CASCADE STOCHASTIC DIFFERENTIAL SYSTEMS: ASYMPTOTIC STABILIZATION VIA A JURDJEVIC-QUINN APPROACH.

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Abstract

The purpose of this paper is to apply the stochastic version of La Salle's invariance principle in order to obtain sufficient conditions for the asymptotic stabilization in probability of cascade nonlinear stochastic differential systems. This result extends the one obtained in Florchinger [4] for partially linear stochastic cascade differential systems.

1 Introduction

The aim of this paper is to study the asymptotic stabilization in probability of cascade nonlinear stochastic differential systems by means of state feedback laws.

In connexion with various practical problems, the questions of stability and stabilizability of nonlinear stochastic differential systems have been considered by different authors in the last past years. A fundamental tool consists in the stochastic Lyapunov machinery developed by Khasminskii in [8]. See for example the papers [6] and [3] and the references therein. An extension of the well-known result of Jurdjevic-Quinn [7] allowing to compute explicitely state feedback stabilizers for stochastic differential systems affine in the control has been established by Florchinger in [5].

In general, the cascade connexion of two globally asymptotically stable in probability stochastic differential systems does not yield an asymptotically stable in probability stochastic differential system. The resulting stabilization problem has been investigated by Florchinger in [4] and Boulanger and Florchinger in [1]. The construction of the stabilizing control laws in the above cited papers is an extension of the cancellation procedure used in deterministic control theory, and make use of a composite Lyapunov function like that introduced in Saberi, Kokotovic and Sussmann [11] for deterministic systems.

In this paper, the asymptotic stabilization in probability of cascade systems obtained when connecting two nonlinear stochastic differential sytems stable (but not necessarily both asymptotically stable) in probability is obtained by mean of explicit state feedback laws. The paper is divided in three sections and is organized as follows. In section two, we briefly recall some results about stochastic stability and stabilization which are closely related to the present paper. In section three, we introduce the class of nonlinear stochastic differential systems we are deeling with. In section four, we state and prove the main result of the paper.

2 Stochastic stability and stabilization.

Let $(w_t)_{t\geq 0}$ be a standard Wiener process with values in $\mathbb{R}^q$ defined on some complete probability space $(\Omega, \mathcal{F}, P)$.

Denote by $\left(x_t^\xi\right)_{t\geq 0}$ the stochastic process solution in $\mathbb{R}^n$ of the stochastic differential equation written in the sense of Itô:

$$x_t^\xi = \xi + \int_0^t b\left(x_s^\xi\right) ds + \int_0^t \sigma\left(x_s^\xi\right) dw_s \tag{1}$$

where b and σ are meaurable functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times q}$, respectively, vanishing in the origin.

The infinitesimal generator of the solution $\left(x_t^\xi\right)_{t\geq 0}$ of the stochastic differential equation (1) is the second order differential operator L defined by:

$$L = \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n\sum_{k=1}^q (\sigma_{i,k}\sigma_{j,k})(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

The following facts, proven in Khasminskii [8] and Kushner [9] will be used in the sequel.

Assume that there exist a positive constant c and a Lyapunov function V, that is a proper and positive definite C^2 function mapping $\mathbb{R}^n$ into $\mathbb{R}^n$, such that:

$$LV(x) \leq cV(x),$$

for all $x \in \mathbb{R}^n$.

Then, if the functions f and σ satisfy Lipschitz conditions on any ball in $\mathbb{R}^n$, the stochastic differential equation (1) has a unique solution on the time interval $[0, +\infty[$ for any initial condition ξ in $\mathbb{R}^n$.

If in addition $LV(x) \leq 0$ for all x in $\mathbb{R}^n$, then the equilibrium solution $x_t^0 \equiv 0$ of the stochastic differential equation

(1) is stable in probability. This means that:

$$\lim_{|\xi|\to 0} P\left\{\sup_{t\geq 0}\left|x_t^\xi\right| > r\right\} = 0$$

for any $r > 0$.

The equilibrium solution $x_t^0 \equiv 0$ of the stochastic differential equation (1) is said to be asymptotically stable in probability if, and only if, it is stable in probability and:

$$P\left\{\lim_{t\to+\infty} x_t^\xi = 0\right\} = 1$$

for any initial condition ξ in $\mathbb{R}^n$.

A sufficient condition for the latter to hold is that $LV(x) < 0$ for all x in $\mathbb{R}^n \setminus \{0\}$.

Another powerfull tool to investigate the asymptotic behavior of the stochastic process x_t^ξ is the following stochastic version of La Salle's theorem proved by Kushner in [10].

Theorem 2.1 *Assume that there exists a Lyapunov function V such that*

$$LV(x) \leq 0$$

for any $x \in \mathbb{R}^n$. Then, the stochastic process x_t^ξ solution of the stochastic differential equation (1) tends in probability to the largest invariant set whose support is contained in the locus $LV(x_t^\xi) = 0$ for any $t \geq 0$.

From this result, an extension to stochastic differential systems of Jurdjevic-Quinn's theorem [7] has been obtained by Florchinger in [5].

Consider the stochastic differential system described by the Itô equation:

$$\begin{aligned} x_t^{\xi,u} &= \xi + \int_0^t \left[b\left(x_s^{\xi,u}\right) + h\left(x_s^{\xi,u}\right)u\right]ds \\ &\qquad + \int_0^t \sigma\left(x_s^{\xi,u}\right)dw_s, \end{aligned} \tag{2}$$

where u is some measurable control law with values in $\mathbb{R}^r$ and h is a function mapping $\mathbb{R}^n$ into $\mathbb{R}^{n\times r}$, whose columns will be denoted by h^l, $1 \leq l \leq r$.

Denote by L_0 the infinitesimal generator of the stochastic process solution of the stochastic differential system deduced from (2) by setting $u \equiv 0$, and by $\mathcal{G}_j$, $1 \leq j \leq q$, the first order differential operators defined by:

$$\mathcal{G}_j = \sum_{i=1}^n \sigma_i^j(x)\frac{\partial}{\partial x_i}.$$

Define also the first order differential operators Λ_l, $1 \leq l \leq r$, by:

$$\Lambda_l = \sum_{i=1}^n h_i^l(x)\frac{\partial}{\partial x_i},$$

Then, the following result holds. (See also [2] for a more general dependance of the coefficients on the control law)

Theorem 2.2 *Assume that there exists a smooth Lyapunov function V defined on $\mathbb{R}^n$ such that:*

1. *$L_0V(x) \leq 0$ for all x in $\mathbb{R}^n$.*
2. *The set $\mathcal{K} =$*

$$\left\{\begin{array}{l} x \in \mathbb{R}^n / L_0^{\alpha_0}\mathcal{G}_{j_0}^{\beta_0}\cdots L_0^{\alpha_k}\mathcal{G}_{j_k}^{\beta_k}L_0V(x) = 0, \\ \qquad L_0^{\alpha_0}\mathcal{G}_{j_0}^{\beta_0}\cdots L_0^{\alpha_k}\mathcal{G}_{j_k}^{\beta_k}\Lambda_l V(x) = 0, \\ \forall l \in \{1,\ldots,p\}, \\ \forall k \in \mathbb{N}, \forall j_0,\ldots,j_k \in \{1,\ldots,q\}, \\ \forall \alpha_0,\beta_0,\ldots,\alpha_k,\beta_k \in \{0,\ldots,k\} \\ \text{such that } \sum_{i=0}^k \alpha_i + \beta_i = k \end{array}\right\}$$

is reduced to $\{0\}$.

Then, the control law u defined on $\mathbb{R}^n$ by

$$u_l(x) = -h_i^l(x)\frac{\partial V}{\partial x_i}(x)$$

renders the stochastic differential system (2) asymptotically stable in probability.

3 Problem statement.

Consider the stochastic process (x_t, y_t) solution in $\mathbb{R}^n \times \mathbb{R}^m$ of the stochastic differential system

$$\left\{\begin{array}{rcl} dx_t &=& f_1(x_t,y_t)dt + g_1(x_t,y_t)dw_t \\ dy_t &=& \left(f_2(y_t) + h(y_t)u\right)dt + g_2(y_t)dv_t \end{array}\right. \tag{3}$$

where

1. x_0 and y_0 are given in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.
2. $(v_t)_{t\geq 0}$ and $(w_t)_{t\geq 0}$ are independant standard Wiener processes defined on the probability space $(\Omega, \mathcal{F}, P)$, with values in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.
3. u is a an $\mathbb{R}^r$-valued measurable control law.
4. f_1 and g_1 are smooth functions mapping $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times q}$, respectively, vanishing in the origin.
5. f_2, h and g_2 are smooth functions mapping $\mathbb{R}^m$ into $\mathbb{R}^m$, $\mathbb{R}^{m\times r}$ and $\mathbb{R}^{m\times p}$, respectively, vanishing in the origin.

Furthermore, assume that the following conditions are satisfied.

(A1) The unforced dynamics of the stochastic process $(y_t)_{t\geq 0}$ are stable in probability. More precisely, there exist a Lyapunov function V_2 defined on $\mathbb{R}^m$ such that for all $y \in \mathbb{R}^m$:

$$\begin{aligned} L_2 V_2(y) &= (\nabla_y V_2 f_2)(y) \\ &\quad + \frac{1}{2}\text{Tr}\left(\left(g_2 g_2^\star \nabla^2_{yy} V_2 \right)(y) \right) \\ &\leq 0. \end{aligned}$$

(A2) There exist smooth functions f_1^i and g_1^i, $1 \leq i \leq r$, mapping $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times q}$, respectively, vanishing in the origin, such that for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$:

1. $f_1(x,y) = f_1(x,0) + \sum_{i=1}^r z_i(y) f_1^i(x,y).$
2. $$\begin{aligned} g_1(x,y)g_1(x,y)^\star &= g_1(x,0)g_1(x,0)^\star \\ &\quad + \sum_{i=1}^r z_i(y) g_1^i(x,y) g_1^i(x,y)^\star \end{aligned}$$

where $z(y) = (\nabla_y V_2(y) h(y))^\star$.

(A3) There exists a Lyapunov function V_1 defined on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$:

$$\begin{aligned} L_1 V_1(x) &= \nabla_x V_1(x) f_1(x,0) \\ &\quad + \frac{1}{2}\text{Tr}\left(g_1(x,0) g_1(x,0)^\star \nabla^2_{xx} V_1(x) \right) \\ &\leq 0. \end{aligned}$$

4 Asymptotic stabilization of the composite system.

In this section, the stochastic version of La Salle's theorem (theorem 2.1) will be used to obtain the asymptotic stabilization of the stochastic differential system (3).

First, note that defining the components of the state feedback law $u = \alpha(x,y)$ by:

$$\begin{aligned} \alpha_i(x,y) &= -\nabla_x V_1(x) f_1^i(x,y) \\ &\quad - \tfrac{1}{2}\text{Tr}\left(g_1^i(x,y) g_1^i(x,y)^\star \nabla^2_{xx} V_1(x) \right) \end{aligned} \tag{4}$$

$1 \leq i \leq r$, one renders the equilibrium solution $(0,0)$ of the stochastic differential system (3) stable in probability, as stated in the following proposition.

Proposition 4.1 *Under the assumptions (A1)-(A3), the state feedback law $u = \alpha(x,y)$ defined by (4) renders the stochastic differential system (3) stable in probability.*

Proof. Setting

$$W(x,y) = V_1(x) + V_2(y)$$

and denoting by $\mathcal{L}_\alpha$ the infinitesimal generator of the closed-loop system obtained in this case, one has for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$\begin{aligned} \mathcal{L}_\alpha W(x,y) &= \nabla_x V_1(x) f_1(x,y) + \nabla_y V_2(y) f_2(y) \\ &\quad + \nabla_y V_2(y) h(y) \alpha(x,y) \\ &\quad + \tfrac{1}{2}\text{Tr}\left((g_1 g_1^\star)(x,y) \nabla^2_{xx} V_1(x) \right) \\ &\quad + \tfrac{1}{2}\text{Tr}\left(g_2(y) g_2(y)^\star \nabla^2_{yy} V_2(y) \right). \end{aligned}$$

From the assumption (A2), we know that

$$\begin{aligned} \nabla_x V_1(x) f_1(x,y) &= \nabla_x V_1(x) f_1(x,0) \\ &\quad + \sum_{i=1}^r \left((\nabla_y V_2(y) h(y))^\star \right)_i \\ &\qquad \cdot \nabla_x V_1(x) f_1^i(x,y) \end{aligned}$$

and

$$\begin{aligned} &\text{Tr}\left(g_1(x,y) g_1(x,y)^\star \nabla^2_{xx} V_1(x) \right) = \\ &\text{Tr}\left(g_1(x,0) g_1(x,0)^\star \nabla^2_{xx} V_1(x) \right) \\ &+ \sum_{i=1}^r \Big[\left((\nabla_y V_2(y) h(y))^\star \right)_i \\ &\quad \cdot \text{Tr}\left(g_1^i(x,y) g_1^i(x,y)^\star \nabla^2_{xx} V_1(x) \right) \Big]. \end{aligned}$$

The state feedback law $u = \alpha(x,y)$ is defined in such a way that:

$$\begin{aligned} &\nabla_y V_2(y) h(y) \alpha(x,y) = \\ &- \sum_{i=1}^r \left((\nabla_y V_2(y) h(y))^\star \right)_i \nabla_x V_1(x) f_1^i(x,y) \\ &- \frac{1}{2} \sum_{i=1}^r \Big[\left((\nabla_y V_2(y) h(y))^\star \right)_i \\ &\quad \cdot \text{Tr}\left(g_1^i(x,y) g_1^i(x,y)^\star \nabla^2_{xx} V_1(x) \right) \Big]. \end{aligned}$$

Thus, one gets:

$$\begin{aligned} \mathcal{L}_\alpha W(x,y) &= \nabla_x V_1(x) f_1(x,0) \\ &\quad + \tfrac{1}{2}\text{Tr}\left(g_1(x,0) g_1(x,0)^\star \nabla^2_{xx} V_1(x) \right) \\ &\quad + \nabla_y V_2(y) f_2(y) \\ &\quad + \tfrac{1}{2}\text{Tr}\left(g_2(y) g_2(y)^\star \nabla^2_{yy} V_2(y) \right) \\ &= L_1 V_1(x) + L_2 V_2(y). \end{aligned} \tag{5}$$

Taking into account (A1) and (A3), this implies that

$$\mathcal{L}_\alpha W(x,y) \leq 0 \tag{6}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and the desired conclusion follows.□

Introduce now for $1 \leq l \leq p$ the first order differential operators defined by

$$\mathcal{G}_l\varphi(y) = \nabla_y\varphi(y)g_2^l(y)$$

where $g_2^l(y)$, $1 \leq l \leq p$, denotes the columns of the matrix $g_2(y)$.

Then, the main result of this paper can be stated as follows.

Theorem 4.2 *Assume that the coefficients of the stochastic differential system (3) are smooth functions satisfying to the assumptions (A1) and (A2) and that the sets*

$$\left\{x \in \mathbb{R}^n / L_1^k V_1(x) = 0, k \in \mathbb{N}^\star\right\} \tag{7}$$

$$\left\{y \in \mathbb{R}^m / \mathcal{G}_l^k z_i(y) = 0, 1 \leq i \leq r, 1 \leq l \leq p, k \in \mathbb{N}\right\} \tag{8}$$

are reduced to the origin in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.
Then, the stochastic differential system (3) is asymptotically stabilized in probability by the feedback law defined on $\mathbb{R}^n \times \mathbb{R}^m$ by:

$$u(x, y) = \alpha(x, y) - z(y). \tag{9}$$

Proof. Let u be the state feedback control law defined on $\mathbb{R}^n \times \mathbb{R}^m$ by (9).

It is convenient to define the function $\tilde{f}_2$ on $\mathbb{R}^n \times \mathbb{R}^m$ by:

$$\tilde{f}_2(x, y) = f_2(y) + h(y)\alpha(x, y).$$

The infinitesimal generator $\mathcal{L}_{\alpha-z}$ of the composite stochastic differential system

$$d\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} f_1(x_t, y_t) \\ \tilde{f}_2(x_t, y_t) \end{pmatrix} dt - \begin{pmatrix} 0 \\ h(y_t)z(y) \end{pmatrix} dt + \begin{pmatrix} g_1(x_t, y_t) & 0 \\ 0 & g_2(y_t) \end{pmatrix} d\begin{pmatrix} w_t \\ v_t \end{pmatrix}$$

can now be written as:

$$\mathcal{L}_{\alpha-z} = \mathcal{L}_\alpha - z(y)^\star h(y)^\star \nabla_y.$$

Applying this differential operator to the Lyapunov function W yields

$$\mathcal{L}_{\alpha-z}W(x, y) = \mathcal{L}_\alpha W(x, y) - \left|(\nabla_y V_2(y)h(y))^\star\right|^2 \leq 0, \tag{10}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$.

Let $(x_t, y_t)_{t\geq 0}$ be a trajectory of the composite system such that

$$\mathcal{L}_{\alpha-z}W(x_t, y_t) = 0$$

for all $t \geq 0$.

In view of inequalities (6) and (10), this implies that

$$L_1V_1(x_t) = L_2V_2(y_t) = 0 \tag{11}$$

as well as

$$z(y_t) = 0 \tag{12}$$

for all $t \geq 0$.

In view of (12), it can be seen that the stochastic process $(x_t)_{t\geq 0}$ obeys to the stochastic differential equation

$$dx_t = f_1(x_t, 0)dt + g_1(x_t, 0)dw_t.$$

Starting with (11), recursive applications of Itô's formula yield for any $k \in \mathbb{N}^\star$:

$$L_1^{k+1}V_1(x_t) \equiv 0.$$

Moreover, other conditions have to be satisfied by the pair (x_t, y_t) . In particular, one has for any $k \in \mathbb{N}$ and $1 \leq l \leq p$:

$$\mathcal{G}_l^k z(y_t) = 0,$$

as it can be again verified by recursive applications of Itô's formula.

Therefore, we may conclude from (7) and (8) that the stochastic process (x_t, y_t) verifying $\mathcal{L}_{\alpha-z}W(x_t, y_t) = 0$ for all $t \geq 0$ is identically $(0, 0)$. The result follows from theorem 2.1. □

Remark 4.3 *The assumptions (7) and (8) in theorem 4.2 seem to be quite natural in the sense that they discribe properties of the components of the cascade stochastic differential system (2) when these components are uncoupled and unforced. Nevertheless, it is clear that the asymptotic stabilization of (2) can be achieved by a direct and full application of the sufficient condition stated in theorem 2.2.*

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