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**Citation for published version:**

Righetto, GM, Morabito, R & Alem, D 2020, 'Cash flow management by risk-neutral and risk-averse stochastic approaches', *Journal of the Operational Research Society*, vol. 71, no. 1, pp. 55-68.  
<https://doi.org/10.1080/01605682.2018.1524350>

**Digital Object Identifier (DOI):**

[10.1080/01605682.2018.1524350](https://doi.org/10.1080/01605682.2018.1524350)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

Journal of the Operational Research Society

**Publisher Rights Statement:**

This is an Accepted Manuscript of an article published by Taylor & Francis in the Journal of the Operational Research Society on 17 Feb 2019, available online:  
<https://www.tandfonline.com/doi/full/10.1080/01605682.2018.1524350> .

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# Cash flow management by risk-neutral and risk-averse stochastic approaches

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## Abstract

This paper presents a dynamic cash flow management problem with uncertain parameters in a finite planning horizon via two-stage stochastic programming. We propose a risk-neutral mixed-integer two-stage stochastic programming model and risk-averse versions based on the minimax regret and conditional value-at-risk criteria. The models support decisions in cash management that deals with different grace periods, piecewise linear yields and uncertainty in the exchange rate of external sales. The developed approach is applied to a real-world stationary company in Brazil. Numerical results assess the trade-off between risk and return, showing that the optimization models generate effective solutions for the company's treasury with reduced risks, which might be appealing for companies from other sectors as well.

Keywords: Cash flow management, mixed-integer programming, stochastic programming, minimax with regret, conditional value-at-risk, stationary industry.

## 1. Introduction

One of the main challenges faced by financial managers is to establish effective policies to perform operations involving foreign exchange transactions, which are usually based on exchange rate forecasts with inherent uncertain values (Perdomo and Botelho, 2007). This type of uncertainty is one of the main risk factors in the international market, as the fluctuations in the exchange rate may generate significant variations in financial investments. The exchange rate is one of the most difficult economic variables to predict and it depends on the evolution of economic fundamentals – trade balance, productivity and overall growth which itself is already uncertain. Furthermore, as it is a financial asset price, it also depends on the mood of the market and the perception of the country risk.

Most companies depend on the exchange rate in some way, either because they export part of their production or because they use imported inputs. Setting the forecasts for the exchange rate, even based on economic theory and a detailed observation of the economic situation, is a difficult task. In recent years, this task was particularly painful for some companies in Brazil. For example, in 2014, it was expected that the exchange rate in Brazil in 2016 would be at BRL 2.55 per US dollar (USD), according to the Focus survey of the Central Bank. In 2015, the forecast for 2016 was BRL 3.20 per dollar and earlier in 2016, the forecast for 2016 was BRL 4.10 per dollar. This did not happen - the exchange rate in Brazil in 2016 was BRL 3.25 per dollar. This high unpredictability incurred losses for many sectors of the economy, thereby reducing the confidence of entrepreneurs and consumers.

During the 2008 financial crisis, the Federal Reserve (Fed) cut interest rates rapidly to zero, resulting in the dollar being at its most undervalued level in decades. In 2014, when the economy was recovering, the Fed began to show signs that it would normalize its monetary policy, causing an intensive realignment of exchange rates. In less than a year, the dollar appreciated no less than 20% against the major global currencies, a trend that naturally affected emerging currencies such as the Brazilian Real. Thus, large exchange variations have significant impacts on companies, especially when considering the predictability of their cash flows. There are many risks and uncertainties on the horizon and the volatility of the exchange rate will continue at high levels.

In order to handle uncertainty and risk in this context, this paper develops risk-neutral and risk-averse stochastic programming (SP) models to support tactical decisions in cash-flow management problems that encompass grace periods, piecewise linear yields and uncertainty in the exchange rate of external sales. The risk-averse models are based on the minimax with regret and on the conditional value-at-risk (CVaR) criteria. Whereas the minimax with regret model provides solutions from a worst-case perspective when the probabilities of the scenarios are not known (or not reliable), the CVaR model aims at minimizing the risk of a solution influenced by a bad scenario with a low probability of occurrence. The risk-averse models are particularly appealing for providing less *risky* solutions, which might be reflected by the mitigation of the profit dispersion across a finite set of scenarios. The performance of the models is analyzed vis-a-vis a real-world stationary company in Brazil.

The developed risk-neutral and risk-averse stochastic programming models are based on deterministic network flow models with gains and losses, which have been used to represent cash flow management problems in different settings, as discussed in Golden et al. (1979), Crum et al. (1979), Srinivasan and Kim (1986), Pacheco and Morabito (2011) and Righetto et al. (2016). The aim of these network flow models is to maximize the cash return of the financial resources at the end of a multi-period and finite planning horizon. The importance of addressing uncertainty in cash flow management problems has also been addressed in the specialized literature (see, e.g., Mulvey and Vladimirou, 1992; Mulvey et al., 1995; Gitman and Chad, 2014 and the references therein).

It is worth mentioning that the first attempt in addressing uncertainty in cash-flow management problems – with the practical characteristics aforementioned – was due to Righetto et al. (2016). Differently from this current paper in which uncertainty is addressed via stochastic programming, Righetto et al. (2016) applies Robust Optimization (RO) to handle cash multipliers and exchange rates uncertainties in which the main goal is to analyze the trade-off between the profit deterioration and the *probability of constraint violation*. On the other hand, the stochastic-induced model that we develop here is always feasible for all outcomes (scenarios) simultaneously, since the recourse actions adjust the first-stage decisions to the materialized scenarios. Therefore, we evaluate the probability of a given solution falling within a pre-determined interval, instead of determining the probability of constraint violation. This ends up being more appealing from the decision-maker viewpoint because s(he) can adopt a solution according to its profitability range. The CVaR analysis we examine here also help us to understand what a risk-averse solution implies in terms of stability across the scenarios, worst-case/best-case performance, and so forth, providing additional rich managerial insights for the cash-flow management problem. As the performance metrics in Righetto et al. (2016) and here are built upon rather different modelling paradigms, a vis-à-vis comparison is not straightforward and, for this

reason, we solely pointed out the potential advantages of the methodologies presented in both papers in the conclusions.

The remainder of this paper is organized as follows. Section 2 develops the risk-neutral two-stage stochastic programming model for a cash flow management problem. Section 3 develops the minimax regret model and a CVaR formulation for this problem. Section 4 presents the scenario generation method. The performance and analysis of the models based on the considered company are discussed in Section 5. Final remarks and future research are pointed out in Section 6.

## 2. Risk-neutral cash flow optimization model

In this section, we present the risk-neutral two-stage stochastic programming model for the cash flow management under exchange rate (BRL/USD) uncertainty. The stochastic model is an extension of the deterministic mixed-integer programming model to cash-flow management presented in Righetto et al. (2016) with the addition of scenarios.

When cash inflows and outflows have a deterministic characteristic, i.e., there is a low probability that the expected cash inflows and outflows are not carried out, the deterministic model serves the purpose of maximizing cash resources at the end of the planning horizon. Nevertheless, there are companies that show significant uncertainties in cash inflows and outflows. In the cash budget, cash-in (accounts receivable) events are forecasted and some uncertainty factors may be present, such as demand itself, which may or may not occur, affecting the expected cash-in sales and the cash inflows from sales made overseas, for example (see Corcoran, 1978; Tsai, 2011 and Tangsucheeva and Prabhu, 2014). These cash inflows depend on the exchange rate that may be quite different from that anticipated in the cash budget, affecting the cash balance of a certain period of time and, as a consequence, the remaining periods. The exchange rate is the uncertainty that this paper will incorporate into the deterministic model using stochastic programming. The exchange rate causes significant uncertainties in Brazilian companies, especially in cash inflows, as foreign market sales in USD have to be converted to BRL in order to be transferred to the exporter's current account in Brazil.

In this paper, the timing of payments to suppliers and from customers are not being considered because they are indirectly already being treated when we considered both the cash-in and cash-out amount. We considered cash-in and cash-out to be the money that actually entered and left the bank account, and not the face value of an invoice. For this reason, we do not discuss the timing of payments to suppliers and from customers. We are not concerned with calculating the cash cycle of the company, but rather with the investment and financing decisions, considering the cash-in and cash-out that really modified the balance of bank account.

Basically, the optimization model supports tactical decisions in a dynamic cash management considering grace periods for investments and piecewise linear yields that depend on the amount of investment. A grace period is considered as a period that the investors do not withdraw the money from the bank. For example, when the investor invests some money in an asset with a specific remuneration, but this remuneration is just paid if he/she leaves the money invested for two months with no withdraw during this period. It is possible to define a grace period as a provision in most loan contracts which allows payment to be received for a certain period of time after the actual due date. In

this paper, we consider that the problem has grace periods represented by piecewise linear functions with only two segments, as the present case study uses only two types of yields. However, the optimization model presented in the following can be easily extended to consider piecewise linear functions with more than two segments.

Various assets of different levels of liquidity are taken into account, as well as the possibility of making conversions between them. Figure 1 presents an illustrative cash flow example considering a planning horizon of  $n$  time periods and three assets:  $a$ ,  $b$  and  $c$ . Asset  $a$  is money, while assets  $b$  and  $c$  are financial investments easily converted into cash, but  $a$  is assumed to be more liquid than  $b$ , which consequently is assumed to be more liquid than  $c$ , and cash disbursements are made only with asset  $a$ . The initial balances of assets  $a$ ,  $b$  and  $c$  are known and it is supposed that all transactions take place at the beginning of each time period and their returns are available at the end of each period. In the optimization model we consider only three assets, as this is the case of the studied company, but the model can be easily extended to consider an arbitrary number of asset classes.

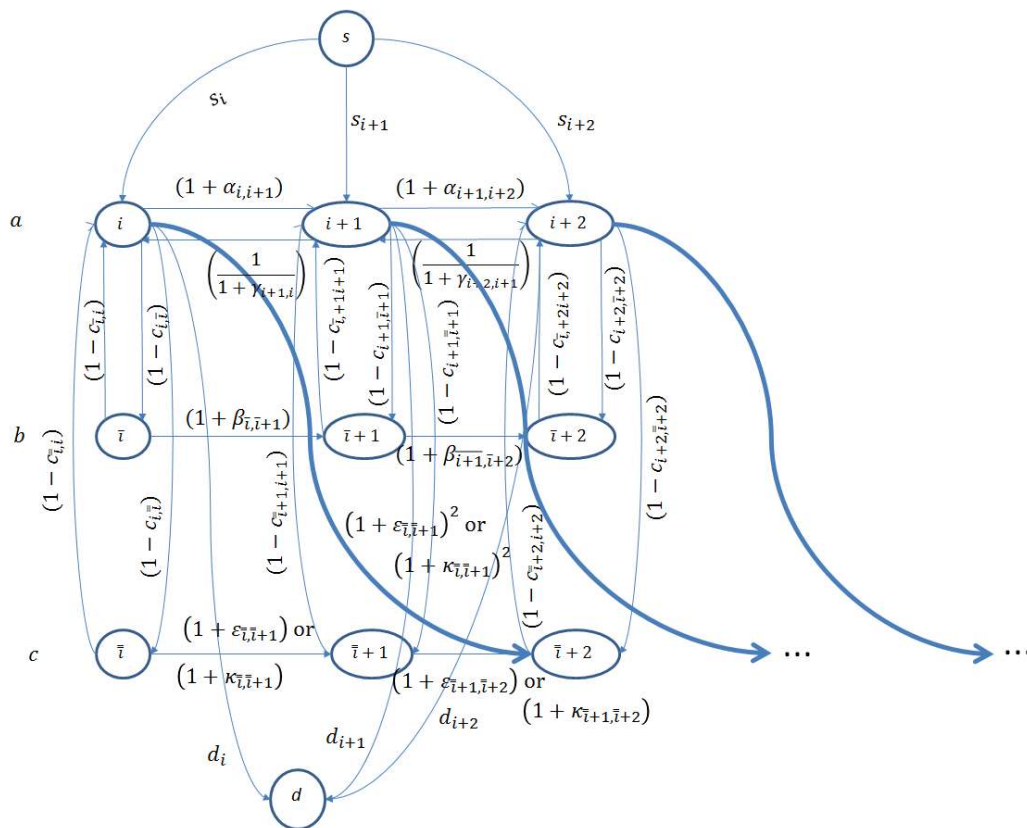
The problem is thus represented in a graph  $G = (N, A, W)$ , in which  $N$  is the set of nodes,  $A$  is the set of arcs connecting two nodes in  $N$  and  $W = [w_{ij}]$  is the matrix of multipliers for each arc  $(i, j)$  in  $A$ , with the set of nodes  $N = \{s, d, z, 1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}, \bar{\bar{1}}, \bar{\bar{2}}, \dots, \bar{\bar{n}}\}$ . Nodes  $s$  and  $d$  are the supply (accounts receivable or *cash-in*) and the demand (accounts payable or *cash-out*) nodes, respectively. We also define  $N^- = N \setminus \{s, d\}$ . Nodes 1 to  $n$  represent the cash nodes (asset  $a$ ) in periods  $t = 1$  to  $n$ , respectively (in the figure, periods  $t = i$  to  $i+2$ ). Nodes  $\bar{1}$  to  $\bar{n}$  and nodes  $\bar{\bar{1}}$  to  $\bar{\bar{n}}$  correspond to two investment options: without a grace period (asset  $b$ ) and with two grace periods (asset  $c$ ) in periods  $t = 1$  to  $n$ , respectively.

The horizontal reverse arcs  $(i + 1, i)$  are the loan banks, while the horizontal arcs  $(i, i + 1)$ , for  $i = 1, 2, \dots, n - 1$ ;  $(\bar{i}, \bar{i} + 1)$ , for  $\bar{i} = \bar{1}, \bar{2}, \dots, \bar{n} - 1$ ; and  $(\bar{\bar{i}}, \bar{\bar{i}} + 1)$ , for  $\bar{\bar{i}} = \bar{\bar{1}}, \bar{\bar{2}}, \dots, \bar{\bar{n}} - 1$ , show the investment flows of the funds for each asset and period. Note that the horizontal arcs  $(\bar{\bar{i}}, \bar{\bar{i}} + 1)$ , for  $\bar{\bar{i}} = \bar{\bar{1}}, \bar{\bar{2}}, \dots, \bar{\bar{n}} - 1$ , and the diagonal arcs  $(i, \bar{i} + 2)$ , for  $i = 1, 2, \dots, n - 2$ , represent investment flows in the asset with a grace period. These flows are considered as an investment decision because there could be a decision to withdraw resources from this asset by the model. Therefore, maintaining the money in this specific class of asset that generates interest is an investment decision. There is no investment decision for asset  $b$  because the model, considering interest paid on this asset, does not allocate funds to it. In Figure 1, the vertical arcs  $(i, \bar{i})$ ,  $(i, \bar{\bar{i}})$ ,  $(\bar{i}, i)$  and  $(\bar{\bar{i}}, i)$  represent the conversion flows of funds between the three levels of liquidity assets,  $a$ ,  $b$ , and  $c$ . The arcs in set  $A$ ,  $(n, z)$ ,  $(\bar{n}, z)$  and  $(\bar{\bar{n}}, z)$  correspond to the inflows of the last period regarding assets  $a$ ,  $b$  and  $c$  to node  $z$ , respectively. Node  $z$  is the goal node. The objective is to maximize the total inflow for this node.

Note in the figure that the multipliers in the horizontal arcs  $(i, i + 1)$ ,  $(\bar{i}, \bar{i} + 1)$ , and  $(\bar{\bar{i}}, \bar{\bar{i}} + 1)$  correspond to the percentage yields for each asset  $a$ ,  $b$  and  $c$ , respectively, for each period. The multipliers in the diagonal arcs  $(i, \bar{i} + 2)$  are the percentage yields for asset  $c$  with a grace period, while the ones in the horizontal reverse arcs  $(i + 1, i)$  are the percentage interest paid by bank loans. The multipliers in the vertical arcs  $(i, \bar{i})$ ,  $(i, \bar{\bar{i}})$  are the percentage cost of conversion from asset  $a$  to  $b$ , and from  $a$  to  $c$ , while the ones in arcs  $(\bar{i}, i)$  and  $(\bar{\bar{i}}, i)$  are the percentage cost of conversion from asset  $b$  to

$a$ , and from  $c$  to  $a$ . The values in arcs  $(s, i)$  and  $(i, d)$  represent the total inflows and outflows of cash, respectively, in each period.

Figure 1. The network flow with both the loans and investments with a grace period for three assets  $a$ ,  $b$  and  $c$ , and three generic periods  $i$ ,  $i+1$  and  $i+2$  (Adapted from Righetto et al., 2016).



Following the two-stage stochastic programming framework, we consider that both cash-in and cash-out are random variables approximately by a set of realizations or scenarios  $\mathcal{C} = \{1, 2, \dots, C\}$ , with corresponding probability of occurrence given by  $\pi_c$ , such that  $\sum_{c \in \mathcal{C}} \pi_c = \mathbf{1}$  and  $\pi_c > \mathbf{0}$  hold. In this paper, the first-stage variables are the decisions related to investments, loans or keeping the money in cash for the first periods and these decisions are independent from the scenarios. The second-stage decision variables refer to the same set of decisions, but from the remaining periods. The complete list of parameters and decision variables are depicted as follows.

Parameters:

- $a$  cash (the highest liquidity asset);
- $b$  investment without grace periods;
- $c$  investment with two grace periods;
- $\alpha_{ij}$  yield for asset  $a$ ,  $(i, j) = (i, i + 1), i = 1, 2, \dots, n - 1$  and  $(n, Z)$ ;
- $\beta_{ij}$  yield for asset  $b$ ,  $(i, j) = (\bar{i}, \bar{i} + 1)$  and  $(\bar{n}, Z), \bar{i} = \bar{1}, \bar{2}, \dots, \bar{n} - 1$ ;
- $\varepsilon_{ij}$  yield for asset  $c$ ,  $(i, j) = (\bar{i}, \bar{i} + 1), (i, \bar{i} + 2)$  and  $(\bar{n}, Z), 0 \leq f_{ijc} < lm_{ij}, \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, 2, \dots, n - 2$  ( $f_{ijc}$  is a decision variable defined below);
- $\kappa_{ij}$  yield for asset  $c$ ,  $(i, j) = (\bar{i}, \bar{i} + 1), (i, \bar{i} + 2)$  and  $(\bar{n}, Z), lm_{ij} \leq f_{ijc} < M, \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, 2, \dots, n - 2$ ;
- $\gamma_{ij}$  interest paid by bank loans,  $(i, j) = (i + 1, i), i = 2, \dots, n$ ;
- $c_{ab}$  unit cost of conversion from asset  $a$  to  $b$ ;
- $c_{ac}$  unit cost of conversion from asset  $a$  to  $c$ ;
- $c_{ba}$  unit cost of conversion from asset  $b$  to  $a$ ;
- $c_{ca}$  unit cost of conversion from asset  $c$  to  $a$ ;
- $a_0$  initial balance of asset  $a$ ;
- $b_0$  initial balance of asset  $b$ ;
- $c_0$  initial balance of asset  $c$ ;
- $s_{ic}$  inflow of cash in period  $i = 1, 2, \dots, n$  and in scenario  $c \in \mathcal{C}$ ;
- $d_{ic}$  outflow of cash in period  $i = 1, 2, \dots, n$  and in scenario  $c \in \mathcal{C}$ ;
- $\pi_c$  probability of scenario  $c \in \mathcal{C}$ ;
- $w_{ij}$  multipliers of flows in all arcs  $(i, j)$  (except the arcs for  $w_{ij}^2$  defined just below);
- $w_{ij}^2$  multipliers of flows in arcs  $(i, j)$  used only for asset  $c$ ,  $(i, j) = (\bar{i}, \bar{i} + 1), (i, \bar{i} + 2)$  and  $(\bar{n}, Z), lm_{ij} \leq f_{ijc} < M, \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, 2, \dots, n - 2$ ;
- $u$  maximum bank loan limit;
- $m$  minimum requirement of asset  $a$ ;
- $q$  minimum requirement of asset  $b$ ;
- $lm_{ij}$  maximum limit to asset  $c$  receive yield  $\varepsilon_{ij}$  and minimum to asset  $c$  receive yield  $\kappa_{ij}$ ,  $(i, j) = (\bar{i}, \bar{i} + 1), (i, \bar{i} + 2)$  and  $(\bar{n}, Z), \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, 2, \dots, n - 2$ ;
- $M$  is a large enough number that is bound for the active  $c$  receive yield  $\kappa_{ij}$ ,  $(i, j) = (\bar{i}, \bar{i} + 1), (i, \bar{i} + 2)$  and  $(\bar{n}, Z), \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, 2, \dots, n - 2$ .

Decision Variables:

- $f_{ijc}$  Flow of financial resources that node  $i$  receives in scenario  $c \in \mathcal{C}$ ,  $(i, j) \in A$ ,  $i \in N^-$ ,  $j \in N^-$ ;
- $f_{ijc}^1$  Amount invested less than or equal to  $lm_{ij}$  in scenario  $c \in \mathcal{C}$ , whose yield is represented by multiplier  $w_{ij}$ ;
- $f_{ijc}^2$  Amount invested higher than  $lm_{ij}$  in scenario  $c \in \mathcal{C}$ , whose yield is represented by multiplier  $w_{ij}^2$ ;
- $x_{ijc}^1$  Binary variable equals 1, if  $0 \leq f_{ijc} < lm_{ij}$ ,  $(\bar{i}, \bar{i} + 1)$ ,  $\bar{i} = \bar{1}, \bar{2}, \dots, \bar{n} - 1$ , and  $(i, \bar{i} + 2)$ ,  $i = 1, 2, \dots, n - 2$ , for all  $c \in \mathcal{C}$ ;
- $x_{ijc}^2 = 1 - x_{ijc}^1$  binary variable equals 1, if  $lm_{ij} \leq f_{ijc} < M$ ,  $(\bar{i}, \bar{i} + 1)$ ,  $\bar{i} = \bar{1}, \bar{2}, \dots, \bar{n} - 1$ , and  $(i, \bar{i} + 2)$ ,  $i = 1, 2, \dots, n - 2$ , for all  $c \in \mathcal{C}$  (to simplify the model presentation, we explicitly define variables  $x_{ijk}^1$  and  $x_{ijk}^2$  such that  $x_{ijk}^1 + x_{ijk}^2 = 1$ , instead of replacing  $x_{ijc}^2$  by  $1 - x_{ijc}^1$  in the model. In cases in which the asset has more than two grace periods, i.e. more than two segments/slopes, one could define additional variables  $x_{ijk}^3, x_{ijk}^4, \dots$ , such that  $x_{ijk}^1 + x_{ijk}^2 + x_{ijk}^3 + x_{ijk}^4 + \dots = 1$ , in a similar way).

The first-stage decision variables are those  $f_{ijc}$  in which  $i \in P$  and  $P$  is a subset of *here-and-now* periods, i.e., those associated to the decisions that must be taken under partial information or before uncertainties are revealed. For example, if  $P = \{1, \bar{1}, \bar{1}\}$ , then the variables of the first-stage are those of the first period, which is equivalent to define  $f_{ijc}$  for all  $i$ , such that  $(i, j) \in A$ ,  $i = 1, \bar{1}, \bar{1}$ , satisfying the so-called nonanticipativity constraints given by  $f_{ij1} = f_{ij2} = \dots = f_{ijc}$ . In this case, the remaining decision variables  $f_{ijc}$  are second-stage ones.

Similarly, other sets of  $P$  can be used to define the first-stage, for example, if  $P = \{1, \bar{1}, \bar{1}, 2, \bar{2}, \bar{2}\}$ , the decisions of the first two periods are variables of the first-stage and we should impose  $f_{ij1} = f_{ij2} = \dots = f_{ijc}$ , for  $(i, j) \in A$ ,  $i = 1, \bar{1}, \bar{1}, 2, \bar{2}, \bar{2}$ . Thereby, the variables of one or more consecutive periods may be considered as first-stage variables using this formulation. The same rationale is applied to the decision variables  $f_{ijc}^1, f_{ijc}^2, x_{ijc}^1$  and  $x_{ijc}^2$ .

It should be noted that an alternative way to define the first-stage variables would be simply by  $f_{ij}$ ,  $(i, j) \in A$  and  $i \in P$ , while the second-stage variables  $f_{ijc}$ ,  $(i, j) \in A$  and  $i \notin P$ . However, for the sake of simplicity considering presentation and computational implementation of this model and other models of this section, we did not choose this representation in this paper.

Finally, the risk-neutral two-stage stochastic programming with recourse, where the recourse represents the decisions of investments and hire bank loans (to contract loans from the bank) after the uncertainty of the exchange rate is revealed, can be posed as follows.

*Maximize*

$$\sum_{c \in \mathcal{C}} \pi_c [w_{n-1,z} f_{n-1,z,c}^1 + w_{n-1,z}^2 f_{n-1,z,c}^2 + w_{nz} f_{nzc}^1 + w_{nz}^2 f_{nzc}^2 + w_{\bar{n}z} f_{\bar{n}zc} + w_{\bar{n},z} f_{\bar{n}zc}^1 + w_{\bar{n}z}^2 f_{\bar{n}zc}^2] \quad (1)$$

in which the multipliers are given by



$$w_{ij} = \begin{cases} 1 + \alpha_{ij}, (i, j) = (i, i + 1) \text{ and } (n, Z) \\ 1 + \beta_{ij}, (i, j) = (\bar{i}, \bar{i} + 1) \text{ and } (\bar{n}, Z) \\ 1 + \varepsilon_{ij}, (i, j) = (\bar{i}, \bar{i} + 1), \text{ and } (\bar{n}, Z), 0 \leq f_{ijc}^1 < lm_{ij}, c \in \mathcal{C} \\ (1 + \varepsilon_{ij})^2, (i, j) = (i, \bar{i} + 2), 0 \leq f_{ijc}^1 < lm_{ij}, c \in \mathcal{C} \\ 1 - C_{ab}, (i, j) = (i, \bar{i}) \\ 1 - C_{ac}, (i, j) = (i, \bar{i} + 2) \\ 1 - C_{ba}, (i, j) = (\bar{i}, i) \\ 1 - C_{ca}, (i, j) = (\bar{i}, i) \\ \frac{1}{1 + \gamma_{ij}}, (i, j) = (i + 1, i) \\ 0, \text{ otherwise,} \end{cases} \quad (2)$$

and

$$w_{ij}^2 = \mathbf{1} + \kappa_{ij}, (i, j) = (\bar{i}, \bar{i} + 1), (i, \bar{i} + 2) \text{ and } (\bar{n}, Z), lm_{ij} \leq f_{ijc}^1 < M, c \in \mathcal{C} \quad (3)$$

Subject to the following constraints:

$$\sum_{j \in N^-} w_{ji} f_{jic} - \sum_{j \in N^-} f_{ijc} \geq -a_0 - s_{ic} + d_{ic}, i = 1, c \in \mathcal{C} \quad (4)$$

$$\sum_{j \in N^-} w_{ji} f_{jic} - \sum_{j \in N^-} f_{ijc} \geq -b_0, i = \bar{1}, c \in \mathcal{C} \quad (5)$$

$$\sum_{j \in N^-} w_{ji} f_{jic} - \sum_{j \in N^-} f_{ijc} \geq -c_0, i = \bar{\bar{1}}, c \in \mathcal{C} \quad (6)$$

$$\sum_{j \in N^-} w_{ji} f_{jic} - \sum_{j \in N^-} f_{ijc} \geq -s_{ic} + d_{ic}, i = 2, 3, \dots, n, c \in \mathcal{C} \quad (7)$$

$$\sum_{j \in N^-} w_{ji} f_{jic} - \sum_{j \in N^-} f_{ijc} \geq 0, i = \bar{2}, \bar{3}, \dots, \bar{n}, c \in \mathcal{C} \quad (8)$$

$$\sum_{j \in N^-} w_{ji} f_{jic}^1 + \sum_{j \in N^-} w_{ji}^2 f_{jic}^2 - \sum_{j \in N^-} f_{ijc} \geq 0, i = \bar{\bar{2}}, \bar{\bar{3}}, \dots, \bar{\bar{n}}, c \in \mathcal{C} \quad (9)$$

$$f_{n-1,z,c}^1 + f_{n-1,z,c}^2 = f_{n-1,z,c}, c \in \mathcal{C} \quad (10)$$

$$f_{nzc}^1 + f_{nzc}^2 = f_{nzc}, c \in \mathcal{C} \quad (11)$$

$$f_{\bar{n}zc}^1 + f_{\bar{n}zc}^2 = f_{\bar{n}zc}, c \in \mathcal{C} \quad (12)$$

$$0 \leq f_{n-1,z,c}^1 < lm_{n-1,z,c} x_{n-1,z,c}^1, c \in \mathcal{C} \quad (13)$$

$$lm_{n-1,z,c} x_{n-1,z,c}^2 \leq f_{n-1,z,c}^2 < M x_{n-1,z,c}^2, c \in \mathcal{C} \quad (14)$$

$$0 \leq f_{nzc}^1 < lm_{nzc} x_{nzc}^1, c \in \mathcal{C} \quad (15)$$

$$lm_{nzc} x_{nzc}^2 \leq f_{nzc}^2 < M x_{nzc}^2, c \in \mathcal{C} \quad (16)$$

$$0 \leq f_{\bar{n}zc}^1 < lm_{\bar{n}zc} x_{\bar{n}zc}^1, c \in \mathcal{C} \quad (17)$$

$$lm_{\bar{n}zc} x_{\bar{n}zc}^2 \leq f_{\bar{n}zc}^2 < M x_{\bar{n}zc}^2, c \in \mathcal{C} \quad (18)$$

$$x_{n-1,z,c}^1 + x_{n-1,z,c}^2 = 1, c \in \mathcal{C} \quad (19)$$

$$x_{nzc}^1 + x_{nzc}^2 = 1, c \in \mathcal{C} \quad (20)$$

$$x_{\bar{n}zc}^1 + x_{\bar{n}zc}^2 = 1, c \in \mathcal{C} \quad (21)$$

$$f_{ijc}^1 + f_{ijc}^2 = f_{ijc}, (i, j) = (\bar{i}, \bar{i} + 1) \text{ and } (i, \bar{i} + 2), \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, \dots, n - 2, c \in \mathcal{C} \quad (22)$$

$$0 \leq f_{ijc}^1 < lm_{ijc} x_{ijc}^1, (i, j) = (\bar{i}, \bar{i} + 1) \text{ and } (i, \bar{i} + 2), \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, \dots, n - 2, c \in \mathcal{C} \quad (23)$$

$$lm_{ijc} x_{ijc}^2 \leq f_{ijc}^2 < Mx_{ijc}^2, (i, j) = (\bar{i}, \bar{i} + 1) \text{ and } (i, \bar{i} + 2), \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, \dots, n - 2, c \in \mathcal{C} \quad (24)$$

$$x_{ijc}^1 + x_{ijc}^2 = 1 \quad (25)$$

$$0 \leq f_{i+1,ic} \leq u, i = 1, \dots, n - 1, c \in \mathcal{C} \quad (26)$$

$$f_{i,i+1,c} \geq m, i = 1, 2, \dots, n, c \in \mathcal{C} \quad (27)$$

$$f_{nzc} \geq m, c \in \mathcal{C} \quad (28)$$

$$f_{i,i+1,c} \geq m, i = \bar{1}, \bar{2}, \dots, \bar{n}, c \in \mathcal{C} \quad (29)$$

$$f_{\bar{n}zc} \geq q, c \in \mathcal{C} \quad (30)$$

$$f_{ijc} \geq 0, (i, j) \in A, c \in \mathcal{C} \quad (31)$$

$$f_{ij1} = f_{ij2} = \dots = f_{ijc}, (i, j) \in A, i \in P \quad (32)$$

$$x_{n-1,z,c}^1, x_{n-1,z,c}^2, x_{nzc}^1, x_{nzc}^2, x_{\bar{n}zc}^1, x_{\bar{n}zc}^2 \in [0,1], c \in \mathcal{C} \quad (33)$$

$$x_{ijc}^1, x_{ijc}^2 \in [0,1], (i, j) = (\bar{i}, \bar{i} + 1) \text{ and } (i, \bar{i} + 2), \bar{i} = \bar{1}, \dots, \bar{n} - 1, i = 1, \dots, n - 2, c \in \mathcal{C}. \quad (34)$$

The objective function (1) maximizes the expected amount of financial resources on node  $z$ . The first two terms of the objective function,  $w_{n-1,z} f_{n-1,z,c}^1 + w_{n-1,z}^2 f_{n-1,z,c}^2$ , are the flow of resources that reach the terminal node  $z$  through the penultimate node of asset  $a$ . These flows reach node  $z$ , multiplied by  $w_{n-1,z}$  or by  $w_{n-1,z}^2$ , depending on the amount invested. The constraints that determine the yield of flow  $f_{n-1,z,c}$  are (10), (13), (14), (20) and (33). The third and fourth terms of the objective function  $w_{nzc} f_{nzc}^1 + w_{nzc}^2 f_{nzc}^2$ , are the flow of resources that achieve the terminal node  $z$  through the last node of asset  $a$ . They also depend on the amount invested and the constraints that determine the yield of flow  $f_{nzc}$  are (11), (15), (16), (21) and (33). The fifth term of the objective function (1) is the flow of resources that achieve node  $z$  through asset  $b$ . The last two terms of the objective function  $w_{\bar{n},z} f_{\bar{n}zc}^1 + w_{\bar{n},z}^2 f_{\bar{n}zc}^2$ , are the flows that achieve the terminal node  $z$  through asset  $c$ . These flows depend on the amount invested for the determination of their yield and their constraints are (12), (17), (18), (22) and (33).

Constraint (4) represents the balance of flow for node 1. This constraint considers the initial balance for asset  $a$ , the cash inflow  $s_{ic}$  and the cash outflow  $d_{ic}$ , the movement of the asset over time and the bank loan. Constraint (5) ensures the balance of the flow for node  $\bar{1}$  and the initial balance for asset  $b$ . Constraint (6) is the balance of the flow for node  $\bar{1}$  in which the movements of asset  $c$ , as well its initial balance, are taken into account. Constraint (7) is the balance of the flow for the remaining nodes  $2, 3, \dots, n$ , which entails the inflow and outflow of cash,  $s_{ic}$  and  $d_{ic}$ , respectively, the movement of asset  $a$  and the bank loans. Constraints (8) and (9) are the balance of flow for nodes  $\bar{2}, \bar{3}, \dots, \bar{n}$  and  $\bar{2}, \bar{3}, \dots, \bar{n}$ , respectively.

Constraints (23), (24), (25) and (34) model the piecewise linear convex function. Constraint (26) is the maximum indebtedness limit and the non-negativity condition of the bank loans. Constraints (27) and (28) ensure the minimum balance of cash, while expressions (29) and (30) indicate the minimum balance for investments in asset  $b$ . As stated before, these minimum balances are necessary to guarantee a minimum liquidity condition for any type of eventuality in the economy, such as a

restriction of credit by the bank sector. Constraint (31) reflects the non-negativity of all the flow variables for arcs  $(i, j)$ .

Constraint (32) is the nonanticipativity constraint that defines the first-stage variables. This constraint ensures that for all scenarios  $c \in \mathcal{C}$ , flow  $f_{ijc}$  for assets  $a, b$  and  $c$  for the first periods of the planning horizon in set  $P$  are the same.

### 3. Risk-averse cash flow optimization models

Here, we develop a minimax regret model and a CVaR formulation for the considered cash flow management problem. The minimax with regret model aims to determine the best deviation of the worst-case of optimality among all possible decisions in all scenarios considered (Kouvelis and Yu, 1997). This model is indicated to represent conservative decisions of risk aversion and/or when the probabilities of the scenarios are not known.

In order to formulate the minimax with regret model for the cash flow problem, the wait-and-see solutions  $W_c^*$  for all scenarios  $c = 1, 2, \dots, \mathcal{C}$  need to be determined. After this, the maximum difference among the number of terminal nodes of the two-stage with recourse model and the solutions  $W_c^*$  taking into account all scenarios  $c$ , is minimized. The minimax with regret problem can be formulated as follows:

$$\text{Minimize } \Theta \tag{36}$$

Subject to the following constraints: (2)–(34)

$$\Theta \geq W_c^* - (w_{n-1,z} f_{n-1,z,c}^1 + w_{n-1,z}^2 f_{n-1,z,c}^2 + w_{nz} f_{nzc}^1 + w_{nz}^2 f_{nzc}^2 + w_{\bar{n}z} f_{\bar{n}zc} + w_{\bar{n},z} f_{\bar{n}zc}^1 + w_{\bar{n}z}^2 f_{\bar{n}zc}^2), \tag{37}$$

$$c \in \mathcal{C}$$

$$\Theta \geq 0 \tag{38}$$

The objective function plus constraints (37) and (38) ensure that the maximum deviation (regret) between the value of the objective function of the wait-and-see problem ( $W_c^*$ ) and the value of the two-stage problem (the second term of the right-hand-side of constraint (37)) is minimized. Note that this is accomplished by minimizing variable  $\Theta \geq 0$  in (36) under constraint (37). Note also that  $\Theta \geq 0$  ensures that just the most unfavorable deviations are considered.

The minimax with regret does not take into account the probability of occurrence of each scenario and because of this, the solution of the model may be influenced by a scenario of low probability of occurrence. In order to overcome this issue, this paper also develops a measure widely used as a risk measure called conditional value-at-risk (CVaR) (Rockafellar and Uryasev, 2002). The CVaR is also known as mean excess loss, mean shortfall or tail VaR (value-at-risk). The CVaR model for the cash flow problem maximizes the difference among the value-at-risk, VaR, represented by  $\eta$ , and a set of weighted expected losses lower than VaR with confidence level  $\alpha$ . The  $\mathbf{CVaR}_\alpha$  optimization problem may be formulated as follows:

$$\text{Maximize } \eta - \frac{1}{1-\alpha} \sum_{c=1}^c \pi_c v_c \tag{39}$$

Subject to the following constraints: (2)–(34)

$$v_c \geq \eta - (w_{n-1,z} f_{n-1,z,c}^1 + w_{n-1,z}^2 f_{n-1,z,c}^2 + w_{nz} f_{nz,c}^1 + w_{nz}^2 f_{nz,c}^2 + w_{\bar{n}z} f_{\bar{n}z,c}^1 + w_{\bar{n}z}^2 f_{\bar{n}z,c}^2),$$

$$c \in \mathcal{C} \quad (40)$$

$$v_c \geq 0, c \in \mathcal{C} \quad (41)$$

$$\eta \in \mathbb{R}. \quad (42)$$

The  $v_c$  is a continuous variable that represents the excess of loss beyond the VaR,  $\eta$ , for each scenario  $c \in \mathcal{C}$ . Confidence level  $\alpha$  is used to reflect the risk preference of the decision maker. Higher values of  $\alpha$  indicate greater risk aversion and CVaR controls the largest deviations of expected losses in relation to VaR.

There are other two-stage stochastic programming models that could be used to evaluate the behavior of optimal solutions, especially the trade-off between risk and return, such as mean-risk models or restricted recourses (see, e.g., Shapiro (2012), Alem and Morabito (2013), Homem-de-Mello and Pagnoncelli (2015) and the authors cited therein). However, the choice of the CVaR model in this paper was an attempt to implement a stochastic optimization approach for the cash flow problem that is adopted in many practical applications, but which has not yet been applied to a cash management problem in a non-financial company. Studies on these other stochastic models are interesting topics for future research.

#### 4. Scenario generation

For the exchange rate scenario generation, each simulated scenario is considered a trajectory over time that follows a stochastic Markov process in which a specific variable has unpredictable values. In this way, the present state of the process is what matters to predict the future value of these variable. Application of Markov process may be seen in Cyert, Davidson, and Thompson (1962). This means that the price on the spot market contains all the relevant information to study the future behavior of it. The Wiener process or Brownian motion is a particular case of the Markov process whose increments are independent from any other time interval and the variation of a process, in a finite time interval, follows a normal distribution with a variance that increases linearly in this interval (Hull, 2003). The scenario generation for the exchange rate is based on the Monte Carlo simulation using the Geometric Brownian Motion (GMB) according to the follow stochastic equation:

$$dx = \alpha x dt + \sigma x dz \quad (43)$$

in which  $dz$  is an incremental Wiener process,  $dz = \varepsilon_t \sqrt{dt}$ ,  $\varepsilon_t$  is a random variable which follows a standard normal distribution,  $N(\mathbf{0}, \mathbf{1})$ ,  $\alpha$  is the-risk free interest rate historically and  $\sigma$  is the volatility of the exchange rate. Applying Itô's lemma and a logarithmic transformation, the discrete equation of price simulation can be obtained in any future moment  $t$ , of exchange rate  $X_t$  as follows:

$$X_t = X_0 e^{\left[ \left( \alpha - \frac{\sigma^2}{2} \right) \Delta t + \sigma \sqrt{\Delta t} N(0,1) \right]} \quad (44)$$

The Monte Carlo simulation of the exchange rate prices using the above equation is done by choosing random values and obtaining a standard normal distribution  $N(0,1)$ , thus yielding the corresponding

price  $\mathbf{X}_t$ . An important feature of Equation (44) is that the discretization in relation to its continuous form is exact and precise. In other words, small increments of time  $\Delta t$  are not necessary to obtain a good approximation. Any size of  $\Delta t$  can be used in which the simulation equation (75) remains valid (Hull, 2003). This technique for scenario generation has been used in various areas of knowledge, such as financial engineering (Breedon and Ingram, 2010; Brandimarte, 2014).

The uncertain parameters in both cash-in and cash-out are the exchange rates, which are used to determine the amount (BRL) of receipts and payments in US dollars (USD). That is, the variation of the exchange rate changes the cash-in and cash-out arising from sales in foreign markets. The initial exchange rate of the simulation was BRL 2.40. The volatility of the exchange rate for the scenario generation follows a uniform distribution between [9%, 34%], and this interval was obtained from historical data taken from BM&FBovespa statistics ([www.bmfbovespa.com.br](http://www.bmfbovespa.com.br)).

The risk-free interest rate for the scenario generation follows a uniform distribution between the range of [7.10%, 32.40%], which was also obtained from BM&FBovespa statistics. Note that the scenario generation method used in this paper provides the achievements for the stochastic parameters but does not determine the number of scenarios that will be incorporated in the stochastic programming model. Especially in mixed-integer problems, it is essential to determine the tradeoff between accuracy and computational tractability of the scenarios generation method. On the one hand, incorporating a large number of scenarios improves the accuracy of the stochastic programming model, but may increase the computational difficulty. Moreover, dealing with few scenarios is computationally simpler, but the random variable may be misrepresented. To deal with this compromise, we used the internal stability analysis, which provides the minimum number of scenarios that must be generated to observe the stabilization of the value of the objective function of the stochastic programming model. First, samples of increasing and varying sizes are generated, for example, 10, 50, 100, ..., 600. Afterwards, various stochastic programming models are solved, and the corresponding optimal values of the objective function are analyzed. Figure 2 shows that from 200 scenarios on the optimal value of the model (2)-(34) varies no more than 0.2%, suggesting that from this number of scenarios there is a stabilization of the objective function value. Therefore, in this study, 200 scenarios were established to generate the numerical tests.

Figure 2. Internal stability analysis for the scenario generation test.

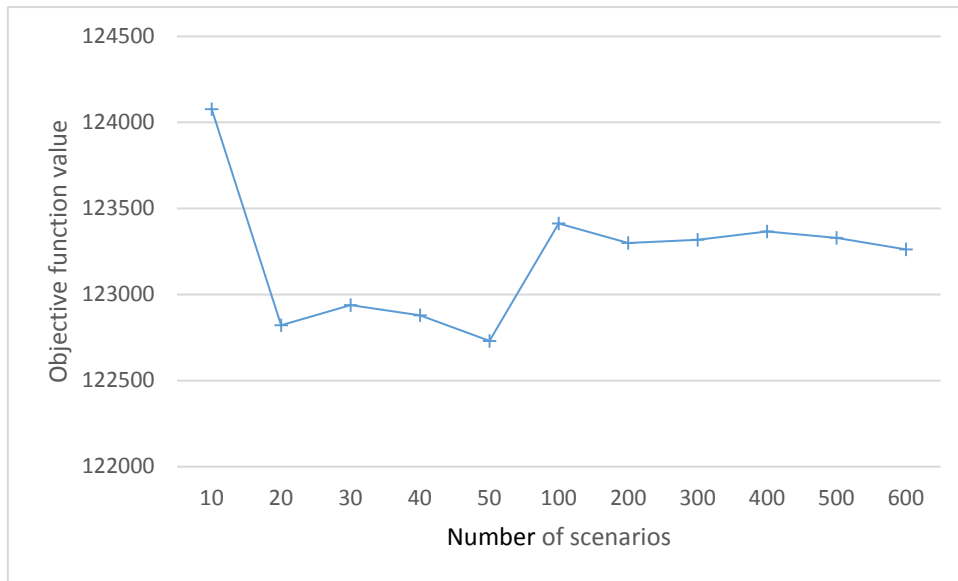


Figure 3 shows the evolution of the 200 scenarios generated with a unit step of one working day for the exchange rate between BRL and USD using the method described in this section. As the cash flow is managed by month, the monthly exchange rate of each scenario was established obtaining the exchange rate generated every 21 days from each scenario, which is the average number of working days of each month. Figure 4 shows the 200 scenarios for the 12 months used in the numerical tests. The scenarios were considered equiprobable.

Figure 3. Scenario evolution for the exchange rate BRL/USD.

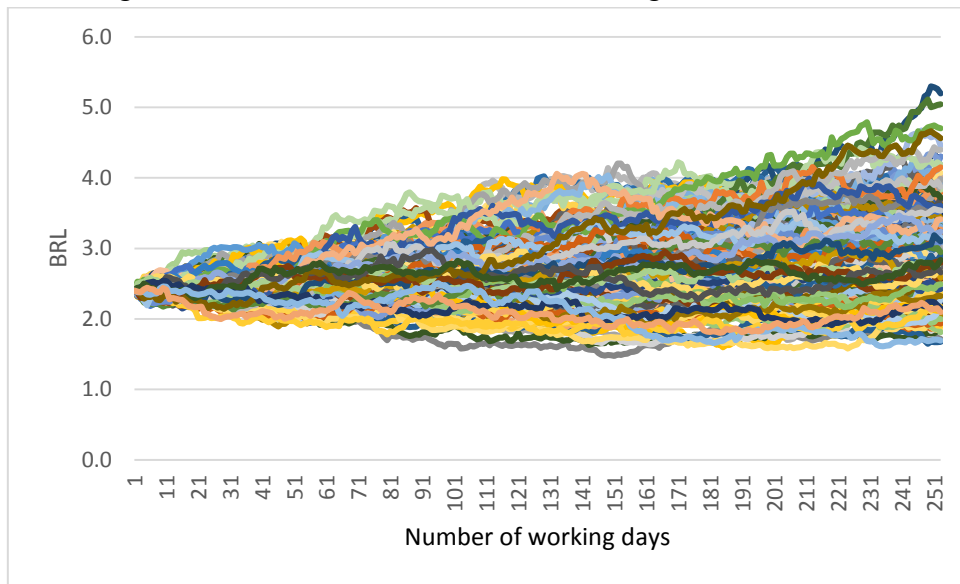
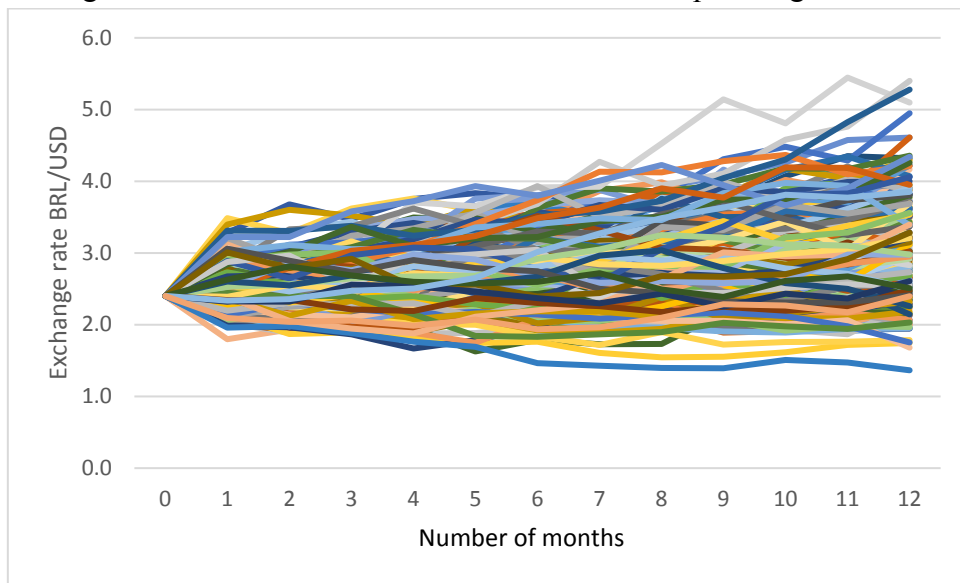


Figure 4. Scenarios for the 12 months of the cash planning horizon.



## 5. Numerical Results

In this section, we evaluate the neutral and risk-averse approaches using real data extracted from a benchmark company of the stationary sector studied in Righetto et al (2016). The models were coded in the General Algebraic Modeling System (GAMS 23.0.2) and solved via the CPLEX 12.1 optimization system with all parameters fixed at their default values. The computation experiments were run in an Intel(R) Core(TM) i3-3110M, 2.40GHz and 4GB of RAM. For the sake of simplicity, the monetary amounts are presented in thousands of BRL.

## 5.1. Case study and data description

As mentioned, our case study is based on a stationery company in São Paulo State, Brazil. This case study was first presented in Righetto et al. (2016) and, for the sake of brevity, we summarize only the most important information regarding the company and other details of the data set are provided in Appendix A.

Basically, the studied company has more than two thousand employees, with an annual revenue from Brazilian and international markets of approximately five hundred million BRL. The commercial portfolio in the Brazilian market has more than five thousand customers. Domestic sales have a typical seasonality, which is the concentration of cash-in in January, February, March and April of each year, which is the so-called “back-to-school” period. The amount of cash-in in these months depends mostly on the sales plan policy that the company adopts. The main sales period of the company is from August to December. To manage inventory and customer supply logistics problems, the company finances its customers through payment terms, facilitating purchases by customers in those months. This strategy is critical to supply customers for the main sales period.

The first-stage decision variables correspond to the first period of the planning horizon. Both cash-in and cash-out come from the internal market quoted in BRL and the external market quoted in USD. The exchange rates to convert the dollars received from the foreign market are from the scenarios generated in the previous section. Table 1 presents the cash-in and cash-out in BRL and USD used as parameters to run the stochastic programming models. The computational time to run the stochastic models was less than one minute.

Table 1. Cash-in and cash-out in thousands of BRL and USD.

Period	Cash-in		Cash-out	
	BRL	USD	BRL	USD
1	11,088	1,540	62,411	11,144
2	6,952	1,557	5,479	978.50
3	9,371	1,484	12,606	2,251
4	10,192	1,455	13,440	2,400
5	9,381	1,345	10,751	1,919
6	15,070	2,127	29,999	5,357
7	9,400	1,424	6,505	1,161
8	15,883	2,214	12,132	2,166
9	13,221	1,839	4,211	751.98
10	105,591	14,739	35,482	6,336
11	86,818	12,500	66,753	11,920
12	93,588	13,020	98,024	17,504

The values of cash-in and cash-out in USD are multiplied by each exchange rate generated by scenarios in order to generate both cash-in and cash-out in BRL of each scenario. For the computational results, two hundred scenarios generated by the methodology developed in the previous section were considered and they were used for risk neutral two-stage stochastic programming model, called RN, minimax with regret, called minimax and CVaR. For the latter,  $\alpha \in \{0.7; 0.75; 0.80; 0.85; 0.95; 0.96;$



0.97; 0.98; 0.99} was considered.

## 5.2. Performance measures

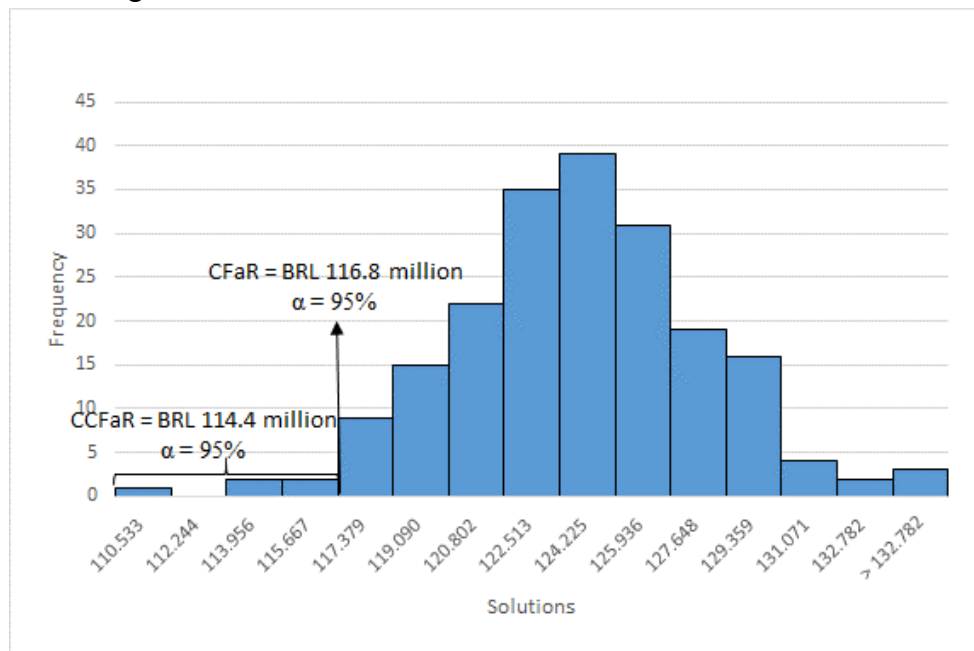
As the company's treasury does not use stochastic models for cash management neither does it include statistical analyses in its reports, it is important to show to the company the attractiveness of the proposed framework based on stochastic models. One way to do this is by showing the financial manager that the solutions generated by the stochastic models outperform the deterministic ones and that the consequent implications of using them to support tactical decisions of the company may result in rewards.

For this purpose, we analyze the expected profit (EP) given by the stochastic programming model vis-à-vis some performance measures that are typically evaluated considering the independent solutions of the scenarios, such as the worst-case (WC) and the best-case (BC) profits, the profit standard deviation (SD), and the probability "Pr" that the "Profit" is less than a specified threshold, say,  $t$ , i.e.,  $\wp_t = Pr[\text{Profit} < t]$ . In this case,  $\wp_t$  is simply defined by the relative frequency of the solutions in the sample of 200 scenarios whose optimal value (profit) is less than the threshold. In particular, we define a threshold of BRL 120 million, which is the value set as a goal for the company's treasury at the end of the planning horizon.

We also present the  $\beta$ -index analysis to compare two sets of solutions regarding risk-aversion. We evaluate  $\beta = \text{COV}/\text{VAR}$ , in which COV is the covariance between the solution set comprising two hundred scenarios generated by the RN model and the solution set composed by the two hundred scenarios generated by a given risk-averse model. VAR is the variance of the two hundred scenario set generated by the RN model (Gitman, 2014). When  $\beta$  is less than 1 (greater than 1), we say that the corresponding risk-averse model provides less (more) risky solutions in comparison to the reference RN solutions. If  $\beta = 1$ , both models provide solutions within the same risk level.

Finally, we also propose two additional metrics, the so-called *Cash Flow at Risk* (CFaR) and the *Conditional Cash Flow at Risk* (CCFaR), attempting to provide useful managerial insights. The CFaR is the maximum decrease of the amount of cash flow generated in the planning horizon, which occurred due to the impact of changes in market rates in a given set of exposures and a certain confidence level. In the specific case of this paper, the exposures are both cash-in and cash-out in USD. The exchange rate BRL by USD is the market rate and the confidence level is 95%. To calculate the CFaR, the stochastic solutions corresponding to each exchange rate scenario were sorted in ascending order and the solution corresponding to the desired 95% confidence level (percentile) was identified. Figure 6 shows that the value of BRL 116.8 million represents the  $\text{CFaR}_{0.95}$  for a set of cash flow solutions. The CCFaR is evaluated as the average of values that exceed the CFaR within a desired confidence level. In Figure 5, the value of  $\text{CCFaR}_{0.95}$  is BRL 114.4 million. More details concerning these measures are found in Lee (1999) and Fonseca (2006).

Figure 5. CFaR<sub>0.95</sub> and CCFaR<sub>0.95</sub> for a cash flow solution set.



### 5.3. EVPI and VSS analysis

For the risk neutral model, both the Expected Value of Perfect Information (EVPI) and the Value of Stochastic Solution (VSS) were evaluated to verify if the randomness of the parameters is important for the decision maker. The results in thousands of BRL are as follows (Table 2): RN = 123,133; WS = 124,347; EEV = 121,983; EVPI = 1,213; and VSS = 1,149, revealing that it is possible to obtain a significant gain through the acquisition of perfect information about the future (it means investing in more information, news, demand forecast and so on). In absolute terms, for the studied company, the amount of BRL 1.2 million is a reduction in its financial expenses of around 10%, which generates a profit increase in this percentage, which is representative for the studied company. Furthermore, the worst-case value for the wait-and-see solutions is 5.44% better in comparison to the solutions provided by the RN model. This figure represents a difference of around BRL 6 million. The expected profit of the WS solutions is slightly better than in the RN approach though, showing that the WS solutions may improve more pessimistic situations, but not necessarily have advantages from the best-case point of view. The results of the standard deviation also reveal that both the WS and EEV solutions have a lower profit dispersion in comparison to the RN solutions.

Another important indicator for the financial manager is the probability of the solutions of the models being below a certain target set by the company's management. Solution values below, for example, BRL 120 million are considered bad by the company's management and measuring the probability of these events happening is important for the financial manager. As expected, under uncertainty, this probability is substantially higher than under perfect information about the future. In particular, the usage of the expected value solution as the first-stage decision variables (EEV) results in overall bad performances. In particular, the risk of not achieving the profit's goal increases by 50% in this case.

Table 2. Statistical indicators of RN, WS and EEV solutions in amount in thousands of BRL.

	Expected Profit (EP)	Worst Case Profit (WC)	Best Case Profit (BC)	Profit Standard Deviation (SD)	Probability (%) Profit less than BRL 120 million ( $\rho_{120}$ )
RN	123,133	110,532	134,493	3,955	14.50
WS	124,347	116,544	130,979	2,741	6.500
EEV	121,983	110,552	130,973	3,195	21.00

The overall results suggest that it might be worth investing in more accurate information on the future exchange rates. However, under uncertainty, the incorporation of all the scenarios simultaneously in the problem via the RN approach result in a more effective cash management policy, particularly if we adopt the expected value approach given by the EEV methodology.

#### 5.4 Results and statistics of RN and risk aversion models

Table 3 summarizes the main results and performance measures for the RN, minimax and CVaR models. The profit distributions across the 200 scenarios for the RN, minimax and CVaR under a confidence level 0.70 and 0.95 are also depicted in Figure 6. As expected, there is a profit reduction as robustness is enforced either via the minimax or the CVaR criteria. However, it is possible to obtain much less risky solutions, as observed by both the standard deviation and the  $\beta$ -index. The profit distributions also revealed that risk-aversion is enforced mostly by reducing the difference between profit worst- and best-cases, leading to narrower profit distribution ranges. In many cases, the profit reduction is only marginal given the improvement in terms of the profit standard deviation reduction for both risk-averse approaches.

Concerning the *RN approach*, the results reveal that most solutions are between BRL 120 and BRL 125 million. Moreover, 37.5% of the solutions are above BRL 125 million and only 14.5% of the solutions are below the target set by company. Considering a 95% confidence level, the worst cash flow result is BRL 116.8 million and the average cash results lower than the CFaR are BRL 114.4 million. Although the RN best-case profit is substantially greater than those provided by the risk-averse models, one needs to consider that the profit standard deviation is also greater. The profit distribution also shows that due to its greater dispersion of solutions, the RN model has the highest probability of obtaining values above BRL 125 million.

As the *minimax approach* takes into account the most unfavorable deviation to compute the optimal values, it often leads to worse expected profits in comparison to those from the risk-neutral approach. Surprisingly, in our particular application, the profit reduction of 8.49% might be compensated by the decrease of 10.59% in the profit standard deviation and a smaller  $\beta$ -index, suggesting that minimax indeed provides more stable solution across the 200 scenarios. The occurrence of values below the target of BRL 120 million is 95.5%, which is much higher than the same measure of the RN model. The probability of the values being between BRL 120 and BRL 125 million is only 4.5%. Both metrics CFaR and CCFaR are also worse in comparison to the RN values.

Remind that the confidence level has a risk-aversion role for the CVaR $_{\alpha}$  approach. In fact, larger values

of  $\alpha$  give more weight to worse (lower-profit) scenarios, whose implication is twofold. First, expected profit is reduced because of the incorporation of more scenarios with lower potential for financial gains. Second, the risk of larger deviations is naturally mitigated because  $v_c$  is deeper penalized. For example, worse deviations are 33 times more penalized in  $\text{CVaR}_{0.99}$  than in  $\text{CVaR}_{0.70}$ , approximately, which explains the profit deterioration of 7.14% for  $\text{CVaR}_{0.99}$ .

In general,  $\text{CVaR}_\alpha$  yields a good tradeoff between profit and risk, which can be confirmed by the dramatic reduction in the profit standard deviation and by the  $\beta$ -index that does not lead to relevant profit losses. On average, 4% of the profit loss is necessary to obtain a 34% of dispersion reduction. In particular, notice that  $\text{CVaR}_{0.70}$  mitigates the risk (standard-deviation) by 26.22% in exchange of an expected profit only 1.35% lower, showing the attractiveness of the CVaR approach, even for less conservative decision-makers. Moreover, in this case, the worst-case performance, as well as both CFaR and CCFaR, have also a slightly better performance in comparison to the RN solution and the company can ensure that the probability of generating a profit within target is exactly  $\bar{\rho}_{120} = 69.5\%$ .

Table 3. Results and performance measures for the RN, minimax and CVaR approaches.

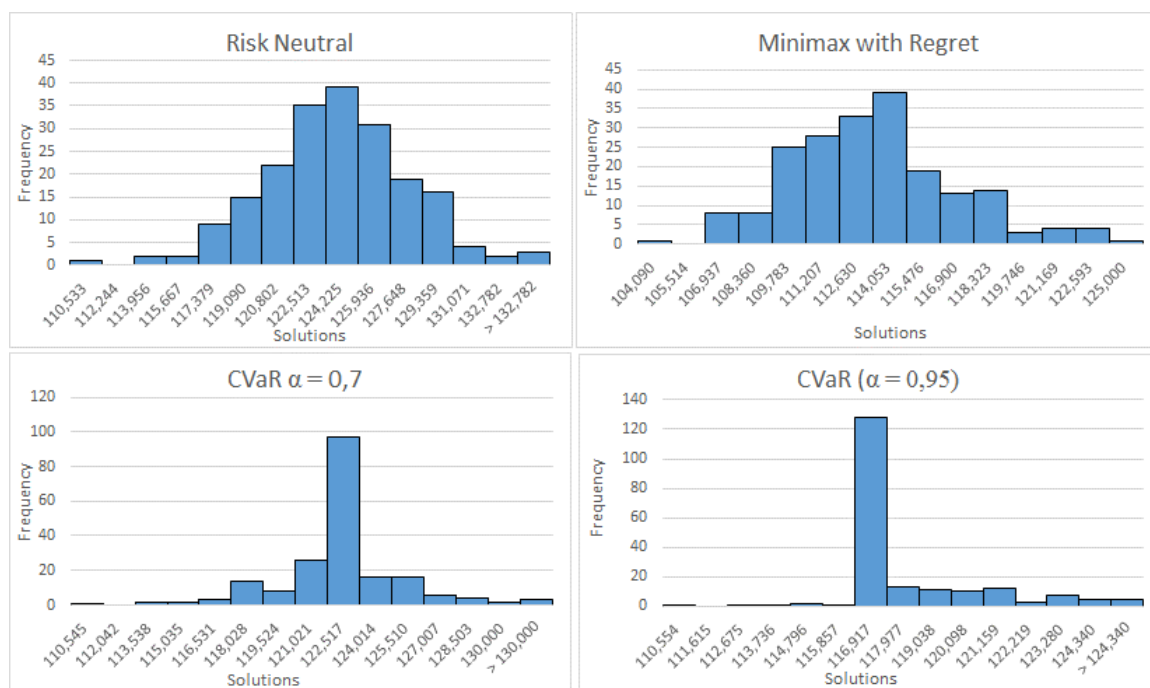
Approach	EP	Profit				Risk						
		reduction (%)	$\rho_{120}$	$\bar{\rho}_{120}$	$\bar{\rho}_{125}$	WC	BC	SD	reduction (%)	$\beta$	CFaR <sub>0.95</sub>	CCFaR <sub>0.95</sub>
RN	123,133	*	14.5	48.0	37.5	110,532	134,493	3,955	*	1	116,776	114,387
Minimax	112,677	8.49	95.5	4.50	*	104,090	124,016	3,536	10.59	0.88	106,994	106,034
CVaR <sub>0.70</sub>	121,469	1.35	15.0	69.5	15.5	110,545	131,496	2,918	26.22	0.71	116,789	114,399
CVaR <sub>0.75</sub>	120,554	2.09	18.0	73.0	9.0	107,388	130,977	3,206	18.94	0.76	114,532	112,107
CVaR <sub>0.80</sub>	120,946	1.78	15.0	70.5	14.5	110,548	131,329	2,856	27.79	0.69	116,791	114,402
CVaR <sub>0.85</sub>	119,631	2.84	14.5	82.0	3.5	110,550	130,436	2,237	43.44	0.50	116,793	114,404
CVaR <sub>0.90</sub>	119,059	3.31	70.0	22.5	7.5	110,552	127,719	2,560	35.27	0.58	116,795	114,406
CVaR <sub>0.95</sub>	117,870	4.27	79.0	20.0	1.0	110,554	125,400	2,312	41.54	0.50	116,796	114,408
CVaR <sub>0.96</sub>	117,471	4.60	83.5	14.5	2.0	110,555	126,703	2,377	39.90	0.50	116,360	114,389
CVaR <sub>0.97</sub>	116,765	5.17	89.0	11.0	*	110,556	124,469	2,093	47.08	0.43	115,844	114,252
CVaR <sub>0.98</sub>	115,733	6.01	88.5	11.0	0.5	110,560	126,041	2,587	34.59	0.55	114,219	113,492
CVaR <sub>0.99</sub>	114,336	7.14	94.5	5.5	*	110,561	123,277	2,690	31.98	0.59	112,423	112,216

Notes. Profits are given in thousands of BRL. *Profit reduction (Risk reduction)* is evaluated with reference to the RN profit (standard deviation).  $\bar{\rho}_{120} = Pr[120 < \text{Profit} \leq 125]$  and  $\bar{\rho}_{125} = Pr[\text{Profit} > 125]$ .

For  $\alpha$  varying between 0.7 and 0.85, the CVaR solutions are mostly concentrated between BRL 120 million and BRL 125 million. Interestingly, for  $\alpha$  values varying in this range, values below BRL 120 million have occurrence probabilities almost identical to those of the RN model. For  $\alpha$  values between 0.90 and 0.97, the highest concentration of the solutions is between BRL 115 million and BRL 120 million, while for  $\alpha = 0.98$  and  $\alpha = 0.99$ , the concentration of the solutions is between BRL 110 million and BRL 115 million. These results confirm that it is not worth increasing the  $\alpha$  values more than 0.85, despite the substantial improvements in the dispersion of the solutions, as the probability of not achieving the company's financial goals increase. All in all, the CVaR model provides solutions with reduced standard deviation, which helps the financial manager to take more assertive decisions on the cash-flow under uncertainty.

Depending on the risk policy of a particular company, the CVaR model may or may not be favored over the RN model, as the great advantage of using it, in the case of the cash flow problem, is its assertiveness. However, these results also indicate that the risk-neutral approach might be a good choice for supporting cash-flow decisions under exchange-rate uncertainty since it practically has the same probability of occurrence of values below the target of BRL 120 million as the CVaR model, with  $\alpha$  varying between 0.7 and 0.85, and has a high probability of occurrence of values above BRL 125 million, which is not the case of the CVaR model. The minimax regret model proved to be very conservative and did not arouse interest in the company's treasury. All analyses were performed with  $P = \{1, \bar{1}, \bar{1}\}$ , i.e., with the first stage variables equal to the first period. Further analysis could have been performed with  $P = \{1, \bar{1}, \bar{1}, 2, \bar{2}, \bar{2}\}$  similarly. However, some preliminary investigations along these lines did not show significant advantages when using more periods as the first-stage variables.

Figure 6. Profit distributions of the RN, minimax and CVaR models, for  $\alpha = 0.70$  and 0.95.



## 6. Conclusion

This study proposed stochastic programming models to properly represent and solve the problem of maximizing financial resources available at the end of a cash planning horizon in companies under exchange rates uncertainties. In addition to the risk-neutral model, we also presented a minimax regret and a CVaR version to provide less risky solutions, which might be appealing in volatile markets. The overall results revealed that the exchange rate randomness is representative for the proposed cash flow management problem. The comparisons amongst the stochastic models suggested that there is no reason to apply the minimax approach, as its solutions are completely dominated by the CVaR solutions in terms of profits and lower dispersion. On the other hand, CVaR is a good choice for supporting tactical decisions in an uncertain and risky environment, because it reduces the dispersion of the solution values at negligible profit losses, and it possesses the advantage of generating more

assertiveness for the financial manager. However, depending on the risk policy of a company, the CVaR model may or may not be favored over the RN model, as the latter presents similar results in terms of probability of occurrence of values below the targets as the CVaR model, and it has a high probability of occurrence of values above higher amounts.

According to the best of our knowledge, there is no *optimization under uncertainty* methodology (RO or SP) that is unrestrictedly recommended for general problems. In general, both SP and RO may present advantages and disadvantages depending on the problem that is being modelled, the available input data, and the type of analysis that is required. When the problem naturally poses hard constraints in such a way data variation due to uncertainty might imply infeasible solutions, RO can be more appealing than SP. But if we are interested in providing relatively good decisions regardless the outcome and, at the same time, we can guarantee that any outcome could be properly accommodated using contingency decisions, SP might be a suitable modelling paradigm. In addition, we know that RO does not require any knowledge on the underlying distribution of the random variables, whereas SP does. Investing in a good scenario generating technique to provide a set of the realizations for the random variables, as we have done in this paper, might be advantageous, though, as more accurate approximations for the uncertainty may result in more appealing and assertive decisions, mainly if we compare with (overconservative) robust-induced solutions based on worst-case perspectives.

In this context, it would be interesting to study a systematic way to compare our stochastic programming approach for the cash-flow problem under uncertainty to an alternate robust optimization method (as in Righetto et al. 2016) in order to understand the relative strengths and weaknesses of each approach in terms of managerial implications (Alem et al., 2018; Cuvelier et al.; 2018). Other interesting research perspectives arose from this study. For example, additional computational experiments with examples from other companies using our proposed framework could help to obtain more conclusive evaluations on the performance of the cash flow models, as well as to make comparisons between the models and the solutions used by the treasurer. Moreover, the extension of the model for considering the possibility of correlation (possibly lagged) between accounts receivable and accounts payable, as well as the development of a multistage stochastic programming approach for the cash flow problem can bring relevant contributions to the analysis of this problem.

*Acknowledgements:* We would like to thank the anonymous reviewers for their useful comments and suggestions of revisions. We also thank the Brazilian stationery company for its important collaboration with this study, as well as the National Council of Technological and Scientific Development (CNPq) and the São Paulo State Research Foundation (FAPESP) in Brazil for their financial support.

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## Appendix

Table A.1 shows the yield for assets  $b$  and  $c$  per month (net of taxes) and the interest rate per month for the bank loan. For  $\alpha_{ij}$ , the yield is zero. The initial balance for asset  $a$  is BRL 1.994. For asset  $b$ , the initial balance is zero. For asset  $c$ , the initial balance is BRL 100,000. The unit conversion costs between the assets are considered null, i.e.,  $C_{(a,b)} = C_{(a,c)} = 0$ ,  $C_{(b,a)} = C_{(c,a)} = 0$ ,  $C_{(a,b)} = C_{(a,c)} = 0$ , and  $C_{(b,a)} = C_{(c,a)} = 0$ . The bank loan limit for the company is BRL 100,000. It is assumed that once the loan has been taken out, the payment amount plus the interest rate should be made immediately. Therefore, the options for financial transactions of this planning horizon are threefold: (i) to maintain the current account balance; (ii) to allocate funds available for financial investments; and (iii) to use the line of credit to cover negative net cash flows. If there is a negative cash flow, the cash needs should be covered using the available options: (i) previous cash balance; (ii) financial investment redemption; or (iii) loans. As input data, it is assumed that all transactions occur at the beginning of each period.

Table A.1. Monthly yields and interest rates for each type of assets (in percentage).

Period	1	2	3	4	5	6	7	8	9	10	11	12
$\beta_{ij}$	0.54	0.56	0.49	0.52	0.53	0.41	0.47	0.42	0.42	0.47	0.38	0.42



$\varepsilon_{ij}$	0.57	0.59	0.52	0.55	0.56	0.43	0.49	0.45	0.45	0.49	0.40	0.45
$\kappa_{ij}$	0.58	0.61	0.53	0.56	0.57	0.45	0.51	0.46	0.46	0.51	0.41	0.46
$\gamma_{ij}$	0.76	0.80	0.70	0.74	0.75	0.59	0.66	0.60	0.60	0.66	0.54	0.60

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