}(w_1)\wedge s}F\Big(\frac{1}{M_{u-}(w_1)}\cdot\mu_u(w_1)\Big)\Big]$$

$$=E_{M_t}\Big[\sum_{u\leqq\sigma_{xd}\wedge s}F\Big(\frac{1}{M_{u-}}\cdot\mu_u\Big)\Big]$$

since $F\in B^{\epsilon}_d$ and

$$M\Big(\frac{1}{M_{u-}(w_1)}\cdot\mu_u(w_2)\Big)=\frac{M_u(w_2)}{M_{u-}(w_1)}<\frac{x(1-d)}{xd}=\frac{1-d}{d}.$$

Thus, we have

$$\mathrm{II}=E_x\Big[E_{M_t}\Big[\sum_{u\leqq s\wedge\tau_p}F\Big(\frac{1}{M_{u-}}\cdot\mu_u\Big)\Big]_{p=xd/M_t};\ t<\tau\Big].$$

Applying the same argument as above, we have

$$\mathrm{III}=E_x[E_{\mu_t}[A(s-\sigma_{xd});\ s\geqq\sigma_{xd}];\ t<\tau]$$

$$=E_x[E_{M_t}[A(s-\tau_p);\ s\geqq\tau_p]_{p=xd/M_t};\ t<\tau].$$

Therefore, we have

$$A(t+s)=B(t)+E_x[A(t+s-\tau);\ t\geqq\tau]$$

$$+E_x\Big[\Big\{E_y\Big[\sum_{u\leqq s\wedge\tau_p}F\Big(\frac{1}{M_{u-}}\cdot\mu_u\Big)\Big]+E_y[A(s-\tau_p);\ s\geqq\tau_p]\Big\}_{\substack{y=M_t\\ p=xd/M_t}};\ t<\tau\Big].$$

Since the integrand of the third term is equal to $A(s)$ by Lemma 4.3 because $p\geqq d$, we have

$$A(t+s)=B(t)+E_x[A(t+s-\tau);\ t\geqq\tau]+E_x[A(s);\ t<\tau],$$

or

$$A(t+s)-A(t)-A(s)=E_x[A(t+s-\tau)-A(t-\tau)-A(s);\ t\geqq\tau].$$

Now, if we fix s and put $u(t)=|A(t+s)-A(t)-A(s)|$, then $u(t)$ satisfies the inequality:

$$u(t)\leqq E_x[u(t-\tau);\ t\geqq\tau].$$

By Lemma 2. 6, $u(t)$ is identically zero and this verifies

$$A(t+s)=A(t)+A(s).$$

By Lemma 4. 1 and Lemma 4. 4, $A_d^\varepsilon(t;\ F)$ is a linear function of t and hence can be written as

$$A_d^\varepsilon(t;\ F)=tL_d^\varepsilon(F) \tag{4. 6}$$

for all $F\in B_d^\varepsilon$. It follows from (4. 4), (4. 5), and Lemma 4. 2 that $L_d^\varepsilon(F)$ has the following properties:

(4. 7)
- (i) $L_d^\varepsilon(F)\geqq 0$ for all $F\in B_d^\varepsilon$,
- (ii) $L_d^\varepsilon(F+G)=L_d^\varepsilon(F)+L_d^\varepsilon(G)$ for all $F,\ G\in B_d^\varepsilon$,
- (iii) for any sequence $\{F_n\}$ in B_d^ε such that $F_n(\mu)\downarrow 0$ as $n\rightarrow\infty$, $L_d^\varepsilon(F_n)\downarrow 0$ as $n\rightarrow\infty$.

Moreover, we have by Lemma 4. 3 and by the definition of B_d^ε,

(4. 8)
- (iv) $L_d^\varepsilon(F)=L_{d'}^\varepsilon(F)$ for $F\in B_d^\varepsilon$, if $d\leqq d'$,
- (v) $L_d^\varepsilon(F)=L_d^{\varepsilon'}(F)$ for $F\in B_d^\varepsilon$, if $\varepsilon'\leqq\varepsilon$.

By (4. 7), there exists a finite measure Π_d^ε on $(\boldsymbol{M}_2, \varphi_d^{-1}(\mathscr{B}_2^d))$ such that

$$\begin{cases}\Pi_d^\varepsilon(\cdot\cap\boldsymbol{M}_{2,\varepsilon})=\Pi_d^\varepsilon(\cdot),\\ L_d^\varepsilon(F)=\displaystyle\int_{\boldsymbol{M}_2}F(\mu)\Pi_d^\varepsilon(d\mu)\end{cases} \tag{4. 9}$$

for all $F\in B_d^\varepsilon$, where $\boldsymbol{M}_{2,\varepsilon}=\boldsymbol{M}_2\cap\{\mu;\ M(\mu)\leqq 1-\varepsilon\}$. By the property (iv) of $L_d^\varepsilon(F)$, Π_d^ε is a restriction of $\Pi_{d'}^\varepsilon$ on $\varphi_d^{-1}(\mathscr{B}_2^d)$ if $d<d'$. Therefore, we see by Lemma 1. 5 that there exists a unique measure Π^ε on $\mathscr{B}_2$ such that

$$\begin{aligned}&\Pi^\varepsilon(\cdot\cap\boldsymbol{M}_{2,\varepsilon})=\Pi^\varepsilon(\cdot),\\ &\Pi_d^\varepsilon=\Pi^\varepsilon|_{\varphi_d^{-1}(\mathscr{B}_2^d)}.\end{aligned} \tag{4. 10}$$

Since (v) of (4. 8) implies

$$\Pi^\varepsilon(\cdot)=\Pi^{\varepsilon'}(\cdot\cap\boldsymbol{M}_{2,\varepsilon}) \tag{4. 11}$$

if $\varepsilon'<\varepsilon$, $\Pi(\cdot)=\lim_{\varepsilon\downarrow 0}\Pi^{\varepsilon}(\cdot)$ is a σ-finite measure on $(\boldsymbol{M}_2, \mathscr{B}_2)$ which is concentrated on the Borel set $\widetilde{\boldsymbol{M}}_2=\boldsymbol{M}_2\cap\{\mu\colon M(\mu)<1\}$. Moreover, we have

$$(4.12)\qquad L_d^{\varepsilon}(F)=\int_{\boldsymbol{M}_{2,\varepsilon}}F(\mu)\Pi_d^{\varepsilon}(d\mu)=\int_{\boldsymbol{M}_{2,\varepsilon}}F(\mu)\Pi^{\varepsilon}(d\mu)=\int_{\widetilde{\boldsymbol{M}}_2}F(\mu)\Pi(d\mu)$$

for all $F\in B_d^{\varepsilon}$.

Now, since $A_d^{\varepsilon}(t;F)=t\,L_d^{\varepsilon}(F)$ is a solution of the equation (4.2) for $F\in B_d^{\varepsilon}$, we have

$$tL_d^{\varepsilon}(F)=E_x\Big[\sum_{s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]+E_x[(t-\tau)L_d^{\varepsilon}(F);\ t\geqq\tau]$$

and hence

$$(4.13)\qquad E_x\Big[\sum_{s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]=E_x(\tau_t)L_d^{\varepsilon}(F)=E_x(\tau_t)\int_{\widetilde{\boldsymbol{M}}_2}F(\mu)\Pi(d\mu)$$

for all $F\in\bigcup_{0<\varepsilon<1/2}B_d^{\varepsilon}$. Let $B_d(\boldsymbol{M}_2)$ be the set of all nonnegative, bounded and $\varphi_d^{-1}(\mathscr{B}_2^d)$-measurable function F on $\boldsymbol{M}_2$ such that $F(\mu)=0$ if $M(\mu)=1$. Then, (4.13) is valid for all $F\in B_d(\boldsymbol{M}_2)$.

Finally we shall show that the support of Π is on $\boldsymbol{M}_1-\{\delta_1\}$. Set $E_d=\widetilde{\boldsymbol{M}}_2\cap\{\mu\in\boldsymbol{M}_2;\ \|\varphi_d(\mu)\|>1\}$, then $E_d\in\varphi_d^{-1}(\mathscr{B}_2^d)$, $\chi_{E_d}\in B_d(\boldsymbol{M}_2)$[8] and $\sum_{s\leqq\tau_t}\chi_{E_d}((1/M_{s-})\cdot\mu_s)=0$ a.s. (P_x) because for $s\leqq\tau_d$

$$\Big\|\varphi_d\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big\|\leq\frac{1}{M_{s-}}\int_{(x(1-d),1]}\mu_s(dx)\leqq 1\qquad\text{a.s. }(P_x).$$

Therefore, by putting $F=\chi_{E_d}$ in (4.13), we have $\Pi(E_d)=0$ because $E_x(\tau_t)>0$ for $t>0$. Thus, it follows $\Pi(\lim_{d\uparrow 1}E_d)=0$, where $\lim_{d\uparrow 1}E_d=\bigcup_{2/3<d<1}E_d=\widetilde{\boldsymbol{M}}_2\wedge\{\mu\in\boldsymbol{M}_2;\ \|\mu\|>1\}$. Therefore, the measure Π is concentrated on the set $\boldsymbol{M}_1-\{\delta_1\}=\widetilde{\boldsymbol{M}}_2\cap\{\mu\in\boldsymbol{M}_2;\ \|\mu\|\leqq 1\}$, or we can consider Π as a Borel measure on $\boldsymbol{M}_1-\{\delta_1\}$.

THEOREM 4.1. *There exists a Borel measure $\Pi(d\mu)$ on $\boldsymbol{M}_1-\{\delta_1\}$ such that*

$$(4.14)\qquad E_x\Big[\sum_{s\leqq\tau_d\wedge t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]=E_x(\tau_d\wedge t)\int_{\boldsymbol{M}_1-\{\delta_1\}}F(\mu)\Pi(d\mu)$$

for all $F\in B_d(\boldsymbol{M}_2)$ and for d $(2/3<d<1)$. In addition, for any nonnegative function $g\in B(\bar{S})$ satisfying $g(1)=0$,

$$(4.15)\qquad \int_{\boldsymbol{M}_1-\{\delta_1\}}g(M(\mu))\Pi(d\mu)=\int_{[0,1)}g(a)k(da),$$

or $k(da)=\int_{M(\mu)\in da}\Pi(d\mu)$, where $k(da)$ is the measure on $[0,1)$ defined in Proposition

8) $\chi_E(\mu)$ is an indicator function of a set E, i.e. $\chi_E(\mu)=1$ if $\mu\in E$, $=0$ if $\mu\notin E$.

3. 1. *Moreover,*

$$\int_{M_1-\{\delta_1\}}(1-M(\mu))\Pi(d\mu)<+\infty. \tag{4.16}$$

Proof. The first statement (4. 14) is already proved. Since the condition (4. 16) follows directly from (4. 15) and (3. 15), we shall show (4. 15) only. First of all, we choose a nonnegative function $g\in B(\bar{S})$ such that $g(x)=0$ for $1-\varepsilon<x\leqq 1$ and $g(x)$ is a constant for $0\leqq x\leqq(1-d)/d$. Then, $g(M(\mu))\in B_d(\boldsymbol{M}_2)$ and (4. 14) implies

$$E_x\left[\sum_{s\leqq\tau_t}g\left(\frac{M_s}{M_{s-}}\right)\right]=E_x\left[\sum_{s\leqq\tau_t}g\left(M\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right)\right]=E_x(\tau_t)\int_{M_1-\{\delta_1\}}g(M(\mu))\Pi(d\mu).$$

On the other hand, we have

$$E_x\left[\sum_{s\leqq\tau_t}g\left(\frac{M_s}{M_{s-}}\right)\right]=E_x^0\left[\sum_{s\leqq\tau_t^0}g\left(\frac{x_s}{x_{s-}}\right)\right]$$

and $E_x(\tau_t)=E_x^0(\tau_t^0)$ since the two processes $(x_t, P_x^0, t\leqq\tau_d^0)$ and $(M_t, P_x, t\leqq\tau_d)$ are equivalent as seen in §3. Moreover, since (3. 17) implies

$$E_x^0\left[\sum_{s\leqq\tau_t^0}g\left(\frac{x_s}{x_{s-}}\right)\right]=E_x^0(\tau_t^0)\int_{[0,1)}g(a)k(da),$$

we have (4. 15). Finally, since (4. 15) does not depend explicitly on d and ε, we have (4. 15), by letting $d\uparrow 1$ and then $\varepsilon\downarrow 0$, for all nonnegative functions $g\in B(\bar{S})$ satisfying $g(1)=0$.

Π is in general σ-finite and uniquely determined by (4. 14). We shall call Π a *branching measure* of the cascade process (μ_t).

§ 5. Fundamental equation for a cascade semigroup.

In the preceding sections we have constructed the underlying process (x_t) and the branching measure Π of a given cascade semigroup $\{T_t\}$. We intend here to obtain a system of integral equations which are satisfied by $T_t\hat{f}$ in terms of (x_t) and Π.

It has been proved in the proof of Lemma 4. 3 that $A_d^\varepsilon(t;\, F)=tL_d^\varepsilon(F)$ satisfies the equation:

$$A_d^\varepsilon(t;\, F)=E_x\left[\sum_{s\leqq\rho\wedge t}F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]+E_x[A_d^\varepsilon(t-\rho;\, F);\, t\geqq\rho]$$

for $F\in B_d^\varepsilon$ and for any Markov time ρ such as $\rho\leqq\tau_d$. Hence we have

$$E_x\left[\sum_{s\leqq\rho\wedge t}F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]=E_x(\rho\wedge t)\Pi(F)$$

for such ρ and $F\in B_d(\boldsymbol{M}_2)$, where

$$\Pi(F)=L_d^{\varepsilon}(F)=\int_{\boldsymbol{M}_1-\{\delta_1\}} F(\mu)\Pi(d\mu).$$

Thus, by letting $t\to\infty$, we have

$$E_x\Big[\sum_{s\leqq\rho} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]=E_x(\rho)\Pi(F). \tag{5.1}$$

LEMMA 5.1. *For $F\in B_d(\boldsymbol{M}_2)$ and $g\in B(\bar{S})$,*

$$E_x\Big[\sum_{s\leqq\tau_t} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)g(M_{s-})\Big]=E_x\Big[\int_0^{\tau_t} g(M_s)ds\Big]\Pi(F). \tag{5.2}$$

Proof. It is sufficient to show (5.2) for $F\in B_d^{\varepsilon}$ and $g\in C(\bar{S})$, $g\geqq 0$. We define for any $\varepsilon>0$ a sequence $\{\rho_n\}_{n\geqq 0}$ of Markov times by

$$\begin{cases} \rho_0\equiv 0, \\ \rho_1=\begin{cases}\inf\ \{s\colon\ |g(M_s)-g(M_0)|\geqq\varepsilon\} \\ +\infty, \qquad \text{if}\ \ \{\cdots\}=\phi,\end{cases} \\ \rho_{n+1}(w)=\rho_n(w)+\rho_1(w_{\rho_n}^+)\ \text{ for } n\geqq 1,\end{cases}$$

where $w_{\rho_n}^+$ means $w_{\rho_n}^+(t)=w(\rho_n+t)$ for $t\geqq 0$. Since $g(M_s)$ has a left limit at each $s\geqq 0$, we have $\rho_n\to\infty$ as $n\to\infty$. Now, we decompose the left-hand side of (5.2) into the sum:

$$E_x\Big[\sum_{s\leqq\tau_t} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)g(M_{s-})\Big]=\sum_{n=0}^{\infty}E_x\Big[\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)g(M_{s-})\Big],$$

while

$$\begin{aligned} I_n &\equiv E_x\Big[g(M_{\rho_n})\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big] \\ &=E_x\Big[g(M_{\rho_n})\sum_{\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big);\ \rho_n<\tau_t\Big] \\ &=E_x\Big[g(M_{\rho_n})E_{\mu_{\rho_n}}\Big[\sum_{s\leqq((p+\tau_q)\wedge t)\wedge(p+\rho_1)-p} F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)\Big]_{\substack{p=\rho_n\\ q=xd/M_{\rho_n}}};\ \rho_n<\tau_t\Big]\end{aligned}$$

since $\tau(w)=\rho_n(w)+\tau_q(w_{\rho_n}^+)$, $q=xd/M_{\rho_n}$ if $\rho_n(w)<\tau_t(w)$. Since we can replace $E_{\mu_{\rho_n}}$ in the above integrand by $E_{M_{\rho_n}}$ because $F\in B_d(\boldsymbol{M}_2)$ and since we have $((p+\tau_q)\wedge t)\wedge(p+\rho_1)-p\leqq\tau_q\leqq\tau_d$, if follows from (5.1) that

$$I_n=E_x[g(M_{\rho_n})E_{M_{\rho_n}}[((p+\tau_q)\wedge t)\wedge(p+\rho_1)-p]_{\substack{p=\rho_n\\ q=xd/M_{\rho_n}}}\cdot\Pi(F);\ \rho_n<\tau_t].$$

Now, replacing $E_{M_{\rho_n}}$ by $E_{\mu_{\rho_n}}$ again and using a Markov property at time ρ_n,

$$I_n = E_x[g(M_{\rho_n})(\rho_{n+1}\wedge\tau_t-\rho_n)\cdot\Pi(F);\ \rho_n<\tau_t]$$
$$= E_x\left[\int_{\rho_n\wedge\tau_t}^{\rho_{n+1}\wedge\tau_t} g(M_{\rho_n})ds\right]\cdot\Pi(F).$$

Since we have in addition

$$\sum_{n=0}^{\infty} E_x\left[\int_{\rho_n\wedge\tau_t}^{\rho_{n+1}\wedge\tau_t} g(M_s)ds\right]\cdot\Pi(F)=E_x\left[\int_0^{\tau_t} g(M_s)ds\right]\cdot\Pi(F),$$

we can carry out the following estimation:

$$\left|E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})\right]-E_x\left[\int_0^{\tau_t} g(M_s)ds\right]\cdot\Pi(F)\right|$$
$$\leqq\left|\sum_{n=0}^{\infty} E_x\left[\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})\right]\right.$$
$$\left.-\sum_{n=0}^{\infty} E_x\left[g(M_{\rho_n})\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]\right|$$
$$+\left|\sum_{n=0}^{\infty} E_x\left[g(M_{\rho_n})\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]-E_x\left[\int_0^{\tau_t} g(M_s)ds\right]\Pi(F)\right|$$
$$\leqq\sum_{n=0}^{\infty} E_x\left[\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)|g(M_{s-})-g(M_{\rho_n})|\right]$$
$$+\sum_{n=0}^{\infty} E_x\left[\int_{\rho_n\wedge\tau_t}^{\rho_{n+1}\wedge\tau_t} |g(M_{\rho_n})-g(M_s)|ds\right]\cdot\Pi(F)$$
$$\leqq\varepsilon\sum_{n=0}^{\infty} E_x\left[\sum_{\tau_t\wedge\rho_n<s\leqq\tau_t\wedge\rho_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]+\varepsilon\sum_{n=0}^{\infty} E_x\left[\int_{\rho_n\wedge\tau_t}^{\rho_{n+1}\wedge\tau_t} ds\right]\Pi(F)$$
$$=\varepsilon E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)\right]+\varepsilon E_x(\tau_t)\cdot\Pi(F)$$
$$=2\varepsilon E_x(\tau_t)\Pi(F)$$
$$\leqq\varepsilon(2t\Pi(F)).$$

Since ε is arbitrary, we have (5. 2).

Define a mapping ψ on $\boldsymbol{M}_2^d$ into $\boldsymbol{M}_2$ such that $\varphi_d(\psi(\nu))=\nu$ and $\psi(\nu)(S-S_d)=0$ for $\nu\in\boldsymbol{M}_2^d$.

LEMMA 5. 2. *The mapping* ψ: $(\boldsymbol{M}_2^d, \mathcal{B}_2^d)\to(\boldsymbol{M}_2, \mathcal{B}_2)$ *is measurable.*

Proof. Put $\mathcal{A}=\{B\in\mathcal{B}_2: \psi^{-1}(B)\in\mathcal{B}_2^d\}$, then $\mathcal{A}$ is a sub σ-field of $\mathcal{B}_2$. To show the measurability of ψ, it is sufficient to show $\mathcal{A}\supset\mathcal{B}_2$. For $f\in C_0^*$ and $\nu\in \boldsymbol{M}_2^d$,

$$\hat{f}(\psi(\nu))=\exp\Big(\int_S \frac{1}{x}\log f(x)\psi(\nu)(dx)\Big)$$
$$=\exp\Big(\int_{S_d}\frac{1}{x}\log f(x)\nu(dx)\Big),$$

and hence $\hat{f}(\psi(\nu))$ is a $\mathcal{B}_2^d$-measurable function of ν. Thus, it follows from Proposition 1.2 that $F\circ\psi$ is $\mathcal{B}_2^d$-measurable for all $F\in C(\boldsymbol{M}_2)$. Since $\psi^{-1}(F^{-1}(E))=(F\circ\psi)^{-1}(E)\in\mathcal{B}_2^d$ for $E\in\mathcal{B}(R^1)$, we have $F^{-1}(E)\in\mathcal{A}$ and hence $F\in C(\boldsymbol{M}_2)$ is always $\mathcal{A}$-measurable, and this implies $\mathcal{B}_2\subset\mathcal{A}$.

By Lemma 5.2, $\bar{\varphi}_d=\psi\circ\varphi_d$ is a measurable mapping of $(\boldsymbol{M}_2, \varphi_d^{-1}(\mathcal{B}_2^d))$ into $(\boldsymbol{M}_2, \mathcal{B}_2)$. Therefore, $B_d(\boldsymbol{M}_2)$ is coincident with the totality of nonnegative bounded Borel measurable functions F on $\boldsymbol{M}_2$ which satisfy the relation $F=F\circ\bar{\varphi}_d$ and $F(\mu)=0$ if $M(\mu)=1$.

LEMMA 5.3.

$$E_x\Big[\sum_{s\leqq\tau_d\wedge t}F\Big(\bar{\varphi}_d\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big), M_{s-}, s\Big)\Big]=E_x\Big[\int_0^{\tau_d\wedge t}ds\int_{\boldsymbol{M}_1-\{\delta_1\}}\Pi(d\mu)F(\bar{\varphi}_d(\mu), M_s, s)\Big] \tag{5.3}$$

for any nonnegative bounded measurable function $F(\mu, y, s)$ on the product measurable space $\boldsymbol{M}_2\times\bar{S}\times[0,\infty]$ which satisfies $F(\mu, y, s)=0$ if $M(\mu)=1$.

Proof. Since $F\circ\bar{\varphi}_d\in B_d(\boldsymbol{M}_2)$ is valid for a nonnegative bounded Borel measurable function F on $\boldsymbol{M}_2$ which satisfies $F(\mu)=0$ if $M(\mu)=1$, we have the equality (5.3) if we show the following:

$$E_x\Big[\sum_{s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)g(M_{s-})h(s)\Big]=E_x\Big[\int_0^{\tau_t}g(M_s)h(s)ds\Big]\Pi(F) \tag{$*$}$$

for $F\in B_d^s$, $g\in B(\bar{S})$, $g\geqq 0$, and $h\in C([0,\infty])$, $h\geqq 0$.

Take $\delta>0$ for a given $\varepsilon>0$ such that $|h(s_1)-h(s_2)|<\varepsilon$ whenever $|s_1-s_2|<\delta$. Then, we divide the time axis into $0=t_0<t_1<t_2<\cdots<t_n<\cdots\to\infty$ such that $t_{n+1}-t_n<\delta$ for all $n\geqq 0$, and decompose the both sides of (*) as

$$E_x\Big[\sum_{s\leqq\tau_t}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)g(M_{s-})h(s)\Big]=\sum_{n=0}^{\infty}E_x\Big[\sum_{\tau_t\wedge t_n<s\leqq\tau_t\wedge t_{n+1}}F\Big(\frac{1}{M_{s-}}\cdot\mu_s\Big)g(M_{s-})h(s)\Big],$$

$$E_x\Big[\int_0^{\tau_t}g(M_s)h(s)ds\Big]\Pi(F)=\sum_{n=0}^{\infty}E_x\Big[\int_{\tau_t\wedge t_n}^{\tau_t\wedge t_{n+1}}g(M_s)h(s)ds\Big]\Pi(F).$$

Since, by the equality (5.2),

$$E_x\left[\sum_{\tau_t\wedge t_n<s\leqq\tau_t\wedge t_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})h(t_{n+1})\right]$$

$$=h(t_{n+1})\left\{E_x\left[\int_0^{\tau_t\wedge t_{n+1}} g(M_s)ds\right]\Pi(F)-E_x\left[\int_0^{\tau_t\wedge t_n} g(M_s)ds\right]\Pi(F)\right\}$$

$$=E_x\left[\int_{\tau_t\wedge t_n}^{\tau_t\wedge t_{n+1}} g(M_s)h(t_{n+1})ds\right]\Pi(F),$$

we can estimate the following:

$$\left|E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})h(s)\right]-E_x\left[\int_0^{\tau_t} g(M_s)h(s)ds\right]\Pi(F)\right|$$

$$\leqq\left|E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})h(s)\right]-\sum_{n=0}^{\infty}E_x\left[\sum_{\tau_t\wedge t_n<s\leqq\tau_t\wedge t_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})h(t_{n+1})\right]\right|$$

$$+\left|\sum_{n=0}^{\infty}E_x\left[\sum_{\tau_t\wedge t_n<s\leqq\tau_t\wedge t_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})h(t_{n+1})\right]-E_x\left[\int_0^{\tau_t} g(M_s)h(s)ds\right]\Pi(F)\right|$$

$$\leqq\sum_{n=0}^{\infty}E_x\left[\sum_{\tau_t\wedge t_n<s\leqq\tau_t\wedge t_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})|h(s)-h(t_{n+1})|\right]$$

$$+\sum_{n=0}^{\infty}E_x\left[\int_{\tau_t\wedge t_n}^{\tau_t\wedge t_{n+1}} g(M_s)|h(t_{n+1})-h(s)|ds\right]\Pi(F)$$

$$\leqq\varepsilon\sum_{n=0}^{\infty}E_x\left[\sum_{\tau_t\wedge t_n<s\leqq\tau_t\wedge t_{n+1}} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})\right]+\varepsilon\sum_{n=0}^{\infty}E_x\left[\int_{\tau_t\wedge t_n}^{\tau_t\wedge t_{n+1}} g(M_s)ds\right]\Pi(F)$$

$$=\varepsilon E_x\left[\sum_{s\leqq\tau_t} F\left(\frac{1}{M_{s-}}\cdot\mu_s\right)g(M_{s-})\right]+\varepsilon E_x\left[\int_0^{\tau_t} g(M_s)ds\right]\Pi(F)$$

$$=2\varepsilon E_x\left[\int_0^{\tau_t} g(M_s)ds\right]\Pi(F)$$

$$\leqq\varepsilon(2t\|g\|\Pi(F)),$$

where $\|g\|=\sup_{0\leqq x\leqq 1}|g(x)|<+\infty$. Since $\varepsilon>0$ is arbitrary, we have the equality (*).

COROLLARY.

$$E_x\left[\sum_{s\leqq\tau_d\wedge t} G(\bar{\varphi}_d(\mu_s),\,s)\chi(M_s<xd\leqq M_{s-})\right]$$

(5.4)

$$=E_x\left[\int_0^{\tau_d\wedge t} ds\int_{\boldsymbol{M}_1-\{\delta_1\}} \Pi(d\mu)G(\bar{\varphi}_d(M_s\cdot\mu),\,s)\chi(M_sM(\mu)<xd)\right]$$

for any nonnegative bounded measurable function G on the product measurable space $\boldsymbol{M}_2\times[0,\infty]$.

Proof. Putting $F(\mu, y, s)=G(\bar{\varphi}_d(y\cdot\mu), s)\chi(M(y\cdot\mu)<xd\leqq y)$ in (5. 3) and noting that $\bar{\varphi}_d(y\cdot\mu)=\bar{\varphi}_d(y\cdot\bar{\varphi}_d(\mu))$ and $\chi(M(y\cdot\mu)<xd)=\chi(M(y\cdot\bar{\varphi}_d(\mu))<xd)$, (5. 4) is easily shown.

THEOREM 5. 1. *For $f\in B_d^*$, $u_t(x)=T_t\hat{f}(x\delta_x)$ is a solution of the following (S_d)-equation*:

$$(S_d)\qquad \begin{aligned} u_t(x)=&E_x^0[f(x_t);\ t<\tau_d^0]+E_x^0[u_{t-\tau_d^0}(x_{\tau_d^0});\ x_{\tau_d^0}=xd,\quad \tau_d^0\leqq t]\\ &+E_x^0\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\hat{u}_{t-s}(x_s\cdot\mu)\right]. \end{aligned}$$

Proof. Put $\tau_d=\tau$ and $\tau_d^0=\tau^0$.

$$\begin{aligned} T_t\hat{f}(x\delta_x)&=E_x[\hat{f}(\mu_t)]\\ &=E_x[\hat{f}(\mu_t);\ t<\tau]+E_x[\hat{f}(\mu_t);\ t\geqq\tau]\\ &=\mathrm{I}+\mathrm{II},\quad \text{say.} \end{aligned}$$

From $f\in B_d^*$,

$$\begin{aligned} \mathrm{I}&=E_x[\hat{f}(M_t\delta_{M_t});\ t<\tau]=E_x[f(M_t);\ t<\tau]\\ &=E_x^0[f(x_t);\ t<\tau^0] \end{aligned}$$

by Lemma 3. 7.

$$\begin{aligned} \mathrm{II}&=E_x[E_{\mu_\tau}[\hat{f}(\mu_{t-s})]_{s=\tau};\ t\geqq\tau]\\ &=E_x[T_{t-\tau}\hat{f}(\mu_\tau);\ t\geqq\tau]\\ &=E_x[T_{t-\tau}\hat{f}(\mu_\tau);\ M_\tau=xd,\ \tau\leqq t]+E_x[T_{t-\tau}\hat{f}(\mu_\tau);\ M_\tau<xd,\ \tau\leqq t]\\ &=\mathrm{II}_1+\mathrm{II}_2,\quad \text{say.} \end{aligned}$$

Since $(T_t\hat{f})|_S\in B_d^*$ and $T_t\hat{f}=\widehat{(T_t\hat{f})|_S}$ for $f\in B_d^*$, we have by Lemma 3. 8

$$\begin{aligned} \mathrm{II}_1&=E_x[T_{t-\tau}\hat{f}(M_\tau\delta_{M_\tau});\ M_\tau=xd,\ \tau\leqq t]\\ &=E_x^0[T_{t-\tau^0}\hat{f}(x_{\tau^0}\delta_{x_{\tau^0}});\ x_{\tau^0}=xd,\ \tau^0\leqq t]. \end{aligned}$$

Finally, we begin with rewriting the term II_2:

$$\begin{aligned} \mathrm{II}_2&=E_x[T_{t-\tau_t}\hat{f}(\mu_{\tau_t});\ M_{\tau_t}<xd]\\ &=E_x\left[\sum_{s\leqq\tau_t}T_{t-s}\hat{f}(\mu_s)\chi(M_s<xd)\right]. \end{aligned}$$

Since $T_t\hat{f}\circ\bar{\varphi}_d=T_t\hat{f}$,

$$\mathrm{II}_2=E_x\left[\sum_{s\leqq\tau_t} T_{t-s}\hat{f}(\bar{\varphi}_d(\mu_s))\chi(M_s<xd\leqq M_{s-})\right],$$

and hence applying (5. 4) for $G(\mu, s)=T_{t-s}\hat{f}(\mu)$,

$$\mathrm{II}_2 = E_x\left[\int_0^{\tau_t} ds\int_{M_1-\{\delta_1\}} \Pi(d\mu)T_{t-s}\hat{f}(\bar{\varphi}_d(M_s\cdot\mu))\chi(M_sM(\mu)<xd)\right]$$

$$= E_x\left[\int_\delta^{\tau_t} ds\int_{M_1-\{\delta_1\}} \Pi(d\mu)T_{t-s}\hat{f}(M_s\cdot\mu)\chi(M_sM(\mu)<xd)\right]$$

$$= E_x^0\left[\int_0^{\tau_t^0} ds\int_{M_1-\{\delta_1\}} \Pi(d\mu)T_{t-s}\hat{f}(x_s\cdot\mu)\chi(x_sM(\mu)<xd)\right]$$

$$= E_x^0\left[\int_0^{t} ds\int_{M_1-\{\delta_1\}} \Pi(d\mu)T_{t-s}\hat{f}(x_s\cdot\mu)\chi(x_sM(\mu)<xd<x_s)\right]$$

by Lemma 3. 7.

Since $\hat{u}_t(\mu)=T_t\hat{f}(\mu)$ for $u_t(x)=T_t\hat{f}(x\delta_x)$, it follows from the above arguments that $u_t(x)$ satisfies the (S_d)-equation.

By putting $f\equiv 1$ in Theorem 5. 1, we have

$$P_x^0(x_{\tau_d^0}=xd,\ \tau_d^0\leqq t)+E_x^0\left[\int_0^t ds\int_{M_1-\{\delta_1\}} \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\right]=P_x^0(\tau_d^0\leqq t). \tag{5. 5}$$

Put $C_d^*=\{f\in C_0^*;\ f(x)=1 \text{ for } 0<x\leqq(1-d)/d\}$, then $C_d^*\subset B_d^*$ and $\bigcup_{2/3<d<1} C_d^*=C_0^*$.

THEOREM 5. 2. *Two cascade semigroups* $\{T_t^{(1)}\}$ *and* $\{T_t^{(2)}\}$ *coincide*, i.e. $T_t^{(1)}=T_t^{(2)}$ *for all* $t\geqq 0$, *if the underlying processes and the branching measures are both coincident, respectively.*

Proof. Let (x_t, P_x^0) and Π be the underlying process and the branching measure, respectively. By putting $\tau_d^0=\tau^0$, it follows from Theorem 5. 1 that

$$|T_t^{(1)}\hat{f}(x\delta_x)-T_t^{(2)}\hat{f}(x\delta_x)|$$

$$\leqq E_x^0[|T_{t-\tau^0}^{(1)}\hat{f}(x_{\tau^0}\delta_{x_{\tau^0}})-T_{t-\tau^0}^{(2)}\hat{f}(x_{\tau^0}\delta_{x_{\tau^0}})|;\ x_{\tau^0}=xd,\ \tau^0\leqq t]$$

$$+E_x^0\left[\int_0^t ds\int_{M_1-\{\delta_1\}} \Pi(d\mu)|T_{t-s}^{(1)}\hat{f}(x_s\cdot\mu)-T_{t-s}^{(2)}\hat{f}(x_s\cdot\mu)|\chi(x_sM(\mu)<xd<x_s)\right]$$

for any $f\in C_d^*$. Since $\bar{\varphi}_d(\mu)=\sum_{x_i\in S_d} x_i\delta_{x_i}$ for $\mu=\sum x_i\delta_{x_i}\in \boldsymbol{M}_1$ and the number of x_i's in S_d is not greater than $d/(1-d)$, and since in general

$$\left|\prod_{i=1}^n f(x_i)-\prod_{i=1}^n g(x_i)\right|\leqq n\|f-g\|$$

for any f and g such as $\|f\|\leqq 1$ and $\|g\|\leqq 1$, we have

$$\sup_{\substack{\mu\in \boldsymbol{M}_1 \\ s\leqq t\leqq t_0}} |T_{t-s}^{(1)}\hat{f}(x_s\cdot\mu)-T_{t-s}^{(2)}\hat{f}(x_s\cdot\mu)|\leqq \frac{d}{1-d}a_{t_0},$$

where

$$a_{t_0}=\sup_{\substack{x\in S \\ 0\leqq t\leqq t_0}} |T_t^{(1)}\hat{f}(x\delta_x)-T_t^{(2)}\hat{f}(x\delta_x)|.$$

Therefore, we have by (5. 5)

$$a_{t_0}\leqq \frac{d}{1-d}a_{t_0}\cdot P_x^0(\tau^0\leqq t_0),$$

and in addition $a_{t_0}=0$ for a sufficiently small $t_0>0$ because $\lim_{t_0\downarrow 0} P_x^0(\tau^0\leqq t_0)=0$. Thus, it holds $T_t^{(1)}\hat{f}(x\delta_x)=T_t^{(2)}\hat{f}(x\delta_x)$, and hence

$$T_t^{(1)}\hat{f}=(\widehat{T_t^{(1)}\hat{f})|_S}=(\widehat{T_t^{(2)}\hat{f})|_S}=T_t^{(2)}\hat{f}$$

for all t such as $0\leqq t\leqq t_0$, where t_0 depends only on d. Since both $\{T_t^{(1)}\}$ and $\{T_t^{(2)}\}$ are branching semigroups (i.e. semigroups with branching property) and $(T_t^{(i)}\hat{f})|_S\in C_d^*$ $(i=1, 2)$ if $f\in C_d^*$, the equality $T_t^{(1)}\hat{f}=T_t^{(2)}\hat{f}$ holds for all $0\leqq t<\infty$ and $f\in C_d^*$. Moreover, it holds $T_t^{(1)}F=T_t^{(2)}F$ for all $F\in C(\boldsymbol{M}_1)$ and $0\leqq t<\infty$ by Proposition 1. 2 and the fact $C_0^*=\bigcup_{2/3<d<1} C_d^*$.

§ 6. Uniqueness of the underlying process and the branching measure.

We shall show in this section that the underlying process and the branching measure of a given cascade semigroup are uniquely determined by the system of (S_d)-equations. The meaning of the assertion is formulated in the following (Theorem 6. 1).

Let $\tilde{\Pi}(d\mu)$ be a measure on $\boldsymbol{M}_1-\{\delta_1\}$ such that

$$\int_{\boldsymbol{M}_1-\{\delta_1\}}(1-M(\mu))\tilde{\Pi}(d\mu)<+\infty \tag{6. 1}$$

and $\tilde{m}$ a nonnegative constant. Then, define a Markov process $(\tilde{x}_t, \tilde{P}_x)$ on $\bar{S}=[0, 1]$ whose infinitesimal generator $\tilde{A}$ on $C[0, 1]$ has a core $C^1[0, 1]$ and is given by

$$\tilde{A}f(x)=-\tilde{m}xf'(x)+\int_{[0,1)}\tilde{k}(da)(f(xa)-f(x)) \tag{6. 2}$$

for all $f\in C^1[0, 1]$, where

$$\tilde{k}(da)=\int_{\{\mu:\ M(\mu)\in da\}}\tilde{\Pi}(d\mu). \tag{6. 3}$$

Note that the underlying process and the branching measure satisfy all of the above conditions (see Proposition 3.1, (4.15) and (4.16)). Moreover we remark two relations:

$$\tilde{E}_{ax}[f(\tilde{x}_t)]=\tilde{E}_x[f(a\tilde{x}_t)] \tag{6.4}$$

for $a\in\bar{S}$ and $f\in B(\bar{S})$, and

$$\tilde{E}_x\left[\sum_{s\leqq\rho} g(\tilde{x}_{s-}, \tilde{x}_s)\right]=\tilde{E}_x\left[\int_0^\rho ds\int_{[0,1)} g(\tilde{x}_s, a\tilde{x}_s)\tilde{k}(da)\right] \tag{6.5}$$

for a Markov time ρ and $g\in B([0, 1]\times[0, 1])$ such as $g\geqq 0$ and $g(x, x)=0$.

Set

$$\begin{aligned}\tilde{S}_d(x, t; f, u)&=\tilde{E}_x[f(\tilde{x}_t); t<\tilde{\tau}_d]+\tilde{E}_x[u_{t-\tilde{\tau}_d}(\tilde{x}_{\tilde{\tau}_d}); \tilde{x}_{\tilde{\tau}_d}=xd, \tilde{\tau}_d\leqq t]\\ &\quad+\tilde{E}_x\left[\int_0^t ds\int_{M_1-\{\delta_1\}} \tilde{\Pi}(d\mu)\chi(\tilde{x}_s M(\mu)<xd<\tilde{x}_s)\hat{u}_{t-s}(\tilde{x}_s\cdot\mu)\right],\end{aligned} \tag{6.6}$$

where

$$\tilde{\tau}_d=\inf\{s: \tilde{x}_s/\tilde{x}_0\leqq d\}, \quad =+\infty \text{ if } \{\cdots\}=\phi. \tag{6.7}$$

THEOREM 6.1. *Let* $\{(\tilde{x}_t, \tilde{P}_x), \tilde{\Pi}(d\mu)\}$ *be a pair of a Markov process on* $\bar{S}$ *and a measure on* $M_1-\{\delta_1\}$ *which satisfy* (6.1), (6.2), *and* (6.3), *and* T_t *be a cascade semigroup. Then,* $(\tilde{x}_t, \tilde{P}_x)$ *is the underlying process and* $\tilde{\Pi}(d\mu)$ *is the branching measure of the cascade semigroup* T_t *if, for any* d $(2/3<d<1)$ *and for any* $f\in C_d^*$, $u_t(x; f)=T_t\hat{f}(x\delta_x)$ *is a solution of the equation*:

$$u_t(x; f)=\tilde{S}_d(x, t; f, u). \tag{6.8}$$

In view of Proposition 3.1, Theorem 4.1, and Theorem 5.1, it is sufficient for the proof of Theorem 6.1 to show: Let $\{(x_t^{(i)}, P_x^{(i)}), \Pi^{(i)}(d\mu)\}$ $(i=1, 2)$ be two pairs satisfying (6.1), (6.2), and (6.3). Then $(x_t^{(1)}, P_x^{(1)})=(x_t^{(2)}, P_x^{(2)})$ and $\Pi^{(1)}(d\mu)=\Pi^{(2)}(d\mu)$ if, for any d $(2/3<d<1)$ and for any $f\in C_d^*$, $u_t(x; f)=T_t\hat{f}(x\delta_x)$ satisfies two equations: $u_t(x; f)=S_d^{(i)}(x, t; f, u)$ $(i=1, 2)$ where T_t is a given cascade semigroup.

We need several lemmas for the proof.

LEMMA 6.1. $P_x^{(1)}(\tau_d^{(1)}\leqq t)=P_x^{(2)}(\tau_d^{(2)}\leqq t)$ *for all* $t\geqq 0$, *where* $\tau_d^{(i)}$*'s are defined by* (6.7) *for* $x_s^{(i)}$*'s.*

Proof. Put $\tau_d^{(i)}=\tau^{(i)}$ $(i=1, 2)$ for a fixed d $(2/3<d<1)$. Since $\tau^{(i)}(a\cdot w^{(i)})=\tau^{(i)}(w^{(i)})$ by the definition, $P_x^{(i)}(\tau^{(i)}\leqq t)$ does not depend on $x\in S$ by (6.4). Hence, it suffices to prove the lemma for $x=1$. Taking a sequence $\{f_n\}\subset C_d^*$ such that

$$f_n(x)\downarrow f(x)=\begin{cases}1, & 0<x\leqq d,\\ 0, & d<x\leqq 1,\end{cases}$$

we have

$$E_1^{(i)}[f_n(x_t^{(i)});\ t<\tau^{(i)}] \downarrow E_1^{(i)}[f(x_t^{(i)});\ t<\tau^{(i)}]=0.$$

Since $u_t(x;\ f_n)=T_t\hat{f}_n(x\delta_x)=1$ for $x\leqq d$, we have

$$E_1^{(i)}[u_{t-\tau^{(i)}}(x_{\tau(i)}^{(i)};\ f_n);\ x_{\tau(i)}^{(i)}=d,\ \tau^{(i)}\leqq t]=P_1^{(i)}(x_{\tau(i)}^{(i)}=d,\ \tau^{(i)}\leqq t)$$

and

$$E_1^{(i)}\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<x_s^{(i)})\,\widehat{u_{t-s}(\cdot\,;\ f_n)}\,(x_s^{(i)}\cdot\mu)\right]$$

$$=E_1^{(i)}\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<x_s^{(i)})\right]$$

$$=E_1^{(i)}\left[\int_0^t ds\int_{[0,1)}k^{(i)}(da)\chi(x_s^{(i)}a<d<x_s^{(i)})\right]$$

$$=E_1^{(i)}\left[\int_0^{\tau^{(i)}\wedge t} ds\int_{[0,1)}k^{(i)}(da)\chi(x_s^{(i)}a<d\leqq x_s^{(i)})\right]$$

$$=E_1^{(i)}\left[\sum_{s\leqq\tau^{(i)}\wedge t}\chi(x_s^{(i)}<d\leqq x_{s-}^{(i)})\right]$$

$$=P_1^{(i)}(x_{\tau(i)}^{(i)}<d,\quad \tau^{(i)}\leqq t).$$

Therefore, the relation $S_d^{(1)}(1,t;\ f_n,u)=S_d^{(2)}(1,t;\ f_n,u)$ for all n implies

$$P_1^{(1)}(x_{\tau(1)}^{(1)}=d,\quad \tau^{(1)}\leqq t)+P_1^{(1)}(x_{\tau(1)}^{(1)}<d,\quad \tau^{(1)}\leqq t)$$
$$=P_1^{(2)}(x_{\tau(2)}^{(2)}=d,\quad \tau^{(2)}\leqq t)+P_1^{(2)}(x_{\tau(2)}^{(2)}<d,\quad \tau^{(2)}\leqq t),$$

i.e. $P_1^{(1)}(\tau^{(1)}\leqq t)=P_1^{(2)}(\tau^{(2)}\leqq t)$.

LEMMA 6.2. *For any* d $(2/3<d<1)$,

$$(x_t^{(1)},\ P_x^{(1)},\ t<\tau_d^{(1)};\ x\in\bar{S})=(x_t^{(2)},\ P_x^{(2)},\ t<\tau_d^{(2)};\ x\in\bar{S}).$$

Proof. When $f\in C_d^*$ satisfies $f(x)=1$ for $x\leqq d$, we have by (6.8)

$$T_t\hat{f}(\delta_1)=E_1^{(i)}[f(x_t^{(i)});\ t<\tau^{(i)}]+P_1^{(i)}(\tau^{(i)}\leqq t)$$

for $i=1,2$, and hence by Lemma 6.1

$$E_1^{(1)}[f(x_t^{(1)});\ t<\tau^{(1)}]=E_1^{(2)}[f(x_t^{(2)});\ t<\tau^{(2)}].$$

This is easily shown to be valid for all $f\in B(\bar{S})$. Moreover, Since

$$E_x^{(i)}[f(x_t^{(i)});\ t<\tau^{(i)}]=E_1^{(i)}[\theta_x f(x_t^{(i)});\ t<\tau^{(i)}],$$

we have

$$E_x^{(1)}[f(x_t^{(1)});\ t<\tau_d^{(1)}]=E_x^{(2)}[f(x_t^{(2)});\ t<\tau_d^{(2)}]$$

for any d $(2/3<d<1)$, any $x\in\bar{S}$, and for any $f\in B(\bar{S})$.

Now, it is shown by induction

$$(*)\qquad \begin{aligned}&E_x^{(1)}[f_1(x_{t_1}^{(1)})f_2(x_{t_2}^{(1)})\cdots f_n(x_{t_n}^{(1)});\ t_n<\tau_d^{(1)}]\\ &=E_x^{(2)}[f_1(x_{t_1}^{(2)})f_2(x_{t_2}^{(2)})\cdots f_n(x_{t_n}^{(2)});\ t_n<\tau_d^{(2)}]\end{aligned}$$

for all $n\geqq 1$, $0\leqq t_1<t_2<\cdots<t_n$, and $f_k\in B(\bar{S})$ $(k=1,2,\cdots,n)$, since, if we assume (*), we have for another $f_{n+1}\in B(\bar{S})$ and $t_{n+1}>t_n$

$$\begin{aligned}&E_x^{(1)}[f_1(x_{t_1}^{(1)})\cdots f_n(x_{t_n}^{(1)})f_{n+1}(x_{t_{n+1}}^{(1)});\ t_{n+1}<\tau_d^{(1)}]\\
&=E_x^{(1)}[f_1(x_{t_1}^{(1)})\cdots f_n(x_{t_n}^{(1)})E^{(1)}_{x_{t_n}^{(1)}}[f_{n+1}(x_{t_{n+1}-t_n}^{(1)});\ t_{n+1}-t_n<\tau_p^{(1)}]_{p=xd/x_{t_n}^{(1)}};\ t_n<\tau_d^{(1)}]\\
&=E_x^{(1)}[f_1(x_{t_1}^{(1)})\cdots f_n(x_{t_n}^{(1)})E^{(2)}_{x_{t_n}^{(1)}}[f_{n+1}(x_{t_{n+1}-t_n}^{(2)});\ t_{n+1}-t_n<\tau_p^{(2)}]_{p=xd/x_{t_n}^{(1)}};\ t_n<\tau_d^{(1)}]\\
&=E_x^{(2)}[f_1(x_{t_1}^{(2)})\cdots f_n(x_{t_n}^{(2)})E^{(2)}_{x_{t_n}^{(2)}}[f_{n+1}(x_{t_{n+1}-t_n}^{(2)});\ t_{n+1}-t_n<\tau_p^{(2)}]_{p=xd/x_{t_n}^{(2)}};\ t_n<\tau_d^{(2)}]\\
&=E_x^{(2)}[f_1(x_{t_1}^{(2)})\cdots f_n(x_{t_n}^{(2)})f_{n+1}(x_{t_{n+1}}^{(2)});\ t_{n+1}<\tau_d^{(2)}].\end{aligned}$$

LEMMA 6. 3. *For any* $f\in C_d^*$,

$$\int_{M_1-\{\delta_1\}}\Pi^{(1)}(d\mu)\chi(M(\mu)<d)\hat{f}(\mu)=\int_{M_1-\{\delta_1\}}\Pi^{(2)}(d\mu)\chi(M(\mu)<d)\hat{f}(\mu).$$

Proof. Put $\tau^{(i)}=\tau_d^{(i)}$, $\tau^{(i)\prime}=\tau_{d'}^{(i)}$, and $\tau^{(i)\prime\prime}=\tau_{d''}^{(i)}$ $(i=1,2)$ for $2/3<d''<d<d'<1$. We have by (6. 8) for $f\in C_d^*$

$$\begin{aligned}T_t\hat{f}(\delta_1)&=E_1^{(i)}[f(x_t^{(i)});\ t<\tau^{(i)}]+E_1^{(i)}[T_{t-\tau^{(i)}}\hat{f}(d\delta_d);\ x_{\tau^{(i)}}^{(i)}=d,\ \tau^{(i)}\leqq t]\\
&\quad+E_1^{(i)}\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<x_s^{(i)})T_{t-s}\hat{f}(x_s^{(i)}\cdot\mu)\right]\\
&=\mathrm{I}^{(i)}+\mathrm{II}^{(i)}+\mathrm{III}^{(i)},\ \text{say}.\end{aligned}$$

$$\mathrm{II}^{(i)}=E_1^{(i)}[T_{t-\tau^{(i)}}\hat{f}(d\delta_d);\ x_{\tau^{(i)}}^{(i)}=d,\quad \tau^{(i)}\leqq t,\quad \tau^{(i)}<\tau^{(i)\prime\prime}].$$

$$\begin{aligned}\mathrm{III}^{(i)}&=E_1^{(i)}\left[\int_0^t ds\int\Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<d'<x_s^{(i)})T_{t-s}\hat{f}(x_s^{(i)}\cdot\mu)\right]\\
&\quad+E_1^{(i)}\left[\int_0^t ds\int\Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<x_s^{(i)}\leqq d')T_{t-s}\hat{f}(x_s^{(i)}\cdot\mu)\right]\\
&=\mathrm{III}_1^{(i)}+\mathrm{III}_2^{(i)},\quad\text{say}.\end{aligned}$$

$$\mathrm{III}_2^{(i)}=E_1^{(i)}\left[\int_{\tau^{(i)\prime}}^t ds\int \Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<x_s^{(i)})T_{t-s}\hat{f}(x_s^{(i)}\cdot\mu)\right]$$

$$=E_1^{(i)}\left[E^{(i)}_{x^{(i)}_{\tau^{(i)\prime}}}\left[\int_0^u ds\int \Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<d<x_s^{(i)})T_{u-s}\hat{f}(x_s^{(i)}\cdot\mu)\right]_{u=t-\tau^{(i)\prime}};\ \tau^{(i)\prime}\leqq t,\ \tau^{(i)\prime}<\tau^{(i)}\right].$$

Putting $y=x^{(i)}_{\tau^{(i)\prime}}$ and $p=d/x^{(i)}_{\tau^{(i)\prime}}$ $(\geqq d)$ in the last term, the integrand is equal to

$$E_y^{(i)}\left[\int_0^u ds\int \Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<yp<x_s^{(i)})T_{u-s}\hat{f}(x_s^{(i)}\cdot\mu)\right].$$

On the other hand, we have by (6. 8)

$$T_u\hat{f}(y\delta_y)=E_y^{(i)}[f(x_u^{(i)});\ u<\tau_p^{(i)}]+E_y^{(i)}[T_{u-\tau_p^{(i)}}\hat{f}(yp\delta_{yp});\ x^{(i)}_{\tau_p^{(i)}}=yp,\ \tau_p^{(i)}\leqq u]$$

$$+E_y^{(i)}\left[\int_0^u ds\int \Pi^{(i)}(d\mu)\chi(x_s^{(i)}M(\mu)<yp<x_s^{(i)})T_{u-s}\hat{f}(x_s^{(i)}\cdot\mu)\right].$$

Therefore,

$$\mathrm{III}_2^{(i)}=E_1^{(i)}[T_{t-\tau^{(i)\prime}}\hat{f}(x^{(i)}_{\tau^{(i)\prime}}\delta_{x^{(i)}_{\tau^{(i)\prime}}});\ \tau^{(i)\prime}\leqq t,\ \tau^{(i)\prime}<\tau^{(i)}]$$

$$-E_1^{(i)}[E^{(i)}_{x^{(i)}_{\tau^{(i)\prime}}}[f(x_u^{(i)});\ u<\tau_p^{(i)}]_{\substack{u=t-\tau^{(i)\prime}\\ p=d/x^{(i)}_{\tau^{(i)\prime}}}};\ \tau^{(i)\prime}\leqq t,\ \tau^{(i)\prime}<\tau^{(i)}]$$

$$-E_1^{(i)}[E^{(i)}_{x^{(i)}_{\tau^{(i)\prime}}}[T_{u-\tau_p^{(i)}}\hat{f}(d\delta_d);\ x^{(i)}_{\tau_p^{(i)}}=d,\ \tau_p^{(i)}\leqq u]_{\substack{u=t-\tau^{(i)\prime}\\ p=d/x^{(i)}_{\tau^{(i)\prime}}}};\ \tau^{(i)\prime}\leqq t,\ \tau^{(i)\prime}<\tau^{(i)}]$$

$$=E_1^{(i)}[T_{t-\tau^{(i)\prime}}\hat{f}(x^{(i)}_{\tau^{(i)\prime}}\delta_{x^{(i)}_{\tau^{(i)\prime}}});\ \tau^{(i)\prime}\leqq t,\ \tau^{(i)\prime}<\tau^{(i)}]-E_1^{(i)}[f(x_t^{(i)});\ \tau^{(i)\prime}\leqq t<\tau^{(i)}]$$

$$-E_1^{(i)}[T_{t-\tau^{(i)}}\hat{f}(d\delta_d);\ x^{(i)}_{\tau^{(i)}}=d,\ \tau^{(i)}\leqq t,\ \tau^{(i)\prime}<\tau^{(i)}<\tau^{(i)\prime\prime}].$$

Thus, it follows from Lemma 6. 2 that $\mathrm{I}^{(1)}=\mathrm{I}^{(2)}$, $\mathrm{II}^{(1)}=\mathrm{II}^{(2)}$, and $\mathrm{III}_2^{(1)}=\mathrm{III}_2^{(2)}$, and hence $\mathrm{III}_1^{(1)}=\mathrm{III}_1^{(2)}$. Moreover, we have

$$\lim_{t\downarrow 0}\frac{1}{t}\mathrm{III}_1^{(i)}=\int \Pi^{(i)}(d\mu)\chi(M(\mu)<d)\hat{f}(\mu)$$

by the right continuity of $x_t^{(i)}$ in t and by the inequalities:

$$|\chi(x_s^{(i)}M(\mu)<d<d'<x_s^{(i)})T_{t-s}\hat{f}(x_s^{(i)}\cdot\mu)|\leqq\chi\left(M(\mu)<\frac{d}{d'}<1\right)$$

and

$$\int \Pi^{(i)}(d\mu)\chi\left(M(\mu)<\frac{d}{d'}\right)<+\infty.$$

Therefore,

$$\int \Pi^{(1)}(d\mu)\chi(M(\mu)<d)\hat{f}(\mu)=\int \Pi^{(2)}(d\mu)\chi(M(\mu)<d)\hat{f}(\mu).$$

LEMMA 6.4. *Two finite measures Q_1 and Q_2 on $(\boldsymbol{M}_1, \varphi_d^{-1}(\mathcal{B}_1^d))$ are coincident, if they satisfy*

$$\int_{\boldsymbol{M}_1} \hat{f}(\mu)Q_1(d\mu)=\int_{\boldsymbol{M}_1} \hat{f}(\mu)Q_2(d\mu)$$

for all $f\in C_d^*$.

Proof. Let Q_i^d be the induced measure on $(\boldsymbol{M}_1^d, \mathcal{B}_1^d)$ of Q_i by the mapping φ_d $(i=1, 2)$. By putting $f_d=f|_{S_d}$ for $f\in C_d^*$,

$$\int_{\boldsymbol{M}_1} \hat{f}(\mu)Q_i(d\mu)=\int_{\boldsymbol{M}_1} \hat{f}_d(\varphi_d(\mu))Q_i(d\mu)=\int_{\boldsymbol{M}_1^d} \hat{f}_d(\nu)Q_i^d(d\nu)$$

for $i=1, 2$, because $\hat{f}(\mu)=\hat{f}_d(\varphi_d(\mu))$ for any $\mu\in \boldsymbol{M}_1$. Thus, we have

$$\int_{\boldsymbol{M}_1^d} \hat{f}_d(\nu)Q_1^d(d\nu)=\int_{\boldsymbol{M}_1^d} \hat{f}_d(\nu)Q_2^d(d\nu).$$

Moreover, since we can prove as Proposition 1.2 that $\mathcal{L}\{\hat{f}_d; f\in C_d^*\}$ is dense in $C(\boldsymbol{M}_1^d)$, we have

$$\int_{\boldsymbol{M}_1^d} F(\nu)Q_1^d(d\nu)=\int_{\boldsymbol{M}_2^d} F(\nu)Q_2^d(d\nu)$$

for all $F\in C(\boldsymbol{M}_1^d)$, and this implies $Q_1^d=Q_2^d$. Hence, $Q_1=Q_2$ is obvious.

LEMMA 6.5. $\Pi^{(1)}(d\mu)=\Pi^{(2)}(d\mu)$.

Proof. By Lemma 6.3 and Lemma 6.4, we have

$$\Pi^{(1)}|_{\varphi_d^{-1}(\mathcal{B}_1^d)}=\Pi^{(2)}|_{\varphi_d^{-1}(\mathcal{B}_1^d)} \quad \text{on} \quad \{\mu\in \boldsymbol{M}_1;\ M(\mu)<d\},$$

because $\{\mu\in \boldsymbol{M}_1;\ M(\mu)<d\}\in\varphi_d^{-1}(\mathcal{B}_1^d)$. Since $\{\mu\in \boldsymbol{M}_1;\ M(\mu)<d\}\subset\{\mu\in \boldsymbol{M}_1;\ M(\mu)<d'\}$ for $d<d'$, we have

$$\Pi^{(1)}|_{\varphi_{d'}^{-1}(\mathcal{B}_1^{d'})}=\Pi^{(2)}|_{\varphi_{d'}^{-1}(\mathcal{B}_1^{d'})} \quad \text{on} \quad \{\mu\in \boldsymbol{M}_1;\ M(\mu)<d\}$$

for any d' $(d\leqq d'<1)$, and hence, by Lemma 1.4,

$$\Pi^{(1)}=\Pi^{(2)} \quad \text{on} \quad \{\mu\in \boldsymbol{M}_1;\ M(\mu)<d\}.$$

Thus, we have by letting $d\uparrow 1$

$$\Pi^{(1)}=\Pi^{(2)} \quad \text{on} \quad \boldsymbol{M}_1-\{\delta_1\}.$$

LEMMA 6.6. $(x_t^{(1)}, P_x^{(1)})=(x_t^{(2)}, P_x^{(2)})$.

Proof. Let $T_t^{(i)}$ be the semigroup of the process $(x_t^{(i)})$ $(i=1, 2)$. Putting $\tau_d^{(i)}=\tau^{(i)}$, and for $f\in B(\bar{S})$,

$$\begin{aligned} T_t^{(i)}f(x) &= E_x^{(i)}[f(x_t^{(i)})] \\ &= E_x^{(i)}[f(x_t^{(i)});\ t<\tau^{(i)}]+E_x^{(i)}[f(x_t^{(i)});\ t\geqq\tau^{(i)}] \\ &= \mathrm{I}^{(i)}+\mathrm{II}^{(i)},\quad \text{say.} \end{aligned}$$

$$\begin{aligned} \mathrm{II}^{(i)} &= E_x^{(i)}[T_{t-\tau^{(i)}}^{(i)}f(x_{\tau^{(i)}}^{(i)});\ t\geqq\tau^{(i)}] \\ &= E_x^{(i)}[T_{t-\tau^{(i)}}^{(i)}f(x_{\tau^{(i)}}^{(i)});\ x_{\tau^{(i)}}^{(i)}=xd,\ t\geqq\tau^{(i)}] \\ &\quad +E_x^{(i)}[T_{t-\tau^{(i)}}^{(i)}f(x_{\tau^{(i)}}^{(i)});\ x_{\tau^{(i)}}^{(i)}<xd,\ t\geqq\tau^{(i)}] \\ &= \mathrm{II}_1^{(i)}+\mathrm{II}_2^{(i)},\quad \text{say.} \end{aligned}$$

$$\begin{aligned} \mathrm{II}_2^{(i)} &= E_x^{(i)}\Big[\sum_{s\leqq\tau^{(i)}\wedge t} T_{t-s}^{(i)}f(x_s^{(i)})\chi(x_s^{(i)}<xd\leqq x_{s-}^{(i)})\Big] \\ &= E_x^{(i)}\Big[\int_0^{\tau^{(i)}\wedge t}ds\int_{[0,1)}k^{(i)}(da)T_{t-s}^{(i)}f(x_s^{(i)}a)\chi(x_s^{(i)}a<xd\leqq x_s^{(i)})\Big] \\ &= E_x^{(i)}\Big[\int_0^t ds\int_{[0,1)}k^{(i)}(da)T_{t-s}^{(i)}f(x_s^{(i)}a)\chi(x_s^{(i)}a<xd<x_s^{(i)})\Big]. \end{aligned}$$

By Lemma 6. 2, we have $\mathrm{I}^{(1)}=\mathrm{I}^{(2)}$ and

$$\mathrm{II}_1^{(2)}=E_x^{(1)}[T_{t-\tau^{(1)}}^{(2)}f(xd);\ x_{\tau^{(1)}}^{(1)}=xd,\ t\geqq\tau^{(1)}].$$

Moreover, since $k^{(1)}(da)=k^{(2)}(da)$ by Lemma 6. 5, we have

$$\mathrm{II}_2^{(2)}=E_x^{(1)}\Big[\int_0^t ds\int_{[0,1)}k^{(1)}(da)T_{t-s}^{(2)}f(x_s^{(1)}a)\chi(x_s^{(1)}a<xd<x_s^{(1)})\Big].$$

Thus,

$$\begin{aligned} &T_t^{(1)}f(x)-T_t^{(2)}f(x) \\ &= E_x^{(1)}[T_{t-\tau^{(1)}}^{(1)}f(xd)-T_{t-\tau^{(1)}}^{(2)}f(xd);\ x_{\tau^{(1)}}^{(1)}=xd,\ \tau^{(1)}\leqq t] \\ &\quad +E_x^{(1)}\Big[\int_0^t ds\int_{[0,1)}k^{(1)}(da)\chi(x_s^{(1)}a<xd<x_s^{(1)})(T_{t-s}^{(1)}f(x_s^{(1)}a)-T_{t-s}^{(2)}f(x_s^{(1)}a))\Big] \end{aligned}$$

and

$$\begin{aligned} |T_t^{(1)}f(x)-T_t^{(2)}f(x)| &\leqq A_{t'}P_x^{(1)}(x_{\tau^{(1)}}^{(1)}=xd,\ \tau^{(1)}\leqq t) \\ &\quad +A_{t'}E_x^{(1)}\Big[\int_0^t ds\int_{[0,1)}k^{(1)}(da)\chi(x_s^{(1)}a<xd<x_s^{(1)})\Big] \\ &= A_{t'}P_x^{(1)}(\tau^{(1)}\leqq t) \end{aligned}$$

for $t\leqq t'$, where

$$A_{t'}=\sup_{\substack{x\in\bar{S}\\0\leqq t\leqq t'}}|T_t^{(1)}f(x)-T_t^{(2)}f(x)|.$$

Hence, $A_{t'}\leqq A_{t'}P_x^{(1)}(\tau^{(1)}\leqq t')$. Now, taking $t'>0$ such that $P_x^{(1)}(\tau^{(1)}\leqq t')<1$, we have $A_{t'}=0$, and hence $T_t^{(1)}f=T_t^{(2)}f$ for any $t\leqq t'$. Since both $\{T_t^{(1)}\}$ and $\{T_t^{(2)}\}$ are semigroups and t' depends only on d, we have $T_t^{(1)}=T_t^{(2)}$ for all $t>0$, and this implies $(x_t^{(1)}, P_x^{(1)})=(x_t^{(2)}, P_x^{(2)})$.

The proof of Theorem 6. 1 is completed by Lemma 6. 5 and Lemma 6. 6.

§ 7. Generator of the cascade semigroup.

Let $\{T_t\}$ be a given cascade semigroup and A^0 the infinitesimal generator of its underlying process (x_t, P_x^0). Then, by Proposition 3. 1, $C^1[0, 1]\subset\mathcal{D}(A^0)$ (the domain of A^0), and for $f\in C^1[0, 1]$,

$$A^0f(x)=-mxf'(x)+\int_{[0,1)}k(da)(f(xa)-f(x)),$$

where m is a nonnegative constant and $k(da)$ is a measure on $[0, 1)$ such that

$$k(da)=\int_{\{\mu:\ M(\mu)\in da\}}\Pi(d\mu)$$

with the branching measure $\Pi(d\mu)$ (see (4. 15)).

This section is devoted to the infinitesimal generator of the cascade semigroup $\{T_t\}$.

LEMMA 7. 1. *$P_x^0(\tau_d\leqq t)=0(t)$ as $t\downarrow 0$.*

Proof. The formula (3. 16) gives

$$\lim_{t\downarrow 0}\frac{1}{t}E_1^0[f(x_t)]=\int_{[0,1)}k(da)f(a)$$

for any $f\in C^1[0, 1]$ such that $f(1)=f'(1)=0$. Take a sequence $\{f_n\}$ in $C^1[0, 1]$ such that $f_n(1)=f_n'(1)=0$ and

$$f_n(x)\downarrow\chi(x)\equiv\begin{cases}1, & x\leqq d,\\ 0, & d<x\leqq 1\end{cases}$$

then we have

$$\overline{\lim_{t\downarrow 0}}\frac{1}{t}E_1^0[\chi(x_t)]\leqq\lim_{t\downarrow 0}\frac{1}{t}E_1^0[f_n(x_t)]=\int_{[0,1)}k(da)f_n(a)$$

for all n. Hence, by letting $n\to\infty$,

$$\overline{\lim_{t\downarrow 0}}\frac{1}{t}E_1^0[\chi(x_t)]\leqq\int_{[0,1)}k(da)\chi(a)=\int_{[0,d]}k(da)<+\infty.$$

Since $E_1^0[\chi(x_t)]=P_1^0(x_t\leqq d)=P_x^0(\tau_d\leqq t)$ for $x\in S$, we have the result.

LEMMA 7. 2. *For* $f\in C_0^*\cap\mathcal{D}(A^0)$, $((T_t\hat{f})|_S-f)/t$ *converges weakly to* Bf *in* $C[0,1]$, *where*

$$Bf(x)=A^0f(x)+\int_{M_1-\{\delta_1\}}\Pi(d\mu)(\hat{f}(x\cdot\mu)-f(xM(\mu))). \tag{7.1}$$

Proof. Suppose $f\in C_d^*\cap\mathcal{D}(A^0)$, take d' such that $d<d'<1$, and put $\tau_{d'}^0=\tau'$. Then, by Theorem 5. 1,

$$T_t\hat{f}(x\delta_x)=E_x^0[f(x_t);\ t<\tau']+E_x^0[T_{t-\tau'}f(xd'\delta_{xd'});\ x_{\tau'}=xd',\ \tau'\leqq t]$$
$$+E_x^0\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd'<x_s)T_{t-s}\hat{f}(x_s\cdot\mu)\right].$$

Moreover, we have

$$T_t^0f(x)=E_x^0[f(x_t);\ t<\tau']+E_x^0[T_{t-\tau'}^0f(x_{\tau'});\ t\geqq\tau']$$
$$=E_x^0[f(x_t);\ t<\tau']+E_x^0[T_{t-\tau'}^0f(xd');\ x_{\tau'}=xd',\ \tau'\leqq t]$$
$$+E_x^0\left[\int_0^t ds\int_{[0,1)}k(da)\chi(x_sa<xd'<x_s)T_{t-s}^0f(x_sa)\right],$$

since, in the last term,

$$E_x^0[T_{t-\tau'}^0f(x_{\tau'});\ x_{\tau'}<xd',\ \tau'\leqq t]$$
$$=E_x^0\left[\sum_{s\leqq\tau'\wedge t}T_{t-s}^0f(x_s)\chi(x_s<xd'\leqq x_{s-})\right]$$
$$=E_x^0\left[\int_0^{\tau'\wedge t}ds\int_{[0,1)}k(da)T_{t-s}^0f(x_sa)\chi(x_sa<xd'\leqq x_s)\right]$$

by the formula (3. 17). Hence, we have

$$\frac{T_t\hat{f}(x\delta_x)-f(x)}{t}=\frac{T_t^0f(x)-f(x)}{t}+\mathrm{I}+\mathrm{II},$$

where

$$\mathrm{I}=\frac{1}{t}E_x^0\left[\int_0^t ds\int\Pi(d\mu)\chi(x_sM(\mu)<xd'<x_s)(\hat{f}(x_s\cdot\mu)-f(x_sM(\mu)))\right]$$

and

$$\mathrm{II}=\frac{1}{t}E_x^0[T_{t-\tau'}\hat{f}(xd'\delta_{xd'})-T_{t-\tau'}^0f(xd');\ x_{\tau'}=xd',\ \tau'\leqq t]$$

$$+\frac{1}{t}E_x^0\left[\int_0^t ds\int\Pi(d\mu)\chi(x_sM(\mu)<xd'<x_s)\right.$$

$$\left.\times\{(T_{t-s}\hat{f}(x_s\cdot\mu)-\hat{f}(x_s\cdot\mu))-(T_{t-s}^0f(x_sM(\mu))-f(x_sM(\mu)))\}\right].$$

Since $\hat{f}(x\cdot\mu)=f(xM(\mu))$ if $M(\mu)\geqq d$, and since

$$\chi(x_sM(\mu)<xd<xd'<x_s)=\chi\left(M(\mu)<\frac{d}{d'}\right)\chi(x_sM(\mu)<xd<xd'<x_s),$$

we have

$$\lim_{t\downarrow 0}\mathrm{I}=\lim_{t\downarrow 0}\frac{1}{t}E_x^0\left[\int_0^t ds\int\Pi(d\mu)\chi\left(M(\mu)<\frac{d}{d'}\right)\right.$$

$$\left.\times\chi(x_sM(\mu)<xd<xd'<x_s)(\hat{f}(x_s\cdot\mu)-f(x_sM(\mu)))\right]$$

$$=\int\Pi(d\mu)\chi(M(\mu)<d)(\hat{f}(x\cdot\mu)-f(xM(\mu)))$$

$$=\int\Pi(d\mu)(\hat{f}(x\cdot\mu)-f(xM(\mu))),$$

where the convergence is easily seen to be weak in $C[0,1]$. By the formula (5. 5), we have

$$|\mathrm{II}|\leqq\frac{1}{t}P_x^0(\tau'\leqq t)\left\{\sup_{0\leqq s\leqq t}\|T_s\hat{f}-\hat{f}\|+\sup_{0\leqq s\leqq t}\|T_s^0f-f\|\right\}$$

and hence, by Lemma 7. 1 and the strong continuity of T_t and T_t^0, II is shown to converge weakly to zero when t tends to zero. Finally, the proof is completed by the weak convergence of $(T_t^0f-f)/t$ to A^0f for $f\in\mathcal{D}(A^0)$.

LEMMA 7. 3. *Let A and $\mathcal{D}(A)$ be the infinitesimal generator and its domain of $\{T_t\}$. If $f\in C_0^*\cap\mathcal{D}(A^0)$, then $\hat{f}\in\mathcal{D}(A)$ and*

$$A\hat{f}(\mu)=\sum_i\frac{A\hat{f}(x_i\delta_{x_i})}{f(x_i)}\cdot\hat{f}(\mu) \tag{7. 2}$$

for $\mu=\sum_i x_i\delta_{x_i}\in\boldsymbol{M}_1$, and $A\hat{f}(0)=0$.

Proof. Assume $f\in C_d^*\cap\mathcal{D}(A^0)$. For $\mu=\sum x_i\delta_{x_i}\in\boldsymbol{M}_1$, we pick up x_i's such that $x_i>(1-d)/d$ and rewrite them as $x_1, x_2, \cdots, x_n$, where $n\leqq d/(1-d)$ is obvious. Since $(T_t\hat{f})|_S\in C_d^*$,

$$T_t\hat{f}(\mu)-\hat{f}(\mu)=T_t\hat{f}\left(\sum_{i=1}^{n} x_i\delta_{x_i}\right)-\hat{f}\left(\sum_{i=1}^{n} x_i\delta_{x_i}\right)$$

$$=\prod_{i=1}^{n} T_t\hat{f}(x_i\delta_{x_i})-\prod_{i=1}^{n} f(x_i)$$

$$=\sum_{i=1}^{n}(T_t\hat{f}(x_i\delta_{x_i})-f(x_i))\prod_{j=1}^{i-1} f(x_j)\prod_{k=i+1}^{n} T_t\hat{f}(x_k\delta_{x_k})$$

if $n\geqq 2$, and hence, by Lemma 7. 2,

$$\lim_{t\downarrow 0}\frac{T_t\hat{f}(\mu)-\hat{f}(\mu)}{t}=\sum_{i=1}^{n} Bf(x_i)\prod_{j\neq i} f(x_j) \qquad \text{if } n\geqq 2$$

$$=Bf(x_1) \qquad \text{if } n=1$$

$$=0 \qquad \text{if } n=0,$$

or

$$\lim_{t\downarrow 0}\frac{T_t\hat{f}(\mu)-\hat{f}(\mu)}{t}=\sum_i \frac{Bf(x_i)}{f(x_i)}\hat{f}(\mu),$$

where the sum in the right-hand side is taken over all x_i in $\mu=\sum x_i\delta_{x_i}$ because $Bf(x_i)=0$ if $x_i\leqq(1-d)/d$. Since

$$|T_t\hat{f}(\mu)-\hat{f}(\mu)|=\left|\prod_{i=1}^{n} T_t\hat{f}(x_i\delta_{x_i})-\prod_{i=1}^{n} f(x_i)\right|$$

$$\leqq\frac{d}{1-d}\,\|(T_t\hat{f})|_S-f\|,$$

we have by Lemma 7. 2 the boundedness of $(T_t\hat{f}-\hat{f})/t$ as $t\downarrow 0$. Moreover, since

$$\sum_i \frac{Bf(x_i)}{f(x_i)}\hat{f}(\mu)=\int\frac{Bf(x)}{xf(x)}\mu(dx)\cdot\hat{f}(\mu)$$

is continuous in $\mu\in\boldsymbol{M}_1$ because $Bf/f\in C_0$, $(T_t\hat{f}-\hat{f})/t$ converges weakly, and hence $\hat{f}\in\mathcal{D}(A)$ (see Dynkin [1]) and

$$A\hat{f}(\mu)=\int\frac{Bf(x)}{xf(x)}\mu(dx)\hat{f}(\mu)$$

$$=\sum_i\frac{Bf(x_i)}{f(x_i)}\hat{f}(\mu) \qquad \text{if } \mu\neq 0.$$

By putting $\mu=x\delta_x$ in the above, we have $A\hat{f}(x\delta_x)=Bf(x)$, and this completes the proof since $A\hat{f}(0)=0$ is obvious.

By Lemma 7. 2 and Lemma 7. 3,

(7. 3) $$A\hat{f}(x\delta_x)=A^0f(x)+\int_{M_1-\{\delta_1\}}\Pi(d\mu)(\hat{f}(x\cdot\mu)-f(xM(\mu)))$$

for $f\in C_0^*\cap\mathcal{D}(A^0)$. In addition, if $f\in C^1[0, 1]$, $A^0f(x)$ is given by the formula (3. 16), and hence we have the formula:

(7. 4) $$A\hat{f}(x\delta_x)=-mxf'(x)+\int_{M_1-\{\delta_1\}}\Pi(d\mu)(\hat{f}(x\cdot\mu)-f(x))$$

for $f\in C_0^*\cap C^1[0, 1]$.

Since $\hat{f}\in\mathcal{D}(A)$ for $f\in C_0^*\cap\mathcal{D}(A^0)$, we have

(7. 5) $$\frac{\partial T_t\hat{f}}{\partial t}=AT_t\hat{f}$$

in the strong sense of derivative. Therefore, $u_t(x)\equiv(T_t\hat{f})|_S(x)$ satisfies the equation:

(7. 6) $$\frac{\partial u_t(x)}{\partial t}=A\hat{u}_t(x\delta_x).$$

LEMMA 7. 4. *$u_t=(T_t\hat{f})|_S\in C_0^*\cap C^1[0, 1]$ if $f\in C_0^*\cap C^1[0, 1]$.*

Proof. Since

$$u_t(x)=T_t\hat{f}(x\delta_x)=T_t\widehat{\theta_x f}(\delta_1)=E_1[\widehat{\theta_x f}(\mu_t)]$$
$$=E_1\left[\exp\left(\int\frac{1}{y}\log f(xy)\mu_t(dy)\right)\right],$$

$u_t\in C_0^*$ is clear and that $u_t(x)$ is continuously differentiable in x follows from the expression:

$$\frac{\partial u_t(x)}{\partial x}=E_1\left[\exp\left(\int\frac{1}{y}\log f(xy)\mu_t(dy)\right)\int\frac{f'(xy)}{f(xy)}\mu_t(dy)\right].$$

Let D be the linear hull of all $\hat{f}$'s such that $f\in C_0^*\cap C^1[0, 1]$, then Lemma 7. 4 implies that D is T_t-invariant. Since D is obviously dense in $C(M_1)$, Watanabe's lemma (see [14]) assures that D is a core of the closed operator A.

Let us complete the above arguments by the statement:

THEOREM 7. 1. *The infinitesimal generator A of a cascade semigroup $\{T_t\}$ has a core $D=\mathcal{L}\{\hat{f};\ f\in C_0^*\cap C^1[0, 1]\}$, and if $f\in C_0^*\cap C^1[0, 1]$, then*

$$A\hat{f}(\mu)=\sum_i\frac{A\hat{f}(x_i\delta_{x_i})}{f(x_i)}\hat{f}(\mu)$$

for $\mu=\sum_i x_i\delta_{x_i}\in M_1$, where

$$A\hat{f}(x\delta_x) = -mxf'(x) + \int_{M_1-\{\delta_1\}} \Pi(d\mu)(\hat{f}(x\cdot\mu) - f(x)),$$

and $A\hat{f}(0)=0$.

By putting $u_t \equiv (T_t\hat{f})|_S$, $f \in C_0^* \cap C^1[0,1]$ *implies* $u_t \in C_0^* \cap C^1[0,1]$ *and in addition* $u_t(x)$ *is a solution of the nonlinear equation*:

$$\text{(7.7)} \qquad \begin{cases} \dfrac{\partial u_t(x)}{\partial t} = -mx\dfrac{\partial u_t(x)}{\partial x} + \displaystyle\int_{M_1-\{\delta_1\}} \Pi(d\mu)(\hat{u}_t(x\cdot\mu) - u_t(x)), \\ u_{0+}(x) = f(x). \end{cases}$$

REMARK. Theorem 5.2 can be immediately seen from the first half of Theorem 7.1.

§ 8. Construction of the cascade semigroup.

Let $\{(x_t, P_x^0), \Pi(d\mu)\}$ be any pair defined in the beginning of § 6 which satisfies (6.1), (6.2), and (6.3). Then, all of this section are intended to construct the cascade semigroup $\{T_t\}$ with (x_t, P_x^0) and $\Pi(d\mu)$ as its underlying process and branching measure.

In view of Theorem 6.1, the problem is reduced to solving the system of fundamental (S_d)-equations $(2/3<d<1)$:

$$u_t(x; f) = S_d(x, t; f, u)$$

$$(S_d) \qquad \equiv E_x^0[f(x_t);\ t<\tau_d] + E_x^0[u_{t-\tau_d}(xd; f);\ x_{\tau_d}=xd,\ \tau_d \leqq t]$$

$$+ E_x^0\left[\int_0^t ds \int_{M_1-\{\delta_1\}} \Pi(d\mu)\chi(x_s M(\mu) < xd < x_s)\, \widehat{u_{t-s}(\cdot; f)}(x_s\cdot\mu)\right]$$

for all $f \in C_d^*$, and to construction of the cascade semigroup $\{T_t\}$ such that $u_t(x; f) = T_t\hat{f}(x\delta_x)$. This problem will be answered in Theorem 8.1 at the end of this section. But for this we need many lemmas.

To begin with, it should be remarked that the process (x_t, P_x^0) corresponds to a nondecreasing additive process (y_t, P_1^0) by the transformation $y_t = -\log x_t$ whose Laplace transform has the form $\exp(-t\psi(\alpha))$, where

$$\psi(\alpha) = m\alpha + \int_{(0,\infty]} (1-e^{-\alpha u})l(du) \qquad \text{for} \quad \alpha>0$$

and

$$l(du) = \int_{\{a;\ -\log a \in du\}} k(da).$$

LEMMA 8.1. (i) $P_x^0(\tau_d>0)=1$,

(ii) $E_{ax}^0[f(x_t)]=E_x^0[f(ax_t)]$ *for* $f\in B[0,1]$,

(iii) $P_x^0(x_t$ *is non-increasing in* $t\geqq 0)=1$, *and*

(iv) $P_x^0(\tau_d\leqq t)=P_x^0(x_{\tau_d}=xd,\ \tau_d\leqq t)+E_x^0\left[\int_0^t ds\int_{[0,1)}k(da)\chi(x_s a<xd<x_s)\right]$.

Proof. (i) is obvious by the right continuity of the sample path, and (ii) and (iii) are seen from the above remark. Finally, (iv) is shown by the formula (6.5) as follows:

$$P_x^0(\tau_d\leqq t)-P_x^0(x_{\tau_d}=xd,\ \tau_d\leqq t)$$
$$=P_x^0(x_{\tau_d}<xd,\ \tau_d\leqq t)=E_x^0\left[\sum_{s\leqq\tau_d\wedge t}\chi(x_s<xd\leqq x_{s-})\right]$$
$$=E_x^0\left[\int_0^{\tau_d\wedge t}ds\int_{[0,1)}k(da)\chi(x_s a<xd\leqq x_s)\right]=E_x^0\left[\int_0^t ds\int_{[0,1)}k(da)\chi(x_s a<xd<x_s)\right].$$

By Lemma 8.1 (iv), the (S_d)-equation has always a solution $u_t\equiv 1$ for $f\equiv 1$. The independence of $P_x^0(\tau_d\leqq t)$ on x is seen from Lemma 8.1 (ii) and the definition of τ_d. For simplicity, we denote τ_d^0 by τ_d or τ in this section.

For $f\in B_d^*$, we define $\{u_t^n(x);\ n\geqq 0\}$ by

$$(8.1)\qquad \begin{cases} u_t^0(x)\equiv 1, \\ u_t^{n+1}(x)=S_d(x,t;f,u^n) \qquad \text{for} \quad n\geqq 0. \end{cases}$$

LEMMA 8.2. $0\leqq u_t^{n+1}\leqq u_t^n\leqq 1$ *for all* $n\geqq 0$, *and the limit* $u_t^d(x;f)\equiv u_t^d(x)$ $=\lim_{n\to\infty}u_t^n(x)$ *is a solution of the* (S_d)-*equation such that* $u_t^d\in B_d^*$.

Proof. The first half of the lemma is easily shown by induction. Since the process (x_t) is nonincreasing, it can be proved $u_t^n\in B_d^*$ for all $n\geqq 0$, and hence $u_t^d\in B_d^*$. Since

$$\hat{u}_t^d(\nu)=\hat{u}_t^d\left(\sum_{x_i>(1-d)/d}x_i\delta_{x_i}\right)=\prod_{x_i>(1-d)/d}u_t^d(x_i)$$

is a product of finite number ($\leqq d/(1-d)$) of factors for any $\nu=\sum x_i\delta_{x_i}\in \boldsymbol{M}_1$, u_t^d is shown to satisfy the (S_d)-equation by letting $n\to\infty$ in (8.1).

By Lemma 8.1 (i), there is a constant $t(d)>0$ for each d $(2/3<d<1)$ such that $(d/(1-d))P_x^0(\tau_d\leqq t(d))<1$.

LEMMA 8.3. *The* (S_d)-*equation has a unique solution in* B_d^* *for* $t\leqq t(d)$ *and for* $f\in B_d^*$. *In particular, the unique solution* $u_t^d(\cdot;f)$ *is in* C_d^* *for* $t\leqq t(d)$ *and for* $f\in C_d^*$.

Proof. We first remark the following. Since the number of i's such that $x_i>(1-d)/d$ is less than $d/(1-d)$ for any $\mu=\sum_i x_i\delta_{x_i}\in \boldsymbol{M}_1$, we have

$$|\hat{f}(\mu)-\hat{g}(\mu)|=\left|\prod_{x_i>(1-d)/d} f(x_i)-\prod_{x_i>(1-d)/d} g(x_i)\right|\leqq \frac{d}{1-d}\,||f-g||_S$$

for $f, g\in B_d^*$, and hence

$$||\hat{f}-\hat{g}||_{\boldsymbol{M}_1}\leqq \frac{d}{1-d}\,||f-g||_S.$$

Now, let u_t and v_t be two solutions in B_d^* of the (S_d)-equation for $f\in B_d^*$. Then,

$$\begin{aligned}
|u_t(x)-v_t(x)|&\leqq E_x^0[|u_{t-\tau}(xd)-v_{t-\tau}(xd)|;\ x_\tau=xd,\ \tau\leqq t]\\
&\quad+E_x^0\left[\int_0^t ds\int_{\boldsymbol{M}_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)|\hat{u}_{t-s}(x_s\cdot\mu)-\hat{v}_{t-s}(x_s\cdot\mu)|\right]\\
&\leqq E_x^0[||u_{t-\tau}-v_{t-\tau}||;\ x_\tau=xd,\ \tau\leqq t]\\
&\quad+E_x^0\left[\int_0^t ds\int_{\boldsymbol{M}_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\frac{d}{1-d}\,||u_{t-s}-v_{t-s}||\right],
\end{aligned}$$

where $\tau=\tau_d$. Therefore, by putting $A=\sup_{t\leqq t(d)}||u_t-v_t||$, we have

$$\begin{aligned}
A&\leqq AP_x^0(x_\tau=xd,\ \tau\leqq t(d))+\frac{d}{1-d}\,AE_x^0\left[\int_0^{t(d)} ds\int_{[0,1)}k(da)\chi(x_sa<xd<x_s)\right]\\
&\leqq A\cdot\frac{d}{1-d}\,P_x^0(\tau\leqq t(d)),
\end{aligned}$$

and this implies $A=0$, i.e. $u_t\equiv v_t$ for $t\leqq t(d)$. Thus, the uniqueness assertion has been proved.

Now, we shall prove $u_t^n\in C_d^*$ for all n if $f\in C_d^*$. In fact, since

$$\begin{aligned}
u_t^{n+1}(x)&=S_d(x, t;\ f, u^n)\\
&=E_1^0[f(x\cdot x_t);\ t<\tau]+E_1^0[u_{t-\tau}^n(xd);\ x_\tau=d,\ \tau\leqq t]\\
&\quad+E_1^0\left[\int_0^t ds\int_{\boldsymbol{M}_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<d<x_s)\hat{u}_{t-s}^n(xx_s\cdot\mu)\right]
\end{aligned}$$

and $\hat{u}_{t-s}^n(xx_s\cdot\mu)$ is continuous in x if $u_t^n\in C_d^*$, $u_t^{n+1}(x)$ is continuous in x and $u_t^{n+1}>0$, and hence $u_t^{n+1}\in C_d^*$ if $u_t^n\in C_d^*$ because $u_t^{n+1}\in B_d^*$. Thus, we see $u_t^n\in C_d^*$ for all n by induction.

Since

$$|u_t^{n+1}(x)-u_t^n(x)|\leqq E_x^0[|u_{t-\tau}^n(xd)-u_{t-\tau}^{n-1}(xd)|;\ x_\tau=xd,\ \tau\leqq t]$$

$$+E_x^0\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)|\hat{u}_{t-s}^n(x_s\cdot\mu)-\hat{u}_{t-s}^{n-1}(x_s\cdot\mu)|\right]$$

$$\leqq A_{n-1}\cdot\frac{d}{1-d}P_x^0(\tau\leqq t(d))$$

for $t\leqq t(d)$, where $A_{n-1}=\sup_{t\leqq t(d)}\|u_t^n-u_t^{n-1}\|$, we have

$$A_n\leqq A_{n-1}\cdot\frac{d}{1-d}P_x^0(\tau\leqq t(d)).$$

Thus, $\sum_n A_n$ is convergent and it follows that u_t^n converges uniformly in x as $n\to\infty$ for $t\leqq t(d)$, and hence u_t^d is a continuous function for $t\leqq t(d)$ if $f\in C_d^*$. Moreover, since $u_t^d\in B_d^*$ and $u_t^d(x)\geqq E_x^0[f(x_t);\ t<\tau]>0$ for $t\leqq t(d)$ because $P_x^0(\tau>t(d))>0$, we have $u_t^d\in C_d^*$ for $t\leqq t(d)$ and $f\in C_d^*$.

LEMMA 8.4. *If u_t is a solution in B_d^* of (S_d)-equation for $f\in B_d^*$, then u_t is also a solution of $(S_{d'})$-equation for f and $t\leqq t(d)$, where d' is any number such that $d<d'<1$.*

Proof. Put $\tau_d=\tau$, $\tau_{d'}=\tau'$, and

$$\rho(t,x,d')=u_t(x)-S_{d'}(x,t;\ f,u).$$

Moreover, if we put

$$\rho=\sup_{\substack{0\leq t\leq t(d),\ 0<x\leq 1\\ d\leqq d'<1}}|\rho(t,x,d')|,$$

to show $\rho=0$ is to verify the lemma.

Now, since u_t is a solution of (S_d)-equation, we put

$$u_t(x)=S_d(x,t;\ f,u)$$

$$=E_x^0[f(x_t);\ t<\tau]+E_x^0[u_{t-\tau}(xd);\ x_\tau=xd,\ \tau\leqq t]$$

$$+E_x^0\left[\int_0^t ds\int\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\hat{u}_{t-s}(x_s\cdot\mu)\right]$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III}.$$

$$\mathrm{I}\equiv E_x^0[f(x_t);\ t<\tau']+E_x^0[f(x_t);\ \tau'\leqq t,\ t<\tau]$$

$$=\mathrm{I}_1+\mathrm{I}_2,\quad \text{say}.$$

Since $\tau'(w)<\tau(w)$ in the second term I_2, it holds $\tau(w)=\tau'(w)+\tau_p(w_{\tau'}^+)$ where $p(w)=xd/x_{\tau'}(w)\geqq d$, and we have

$$\mathrm{I}_2=E_x^0[E_{x_{\tau'}}^0[f(x_r);\ r<\tau_p]_{\substack{r=t-\tau'\\ p=xd/x_{\tau'}}};\ \tau'\leqq t,\ \tau'<\tau].$$

$$\begin{aligned}\mathrm{II}&\equiv E_x^0[u_{t-\tau}(xd);\ x_t=xd,\ \tau=\tau'\leqq t]\\&\quad+E_x^0[u_{t-\tau}(xd);\ x_\tau=xd,\ \tau'<\tau\leqq t]\\&=\mathrm{II}_1+\mathrm{II}_2,\quad\text{say.}\end{aligned}$$

$$\mathrm{II}_2=E_x^0[E_{x_{\tau'}}^0[u_{r-\tau_p}(xd);\ x_{\tau_p}=xd,\ \tau_p\leqq r]_{\substack{r=t-\tau'\\ p=xd/x_{\tau'}}};\ \tau'\leqq t,\ \tau'<\tau].$$

$$\begin{aligned}\mathrm{III}&\equiv E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<xd'<x_s)\hat{u}_{t-s}(x_s\cdot\mu)\right]\\&\quad+E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s\leqq xd')\hat{u}_{t-s}(x_s\cdot\mu)\right]\\&=\mathrm{III}_1+\mathrm{III}_2,\quad\text{say.}\end{aligned}$$

$$\mathrm{III}_2=E_x^0\left[E_{x_{\tau'}}^0\left[\int_0^r ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\hat{u}_{r-s}(x_s\cdot\mu)\right]_{r=t-\tau'};\ \tau'\leqq t,\ \tau'<\tau\right].$$

Hence,

$$\begin{aligned}&\mathrm{I}_2+\mathrm{II}_2+\mathrm{III}_2\\&=E_x^0\Bigg[\Bigg\{E_y^0[f(x_r);\ r<\tau_p]+E_y^0[u_{r-\tau_p}(yp);\ x_{\tau_p}=yp,\ \tau_p\leqq r]\\&\quad+E_y^0\left[\int_0^r ds\int \Pi(d\mu)\chi(x_sM(\mu)<yp<x_s)\hat{u}_{r-s}(x_s\cdot\mu)\right]\Bigg\}_{\substack{r=t-\tau'\\ y=x_{\tau'}\\ p=xd/x_{\tau'}}};\ \tau'\leqq t,\ \tau'<\tau\Bigg]\\&=E_x^0[u_{t-\tau'}(x_{\tau'})-\rho(t-\tau',\ x_{\tau'},\ xd/x_{\tau'});\ \tau'\leqq t,\ \tau'<\tau].\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathrm{III}_1-E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd'<x_s)\hat{u}_{t-s}(x_s\cdot\mu)\right]\\&=-E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(xd\leqq x_sM(\mu)<xd'<x_s)\hat{u}_{t-s}(x_s\cdot\mu)\right]\\&=-E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(xd\leqq x_sM(\mu)<xd'<x_s)u_{t-s}(x_sM(\mu))\right]\\&=-E_x^0\left[\int_0^{\tau'\wedge t} ds\int k(da)\chi(xd\leqq x_sa<xd'<x_s)u_{t-s}(x_sa)\right]\\&=-E_x^0\left[\sum_{s\leqq\tau'\wedge t}\chi(xd\leqq x_s<xd'<x_{s-})u_{t-s}(x_s)\right]\end{aligned}$$

$$=-E_x^0[\chi(xd\leqq x_{\tau'\wedge t}<xd')u_{t-(\tau'\wedge t)}(x_{\tau'\wedge t})]$$

$$=-E_x^0[u_{t-\tau'}(x_{\tau'});\ \tau'\leqq t,\ \tau'<\tau]$$

$$+E_x^0[u_{t-\tau'}(xd');\ \tau'<\tau,\ \tau'\leqq t,\ x_{\tau'}=xd']$$

$$-E_x^0[u_{t-\tau'}(xd);\ \tau'=\tau\leqq t,\ x_\tau=xd<xd'].$$

Therefore,

$$\rho(t,x,d')=E_x^0[u_{t-\tau'}(x_{\tau'})-\rho(t-\tau',\ x_{\tau'},\ xd/x_{\tau'});\ \tau'\leqq t,\ \tau'<\tau]$$

$$-E_x^0[u_{t-\tau'}(x_{\tau'});\ \tau'\leqq t,\ \tau'<\tau]$$

$$+E_x^0[u_{t-\tau'}(xd');\ \tau'<\tau,\ \tau'\leqq t,\ x_{\tau'}=xd']$$

$$-E_x^0[u_{t-\tau'}(xd');\ x_{\tau'}=xd',\ \tau'\leqq t]$$

$$=-E_x^0[\rho(t-\tau',\ x_{\tau'},\ xd/x_{\tau'});\ \tau'\leqq t,\ \tau'<\tau],$$

and if $t\leqq t(d)$,

$$|\rho(t,x,d')|\leqq\rho P_x^0(\tau'\leqq t,\ \tau'<\tau).$$

Applying this inequality to the above equality, we have for $t\leqq t(d)$

$$|\rho(t,x,d')|\leqq E_x^0[\rho P_{x_{\tau'}}^0(\tau_p\leqq r,\ \tau_p<\tau)_{\substack{r=t-\tau'\\ p=xd/x_{\tau'}}};\ \tau'\leqq t,\ \tau'<\tau]$$

$$\leqq\rho E_x^0[P_{x_{\tau'}}^0(\tau_p\leqq r)_{\substack{r=t-\tau'\\ p=xd/x_{\tau'}}};\ \tau'\leqq t,\ \tau'<\tau]$$

$$=\rho P_x^0(\tau\leqq t,\ \tau'<\tau)$$

$$\leqq\rho P_x^0(\tau\leqq t(d)),$$

and hence

$$\rho\leqq\rho P_x^0(\tau\leqq t(d)).$$

Since $P_x^0(\tau\leqq t(d))<1$, we have $\rho=0$.

LEMMA 8. 5. *Let $u_t(x;f)$ be a solution in B_d^* of the (S_d)-equation for $f\in B_d^*$. If a positive number r is such that the (S_d)-equation for any $f\in B_d^*$ has always a unique solution in B_d^* for $t\leqq r$, then it holds*

$$u_{t+s}(x;f)=u_t(x;\ u_s(\cdot;f))$$

for $t\leqq r$ and $s\leqq t(d)$.

Proof. Since $u_t(x)\equiv u_t(x;f)$ is a solution of the (S_d)-equation, we have

$$\begin{aligned} u_{t+s}(x) &= S_d(x, t+s; f, u) \\ &= E_x^0[f(x_{t+s});\ t+s<\tau] + E_x^0[u_{t+s-\tau}(xd);\ x_\tau = xd,\ \tau \leqq t+s] \\ &\quad + E_x^0\left[\int_0^{t+s} du \int \Pi(d\mu)\chi(x_u M(\mu) < xd < x_u)\hat{u}_{t+s-u}(x_u \cdot \mu)\right] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned}$$

where $\tau = \tau_d$. Using a Markov property at time t, we develop the terms in the right-hand side as follows.

$$\mathrm{I} \equiv E_x^0[E_{x_t}^0[f(x_s);\ s<\tau_p]_{p=xd/x_t};\ t<\tau].$$

$$\begin{aligned} \mathrm{II} &\equiv E_x^0[u_{t+s-\tau}(xd);\ x_\tau = xd,\ \tau \leqq t] \\ &\quad + E_x^0[u_{t+s-\tau}(xd);\ x_\tau = xd,\ t<\tau \leqq t+s] \\ &= \mathrm{II}_1 + \mathrm{II}_2, \quad \text{say.} \end{aligned}$$

$$\mathrm{II}_2 = E_x^0[E_{x_t}^0[u_{s-\tau_p}(xd);\ x_{\tau_p} = xd,\ \tau_p \leqq s]_{p=xd/x_t};\ t<\tau].$$

$$\begin{aligned} \mathrm{III} &\equiv E_x^0\left[\int_0^t du \int \Pi(d\mu)\cdots\right] + E_x^0\left[\int_t^{t+s} du \int \Pi(d\mu)\cdots\right] \\ &= \mathrm{III}_1 + \mathrm{III}_2, \quad \text{say.} \end{aligned}$$

$$\begin{aligned} \mathrm{III}_2 &= E_x^0\left[\int_0^s du \int \Pi(d\mu)\chi(x_{u+t}M(\mu) < xd < x_{u+t})\hat{u}_{s-u}(x_{u+t}\cdot\mu)\right] \\ &= E_x^0\left[E_{x_t}^0\left[\int_0^s du \int \Pi(d\mu)\chi(x_u M(\mu) < xd < x_u)\hat{u}_{s-u}(x_u\cdot\mu)\right];\ t<\tau\right]. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathrm{I} + \mathrm{II}_2 + \mathrm{III}_2 \\ &= E_x^0\Bigg[\Bigg\{E_y^0[f(x_s);\ s<\tau_p] + E_y^0[u_{s-\tau_p}(yp);\ x_{\tau_p} = yp,\ \tau_p \leqq s] \\ &\quad + E_y^0\left[\int_0^s du \int \Pi(d\mu)\chi(x_u M(\mu) < yp < x_u)\hat{u}_{s-u}(x_u\cdot\mu)\right]\Bigg\}_{\substack{y=x_t \\ p=xd/x_t}};\ t<\tau\Bigg] \\ &= E_x^0[u_s(x_t);\ t<\tau] \end{aligned}$$

for $s \leqq t(d)$, where we applied Lemma 8.4 because $p = xd/x_t \geqq d$. Therefore, for $s \leqq t(d)$,

$$\begin{aligned} u_{t+s}(x) &= E_x^0[u_s(x_t);\ t<\tau] + E_x^0[u_{t+s-\tau}(xd);\ x_\tau = xd,\ \tau \leqq t] \\ &\quad + E_x^0\left[\int_0^t du \int \Pi(d\mu)\chi(x_u M(\mu) < xd < x_u)\hat{u}_{t+s-u}(x_u\cdot\mu)\right]. \end{aligned}$$

When we fix $f \in B_d^*$ and $s \leqq t(d)$ in the above equation and regard $u_{t+s}(x)$ as a function of variables t and x, we have

$$u_{t+s}(x; f) \equiv u_{t+s}(x) = u_t(x; u_s(\cdot; f))$$

for $t \leqq r$ by the assumption of the lemma.

LEMMA 8. 6. (i) *The* (S_d)*-equation for* $f \in B_d^*$ *has always a unique solution* $u_t(x; f)$ *in* B_d^* *for all* $t \geqq 0$, *and* (ii) *the solution satisfies the iteration property*

$$u_{t+s}(x; f) = u_t(x; u_s(\cdot; f)) \tag{8. 2}$$

for all $s, t \geqq 0$. *Moreover,* (iii) *if* $f \in C_d^*$, *then* $u_t(\cdot; f) \in C_d^*$ *also for all* $t \geqq 0$.

Proof. (i) Let $u_t(x; f)$ and $v_t(x; f)$ be two solutions in B_d^* of the (S_d)-equation for $f \in B_d^*$ (such a solution exists by Lemma 8. 2). Then, by Lemma 8. 3 and Lemma 8. 5,

$$u_{t+s}(x; f) = u_t(x; u_s(\cdot; f))$$

and

$$v_{t+s}(x; f) = v_t(x; v_s(\cdot; f))$$

for $s \leqq t(d)$ and $t \leqq t(d)$. However, since the right-hand sides of the above equalities are the same by Lemma 8. 3, we have $u_t(x; f) = v_t(x; f)$ for $t \leqq 2t(d)$. Thus, repeating the same argument, we have $u_t(x; f) = v_t(x; f)$ for all $t \geqq 0$ because $t(d) > 0$.

(ii) By Lemma 8. 5 and (i) above, (8. 2) holds for $s \leqq t(d)$ and any $t \geqq 0$. Therefore, if $s, u \leqq t(d)$ and $t \geqq 0$, then

$$\begin{aligned} u_{t+s+u}(x; f) &= u_{t+s}(x; u_u(\cdot; f)) \\ &= u_t(x; u_s(\cdot; u_u(\cdot; f))) \\ &= u_t(x; u_{s+u}(\cdot; f)), \end{aligned}$$

which shows (8. 2) for $s \leqq 2t(d)$ and $t \geqq 0$. Repeating again the same argument, we have (8. 2) for all $s, t \geqq 0$.

(iii) If $f \in C_d^*$, then $u_t(\cdot; f) \equiv u_t^d(\cdot; f) \in C_d^*$ for $t \leqq t(d)$ by Lemma 8. 3. Using the iteration property (8. 2) for $t, s \leqq t(d)$, we have $u_t(\cdot; f) \in C_d^*$ for $t \leqq 2t(d)$, and hence by applying the result to (8. 2) again and again, we have $u_t(\cdot; f) \in C_d^*$ for all $t \geqq 0$.

LEMMA 8. 7. *If* $f \in B_d^*$ *and* $d < d' < 1$, *then*

$$u_t^d(x; f) \equiv u_t^{d'}(x; f) \text{ for all } t \geqq 0. \tag{8. 3}$$

Proof. First, note that $B_d^* \subset B_{d'}^*$ if $d < d' < 1$. By Lemma 8. 4, $u_t^d(x; f)$ satisfies the $(S_{d'})$-equation for f when $t \leqq t(d)$, and hence, by the uniqueness property of Lemma 8. 6 (i), we have (8. 3) for $t \leqq t(d)$ and any $f \in B_d^*$. Therefore, using the iteration property (8. 2) for $s, t \leqq t(d)$,

$$u_{t+s}^d(x; f) = u_t^d(x; u_s^d(\cdot; f)) = u_t^d(x; u_s^{d'}(\cdot; f))$$
$$= u_t^{d'}(x; u_s^{d'}(\cdot; f)) = u_{t+s}^{d'}(x; f),$$

which shows (8. 3) for $t \leqq 2t(d)$. Thus, the equality (8. 3) for all $t \geqq 0$ follows from the same procedure.

The following lemma was first given by Ikeda, Nagasawa, and Watanabe [5], however for later use, we need some more detailed statement.

LEMMA 8. 8. *For any positive integer n and $0 < p_1, p_2, \cdots, p_n < \infty$, let $Q_{p_i}(i=1, 2, \cdots, n)$ be a given finite Borel measure on $\boldsymbol{M}_{p_i}$. Then, there is a unique finite Borel measure $Q_{p_1+\cdots+p_n}$ on $\boldsymbol{M}_{p_1+\cdots+p_n}$ such that*

$$\int_{\boldsymbol{M}_{p_1+\cdots+p_n}} \hat{f}(\mu) Q_{p_1+\cdots+p_n}(d\mu) = \prod_{i=1}^n \int_{\boldsymbol{M}_{p_i}} \hat{f}(\mu) Q_{p_i}(d\mu) \tag{8. 4}$$

for all $f \in C_0^*$. Moreover, if we write

$$Q_{p_1+\cdots+p_n} \equiv Q_{p_1} \otimes Q_{p_2} \otimes \cdots \otimes Q_{p_n} \equiv \prod_{i=1}^n \otimes Q_{p_i},$$

then

$$Q_{p_1} \otimes Q_{p_2} = Q_{p_2} \otimes Q_{p_1}$$

and

$$\left\| \prod_{i=1}^n \otimes Q_{p_i} \right\| = \prod_{i=1}^n \|Q_{p_i}\|, \tag{8. 5}$$

where $\|Q_p\|$ means the total mass of a measure Q_p.

Proof. Let us consider the case $n=2$. We show the existence of the measure $Q_{p_1+p_2}$ such that

$$\int_{\boldsymbol{M}_{p_1+p_2}} F(\omega) Q_{p_1+p_2}(d\omega) = \int_{\boldsymbol{M}_{p_1}} Q_{p_1}(d\mu) \int_{\boldsymbol{M}_{p_2}} Q_{p_2}(d\nu) F(\mu+\nu)$$

for all $F \in C(\boldsymbol{M}_{p_1+p_2})$ (the uniqueness of such $Q_{p_1+p_2}$ is immediate from Proposition 1. 2). However, the existence is obvious from the fact that the right-hand side of the above equality is a continuous linear functional of $F \in C(\boldsymbol{M}_{p_1+p_2})$.

The lemma can be shown similarly for the case $n \geqq 3$ and the second half of the lemma is obvious.

LEMMA 8. 9. *For $\{u_t^n(x; f); n \geqq 0\}$ defined by (8. 1), there exists a sequence of probability measures $\tilde{Q}_d^n(t, x, d\nu)$ on $\boldsymbol{M}_1$ such that $\tilde{Q}_d^n(t, x, \boldsymbol{M}_x) = 1$,*

$$u_t^n(x;\, f)=\int_{M_x}\hat{f}(\nu)\tilde{Q}_d^n(t,\, x,\, d\nu) \tag{8.6}$$

for all $f\in C_d^*$, *and* $\tilde{Q}_d^n(t, x, E)$ *is measurable in* (t, x) *for* $E\in\mathcal{B}_1$ *fixed.*

Proof. For $n=0$, take $\tilde{Q}_d^0(t, x, d\nu)=\delta_{\{0\}}(d\nu)$ the unit mass concentrated at $0\in M_1$. Thus, the proof is done by induction as follows. Put $u_t^n(x;\, f)\equiv u_t^n(x)$ and assume the statement of the lemma for n. Then, we put

$$\begin{aligned} u_t^{n+1}(x)&=E_x^0[f(x_t);\ t<\tau]+E_x^0[u_{t-\tau}^n(xd);\ x_\tau=xd,\ \tau\leqq t]\\ &\quad+E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\hat{u}_{t-s}^n(x_s\cdot\mu)\right]\\ &=\mathrm{I}+\mathrm{II}+\mathrm{III}, \end{aligned}$$

where $\tau=\tau_d$. By putting

$$\Sigma_d^{(1)}(t,\, x,\, d\nu)=E_x^0[\delta_{\{x_t\delta_{x_t}\}}(d\nu);\ t<\tau]$$

for the first term I, we have

$$\mathrm{I}=\int_{M_x}\hat{f}(\nu)\Sigma_d^{(1)}(t,\, x,\, d\nu)$$

for all $f\in C_d^*$, where $\Sigma_d^{(1)}(t, x, d\nu)$ is a measure in $d\nu$ for fixed (t, x) satisfying $\Sigma_d^{(1)}(t, x, M_1-M_x)=0$ and is measurable in (t, x) for $d\nu$ fixed. For the second term II, since

$$\mathrm{II}=E_x^0\left[\int_{M_{x_\tau}}\hat{f}(\nu)\tilde{Q}_d^n(t-\tau,\ x_\tau,\ d\nu);\ x_\tau=xd,\ \tau\leqq t\right]$$

for $f\in C_d^*$, we have, by putting

$$\Sigma_d^{(2)}(t,\, x,\, d\nu)=E_x^0[\tilde{Q}_d^n(t-\tau,\ x_\tau,\ d\nu);\ x_\tau=xd,\ \tau\leqq t],$$

$$\mathrm{II}=\int_{M_x}\hat{f}(\nu)\Sigma_d^{(2)}(t,\, x,\, d\nu)$$

for $f\in C_d^*$, where $\Sigma_d^{(2)}$ is also a measure with desired property. Finally, for the last term III, let us first define $\tilde{Q}_d^n(t, \varphi_d(\mu), d\nu)$ by

$$\tilde{Q}_d^n(t,\, \varphi_d(\mu),\, d\nu)=\tilde{Q}_d^n(t,\, x_1,\, d\nu)\otimes\cdots\otimes\tilde{Q}_d^n(t,\, x_m,\, d\nu)$$

for $\varphi_d(\mu)=\sum_{i=1}^m x_i\delta_{x_i}$. Then, $\tilde{Q}_d^n(t, \varphi_d(\mu), d\nu)$ is a probability measure on M_1 for $(t, \varphi_d(\mu))$ fixed, satisfying $\tilde{Q}_d^n(t, \varphi_d(\mu), M_{\|\mu\|})=1$ and is measurable in (t, μ) for $d\nu$ fixed. It follows from the assumption that, if $f\in C_d^*$,

$$\hat{u}_t^n(\mu)=\prod_{x_i>(1-d)/d} u_t^n(x_i)=\prod_{x_i>(1-d)/d}\int_{M_{x_i}} \hat{f}(\nu)\tilde{Q}_d^n(t, x_i, d\nu)$$

$$=\int_{M_{||\mu||}} \hat{f}(\nu)\tilde{Q}_d^n(t, \varphi_d(\mu), d\nu),$$

where $\mu=\sum x_i\delta_{x_i}\in M_1$, and hence

$$\text{III}=E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\int_{M_x} \hat{f}(\nu)\tilde{Q}_d^n(t-s,\ \varphi_d(x_s\cdot\mu),\ d\nu)\right].$$

Thus, by putting

$$\Sigma_d^{(3)}(t, x, d\nu)=E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\tilde{Q}_d^n(t-s,\ \varphi_d(x_s\cdot\mu),\ d\nu)\right],$$

we have

$$\text{III}=\int_{M_x} \hat{f}(\nu)\Sigma_d^{(3)}(t, x, d\nu)$$

for $f\in C_d^*$, where $\Sigma_d^{(3)}$ is a measure with the desired property.

Therefore, the sum

$$\tilde{Q}_d^{n+1}(t, x, d\nu)\equiv\Sigma_d^{(1)}(t, x, d\nu)+\Sigma_d^{(2)}(t, x, d\nu)+\Sigma_d^{(3)}(t, x, d\nu)$$

is a measure in $d\nu$ for fixed (t, x) satisfying $\tilde{Q}_d^{n+1}(t, x, M_1-M_x)=0$ and is measurable in (t, x) for fixed $d\nu$, and it satisfies

$$u_t^{n+1}(x)=\int_{M_x} \hat{f}(\nu)\tilde{Q}_d^{n+1}(t, x, d\nu)$$

for all $f\in C_d^*$. Moreover, $\tilde{Q}_d^{n+1}(t, x, M_x)=1$, since $u_t^{n+1}(x;\ 1)\equiv 1$.

LEMMA 8.10. *Let $u_t(x; f)$ be the solution of the (S_d)-equation for $f\in C_d^*$. Then, there exists a probability measure $\tilde{Q}_d(t, \mu, d\nu)$ on M_1 for any $\mu\in M_1$ such that*

$$\widehat{u_t(\cdot\,; f)}(\mu)=\int_{M_1} \hat{f}(\nu)\tilde{Q}_d(t, \mu, d\nu) \tag{8.7}$$

for all $f\in C_d^*$.

Proof. By Lemma 8.9 and the weak*-compactness of a set of probability measures on M_1, there exists a probability measure $\tilde{Q}_d(t, x, d\nu)$ such that

$$u_t(x; f)=\int_{M_x} \hat{f}(\nu)\tilde{Q}_d(t, x, d\nu)$$

for all $f\in C_d^*$. Moreover, the existence of $\tilde{Q}_d(t, \mu, d\nu)$ for any $\mu\in M_1$ and (8.7) follow from Lemma 8.8.

LEMMA 8.11. (i) *There exists a unique probability measure* $Q_d(t, \mu, d\nu)$ *on* $(\boldsymbol{M}_1, \varphi_d^{-1}(\mathscr{B}_1^d))$ *for* $t \geqq 0$ *and* $\mu \in \boldsymbol{M}_1$ *such that*

$$\widehat{u_t(\cdot; f)}(\mu) = \int_{\boldsymbol{M}_1} \hat{f}(\nu) Q_d(t, \mu, d\nu)$$

for all $f \in C_d^*$, *and* (ii), *if* $d < d' < 1$,

$$Q_{d'}(t, \mu, d\nu)|_{\varphi_d^{-1}(\mathscr{B}_1^d)} = Q_d(t, \mu, d\nu).$$

Proof. (i) Since $\hat{f}$ is measurable with respect to $\varphi_d^{-1}(\mathscr{B}_1^d)$ for $f \in C_d^*$ (Lemma 1.4 (iv)), we can take $Q_d(t, \mu, d\nu) \equiv \tilde{Q}_d(t, \mu, d\nu)|_{\varphi_d^{-1}(\mathscr{B}_1^d)}$ in (8.7) instead of $\tilde{Q}_d(t, \mu, d\nu)$. The uniqueness of such a measure is immediate from Lemma 6.4.

(ii) From Lemma 8.7, it follows that

$$\int_{\boldsymbol{M}_1} \hat{f}(\nu) Q_d(t, \mu, d\nu) = \int_{\boldsymbol{M}_1} \hat{f}(\nu) Q_{d'}(t, \mu, d\nu)$$

for all $f \in C_d^*$ $(\subset C_{d'}^*)$, and hence, by Lemma 6.4,

$$Q_{d'}(t, \mu, d\nu)|_{\varphi_d^{-1}(\mathscr{B}_1^d)} = Q_d(t, \mu, d\nu).$$

LEMMA 8.12. *There exists a unique probability measure* $P(t, \mu, d\nu)$ *on* $(\boldsymbol{M}_1, \mathscr{B}_1)$ *for* $t \geqq 0$ *and* $\mu \in \boldsymbol{M}_1$ *which satisfies*

$$\widehat{u_t(\cdot; f)}(\mu) = \int_{\boldsymbol{M}_1} \hat{f}(\nu) P(t, \mu, d\nu) \tag{8.8}$$

for all $f \in C_0^*$.

Proof. It follows from Lemma 8.11 and Lemma 1.5 that there exists a unique probability measure $P(t, \mu, d\nu)$ on $(\boldsymbol{M}_1, \mathscr{B}_1)$ for fixed $t \geqq 0$ and $\mu \in \boldsymbol{M}_1$ such that $P(t, \mu, d\nu)|_{\varphi_d^{-1}(\mathscr{B}_1^d)} = Q_d(t, \mu, d\nu)$. Moreover, the equality (8.8) is obvious for $f \in \cup_d C_d^* = C_0^*$ from Lemma 8.11 and the construction of $P(t, \mu, d\nu)$.

LEMMA 8.13. *Define a family of operators* $\{T_t;\ t \geqq 0\}$ *by*

$$T_t F(\mu) = \int_{\boldsymbol{M}_1} F(\nu) P(t, \mu, d\nu), \tag{8.9}$$

then T_t *is a nonnegative linear operator on* $C(\boldsymbol{M}_1)$ *such that* $T_t 1 = 1$ *and* $\|T_t\| = 1$, *and* $T_0 = I$ *(identity).*

Proof. Since $T_t \hat{f} \in C(\boldsymbol{M}_1)$ for all $f \in C_0^*$ by Lemma 8.12 and Lemma 8.6 (iii), it follows from Proposition 1.2 that $T_t F \in C(\boldsymbol{M}_1)$ for all $F \in C(\boldsymbol{M}_1)$. The remainders of the lemma are obvious.

LEMMA 8.14. *For* $f \in C_0^*$ *and* $t \geqq 0$,

$$T_t \hat{f}(\mu + \nu) = T_t \hat{f}(\mu) \cdot T_t \hat{f}(\nu) \tag{8.10}$$

for $\mu, \nu \in \boldsymbol{M}_1$ *such that* $\mu+\nu \in \boldsymbol{M}_1$ *also.*

Proof. By Lemma 8.12, we have

$$T_t\hat{f}(\mu+\nu)=\widehat{u_t(\cdot;\,f)}(\mu+\nu)$$

$$=\widehat{u_t(\cdot;\,f)}(\mu)\cdot\widehat{u_t(\cdot;\,f)}(\nu)=T_t\hat{f}(\mu)\cdot T_t\hat{f}(\nu).$$

LEMMA 8.15. *For* $F \in C(\boldsymbol{M}_1)$ *and* $t, s \geqq 0$,

$$T_{t+s}F=T_tT_sF. \tag{8.11}$$

Proof. It is enough to show (8.11) for functions $F=\hat{f}$, $f \in C_0^*$. When $\mu=x\delta_x \in \boldsymbol{M}_1$, we have

$$T_{t+s}\hat{f}(x\delta_x)=\widehat{u_{t+s}(\cdot;\,f)}(x\delta_x)$$

$$=u_{t+s}(x;\,f)=u_t(x;\,u_s(\cdot;\,f))$$

$$=T_t\widehat{u_s(\cdot;\,f)}(x\delta_x)=T_tT_s\hat{f}(x\delta_x)$$

by the iteration property (8.2). Therefore, for any $\mu=\sum x_i\delta_{x_i} \in \boldsymbol{M}_1$, since $T_{t+s}\hat{f} \in C(\boldsymbol{M}_1)$,

$$T_{t+s}\hat{f}(\mu)=\lim_{\varepsilon\downarrow 0} T_{t+s}\hat{f}\Big(\sum_{x_i>\varepsilon} x_i\delta_{x_i}\Big)$$

and, by Lemma 8.14,

$$T_{t+s}\hat{f}\Big(\sum_{x_i>\varepsilon} x_i\delta_{x_i}\Big)=\prod_{x_i>\varepsilon} T_{t+s}\hat{f}(x_i\delta_{x_i})=\prod_{x_i>\varepsilon} T_t\widehat{u_s(\cdot;\,f)}(x_i\delta_{x_i})$$

$$=T_t\widehat{u_s(\cdot;\,f)}\Big(\sum_{x_i>\varepsilon} x_i\delta_{x_i}\Big)=T_tT_s\hat{f}\Big(\sum_{x_i>\varepsilon} x_i\delta_{x_i}\Big),$$

we have

$$T_{t+s}\hat{f}(\mu)=\lim_{\varepsilon\downarrow 0} T_tT_s\hat{f}\Big(\sum_{x_i>\varepsilon} x_i\delta_{x_i}\Big)=T_tT_s\hat{f}(\mu).$$

LEMMA 8.16. *For* $F \in C(\boldsymbol{M}_1)$ *and for* $\mu \in \boldsymbol{M}_1$,

$$\lim_{t\downarrow 0} T_tF(\mu)=F(\mu). \tag{8.12}$$

Proof. Let $f \in C_d^*$. Since $u_t(x;\,f)=T_t\hat{f}(x\delta_x)$ is a solution of the (S_d)-equation for f, we put

$$u_t(x;\ f)=E_x^0[f(x_t);\ t<\tau]+E_x^0[u_{t-\tau}(xd;\ f);\ x_\tau=xd,\ \tau\leqq t]$$
$$+E_x^0\left[\int_0^t ds\int_{M_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\widehat{u_{t-s}(\cdot;\ f)}(x_s\cdot\mu)\right]$$
$$=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where $\tau=\tau_d$. By the right continuity of x_t, we have

$$\mathrm{I}\to f(x) \quad \text{as} \quad t\downarrow 0,$$

and

$$\mathrm{II}+\mathrm{III}\leqq P_x^0(\tau\leqq t)\to 0 \quad \text{as} \quad t\downarrow 0.$$

Hence,

$$u_t(x;\ f)\to f(x) \quad \text{as} \quad t\downarrow 0.$$

Therefore, for any $\mu=\sum x_i\delta_{x_i}\in \boldsymbol{M}_1$, noting $u_t(\cdot;\ f)\in C_d^*$,

$$T_t\hat{f}(\mu)=\widehat{u_t(\cdot;\ f)}(\mu)=\widehat{u_t(\cdot;\ f)}\Big(\sum_{x_i>(1-d)/d} x_i\delta_{x_i}\Big)$$
$$=\prod_{x_i>(1-d)/d} u_t(x_i;\ f)\to \prod_{x_i>(1-d)/d} f(x_i)=\hat{f}(\mu) \quad \text{as} \quad t\downarrow 0$$

since the number of i's such that $x_i>(1-d)/d$ is less than $d/(1-d)$. Thus, we have

$$\lim_{t\downarrow 0} T_t\hat{f}(\mu)=\hat{f}(\mu)$$

for all $f\in C_0^*$ and $\mu\in \boldsymbol{M}_1$. Moreover, since T_t is a bounded linear operator, we have (8. 12) for all $F\in C(\boldsymbol{M}_1)$.

LEMMA 8. 17. *For $f\in C_0^*$ and $a\in S$,*

$$T_t\hat{f}(a\delta_a)=T_t\widehat{\theta_a f}(\delta_1). \tag{8. 13}$$

Proof. Let $f\in C_d^*$ and put $\tau_d=\tau$.

$$u_t(ax;\ f)=E_{ax}^0[f(x_t);\ t<\tau]+E_{ax}^0[u_{t-\tau}(axd;\ f);\ x_\tau=axd,\ \tau\leqq t]$$
$$+E_{ax}^0\left[\int_0^t ds\int\Pi(d\mu)\chi(x_sM(\mu)<axd<x_s)\widehat{u_{t-s}(\cdot;\ f)}(x_s\cdot\mu)\right]$$
$$=E_x^0[f(ax_t);\ t<\tau]+E_x^0[u_{t-\tau}(axd;\ f);\ x_\tau=xd,\ \tau\leqq t]$$
$$+E_x^0\left[\int_0^t ds\int\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\widehat{u_{t-s}(\cdot;\ f)}(ax_s\cdot\mu)\right]$$

by the property (ii) of Lemma 8. 1. Therefore,

$$\theta_a u_t(\cdot;\, f)(x)=E_x^0[\theta_a f(x_t);\ t<\tau]+E_x^0[\theta_a u_{t-\tau}(\cdot;\, f)(xd);\ x_\tau=xd,\ \tau\leqq t]$$
$$+E_x^0\left[\int_0^t ds\int \Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\,\widehat{\theta_a u_{t-s}(\cdot;\, f)}\,(x_s\cdot\mu)\right],$$

and hence $\theta_a u_t(\cdot;\, f)(x)$ is a solution of the (S_d)-equation for $\theta_a f\in C_d^*$ such that $\theta_a u_t(\cdot;\, f)\in C_d^*$. By the uniqueness of the solution, we have

$$\theta_a u_t(\cdot;\, f)(x)=u_t(x;\ \theta_a f),$$

or

$$u_t(ax;\, f)\equiv u_t(x;\ \theta_a f).$$

Hence,

$$T_t\hat{f}(a\delta_a)=u_t(a;\, f)=u_t(1;\ \theta_a f)$$
$$=T_t\widehat{\theta_a f}(\delta_1).$$

The family $\{T_t;\ t\geqq 0\}$ defined by (8. 9) is a semigroup on $C(\boldsymbol{M}_1)$ and satisfies (8. 12) and $T_0=I$, so that it is seen to be strongly continuous in $t\geqq 0$ (see, for example, Yosida [17]). Moreover, by Lemmas 8. 14 and 8. 17, $\{T_t\}$ is a cascade semigroup (Definition 2. 1).

Now, we have arrived at the following theorem.

THEOREM 8. 1. *Given a nonnegative constant* m *and a Borel measure* $\Pi(d\mu)$ *on* $\boldsymbol{M}_1-\{\delta_1\}$ *which satisfies*

$$\int_{\boldsymbol{M}_1-\{\delta_1\}}(1-M(\mu))\Pi(d\mu)<+\infty.$$

Let $(x_t,\, P_x^0)$ *be a right continuous strong Markov process on* $[0, 1]$ *generated by* A^0: $\mathcal{D}(A^0)\supset C^1[0, 1]$ *and, for* $f\in C^1[0, 1]$,

$$A^0f(x)=-mx\frac{df}{dx}+\int_{\boldsymbol{M}_1-\{\delta_1\}}\Pi(d\mu)(f(xM(\mu))-f(x)).$$

Then, there exists a cascade semigroup $\{T_t;\ t\geqq 0\}$ *such that* $u_t(x;\, f)\equiv T_t\hat{f}(x\delta_x)$ *is a solution of the following non-linear integral equation for* $f\in C_d^*$ $(2/3<d<1)$:

$$u_t(x;\, f)=E_x^0[f(x_t);\ t<\tau_d]+E_x^0[u_{t-\tau_d}(xd;\, f);\ x_{\tau_d}=xd,\ \tau_d\leqq t]$$
$$+E_x^0\left[\int_0^t ds\int_{\boldsymbol{M}_1-\{\delta_1\}}\Pi(d\mu)\chi(x_sM(\mu)<xd<x_s)\,\widehat{u_{t-s}(\cdot;\, f)}\,(x_s\cdot\mu)\right].$$

Moreover, the cascade semigroup $\{T_t\}$ *has* (x_t, P_x^0) *as its underlying process and* $\Pi(d\mu)$ *as its branching measure.*

The last part of the theorem is due to Theorem 6. 1.

CONCLUDING REMARK. Let (m, Π) be a pair of two quantities m and Π such that m is a nonnegative constant and Π is a Borel measure on $\boldsymbol{M}_1-\{\delta_1\}$ satisfying (4. 16). Then, the preceding theorems 3. 1, 4. 1, 5. 1, 5. 2, 6. 1, and 8. 1 make it clear the fact that *a cascade semigroup is completely characterized by its underlying process and branching measure, or in other words,* (m, Π) *via the system of fundamental* (S_d)*-equations* $(2/3<d<1)$.

REFERENCES

[1] DYNKIN, E. B., Markov processes. Springer (1965).

[2] FUJIMAGARI, T., AND M. MOTOO, On cascade processes. Int. Conference on Func. Analysis and Related Topics (1969), 383-391.

[3] HARRIS, T. E., The theory of branching processes. Springer (1963).

[4] IKEDA, N., AND S. WATANABE, On some relations between the harmonic measure and the Levy measure for a certain class of Markov processes. J. Math. Kyoto Univ. **2** (1) (1962), 79-95.

[5] IKEDA, N., M. NAGASAWA, AND S. WATANABE, Branching Markov processes I; II; III. J. Math. Kyoto Univ. 8 (1968), 233-278; 365-410; **9** (1969), 95-160.

[6] ITO, K., AND H. P. MCKEAN, JR., Diffusion processes and their sample paths. Springer (1965).

[7] JIRINA, M., Stochastic branching processes with continuous state space. Czech. J. Math. **8** (1958), 292-313.

[8] JIRINA, M., Branching processes with measure-valued states. 3rd Prague Conference (1964), 333-357.

[9] LAMPERTI, J., Continuous state branching processes. Bull. Amer. Math. Soc. **73** (3) (1967), 382-386.

[10] MOTOO, M., Additive functionals of Markov processes. Seminar on Prob. **15** (1963). (Japanese)

[11] SATO, K., Semigroup and Markov processes. Lecture notes at University of Minnesota (1968).

[12] SKOROHOD, A. V., Branching diffusion processes. Theory Prob. Appl. **9** (1964), 492-497.

[13] SILVERSTEIN, M. L., Continuous state branching semigroups. Zeits. Wahrscheinlichkeitstheorie **14** (2) (1969), 96-112.

[14] WATANABE, S., A limit theorem of branching processes and continuous state branching processes. J. Math. Kyoto Univ. 8 (1) (1968), 141-167.

[15] WATANABE, S., On two dimensional Markov processes with branching property. Trans. Amer. Math. Soc. **136** (1969), 447-466.

[16] WATANABE, S., On discontinuous additive functionals and Levy measures of a

Markov process. Japan. J. Math. **34** (1964), 53–70.
[17] YOSIDA, K., Functional analysis. Springer (1965).

TOKYO INSTITUTE OF TECHNOLOGY AND
TOKYO UNIVERSITY OF EDUCATION.