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Cash Sub-additive Risk Measures and Interest Rate Ambiguity

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Abstract

A new class of risk measures called cash sub-additive risk measures is introduced to assess the risk of future financial, nonfinancial and insurance positions. The debated cash additive axiom is relaxed into the cash sub-additive axiom to preserve the original difference between the numéraire of the current reserve amounts and future positions. Consequently, cash sub-additive risk measures can model stochastic and/or ambiguous interest rates or defaultable contingent claims. Several practical examples are presented and in such contexts cash additive risk measures cannot be used. Several dual representations of the cash sub-additive risk measures are provided. The new risk measures are characterized by penalty functions defined on a set of sub-linear probability measures and can be represented using penalty functions associated with cash additive risk measures defined on some extended spaces. The issue of the optimal risk transfer is studied in the new framework using inf-convolution techniques. Some examples of dynamic cash sub-additive risk measures are provided via BSDEs. In contrast to the dynamic cash additive risk measures, the dynamic cash sub-additive risk measures are recursive.

Keywords: Risk measures, Fenchel-Legendre transform, model uncertainty, inf-convolution, backward stochastic differential equations.

JEL Classifications: D81, G13.

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1 Introduction

The assessment of financial and nonfinancial risks plays a key role for economic agents when pricing assets or managing their wealths. Consequently, over the last decade several measures of risk have been proposed to assess the riskiness of financial and nonfinancial positions and compute cash reserve amounts for hedging purposes. The axiomatic based monetary risk measures have been largely investigated because most axioms embed desirable economic properties. Coherent risk measures have been introduced by Artzner, Delbaen, Eber, and Heath (1997), Artzner, Delbaen, Eber, and Heath (1999), and further developed by Delbaen (2001), Delbaen (2002); sublinear risk measures by Frittelli (2000); convex risk measures by Föllmer and Schied (2002a), Föllmer and Schied (2002b) and Frittelli and Rosazza Gianin (2002). Examples of convex risk measures related to pricing and hedging in incomplete markets are provided by, for instance, El Karoui and Quenez (1996), Carr, Geman, and Madan (2001), Frittelli and Rosazza Gianin (2004) and Staum (2004). However, while the convexity and the monotonicity axioms have been largely accepted by academics and practitioners, the cash additive axiom has been criticized from an economic viewpoint. A basic reason is that while regulators and financial institutions determine and collet today the reserve amounts to cover future risky positions, the cash additivity requires that risky positions and reserve amounts are expressed in the same numéraire. This is a stringent requirement that limits the applicability of cash additive risk measures. Implicitly it means that risky positions are discounted before applying the risk measure assuming that the discounting process does not involve any additional risk. Unfortunately, when the interest rates are stochastic this procedure does not disentangle the risk of the financial position per sé and the risk associated to the discounting process[1]. Furthermore, payoff functions on risky assets are a priori and contractually determined by economic agents considering different scenarios for the underlying asset. While this procedure is theoretically framed into the cash additive risk measures, the cash additive axiom does not allow to account for ambiguous discount factor. For the correct assessment of the current reserve amount it is equally important to allow for ambiguity on the underlying asset and on the discount factor. This assessment is achieved by relaxing the cash additive axiom and searching for risk measures that preserve the different numéraires of the current reserve amounts and the future risky positions.

[1]Disentangling the different risks is crucial when implementing hedging strategies as different risks are hedged on different markets.

The main contribution of this paper is to propose a new class of risk measures called cash sub-additive risk measures that are directly defined on the future risky positions and provide the reserve amounts in terms of the current numéraire. To reconcile the two different numéraires cash sub-additive risk measures relax the cash additive axiom into the cash sub-additive axiom. This is the minimal requirement to account for the time value of money. Remarkably, the cash sub-additive axiom (together with the monotonicity and convexity axioms) is enough to characterize measures of risk that can be applied also when the cash additive risk measures cannot—as for instance under ambiguous interest rates or defaultable cash flows. Cash sub-additive risk measures turn out to be suitable not only for assessing financial risks but also other kind of risks such as insurance risks. For example, the put option premium investigated by Jarrow (2002) as a measure of the firm insolvency risk defines a cash sub-additive risk measure. Moreover, similarly to the cash additive risk measures, the cash sub-additive risk measures can be represented using penalty functions. In particular, we show that cash sub-additive risk measures are characterized by minimal penalty functions which only depend on finitely additive set functions μ such that $0 \leq \mu(\Omega) \leq 1$, that we call finitely additive sub-probability measures.

The other contributions of this paper are the following. In the framework of cash additive risk measures when the zero-coupon bond is available for the relevant time horizon, we provide the conditions under which discounting the forward risk measure to obtain current reserve amounts defines risk measures additive with respect the current numéraire and vice versa (Section 3).

In Section 4 we introduce the cash sub-additive risk measures (denoted by $\mathcal{R}$) and we give several examples, for instance generalizing the put option premium investigated by Jarrow (2002). In these examples cash sub-additive risk measures are obtained applying cash additive risk measures to the discounted positions and considering the worst case scenario on the ambiguous discount factor. Random convex functions arise naturally and using their Fenchel transforms a dual representation is obtained penalizing discount factors through the Fenchel functionals.

In Section 5 making a minimal enlargement of the sample space we define a cash additive risk measure which is in a one to one correspondence with $\mathcal{R}$. Such a correspondence allows for a rich interpretation of both cash additive and cash sub-additive risk measures. Moreover, this correspondence simplifies the study of cash sub-additive risk measures as it enables to exploit several results on cash additive risk measures. For instance, we characterize cash sub-additive risk measures by showing that the minimal penalty function of $\mathcal{R}$ only depends on sub-linear probability measures.

In Section 6 we provide two other links between cash sub-additive and cash additive risk measures on an enlarged linear space which "contains" the domain of $\mathcal{R}$. The first link allows to represent any cash sub-additive risk measure in terms of ambiguous probability models and ambiguous discount factors, both defined on the original space of definition of $\mathcal{R}$. The second link shows that cash sub-additive risk measures given by compositions of a risk measure and a convex random function are compositions of an unconditional and a conditional cash additive risk measures. The first risk measure accounts for the model uncertainty and the second one for the ambiguity on interest rates or more in general for the risk affecting the numéraire.

In Section 7 using cash sub-additive risk measures we study the problem of designing the optimal transaction between two economic agents in a general framework allowing for ambiguous discount factors. In particular we show that the risk transfer problem can be reduced to an inf-convolution of cash sub-additive risk measures which is again a cash sub-additive risk measure.

Finally, in Section 8 we provide a dynamic example of cash sub-additive risk measures defined as solution of backward stochastic differential equations. In particular, we let the generator of backward differential equations depend on the solution in a monotone way and we obtain recursive dynamic cash sub-additive risk measures. Section 9 concludes.

2 Cash additive risk measures

Coherent risk measures have been introduced by Artzner, Delbaen, Eber, and Heath (1997), Artzner, Delbaen, Eber, and Heath (1999), and further developed by Delbaen (2001), Delbaen (2002); sublinear risk measures by Frittelli (2000); convex risk measures by Föllmer and Schied (2002a), Föllmer and Schied (2002b) and Frittelli and Rosazza Gianin (2002). In the following we recall some key properties of cash additive risk measures and we discuss the cash additive axiom. The following definitions are consistent with the definitions of monetary risk measure in Föllmer and Schied (2002b).

2.1 Definitions and properties of cash additive risk measures

Let $(\Omega, \mathcal{A})$ be a measurable space. The risky positions at the relevant time horizon belong to the linear space of bounded functions including constant functions denoted by $\mathcal{X}$.

Definition 2.1 *A cash additive risk measure is a functional $\rho : \mathcal{X} \to \mathbb{R}$ cash additive, convex and monotone decreasing, i.e.,*

a) Convexity: $\forall \lambda \in [0,1], \quad \rho\big(\lambda X + (1-\lambda)Y\big) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$;

b) Monotonicity: $X \le Y \Rightarrow \rho(X) \ge \rho(Y)$;

c) Cash additivity (or cash invariance): $\forall m \in \mathbb{R}, \quad \rho(X+m) = \rho(X) - m$.

A cash additive risk measure is coherent when

d) Positive homogeneity: $\forall \lambda \in \mathbb{R}^+, \quad \rho\big(\lambda X\big) = \lambda\rho\big(X\big)$.

e) ρ *is* normalized *when* $\rho(0) = 0$.

f) ρ *is* continuous from below (from above) *when*

$$X_n \nearrow X \quad \Rightarrow \quad \rho(X_n) \searrow \rho(X), \qquad (X_n \nearrow X \quad \Rightarrow \quad \rho(X_n) \nearrow \rho(X)).$$

The convexity axiom translates the natural important fact that diversification should not increase risk. In particular, convex combinations of "admissible" risks should be "admissible". Indeed, a major drawback of the well-known Value at Risk measure is its failure to meet this criterion.
To shorten the representation of convex combinations of elements we use the following notation. We denote the *barycenter* (or *convex combination*) of the set $x_I := \{x_{(1)}, x_{(2)}, \ldots, x_{(I)}\},\ I \in \mathbb{N}$,

(2.1) $Bar[x_I] := Bar^{\lambda_I}[x_I] := \sum_{i=1}^{I} \lambda_i x_{(i)}$ where $\lambda_i \in [0,1]$, $i = 1, \ldots, I$, and $\sum_{i=1}^{I} \lambda_i = 1$.

In particular, f is a convex function if and only if $f(Bar[x_I]) \le Bar[f(x)_I]$. The same definition holds for a set X_I of random variables.

2.2 Dual representation of cash additive risk measures

A key property of cash additive risk measures is the dual representation in terms of normalized finitely additive set functions and minimal penalty functional (Föllmer and Schied (2002b, Theorem 4.12)). The dual point of view emphasizes the interpretation in terms of a worst case scenario related to the agent's (or regulator's) beliefs: the agent does not know the true "probability" measure and uses distorted beliefs from a subjective set of normalized additive set functions. Under the additional assumption that risk measures are continuous from below, the dual representation is in term of σ-additive probability measures (Föllmer and Schied (2002b, Proposition 4.17)).

Theorem 2.2 *(a) Let $\mathcal{M}_{1,f}(\mathcal{A})$ be the set of all finitely additive set functions Q on $(\Omega, \mathcal{A})$ normalized to one, $Q(\Omega) = 1$, and α the minimal penalty functional taking values in $\mathbb{R} \cup \{+\infty\}$:*

$$\forall Q \in \mathcal{M}_{1,f}(\mathcal{A}), \quad \alpha(Q) = \sup_{X \in \mathcal{X}} \{\mathbb{E}_Q[-X] - \rho(X)\}, \qquad \big(\geq -\rho(0)\big) \tag{2.2}$$

$$\mathcal{D}om(\alpha) = \{Q \in \mathcal{M}_{1,f}(\mathcal{A}) |\ \alpha\big(Q\big) < +\infty\}. \tag{2.3}$$

The Fenchel duality relation holds:

$$\forall X \in \mathcal{X}, \quad \rho(X) = \sup_{Q \in \mathcal{M}_{1,f}(\mathcal{A})} \{\mathbb{E}_Q[-X] - \alpha\big(Q\big)\}. \tag{2.4}$$

Moreover, for any $X \in \mathcal{X}$ there exists a $Q_X \in \mathcal{M}_{1,f}(\mathcal{A})$, such that $\rho(X) = \mathbb{E}_{Q_X}[-X] - \alpha\big(Q_X\big) = \max_{Q \in \mathcal{M}_{1,f}(A)} \{\mathbb{E}_Q[-X] - \alpha(Q)\}$.

(b) Let $\mathcal{M}_1(\mathcal{A})$ denote the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{A})$. Let ρ be a monetary risk measure continuous from below and suppose that β is any penalty function on $\mathcal{M}_{1,f}(\mathcal{A})$ representing ρ. Then β is concentrated on the class $\mathcal{M}_1(\mathcal{A})$ of probability measures, i.e., $\beta(Q) < \infty$ only if Q is σ-additive.

Henceforth, α in equation (2.2) is the minimal penalty function, denoted by $\alpha_{\min}$ in Föllmer and Schied (2002b).

The following lemma shows that a cash additive risk measure is linear with respect to a (risky) position Y if and only if any set function Q in the domain of the penalty functional satisfies the calibration constraint: $Q(-Y) = \rho(Y)$. This lemma will be used to derive the results in Section 3.

Lemma 2.3 *Let ρ be a normalized cash additive risk measure on $\mathcal{X}$ and $\mathcal{W}$ a linear sub-space of $\mathcal{X}$ containing the constants. The risk measure ρ is linear on $\mathcal{W}$, that is*

$$\forall\, (W_1, W_2) \in \mathcal{W} \times \mathcal{W},\ \forall (\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}, \quad \rho(\lambda_1 W_1 + \lambda_2 W_2) = \lambda_1 \rho(W_1) + \lambda_2 \rho(W_2), \tag{2.5}$$

if and only if $\rho(W) = \mathbb{E}_Q[-W]$ *for any* $Q \in \mathcal{D}om(\alpha)$. *This implies that the risk measure is invariant with respect to* $\mathcal{W}$, *that is* $\forall X \in \mathcal{X},\ \forall W \in \mathcal{W}, \quad \rho(X+W) = \rho(X) + \rho(W)$.

Proof. The dual representation and the linearity of ρ with respect to $\mathcal{W}$ imply that for any $Q \in \mathcal{D}om(\alpha)$, $\lambda \in \mathbb{R}$, $\lambda\rho(W) = \rho(\lambda W) \geq \mathbb{E}_Q[\lambda(-W)] - \alpha(Q)$, where α is the minimal penalty of ρ. Then $\alpha(Q) \geq -\lambda\left(\rho(W) + \mathbb{E}_Q[W]\right)$. As the last inequality holds for any $\lambda \in \mathbb{R}$, $\rho(W) = -\mathbb{E}_Q[W], \ \ \forall Q \in \mathcal{D}om(\alpha)$. The vice versa is evident.

If (2.5) holds then for any $X \in \mathcal{X}$, $W \in \mathcal{W}$, $\rho(X+W) = \sup_{Q \in \mathcal{D}om(\alpha)}\{\mathbb{E}_Q\big[-X-W\big] - \alpha(Q)\} = \sup_{Q \in \mathcal{D}om(\alpha)}\{\rho(W) + \mathbb{E}_Q[-X] - \alpha(Q)\} = \rho(W) + \rho(X)$. □

2.3 Cash additivity and discounting

The cash additive axiom is motivated by the interpretation of $\rho(X)$ as capital requirement. Intuitively, $\rho(X)$ is the amount of cash which should be added to the risky position X in order to make it acceptable (i.e., with non positive measure of risk) by a supervising agency

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$

The amount of cash $\rho(X)$ can also be considered as the opposite of the buyer's indifference price of the position. Paying the amount $-\rho(X)$, the new exposure $X - (-\rho(X))$ does not carry any risk (i.e., the risk measure is non positive) and the agent is indifferent between doing nothing and having the "hedged" exposure. Hence the cash additive property requires that the risky position and the risk measure are expressed in the same numéraire. Then either cash additive risk measures are defined on the discounted value of the future positions (see, for instance, Delbaen (2001) and Föllmer and Schied (2002b)) or cash additive risk measures are defined directly on the future positions and give the forward reserve amount to add to the future position at the future date (see, for instance, Rouge and El Karoui (2000)). When interest rates are stochastic the risk measure on the discounted position and the forward risk measure are different. In the next section, assuming that all the agents use the same discount factor for the maturity of interest and there exists a zero coupon bond for that maturity, we provide a link between cash additive risk measures on the discounted positions and forward cash additive risk measures.

3 Spot and forward risk measures under stochastic interest rates

Let D_T denote the stochastic (non-ambiguous) discount factor for the maturity of interest T used by all the agents on the market. $(\Omega, \mathcal{F}_T)$ is a measurable space and the risky position belongs to $\mathcal{X}$ the linear space of real-valued bounded random variables on $(\Omega, \mathcal{F}_T)$ including constant variables. The riskiness of $X_T \in \mathcal{X}$ is assessed at time $t = 0$ and 1_T denotes one unit of cash available at date T.

Definition 3.1 *a) Let D_T be the $\mathcal{F}_T$-measurable discount factor, $0 \leq D_T \leq 1$. A spot risk measure, ρ_0, is a* cash additive risk measure *defined on the discounted position $D_T X_T$, $X_T \in \mathcal{X}$. The cash additive property is with respect to the cash available at time $t = 0$, $\forall X_T \in \mathcal{X}$,*

$$\forall m \in \mathbb{R}, \quad \rho_0\big(D_T X_T + m\big) = \rho_0(D_T X_T) + \rho_0(m) \quad \text{and} \quad \rho_0(m) = m\rho_0(1) = -m. \tag{3.1}$$

b) A forward risk measure, ρ_T, is a cash additive risk measure defined on the future position $X_T \in \mathcal{X}$. The cash additive property is with respect to cash available at time T, $\forall X_T \in \mathcal{X}$,

$$\forall m \in \mathbb{R}, \quad \rho_T(X_T + m1_T) = \rho_T(X_T) + \rho_T(m1_T) \quad \text{and} \quad \rho_T(m1_T) = m\rho_T(1_T) = -m1_T. \tag{3.2}$$

The spot risk measure ρ_0 is the monetary risk measure defined in Föllmer and Schied (2002b). It represents the cash amount at $t = 0$ to add to the discounted position $D_T X_T$ to make it acceptable. The spot risk measure does not disentangle the discounting risk from the risk of the financial position per sé. Furthermore, to meaningful consider the discounted future position the discount factor cannot be ambiguous.

Rouge and El Karoui (2000) partially solve this problem introducing the forward risk measure ρ_T defined on the future position. ρ_T gives the forward cash amount (evaluated today) to add at time T to the position to make it acceptable. When the zero coupon bond for the maturity of interest is available, the forward reserve $\rho_T(X_T)$ can be easily discounted at time zero. We show that this procedure defines a spot risk measure when ρ_T satisfies a calibration constraint on D_T^{-1} and B_T^{-1}. Similarly, the spot risk measure $\rho_0(D_T X_T)$ capitalized by B_T^{-1} defines a forward risk measure if ρ_0 satisfies a calibration constraint on D_T and B_T. The penalty function of ρ_0 is equal to the penalty function of ρ_T discounted by B_T and the corresponding additive set functions satisfy the usual spot-forward change of measure.

Proposition 3.2 *1) Let ρ_T be a normalized forward risk measure with penalty function α_T. The functional*

$$q_0\big(D_T X_T\big) := B_T \rho_T(X_T) \tag{3.3}$$

is convex and monotone decreasing with respect to $D_T X_T$ and satisfies the following calibration constraint on D_T and B_T: $\forall \lambda \in \mathbb{R}$, $q(\lambda D_T) = B_T \rho_T(\lambda) = -\lambda\, B_T = \lambda q(D_T)$. Moreover, q_0 is a spot risk measure if and only if ρ_T satisfies the calibration constraint on D_T^{-1} and B_T^{-1}

$$\forall \lambda \in \mathbb{R}, \quad \rho_T\left(\lambda {D_T}^{-1}\right) = -\lambda {B_T}^{-1} = \lambda \rho_T({D_T}^{-1}). \tag{3.4}$$

In that case any $Q_T \in \mathcal{D}om(\alpha_T)$ is such that $\mathbb{E}_{Q_T}\left[{D_T}^{-1}\right] = {B_T}^{-1}$ and the minimal penalty functional of q_0, α_0, is given by

$$\alpha_0(Q_0) = B_T\, \alpha_T(Q_T), \forall Q_0 : \ dQ_T = \frac{D_T}{B_T} dQ_0 \in \mathcal{D}om(\alpha_T), \quad \text{and } \alpha_0 = \infty \text{ otherwise.} \tag{3.5}$$

2) Vice versa, let ρ_0 be a normalized spot risk measure. The functional $q_T\big(X_T\big) := B_T^{-1}\rho_0(D_T X_T)$ is convex and monotone decreasing with respect to X_T and satisfies the following calibration constraint on D_T^{-1} and B_T^{-1}: $\forall \lambda \in \mathbb{R}$, $q_T(\lambda D_T^{-1}) = B_T^{-1}\rho_0(\lambda) = -\lambda\, {B_T}^{-1} = \lambda q_T(D_T^{-1})$. Moreover, q_T is a forward risk measure if and only if ρ_0 satisfies the calibration constraint on D_T and B_T: $\forall \lambda \in \mathbb{R}$, $\quad \rho_0\left(\lambda D_T\right) = -\lambda B_T$.

Proof. 1) If ρ_T satisfies (3.4) the cash additive follows directly from Lemma 2.3. Conversely, let q be cash additive. This is equivalent to require that ρ_T satisfies

$$\forall X_T \in \mathcal{X}, \forall \lambda \in \mathbb{R}, \quad \rho_T\left(X_T + \lambda\, {D_T}^{-1}\right) = \rho_T(X_T) - \lambda\, {B_T}^{-1}. \tag{3.6}$$

Setting $X_T = 0$ in (3.6) gives the calibration constraint (3.4). To prove (3.5) we observe that if q in (3.3) is a spot risk measure with minimal penalty function α_0, the definition of the minimal penalty function and Lemma 2.3 give

$$\alpha_0(Q_0) = \sup_{X_T}\{\mathbb{E}_{Q_0}[-D_T X_T] - q(D_T X_T)\} = \sup_{X_T}\{B_T \mathbb{E}_{Q_T}[-X_T] - B_T \rho_T(X_T)\},$$

that is $\alpha_0(Q_0) = B_T \alpha_T(Q_T)$, where $dQ_T = D_T {B_T}^{-1} dQ_0$. It follows that Q_0 is in the domain of α_0 if and only if Q_T is a set function in the domain of α_T and satisfies the calibration constraint in (3.5). Conversely, a risk measure with minimal penalty functional α_0 satisfying (3.5) is of the form $\rho_0(D_T X_T) = B_T \, \rho_T(X_T)$.

2) Similar arguments can be used to prove the vice versa. □

Unfortunately, the procedure of computing current reserve amounts discounting forward risk measures in equation (3.3) is feasible only when the zero coupon bonds for the relevant maturities are available on the market.

Next section contains the major contribution of this paper which is the introduction of a new class of risk measures called cash sub-additive risk measures. These risk measures provide reserve amounts which account for the ambiguity on the discount factor. This result is achieved by simply relaxing the cash additive axiom into the cash sub-additive axiom and preserving the original difference in the numéraires of reserves and future positions. This will be illustrated by several examples in the finance and insurance frameworks.

4 Cash sub-additive risk measures

The following observation provides the intuition for introducing cash sub-additive risk measures. Given a spot risk measure ρ_0 in equation (3.1), the convex, non-increasing functional defined on $\mathcal{X}$ denoted by $\mathcal{R}(X_T) = \rho_0(D_T \, X_T)$ is cash *sub-additive*, that is it satisfies the following inequality: $\forall m \geq 0$,

$$\mathcal{R}(X_T + m 1_T) = \rho_0(D_T \, X_T + D_T \, m) \geq \rho_0(D_T \, X_T + m) = \rho_0(D_T \, X_T) - m = \mathcal{R}(X_T) - m.$$

This inequality is a simple consequence of the time value of the money, i.e. $D_T \, m \leq m$. The functional $\mathcal{R}$ is expressed in terms of the current numéraire but directly defined on the future position expressed in terms of the future numéraire. The function $m \in \mathbb{R} \mapsto \mathcal{R}(X_T 1_T + m 1_T) + m$ is non-decreasing, that is $\mathcal{R}$ is cash sub-additive. This observation highlights the cash sub-additive axiom as the minimal condition (imposed by the time value of the money) that has to be satisfied by risk measures which preserve the two different numéraires of current reserve amounts and future risky positions. Remarkably, replacing the cash additive axiom with the cash sub-additive axiom is sufficient to characterize risk measures that can be used also when cash additive risk measures cannot. For instance under stochastic and/or ambiguous interest rates or assessing the risk of defaultable contingent claims. In the sequel we formally define the cash sub-additive risk measures denoted by $\mathcal{R}$. Then we provide several examples showing the different applications of these new risk measures. The previous considerations and the following examples motivate the study of cash sub-additive risk measures.

4.1 Definition of cash sub-additive risk measures

Definition 4.1 *A cash sub-additive risk measure $\mathcal{R}$ is a functional $\mathcal{R} : \mathcal{X} \to \mathbb{R}$, convex and non increasing satisfying the cash sub-additive axiom:*

$$\forall m \in \mathbb{R},\ \mathcal{R}(X_T + m1_T) + m \text{ is non decreasing in } m.$$

The cash sub-additive axiom can also be expressed:

$$\forall m \in \mathbb{R},\quad \mathcal{R}(X_T + |m|1_T) \geq \mathcal{R}(X_T) - |m| \quad \text{and} \quad \mathcal{R}(X_T - |m|1_T) \geq \mathcal{R}(X_T) + |m|.$$

Cash sub-additive risk measures naturally account for the time value of money. When m dollars are added to the future position X_T, $X_T + m1_T$, the capital requirement at time $t = 0$ is reduced by less than m dollars, that is $\mathcal{R}(X_T 1_T + m1_T) \geq \mathcal{R}(X_T 1_T) - m$.

4.2 Examples of cash sub-additive risk measures

This section provides several examples of cash sub-additive risk measures. All these risk measures can be obtained composing cash additive risk measures and convex real (random) functions. The first example arises naturally considering an ambiguous discount factor.

4.2.1 Cash sub-additive risk measures under ambiguous discount factors

Consider a regulator assessing the risk of a future payoff X_T when the discount factor D_T is ambiguous and ranges between two positive constants, $0 \leq d_L \leq D_T \leq d_H \leq 1$, according to her beliefs. The regulator is endowed with a spot risk measure ρ_0 and adverse to ambiguity on discount factor. Hence she assesses the risk of X_T in the interest rates worst case scenario

$$\mathcal{R}^{\rho_0}(X_T) := \sup_{D_T \in \mathcal{X}} \big\{ \rho_0(D_T\, X_T) \,|\, d_L \leq D_T \leq d_H \big\}. \tag{4.1}$$

Proposition 4.2 *The functional $\mathcal{R}^{\rho_0}$ in equation (4.1) is a cash sub-additive risk measure. $\mathcal{R}^{\rho_0}$ can be rewritten as $\mathcal{R}^{\rho_0}(X_T) = \rho_0(-v(X_T))$, where $v(x) = -(d_L x^+ - d_H\,(-x)^+)$ is convex* *decreasing function with left derivative v_x such that $v_x \in [-1, 0]$ and $x^+ = \sup(x, 0)$.*

Proof. $\mathcal{R}^{\rho_0}$ is a cash sub-additive risk measure as it is the supremum of cash sub-additive, convex and monotone functions with respect to $X_T \in \mathcal{X}$. Moreover, as the $\inf_{D_T \in \mathcal{X}}\{D_T\, X_T | d_L \le D_T \le d_H\}$ is attained, then $\sup_{D_T \in \mathcal{X}} \{\rho_0(D_T X_T) | d_L \le D_T \le d_H\} = \rho_0\big(\inf_{D_T \in \mathcal{X}}\{D_T\, X_T | d_L \le D_T \le d_H\}\big) = \rho_0\big(d_L X_T^+ - d_H(-X_T^+)\big)$, where $v(x) = -(d_L x^+ - d_H\,(-x)^+)$. □

Remark 4.3 *When D_T varies between two random variables D_L and D_H in $\mathcal{X}$, $0 \le D_L \le D_T \le D_H \le 1$, the functional in (4.1) is a cash sub-additive risk measure $\mathcal{R}^{\rho_0}(X_T) = \rho_0(-V(X_T))$, where V is the random function $V(\omega, x) = -(D_L(\omega)x^+ - D_H(\omega)\,(-x)^+)$, convex, decreasing with respect to x, $V_x \in [-1, 0]$, for any given $\omega \in \Omega$, and $\mathcal{F}_T$-measurable for any given $x \in \mathbb{R}$. Notice that when $D_L = D_T$, $\mathcal{R}(X_T) = \rho_0(D_T X_T)$.*

Next example of cash sub-additive risk measure is not related to risk/ambiguous discount factors and is a corollary of Proposition 4.2.

4.2.2 Cash sub-additive risk measures and insurance risks

Taking an insurance point Jarrow (2002) studies the put option premium with zero strike price as a possible measure of the firm insolvency risk. The premium is the discounted expected loss. Let $r \ge 1$ be the risk free gross return from time $t = 0$ to time T of a riskless investment and $\mathbb{P}$ the reference probability measure.

Corollary 4.4 Put premium risk measure. *The premium of a put option with strike price zero and maturity T,*

$$\mathcal{R}_p(X_T) := \frac{1}{r}\, \mathbb{E}_{\mathbb{P}}\big[(-X_T)^+\big], \tag{4.2}$$

is a coherent cash sub-additive risk measure as a function of the underlying asset price X_T.

Proof. The cash sub-additive risk measure in (4.1) coincides with the put option premium $\mathcal{R}_p$ when $\rho_0(\cdot) = \mathbb{E}_{\mathbb{P}}[-\,(\cdot)]$, $d_L = 0$ and $d_H = 1/r$. □

Remark 4.5 *For any given strike price K the premium of a put option, $\mathcal{R}_p(X_T) := \frac{1}{r}\,\mathbb{E}_{\mathbb{P}}\big[(K - X_T)^+\big]$ is a cash sub-additive risk measure. This follows setting in equation (4.1) ρ_0 equals to the non normalized risk measure $\rho_0(X_T) = \mathbb{E}_{\mathbb{P}}[K - X_T]$ and $-v(x) = \frac{1}{r}\max(K - x, 0)$.*

4.3 Composing cash additive risk measures and convex functions

Generalizing the previous examples we show that $\rho_0(-V)$ is a cash sub-additive risk measure, where V denotes a continuous random function $V : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $V(\omega, x)$, such that, for any $\omega \in \Omega$, $V(\omega, \cdot)$ is decreasing, convex and $V(\omega, 0) = 0$, $V_x \in [-1, 0]$, and for any $x \in \mathbb{R}$, $V(\cdot, x)$ is $\mathcal{F}_T$-measurable. Moreover, $\rho_0(-V)$ can be represented in terms of finitely additive measures and $\mathcal{F}_T$-measurable "discount factors" over a set of possible scenarios that can be chosen according to the beliefs of the agent/regulator.

From standard results in convex analysis $V(\omega, x) = \sup_{y\in\mathbb{R}}\{xy - \beta_T(\omega, y)\}$, where β_T is the random convex Fenchel transform of V, $\beta_T(\omega, y) := \sup_{x\in\mathbb{R}}\{xy - V(\omega, x)\}$. Notice that β_T is finite only if $y \in (-1, 0)$ as $V_x > -1$. For example, the Fenchel transform of $v(x) = -(D_L x^+ - D_H\,(-x)^+)$ is $\beta_T(y) = l^{\mathcal{D}}(-y)$, where $l^{\mathcal{D}}$ is the convex indicator function of the set $\mathcal{D} = [D_L, D_H]$. While $V_x > -1$ is a necessary condition to obtain a cash sub-additive functional, the decreasing monotonicity ($V_x < 0$) and convexity of V insure the convexity and decreasing monotonicity of $\rho_0(-V)$.

Proposition 4.6 *Let V be a random convex function as above and β_T the convex Fenchel transform of V. Let ρ_0 be a cash additive risk measure defined on $\mathcal{X}$ with minimal penalty function α_0. $\mathcal{R}^{\rho_0,V}(X_T) := \rho_0(-V(X_T))$ is a cash sub-additive risk measure. $\mathcal{R}^{\rho_0,V}(X_T)$ can be represented in the following two forms:*

$$\mathcal{R}^{\rho_0,V}(X_T) = \sup_{D_T\in\mathcal{X}} \big\{\rho_0\left(D_T\, X_T + \beta_T(-D_T)\right) \,|\, 0 \le D_T \le 1\big\}, \tag{4.3}$$

$$\mathcal{R}^{\rho_0,V}(X_T) = \sup_{Q_0\in\mathcal{M}_{1,f}} \big\{\mathbb{E}_{Q_0}[V(X_T)] - \alpha^{\rho_0,V}(Q_0)\big\}. \tag{4.4}$$

Moreover, $\mathcal{R}^{\rho_0,V}(X_T) = \rho_0(-V(X_T))$ admits the following representation in term of set functions

$Q \in \mathcal{M}_{1,f}(\mathcal{F}_T)$ *and* $\mathcal{F}_T$*-random variables* $D_T \in [0,1]$,

$$\mathcal{R}^{\rho_0,V}(X_T) = \sup_{Q_0 \in \mathcal{M}_{1,f},\, D_T \in \mathcal{X}} \Big\{ \mathbb{E}_{Q_0}[-D_T\, X_T] - \alpha^{\rho_0,V}(Q_0\,, D_T) |\ 0 \le D_T \le 1 \Big\} \tag{4.5}$$

$$\alpha^{\rho_0,V}(Q_0, D_T) :=\ \alpha_0(Q_0) + \mathbb{E}_{Q_0}[\beta_T(-D_T)]. \tag{4.6}$$

For instance, if ρ_0 is the coherent worst case risk measure, that is, $\rho_0(X_T) = \rho_{max}(X_T) = \sup_{Q \in \mathcal{M}_{1,f}} \mathbb{E}_{Q_0}[-X_T]$, then $\mathcal{R}^{\rho_0,V}(X_T) = \rho_{max}(-V(X_T)) = \| - V(X_T)\|_\infty$ and $\alpha^{\rho_0,V}(Q_0, D_T) := \mathbb{E}_{Q_0}[\beta(-D_T)]$.

Remark 4.7 *Allowing for different specifications of the underlying asset model and of the discount factor, representation (4.5)–(4.6) provides a better understanding of the different risks involved in the evaluation of the risky position* X_T*. The scenarios could be exogenously determined, for instance by some regulatory institution. The penalty function* $\alpha^{\rho_0,V}$ *depending on the ambiguous model and ambiguous discount factor could be determined by the preferences of the economic agent on* Q_0 *and* D_T*.*

Remark 4.8 Robust expected utility and cash sub-additive risk measures. *Consider a concave utility function* U *such that* $U_x \in [0,1]$*. From equation (4.4) it follows that the robust expected utilities,* $\sup_{Q_0 \in \mathcal{M}_{1,f}} \big\{ \mathbb{E}_{Q_0}[-U(X_T)] - \alpha^{\rho_0,U}(Q_0) \big\}$ *are cash sub-additive risk measures as functions of* X_T*. Notice that* U *does not satisfies the Inada conditions. For robust robust expected utility see, for instance, Schied (2004) and Maccheroni, Marinacci, and Rustichini (2004).*

Proof. Firstly we prove that $\mathcal{R}^{\rho_0,V}$ is a cash sub-additive risk measure. *Decreasing monotonicity:* The increasing monotonicity of $-V$ and the decreasing monotonicity of ρ_0 imply the decreasing monotonicity of $\mathcal{R}^{\rho_0,V}$.
Convexity: The concavity of $-V$, the decreasing monotonicity and the convexity of ρ_0 imply the convexity of $\mathcal{R}^{\rho_0,V}$.
Cash sub-additivity: $\mathcal{R}^{\rho_0,V}(X_T + m) + m = \rho_0(-V(X_T + m1_T)) + m = \rho_0(-V(X_T + m1_T) - m)$ is increasing in m if $-V(X_T + m1_T) - m$ is decreasing in m. As $V_x > -1$ the result follows.
Representations: To prove (4.3) we observe that

$$\rho_0(-V(X_T)) = \rho_0\big(\inf_{-1 \le y \le 0} \{-X_T y + \beta_T(y)\}\big) = \rho_0\big(\inf_{D_T \in \mathcal{X}} \{D_T X_T + \beta_T(-D_T)\,|\, 0 \le D_T \le 1\}\big).$$

From the decreasing monotonicity of ρ_0, for any $\widetilde{D}_T \in \mathcal{X},\ 0 \le \widetilde{D}_T \le 1$ we have

$$\rho_0\big(\inf_{D_T\in\mathcal{X}}\{D_T X_T + \beta_T(-D_T)\,|\, 0 \le D_T \le 1\}\big) \ge \rho_0\left(\widetilde{D}_T X_T + \beta_T(-\tilde{D}_T)\right). \tag{4.7}$$

The result follows setting $\widetilde{D}_T = D_T^*$ in equation (4.7), where $D_T^* \in \mathcal{X}$ is the element achieving the $\inf_{D_T\in\mathcal{X}}\{D_T X_T + \beta_T(-D_T)\,|\, 0 \le D_T \le 1\}$. Finally, (4.4) is obtained from the dual representation of ρ_0. (4.5)–(4.6) are obtained from the dual representation of ρ_0 and from (4.3). □

The penalty function $\alpha^{\rho_0,v}$ in (4.6) is not the minimal one. As any pair (Q_0, D_T) defines a unique additive set function μ absolutely continuous with respect to Q_0, $d\mu := D_T dQ_0$, $0 \le \mu(\Omega) \le 1$, the functional $\mathcal{R}^{\rho_0,V}$ can be rewritten as

$$\mathcal{R}^{\rho_0,V}(X_T) = \sup_{\mu\in\mathcal{M}_{1,f}(\mathcal{F}_T)} \Big\{\mu(-X_T) - \gamma(\mu)\,|\, 0 \le \mu(\Omega) \le 1\Big\},$$

where $\mu(-X_T) := \int -X_T(\omega)\mu(d\omega)$ and $\gamma(\mu) = \inf_{Q_0\in\mathcal{M}_{1,f}} \left\{\alpha_0(Q_0) + \mathbb{E}_{Q_0}\left[\beta_T\left(-\frac{d\mu}{dQ_0}\right)\right]\right\}$ for any μ such that $d\mu = D_T dQ_0$, $0 \le D_{0,T} \le 1$, and $\gamma = \infty$ otherwise.

Next section gives the dual representation of the cash sub-additive risk measures $\mathcal{R}$ in terms of the minimal penalty function.

5 Minimal cash additive extension of $\mathcal{R}$ and duality

Considering a minimal enlargement of the sample space Ω we define a cash additive risk measure which is in a one to one correspondence with the cash sub-additive risk measure $\mathcal{R}$. This relation provides for a rich interpretation of both cash additive and cash sub-additive risk measures. Moreover this one to one correspondence allows to exploit all the results on the cash additive risk measures to derive the corresponding properties of $\mathcal{R}$. For instance, the dual representation of cash sub-additive risk measures is obtained using the dual representation of cash additive risk measures.

A simple procedure to obtain a cash additive risk measure using a cash sub-additive risk measure $\mathcal{R}$ is as follows. While $\mathcal{R}$ is not cash additive with respect to $X_T \in \mathcal{X}$, the bivariate function $\hat{\rho}(X_T, x) := \mathcal{R}(X_T 1_T - x 1_T) - x$ as a function of the pair (X_T, x) is cash additive. In the sequel we introduce the minimal measurable space where the pair (X_T, x) is the coordinate of a random variable and $\hat{\rho}$ is a cash additive risk measure on these random variables.

5.1 The enlarged space $\widehat{\mathcal{X}}$

Any pair (X_T, x) where $X_T \in \mathcal{X}$ and $x \in \mathbb{R}$ can be viewed as the coordinates of a function defined on the enlarged space $\widehat{\Omega} = \Omega \times \{0,1\}$ with element (ω, θ),

$$\widehat{X}_T(\omega,\theta) := X_T(\omega)1_{\{\theta=1\}} + x1_{\{\theta=0\}}.$$

We endow $\widehat{\Omega}$ with the σ-algebra $\widehat{\mathcal{F}}_T$ generated by the bounded random variables $\widehat{X}_T$. Notice that $\widehat{\mathcal{F}}_T$ is not the product σ-algebra. Let $\widehat{\mathcal{X}}$ be the linear space of all bounded random variable $\widehat{X}_T$. To denote $\widehat{X}_T \in \widehat{\mathcal{X}}$ we use its coordinates $\widehat{X}_T = (X_T, x)$. The constant variables are denoted by $\widehat{m} = (m,m)$ and $\widehat{m} = m1_{\{\theta=1\}} + m1_{\{\theta=0\}} = m$. The event $\{\theta = 1\}$ models the risk affecting the numéraire 1_T. Intuitively, θ is associated with the default time τ of the counterpart. The event $\{\theta = 1\}$ is equivalent to $\{\tau > T\}$. The event $\{\theta = 0\}$ is atomic and all $\widehat{\mathcal{F}}_T$-random variables are constant on this event.

We focus on the normalized finitely additive set functions $\widehat{Q}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}}_T)$. We recall that $\mathcal{M}_{1,f}(\widehat{\mathcal{F}}_T)$ denotes the set of all additive set functions normalized to one on $\widehat{\mathcal{F}}_T$. We formally define the finitely additive sub-probability measures μ on $\mathcal{F}_T$ introduced at the end of the previous section.

Definition 5.1 *A finitely additive sub-probability measure is an additive set function $\mu : \mathcal{F}_T \to \mathbb{R}^+$ such that $0 \leq \mu(\Omega) \leq 1$. $\mathcal{M}_{s,f}(\mathcal{F}_T)$ denotes the set of all finitely additive sub-probability measures and $\mathcal{M}_s(\mathcal{F}_T) \subseteq \mathcal{M}_{s,f}(\mathcal{F}_T)$ the set of σ-additive sub-probability measures.*

The choice of the minimal filtration $\widehat{\mathcal{F}}_T$ implies a one to one correspondence between normalized additive set function $\widehat{Q}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}}_T)$ and sub-probability measure on $(\Omega, \mathcal{F}_T)$. Indeed, any $\widehat{Q}$ in $\mathcal{M}_{1,f}(\widehat{\mathcal{F}}_T)$ can be decomposed as follows, $\forall \widehat{X}_T = (X_T, x) \in \widehat{\mathcal{X}}$,

$$\widehat{Q}(\widehat{X}_T) = \widehat{Q}(X_T 1_{\theta=1}) + x\widehat{Q}(1_{\theta=0}) = \mu(X_T) + x(1-\mu(1)), \tag{5.1}$$

where $\mu(\cdot) := \widehat{Q}(\cdot 1_{\theta=1})$ is an additive sub-probability of $\mathcal{M}_{s,f}(\mathcal{F}_T)$. Equation (5.1) shows how to obtain a probability measure $\widehat{Q}$ from a sub-probability measure μ and vice versa.

5.2 Minimal extension of $\mathcal{R}$ into a cash additive risk measure

The following proposition shows that $\hat{\rho}$ on $\widehat{\mathcal{X}}$ is a cash additive risk measure and that $\hat{\rho}$ and $\mathcal{R}$ are in a one to one correspondence.

Proposition 5.2 *1) A normalized cash sub-additive risk measure $\mathcal{R}$ on $\mathcal{X}$ defines a normalized cash additive risk measure $\widehat{\rho}$ on $\widehat{\mathcal{X}}$,*

$$(5.2) \qquad \forall\ \widehat{X}_T = (X_T, x) \in \widehat{\mathcal{X}}, \quad \hat{\rho}(\widehat{X}_T) := \hat{\rho}\big((X_T, x)\big) := \mathcal{R}\big(X_T - x1_T\big) - x.$$

Notice that $\hat{\rho}(X_T 1_{\{\theta=1\}}) = \mathcal{R}(X_T)$.

2) Any cash additive risk measure on $\widehat{\mathcal{X}}$ restricted to the event $\{\theta = 1\}$ defines a cash sub-additive risk measure which satisfies equation (5.2).

Remark 5.3 *The cash sub-additive risk measure $\mathcal{R}$ can be used to measure the risk of defaultable contingent claims $\widehat{X}_T$ when there is no compensation ($x = 0$) if the default occurs, $\{\theta = 0\}$.*

The proof relies on the cash sub-additive property to obtain a monotone decreasing functional.
Proof. 1) Cash additive: Let $\widehat{X}_T = (X_T, x) \in \widehat{\mathcal{X}}$ and $m \in \mathbb{R}$. By definition, $\hat{\rho}\big(X_T 1_{\theta=1} + x1_{\theta=0} + m1_{\theta=1} + m1_{\theta=0}\big) = \mathcal{R}\big(X_T + m1_T - (x+m)1_T\big) - (x+m) = \hat{\rho}\big(\widehat{X}_T\big) - m$.
Decreasing monotonicity: Let $\widehat{X}_T = (X_T, x)$ and $\widehat{Y}_T = (Y_T, y) \in \widehat{\mathcal{X}}$ such that $\widehat{X}_T \geq \widehat{Y}_T$, that is $X_T \geq Y_T$ and $x \geq y$. From the cash sub-additivity and the decreasing monotonicity of $\mathcal{R}$ it follows that $\hat{\rho}(\widehat{X}_T) = \mathcal{R}(X_T - x1_T) - x \leq \mathcal{R}(X_T - y1_T) - y \leq \mathcal{R}(Y_T - y1_T) - y = \hat{\rho}(\widehat{Y}_T)$.
Convexity: We use the notation in equation (2.1). From the convexity of $\mathcal{R}$, $\mathcal{R}\,(Bar[X_I]) \leq Bar\,[\mathcal{R}(X)_I]$. This implies that $\hat{\rho}(Bar[\widehat{X}_I]) = \mathcal{R}\,\big(Bar\big[X_I - x_I\big]\big) - Bar[x_I] \leq Bar\,\big[\mathcal{R}\big(X - x\big)_I\big] - Bar[x_I] = Bar\left[\hat{\rho}(\widehat{X})_I\right]$, which shows the convexity of $\hat{\rho}$.
2) Let $\check{\rho}$ be a cash additive risk measure on $\widehat{\mathcal{X}}$. We have to show that $\mathcal{R}^{\check{\rho}}(X_T) := \check{\rho}(X_T 1_{\theta=1})$ is a cash sub-additive risk measure. The decreasing monotonicity and convexity follow from the definition. The cash sub-additive property is verified observing that $\mathcal{R}^{\check{\rho}}(X_T + m1_T) + m = \check{\rho}((X_T + m)1_{\theta=1}) + m = \check{\rho}(X_T 1_{\theta=1} - m1_{\theta=0})$ is increasing in m. □

5.3 Dual representation of cash sub-additive risk measures

In the next proposition we use the one to one correspondence in equation (5.2) between $\hat{\rho}$ and $\mathcal{R}$ to characterize cash sub-additive risk measures. We show that the minimal penalty function of $\mathcal{R}$ and the minimal penalty function $\hat{\rho}$ coincide and are concentrated on the set of sub-probability measures $\mathcal{M}_{s,f}(\mathcal{F}_T)$. Moreover, under the additional assumption of continuity from below of

$\mathcal{R}$ the dual representation in terms of σ-additive sub-probability measures is obtained. The same results could be derived using convex analysis tools, however our approach has a richer interpretation.

Theorem 5.4 *(a) Any cash sub-additive risk measure $\mathcal{R}$ on $\mathcal{X}$ can be represented in terms of finitely additive sub-probability measures,*

$$\forall X_T \in \mathcal{X}, \quad \mathcal{R}(X_T 1_T) = \sup_{\mu \in \mathcal{M}_{s,f}(\mathcal{F}_T)} \left\{ \mu(-X_T) - \alpha^{\mathcal{R}}(\mu) \right\}, \quad \alpha^{\mathcal{R}}(\mu) := \hat{\alpha}(\widehat{Q}), \tag{5.3}$$

where $\mu(\cdot) = \widehat{Q}(\cdot 1_{\theta=1})$ and $\hat{\alpha}$ is any penalty function representing $\hat{\rho}$. In particular, if $\hat{\alpha}$ is the minimal penalty function for $\hat{\rho}$ then $\alpha^{\mathcal{R}}$ is the minimal penalty function for $\mathcal{R}$ and $\alpha^{\mathcal{R}}(\mu) =$ $\sup_{X_T \in \mathcal{X}} \{\mu(-X_T) - \mathcal{R}(X_T)\}$.

(b) When $\mathcal{R}$ is a cash sub-additive risk measure continuous from below any penalty function β representing $\mathcal{R}$ is concentrated on the class $\mathcal{M}_s(\mathcal{F}_T)$ of σ-additive sub-probability measures, i.e., $\beta(\mu) < \infty \Rightarrow \mu$ is σ-additive.

Proof. *(a)* From Proposition 5.2, $\mathcal{R}(X_T 1_T) = \hat{\rho}(X_T 1_{\theta=1})$. Equation (5.3) is implied by the dual representation of $\hat{\rho}$ and the one to one correspondence between $\widehat{Q}$ and μ: $\widehat{Q}(\cdot 1_{\theta=1}) = \mu(\cdot)$. Let $\hat{\alpha}$ be the minimal penalty function of $\hat{\rho}$. By definition of the minimal penalty function,

$$\begin{aligned}
\hat{\alpha}(\widehat{Q}) &= \sup_{\widehat{X}_T \in \widehat{\mathcal{X}}_T} \left\{ \mathbb{E}_{\widehat{Q}}[-X_T 1_{\theta=1} - x 1_{\theta=0}] - \hat{\rho}(\widehat{X}_T) \right\} \\
&= \sup_{\widehat{X}_T \in \widehat{\mathcal{X}}} \left\{ \mathbb{E}_{\widehat{Q}}[-(X_T - x) 1_{\theta=1}] - x - \mathcal{R}(X_T - x 1_T) + x \right\} \\
&= \sup_{X_T \in \mathcal{X}_T} \left\{ \mathbb{E}_{\widehat{Q}}[-(X_T) 1_{\theta=1}] - \mathcal{R}(X_T) \right\}, \quad \widehat{Q} \in \mathcal{M}_{1,f}(\widehat{\mathcal{F}}_T).
\end{aligned} \tag{5.4}$$

As $\widehat{Q}(\cdot 1_{\theta=1}) = \mu(\cdot)$, from equation (5.4) we have $\alpha^{\mathcal{R}}(\mu) := \hat{\alpha}(\widehat{Q}) = \sup_{X_T \in \mathcal{X}} \{\mu(-X_T) - \mathcal{R}(X_T)\}$, showing that $\alpha^{\mathcal{R}}$ is the minimal penalty function for $\mathcal{R}$.

(b) If $\mathcal{R}$ is continuous from below the cash additive $\hat{\rho}$ is continuous from below as a function of $\widehat{X}_T = (X_T, x)$. Then from Theorem 2.2 follows that the penalty function of $\hat{\rho}$ is concentrated on the class $\mathcal{M}_1(\widehat{\mathcal{F}}_T)$. This implies that the penalty function of $\mathcal{R}$ is concentrated on the set of σ-additive sub-probability $\mathcal{M}_s(\mathcal{F}_T)$. □

6 Other cash additive extensions of $\mathcal{R}$

The cash additive risk measure $\hat{\rho}$ in equation (5.2) defined using $\mathcal{R}$ cannot asses the risk of $\mathcal{F}_T$-random variables $X_T \in \mathcal{X}$ as $\widehat{\mathcal{X}}$ does not contain $\mathcal{X}$, the domain of definition of $\mathcal{R}$. Hence we extend $\mathcal{R}$ to a larger space which contains $\mathcal{X}$ and we study a dual representation of $\mathcal{R}$ in the enlarged space. For the cash sub-additive risk measures generated via convex functions (introduced in Section 4.3) we propose another extension on the same enlarged space obtained via a conditional risk measure.

6.1 The enlarged space $\widetilde{\mathcal{X}}$

To define a linear space which contains $\mathcal{X}$, the σ-algebra $\widehat{\mathcal{F}}_T$ defined in Section 5.1 is replaced by the product σ-algebra $\mathcal{G}_T$. On $(\Omega \times \{0,1\}, \mathcal{G}_T)$ any bounded $\mathcal{G}_T$-random variable $\widetilde{X}_T$ can be represented as $\widetilde{X}_T(\omega,\theta) = X^1_T(\omega)1_{\theta=1} + X^0_T(\omega)1_{\theta=0}$ and $X^0_T, X^1_T \in \mathcal{X}$. Let $\widetilde{\mathcal{X}}$ be the linear space of all the bounded $\mathcal{G}_T$-random variables $\widetilde{X}_T$. We refer to $\widetilde{X}_T$ using the short notation $\widetilde{X}_T = (X^1_T, X^0_T)$. The diagonal elements $\widetilde{X}_T = (X_T, X_T)$ coincide with X_T and the corresponding σ-algebra with $\mathcal{F}_T$. This identification was not possible for the random variables $\widehat{X} = (X_T, x)$ defined in the previous section.

Now we discuss the probabilistic structure of $(\Omega \times \{0,1\}, \mathcal{G}_T)$. Notice that in this section we consider probability measures and not finite additive set functions.

Definition 6.1 *For any given probability measure $\widetilde{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T)$ let $\mathbb{Q}$ denote the restriction of $\widetilde{\mathbb{Q}}$ to $\mathcal{F}_T$, $\mathbb{Q} := \widetilde{\mathbb{Q}}|\mathcal{F}_T$, and $D_T \in [0,1]$ the $\mathcal{F}_T$-conditional probability of the event $\{\theta = 1\}$, $D_T := \mathbb{E}_{\widetilde{\mathbb{Q}}}[1_{\theta=1}|\mathcal{F}_T]$, also called discount factor. We denote $\overline{\mathbb{Q}}$ the probability measure associated with the restriction of $\widetilde{\mathbb{Q}}$ to the event $\{\theta = 0\}$, which is uniquely determined by $(\mathbb{Q}, D_T)$*

$$\mathbb{Q}(X_T) = \mathbb{Q}(D_T X_T) + (1 - \mathbb{Q}(D_T))\overline{\mathbb{Q}}(X_T). \tag{6.1}$$

$\overline{\mathbb{Q}}$ is a probability measure absolutely continuous with respect to $\mathbb{Q}$, with Radon-Nikodym density given by $\overline{\Delta}_T := \frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} = \frac{1-D_T}{(1-\mathbb{Q}(D_T))}$, $0 \le \overline{\Delta}_T \le 1$, $\mathbb{Q}(\overline{\Delta}_T) = 1$.

For any $\widetilde{X}_T = X^1_T 1_{\theta=1} + X^0_T 1_{\theta=0} \in \widetilde{\mathcal{X}}$,

$$\widetilde{\mathbb{Q}}(X^1_T 1_{\theta=1} + X^0_T 1_{\theta=0}) = \mathbb{Q}(X^1_T D_T) + \mathbb{Q}(X^0_T(1 - D_T)) = \mathbb{Q}(X^1_T D_T) + (1 - \mathbb{Q}(D_T))\overline{\mathbb{Q}}(X^0_T). \tag{6.2}$$

Remark 6.2 The interpretation of D_T in credit risk. *In credit risk, θ is associated with the default time of the counterpart τ, where τ is a positive random variable not $\mathcal{F}_T$-measurable. The event $\{\theta = 1\}$ can be viewed as $\{\tau > T\}$ and $\mathbb{E}_{\widetilde{\mathbb{Q}}}[1_{\theta=1}|\mathcal{F}_T]$ as the conditional survival probability function of τ at time T. $\widetilde{X}_T = X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0} \in \widetilde{\mathcal{X}}$ is a defaultable contingent claim that pays X_T^1 (at time T) if there is no default ($\tau > T$) and X_T^0 otherwise.*

6.2 Cash sub-additive risk measures and ambiguous discounted factors

In the sequel we define a cash additive risk measure on the enlarged space $\widetilde{\mathcal{X}}$. Using its dual representation, any cash sub-additive risk measure is represented in terms of the ambiguous probability model and the ambiguous discount factor both on the original space of definition of $\mathcal{R}$. This representation is similar to the dual representation (see equations (4.5)–(4.6)) of cash sub-additive risk measures generated by convex functions.

To define this cash additive risk measure on $\widetilde{\mathcal{X}}$ we use, as in Section 5.2, the cash additive risk measure $\hat{\rho}$ in (5.2). In this case $\widetilde{X}_T = (X_T^1, X_T^0) \in \widetilde{\mathcal{X}}$ has two risky components and we introduce an a priori risk measure $\overline{\rho}$ assessing the risk of the second component.

Definition 6.3 *Let $\mathcal{R}$ be a cash sub-additive risk measure and $\overline{\rho}$ a cash additive risk measure both normalized and defined on $\mathcal{X}$. The functional on $\widetilde{\mathcal{X}}$*

$$\tilde{\rho}(\widetilde{X}_T) = \tilde{\rho}(X_T^1, X_T^0) := \mathcal{R}(X_T^1 + \overline{\rho}(X_T^0)1_T) + \overline{\rho}(X_T^0) = \hat{\rho}\big(X_T^1, -\overline{\rho}(X_T^0)\big) \tag{6.3}$$

and its restriction on $\mathcal{X}$,

$$\rho_{\mathcal{R},\overline{\rho}}(X_T) := \mathcal{R}(X_T + \overline{\rho}(X_T)1_T) + \overline{\rho}(X_T) = \hat{\rho}\big(X_T, -\overline{\rho}(X_T)\big), \tag{6.4}$$

are cash additive risk measures. Moreover, $\mathcal{R}(X_T 1_T) = \tilde{\rho}(X_T 1_{\theta=1})$.

The following theorem shows that $\mathcal{R}$ can be written as a function of probability measures $\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)$ and $\mathcal{F}_T$-measurable discount factors $D_T \in \mathcal{X}$ using the minimal penalty function of the cash additive risk measure $\tilde{\rho}$. We consider penalty functions concentrated on the class of probabilities measures assuming that $\mathcal{R}$ and $\overline{\rho}$ are continuous from below. This implies that also $\tilde{\rho}$ and $\rho_{\mathcal{R},\overline{\rho}}$ are continuous from below.

Theorem 6.4 *Assume that the convex functionals $\mathcal{R}$ and $\overline{\rho}$ are continuous from below. Let $\alpha^{\mathcal{R}}$ and $\overline{\alpha}$ be the minimal penalty functions of $\mathcal{R}$ and $\overline{\rho}$, respectively. Let $\tilde{\alpha}$ be the minimal penalty function of $\tilde{\rho}$ defined in equation (6.3). For any $\widetilde{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T)$, let $\mathbb{Q}$, D_T and $\overline{\mathbb{Q}}$ be as in Definition 6.1, such that $\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} = \overline{\Delta}_T = \frac{1-D_T}{(1-\mathbb{Q}(D_T))}$.*

1) The cash sub-additive risk measure $\mathcal{R}$ can be represented as

$$\mathcal{R}(X_T) = \tilde{\rho}(X_T 1_{\theta=1}) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathcal{F}_T),\, D_T\in[0,1]} \big\{\mathbb{E}_{\mathbb{Q}}(-D_T X_T) - \tilde{\alpha}\big(D_T, \mathbb{Q}\big)\big\}, \tag{6.5}$$

where the minimal penalty $\tilde{\alpha}$ has the following form

$$\tilde{\alpha}(\widetilde{\mathbb{Q}}) = \tilde{\alpha}(\mathbb{Q}, D_T) = \alpha^{\mathcal{R}}(D_T\cdot\mathbb{Q}) + (1-\mathbb{Q}(D_T))\,\overline{\alpha}(\overline{\mathbb{Q}}), \quad \widetilde{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T). \tag{6.6}$$

Notice that $\widetilde{\mathbb{Q}} \in \mathcal{D}om(\tilde{\alpha})$ if and only if $\mathbb{Q}\cdot D_T \in \mathcal{D}om(\alpha^{\mathcal{R}})$ and $\overline{\mathbb{Q}} \in \mathcal{D}om(\overline{\alpha})$.

2) The minimal penalty function of $\rho_{\mathcal{R},\overline{\rho}}$ in equation (6.4) is given by, for any $\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)$,

$$\alpha_{\mathcal{R},\overline{\rho}}(\mathbb{Q}) = \inf_{D_T,\overline{\mathbb{Q}}} \big\{\alpha^{\mathcal{R}}(D_T\cdot\mathbb{Q}) + (1-\mathbb{Q}(D_T))\,\overline{\alpha}(\overline{\mathbb{Q}}) \;\big|\; \mathbb{Q}(\cdot) = \mathbb{Q}(D_T\cdot) + (1-\mathbb{Q}(D_T))\overline{\mathbb{Q}}(\cdot)\big\}. \tag{6.7}$$

Remark 6.5 *When $\mathcal{R}$ and $\overline{\rho}$ are both coherent risk measures, equation (6.5) reduces to*

$$\mathcal{R}(X_T) = \tilde{\rho}(X_T 1_{\theta=1}) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathcal{F}_T),\, D_T\in[0,1]} \big\{\mathbb{E}_{\mathbb{Q}}(-D_T X_T) | D_T\cdot\mathbb{Q} \in \mathcal{D}om(\alpha^{\mathcal{R}}), \overline{\Delta}_T\cdot\mathbb{Q} \in \mathcal{D}om(\overline{\alpha})\big\}.$$

Proof. 1) The representation (6.5) of $\mathcal{R}$ follows from $\mathcal{R}(X_T 1_T) = \tilde{\rho}(X_T 1_{\theta=1})$ and equation (6.2). To obtain the decomposition of the minimal penalty function in equation (6.6) we use the the representation of $\widetilde{\mathbb{Q}}$ in terms of $\mathbb{Q}(D_T\cdot)$ and $\overline{\mathbb{Q}}$ given in definition 6.1. From the definition of $\tilde{\rho}$ and of the minimal penalty function we have

$$\begin{aligned}
\tilde{\alpha}(\widetilde{\mathbb{Q}}) &= \sup_{(X_T^1, X_T^0)\in\tilde{\mathcal{X}}} \Big\{\widetilde{\mathbb{Q}}(-X_T^1 1_{\theta=1} - X_T^0 1_{\theta=0}) - \mathcal{R}(X_T^1 + \overline{\rho}(X_T^0)1_T) - \overline{\rho}(X_T^0)\Big\} \\
&= \sup_{(X_T^1, X_T^0)\in\tilde{\mathcal{X}}} \Big\{\widetilde{\mathbb{Q}}(-(X_T^1 + \overline{\rho}(X_T^0))1_{\theta=1}) - \mathcal{R}(X_T^1 + \overline{\rho}(X_T^0)1_T) + \widetilde{\mathbb{Q}}(-(X_T^0 + \overline{\rho}(X_T^0))1_{\theta=0})\Big\}.
\end{aligned}$$

Using the change of variable $Y_T := X_T^1 + \overline{\rho}(X_T^0)$ and equations (6.1)–(6.2) give the result

$$\begin{aligned}
\tilde{\alpha}(\widetilde{\mathbb{Q}}) &= \sup_{(Y_T, X_T^0)} \big\{\mathbb{Q}(-Y_T\, D_T) - \mathcal{R}(Y_T) + (1-\mathbb{Q}(D_T))\big[\overline{\mathbb{Q}}(-(X_T^0 + \overline{\rho}(X_T^0))\big]\big\} \\
&= \alpha^{\mathcal{R}}(D_T\cdot\mathbb{Q}) + (1-\mathbb{Q}(D_T)) \sup_{X\in\mathcal{A}^{\overline{\rho}}} \big\{\mathbb{Q}(-X\overline{\Delta}_T)\big\} \\
&= \alpha^{\mathcal{R}}(D_T\cdot\mathbb{Q}) + (1-\mathbb{Q}(D_T))\,\overline{\alpha}(\overline{\Delta}_T\cdot\mathbb{Q}).
\end{aligned}$$

2) To obtain the penalty function $\alpha_{\mathcal{R},\overline{\rho}}$ of $\rho_{\mathcal{R},\rho_0}$ we restrict $\tilde{\rho}$ on $\mathcal{F}_T$ and we use equation (6.1)

$$\begin{aligned}
\rho_{\mathcal{R},\rho_0}(X_T) &= \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathcal{F}_T)} \big\{\mathbb{Q}(-X_T D_T) + (1-\mathbb{Q}(D_T))\overline{\mathbb{Q}}(-X_T) \\
&\qquad -\big(\alpha^{\mathcal{R}}(D_T\cdot\mathbb{Q}) + (1-\mathbb{Q}(D_T)\overline{\alpha}(\overline{\mathbb{Q}})\big)\big\} \\
&= \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathcal{F}_T)} \big\{\mathbb{Q}(-X_T) - \big(\alpha^{\mathcal{R}}(D_T\cdot\mathbb{Q}) + (1-\mathbb{Q}(D_T))\overline{\alpha}(\overline{\mathbb{Q}})\big)\big\}.
\end{aligned}$$

Observing that for a given $\mathbb{Q}\in\mathcal{M}_1(\mathcal{F}_T)$ more then one pair $(D_T,\overline{\mathbb{Q}})$, $D_T\in\mathcal{X}$, $D_T\in[0,1]$, can verify $\mathbb{Q}(-X_T D_T) + (1-\mathbb{Q}(D_T))\overline{\mathbb{Q}}(-X_T)) = \mathbb{Q}(X_T)$ yields the equation (6.7). Similar calculations show that $\alpha_{\mathcal{R},\overline{\rho}}$ is the minimal penalty function. □

6.3 Conditional risk measures and extensions on $\widetilde{\mathcal{X}}$

This section reinterprets the cash sub-additive risk measures $\mathcal{R}^{\rho,V} = \rho(-V)$ studied in Section 4.3. These risk measures are now represented as the composition of the unconditional cash additive risk measure ρ and the *conditional* cash additive risk measure generated by the random function V. We obtain the result introducing a more natural extension of $\mathcal{R}^{\rho,V}$ called $\check{\rho}^V$ to the enlarged space $\widetilde{\mathcal{X}}$. The restriction of $\check{\rho}^V$ to the space $\mathcal{X}$ is ρ itself, and $\check{\rho}^V$ can be obtained composing ρ with a cash additive conditional risk measures. Moreover, we show that any cash additive risk measure on $\widetilde{\mathcal{X}}$ generated from ρ via a conditional cash additive risk measure is associated to a cash sub-additive risk measure generated by a convex function.

As in Section 4.3, in the sequel ρ denotes a normalized cash additive risk measure and $V(\omega,x)$ an $\mathcal{F}_T$-measurable random functional convex monotone decreasing such that $V(0)=0$ and with left derivative $V_x\in[-1,0]$. From Proposition 4.6 we know that $\mathcal{R}^{\rho,V}(X_T) := \rho(-V(X_T))$ is a cash sub-additive risk measure on $\mathcal{X}$.

Proposition 6.6 *On the enlarged space $\widetilde{\mathcal{X}}$ any cash additive risk measure ρ and any random function V define a cash additive risk measure,*

$$(6.8)\quad \check{\rho}^V(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}) := \rho\big(-V(X_T^1 - X_T^0) + X_T^0\big),\quad \widetilde{X}_T = X_T^1 1_{\theta=1} + X_T^0\, 1_{\theta=0}\in\widetilde{\mathcal{X}}.$$

$\check{\rho}^V$ coincides with $\mathcal{R}^{\rho,V}$ on $\{\theta=1\}$ and with ρ on $\mathcal{X}\subset\widetilde{\mathcal{X}}$:

$$\check{\rho}^V(X_T 1_{\theta=1}) = \rho\big(-V(X_T)\big) = \mathcal{R}^{\rho,V}(X_T) \quad \textit{and} \quad \check{\rho}^V((X_T,X_T)) = \rho\big(X_T\big).$$

Requiring V decreasing monotone and such that $V_x \in [-1, 0]$ is crucial to obtain $\check{\rho}^V$ decreasing monotone (see proof below).

Proof. Decreasing monotonicity: $\check{\rho}^V$ is decreasing monotone if $V(X_T^1 - X_T^0) - X_0$ is decreasing monotone with respect to (X_T^1, X_T^0). Let $\widetilde{X}_T = (X_T^1, X_T^0) \geq \widetilde{Y}_T = (Y_T^1, Y_T^0)$, that is $X_T^1 \geq Y_T^1$ and $X_T^0 \geq Y_T^0$. As $V(x+m)+m$ is not decreasing in m, $V(X_T^1 - X_T^0) - X_T^0$ is not increasing in X_T^0, then $V(X_T^1 - X_T^0) - X_T^0 \leq V(X_T^1 - Y_T^0) - Y_T^0 \leq V(Y_T^1 - Y_T^0) - Y_T^0$, where the last inequality is due to the decreasing monotonicity of V.

Cash additivity and *convexity* follow from the definition of $\check{\rho}^V$. □

Now we recall the definition of conditional risk measures that in our setting[2] reads as follows.

Definition 6.7 *1) A cash additive conditional risk measure on $\mathcal{F}_T$ is a monotone decreasing convex functional, $\tilde{\rho}_{\mathcal{F}_T} : \widetilde{\mathcal{X}} \to \mathcal{X}$ which satisfies the $\mathcal{F}_T$-cash additive axiom, that is*

$\forall \widetilde{X} \in \widetilde{\mathcal{X}}, \forall Y \in \mathcal{X}, \quad \tilde{\rho}_{\mathcal{F}_T}(\widetilde{X} + Y) = \tilde{\rho}_{\mathcal{F}_T}(\widetilde{X}) - Y.$

2) $\tilde{\rho}_{\mathcal{F}_T}$ is regular if for any $F_T \in \mathcal{F}_T$, $\widetilde{X}_T \in \widetilde{\mathcal{X}}$, $\tilde{\rho}_{\mathcal{F}_T}(1_{F_T}\widetilde{X}_T) = 1_{F_T}\tilde{\rho}_{\mathcal{F}_T}(\widetilde{X}_T)$.

3) A cash additive risk measure $\check{\rho}$ on $\widetilde{\mathcal{X}}$ is generated from ρ via a conditional risk measure if there exists a cash additive conditional risk measure on $\mathcal{F}_T$, $\tilde{\rho}_{\mathcal{F}_T}$ such that, $\check{\rho}(X_T^1, X_T^0) = \rho(-\tilde{\rho}_{\mathcal{F}_T}((X_T^1, X_T^0))$.

It easy to see that any conditional risk measure on $\mathcal{F}_T$ is completely determined by its value on the set $\{\theta = 1\}$. This observation leads to the following proposition.

Proposition 6.8 *Any $\mathcal{F}_T$-measurable random function V defines a cash additive conditional risk measure on $\mathcal{F}_T$, $\tilde{\rho}^V_{\mathcal{F}_T} : \widetilde{\mathcal{X}} \to \mathcal{X}$, given by*

$$(6.9) \quad \tilde{\rho}^V_{\mathcal{F}_T}(X_T 1_{\theta=1}) := V(X_T) \quad \text{or equivalently by} \quad \tilde{\rho}^V_{\mathcal{F}_T}((X_T^1, X_T^0)) := V(X_T^1 - X_T^0) - X_T^0.$$

Conversely, any regular and continuous from above cash additive conditional risk measure on $\mathcal{F}_T$, $\tilde{\rho}_{\mathcal{F}_T} : \widetilde{\mathcal{X}} \to \mathcal{X}$, generates a convex random function $\widetilde{V}^{\mathcal{F}_T}(\lambda) := \tilde{\rho}_{\mathcal{F}_T}(\lambda 1_{\theta=1})$ which satisfies (6.9).

Proof. Decreasing monotonicity: We refer the reader to the proof of decreasing monotonicity in Proposition 6.6. *$\mathcal{F}_T$-cash invariance* and *convexity* follow respectively from the definition of $\tilde{\rho}^V_{\mathcal{F}_T}$

[2] For conditional risk measures see Bion-Nadal (2004), Detlefsen and Scandolo (2005) and references therein.

and the convexity of V.

Conversely: Define $\widetilde{V}^{\mathcal{F}_T}(\omega,\lambda) := \tilde{\rho}_{\mathcal{F}_T}(\lambda 1_{\theta=1}(\omega))$. $V(\omega,\lambda)$ is $\mathcal{F}_T$-measurable convex and monotone decreasing functional such that $\widetilde{V}^{\mathcal{F}_T}(0) = 0$ and $\widetilde{V}^{\mathcal{F}_T} \in [-1, 0]$. For the regularity of $\tilde{\rho}_{\mathcal{F}_T}$ the previous definition can be extended to all the simple $\mathcal{F}_T$-random variables $\sum \lambda_i 1_{A_i}$, where the sets $A_i \in \mathcal{F}_T$ and $\{A_i\}_{i=1,\dots,n}$ form a partition of Ω. Hence $\tilde{\rho}_{\mathcal{F}_T}(\sum \lambda_i 1_{A_i}) = \sum 1_{A_i} \widetilde{V}^{\mathcal{F}_T}(\lambda_i)$. The continuity from above of $\tilde{\rho}_{\mathcal{F}_T}$ allows to extend the definition to positive $X_T \in \mathcal{X}$ and then to any arbitrary $X_T \in \mathcal{X}$ using standard analysis tools. □

The following theorem states the main result of this section showing that any cash sub-additive risk measure of the form $\mathcal{R}^{\rho,V} = \rho(-V)$ can be extended into a cash additive risk measure which is generated from ρ via a conditional risk measure. Conversely, any cash additive risk measure $\check{\rho}$ on $\widetilde{\mathcal{X}}$ generated from ρ via a conditional risk measure is associated to a cash sub-additive risk measure of type $\mathcal{R}^{\rho,\check{V}_{\mathcal{F}_T}}$.

Theorem 6.9 *The cash additive risk measure $\check{\rho}^V$ in equation (6.8) is generated from ρ via the conditional risk measure $\tilde{\rho}^V_{\mathcal{F}_T}$ in (6.9) associated with V, that is*

$$\check{\rho}^V(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}) = \rho\big(-V(X_T^1 - X_T^0) + X_0\big) = \rho\left(-\tilde{\rho}^V_{\mathcal{F}_T}(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0})\right). \tag{6.10}$$

Moreover,

$$\mathcal{R}^{\rho,V}(X_T) = \check{\rho}^V(X_T 1_{\theta=1}) = \rho\left(-\tilde{\rho}^V_{\mathcal{F}_T}(X_T 1_{\theta=1})\right). \tag{6.11}$$

Conversely, to any cash additive risk measure $\check{\rho}(\cdot) = \rho(-\tilde{\rho}_{\mathcal{F}_T}(\cdot))$ on $\widetilde{\mathcal{X}}$ generated by a cash additive conditional risk measure $\tilde{\rho}_{\mathcal{F}_T}$ on $\mathcal{F}_T$ is associated a cash sub-additive risk measure of the following form $\mathcal{R}^{\rho,\check{V}^{\mathcal{F}_T}}(X_T) = \rho\big(-\check{V}^{\mathcal{F}_T}(X_T)\big)$ where $\check{V}^{\mathcal{F}_T}(X_T) = \tilde{\rho}_{\mathcal{F}_T}(X_T 1_{\theta=1})$.

Proof. The proof follows easily from the previous considerations. □

Equation (6.11) suggests that the risk of the future position X_T depends on the risk/ambiguity on the underlying asset model (the unconditional risk measure ρ) and on the risk/ambiguity on interest rates (the conditional risk measure $\tilde{\rho}_{\mathcal{F}_T}$) or more in general on the risk affecting the numéraire.

7 Optimal derivative design and inf-convolution

The problem of designing the optimal transaction between two economic agents has been largely investigated both in the insurance and in the financial literature. The risk transfer between the agents takes place through the exchange of a derivative contract and the optimal transaction is determined by a choice criterion. For example, in Barrieu and El Karoui (2006) the choice criterion is given by the minimization of the risk of the agent's exposure and the risk is assessed using forward cash additive risk measures. Using cash sub-additive risk measures we study this problem in a general framework that allows for ambiguous discount rates. We focus on the problem of the risk transfer between two agents who determine today the reserve to hedge the future exposure when the discount factor for the maturity of interest is ambiguous. To account for this ambiguity the agents collect the reserve using cash sub-additive risk measures and the decision criterium is the minimization of their reserves.

7.1 Transaction feasibility and optimization program

Let A and B be the two agents and suppose that they are evolving in a uncertain universe modeled by the probability space $(\Omega, \mathcal{F}_T)$. Agent A is exposed towards a non-tradable risk that will impact her wealth $X_T^A \in \mathcal{X}$ at the future date T. To reduce her risk exposure and the reserve associated, A aims at issuing a derivative contract $H_T \in \mathcal{X}$ with maturity T and selling it to the agent B for a price π_0. Agent B will enter the transaction only if this transaction reduces or leaves unchanged the reserve that she has to put aside to hedge her future exposure $X_T^B \in \mathcal{X}$. The objective is to find the optimal structure (H_T, π_0) according to the decision criterion of the agents given by their cash sub-additive risk measure $\mathcal{R}_A$ and $\mathcal{R}_B$.

If the agents agree on the transaction, at time zero B pays π_0 to A. At time T the terminal wealths of the agents A and B are $X_T^A - H_T$ and $X_T^B + H_T$, respectively. A aims at minimizing the current reserve $\mathcal{R}_A\left(X_T^A - H_T\right)$ for the future exposure $X_T^A - H_T$, knowing that today she receives π_0 from B,

$$\inf_{H_T \in \mathcal{X}, \pi_0} \mathcal{R}_A\left(X_T^A - H_T\right) - \pi_0. \tag{7.1}$$

The constraint to the optimization program (7.1) is that B enters the transaction. This happens when buying H_T for π_0 reduces or leaves unchanged the reserve $\mathcal{R}_B\left(X_T^B\right)$ that B would collect not entering the transaction,

$$\mathcal{R}_B\left(X_T^B + H_T\right) + \pi_0 \le \mathcal{R}_B\left(X_T^B\right). \tag{7.2}$$

The pricing rule of the H_T-structure is fully determined by the buyer B simply binding the constraint at the optimum in equation (7.2),

$$\pi_0^* = \pi_0^*\left(H_T\right) = \mathcal{R}_B\left(X_T^B\right) - \mathcal{R}_B\left(X_T^B + H_T\right).$$

This price π_0^* corresponds to an "indifference" pricing rule from the point of view of the agent B as π_0^* gives the maximum amount that agent B is ready to pay to enter the transaction. Given π_0^*, the optimization program in (7.1) becomes

$$\mathcal{R}_{A,B}(X_T^A, X_T^B) := \inf_{H_T \in \mathcal{X}} \mathcal{R}_A\left(X_T^A - H_T\right) + \mathcal{R}_B\left(X_T^B + H_T\right), \tag{7.3}$$

where the optimal transaction H_T^* attains the infimum.

7.2 Optimal transaction and inf-convolution

The risk transfer problem in equation (7.3) can be rewritten as an inf-convolution of cash sub-additive risk measures on $\mathcal{X}$. Indeed defining $F_T := X_T^B + H_T \in \mathcal{X}$ we have

$$\mathcal{R}_{A,B}(X_T^A, X_T^B) = \inf_{F_T \in \mathcal{X}} \left\{\mathcal{R}_A(X_T^A + X_T^B - F_T) + \mathcal{R}_B(F_T)\right\} =: \mathcal{R}_A \Box \mathcal{R}_B(X_T^A + X_T^B), \tag{7.4}$$

where $\Box$ denotes the inf-convolution. The value of $\mathcal{R}_{A,B}(X_T^A, X_T^B)$ can be interpreted as the residual measure of risk after the transaction F_T has occurred. This residual measure of risk depends on the initial exposures X_T^A and X_T^B. The transaction induces an optimal redistribution of the risks of the agents. In the following we show that $\mathcal{R}_A \Box \mathcal{R}_B$ is a cash sub-additive risk measure completely characterized by $\mathcal{R}^A$ and $\mathcal{R}^B$ and we provide its dual representation. Also in this case, instead of using convex analysis tools to prove these results we exploit the one to one correspondence between $\mathcal{R}$ and the cash additive risk measure $\hat{\rho}(\widehat{X}_T) = \mathcal{R}\big(X_T - x1_T\big) - x$ defined on $\widehat{\mathcal{X}}$ and given in equation (5.2). We show that the inf-convolution of cash sub-additive risk measures on $\mathcal{X}$ is equal to the inf-convolution of their corresponding cash additive risk measures $\hat{\rho}$ on $\widehat{\mathcal{X}}$.

Lemma 7.1 *The inf-convolution of $\mathcal{R}_A$ and $\mathcal{R}_B$ on $\mathcal{X}$ in equation (7.4) corresponds to the inf-convolution of the cash additive extensions of $\mathcal{R}_A$ and $\mathcal{R}_B$ on $\widehat{\mathcal{X}}$,*

$$\mathcal{R}_A \Box \mathcal{R}_B(X_T^A + X_T^B) = \hat{\rho}_A \Box \hat{\rho}_B(\widehat{X}_T^A + \widehat{X}_T^B), \ \textit{where}\ \widehat{X}_T^A := X_T^A 1_{\theta=1},\ \widehat{X}_T^B := X_T^B 1_{\theta=1}. \tag{7.5}$$

$\mathcal{R}_A \Box \mathcal{R}_B(X_T^A + X_T^B)$ is the infimum on $F_T \in \mathcal{X}$, while $\hat{\rho}_A \Box \hat{\rho}_B(\widehat{X}_T^A + \widehat{X}_T^B)$ is the infimum on the pairs $(F_T, x) \in \widehat{\mathcal{X}}$.

Proof. The result follows observing that any $F_T \in \mathcal{X}$ can be rewritten as $F_T = G_T - x1_T$, for some $G_T \in \mathcal{X}$ and $x \in \mathbb{R}$, and

$$\begin{aligned}
&\mathcal{R}_A \Box \mathcal{R}_B(X_T^A + X_T^B) = \inf_{F_T \in \mathcal{X}} \big\{\mathcal{R}_A(X_T^A + X_T^B - F_T) + \mathcal{R}_B(F_T)\big\} \\
&= \inf_{(G_T, x) \in \mathcal{X} \times \mathbb{R}} \big\{\mathcal{R}_A(X_T^A + X_T^B - (G_T - x1_T)) + \mathcal{R}_B(G_T - x1_T)\big\} \\
&= \inf_{\widehat{G}_T = (G_T, x) \in \hat{\mathcal{X}}} \big\{\hat{\rho}_A((X_T^A + X_T^B)1_{\theta=1} - \widehat{G}_T) + \hat{\rho}_B(\widehat{G}_T)\big\} = \hat{\rho}_A \Box \hat{\rho}_B(\widehat{X}_T^A + \widehat{X}_T^B). \qquad \Box
\end{aligned}$$

Barrieu and El Karoui (2006, Theorem 3.3) show that the inf-convolution of cash additive risk measures is a cash additive risk measure. We apply this result to $\hat{\rho}_A \Box \hat{\rho}_B$. When $\hat{\rho}_A \Box \hat{\rho}_B(0) > -\infty$, the inf-convolution $\widehat{X} \in \widehat{\mathcal{X}} \longmapsto \hat{\rho}_A \Box \hat{\rho}_B(\widehat{X})$ is a cash additive risk measure[3], continuous from below if one of the two risk measures is continuous from below, and with penalty function the sum of the penalties of $\hat{\rho}_A$ and $\hat{\rho}_B$. We showed that any $\hat{\rho}$ constrained to the event $\theta = 1$ defines a cash sub-additive risk measure with the same penalty function (Proposition 5.2). Then $\mathcal{R}_A \Box \mathcal{R}_B$ in equation (7.5) is a cash sub-additive risk measure. We collect all the previous results in the following theorem.

Theorem 7.2 *Let $\mathcal{R}_A$ and $\mathcal{R}_B$ be two cash sub-additive risk measures with penalty functions α_A and α_B, respectively. Let $\mathcal{R}_{A,B}$ be the inf-convolution of $\mathcal{R}_A$ and $\mathcal{R}_B$*

$$\Psi \to \mathcal{R}_{A,B}(\Psi) := \mathcal{R}_A \Box \mathcal{R}_B(\Psi) = \inf_{H \in \mathcal{X}} \big\{\mathcal{R}_A(\Psi - H) + \mathcal{R}_B(H)\big\} \tag{7.6}$$

and assume that $\mathcal{R}_{A,B}(0) > -\infty$. Then

1) $\mathcal{R}_{A,B}$ is a cash sub-additive risk measure which is finite for all $\Psi \in \mathcal{X}$.

2) The associated penalty function is given by $\forall \mu \in \mathcal{M}_{s,f}(\mathcal{F}_T), \quad \alpha_{A,B}(\mu) = \alpha_A(\mu) + \alpha_B(\mu)$.

3) $\mathcal{R}_{A,B}$ is continuous from below when this property holds for $\mathcal{R}_A$ and/or $\mathcal{R}_B$.

4) The optimal derivative contract is $H^* = F^* - X_T^B$, where F^* attains the infimum in (7.4).

[3]For the interpretation of the condition $\mathcal{R}_A \Box \mathcal{R}_B(0) > -\infty$ see Theorem 3.3 in Barrieu and El Karoui (2006).

8 Dynamic infinitesimal cash sub-additive risk measures

The cash sub-additive risk measures considered so far are static measures assessing the risk of the future position X_T at a given time t. In this section, we give an example of dynamic cash sub-additive risk measure on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented filtration associated to the d-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$. At any time $t \in [0,T]$, the risk measure assesses the riskiness of the future position X_T taking into account the information available, $\mathcal{F}_t$. In particular, following Peng (2004), El Karoui, Peng, and Quenez (1997), Barrieu and El Karoui (2006) and Rosazza Gianin (2006) who link backward stochastic differential equations (BSDEs) and risk measures, we show that BSDEs with suitable coefficients are cash sub-additive risk measures. The main difference with cash additive risk measures generated by BSDEs is that cash sub-additive risk measures are now recursive risk measures. When the dual representation exists, the penalty function of dynamic cash sub-additive risk measures generalizes the penalty function of the static cash sub-additive risk measures in Section 4.2.

Dynamic risk measures not based on BSDEs have been recently studied by several authors such as Cvitanic and Karatzas (1999), Wang (1999), Artzner, Delbaen, Eber, Heath, and Ku (2004) Cheridito, Delbaen, and Kupper (2004), Frittelli and Rosazza Gianin (2004), Riedel (2004), Frittelli and Scandolo (2006), Cheridito, Delbaen, and Kupper (2006), Weber (2006) and Kloeppel and Schweizer (2006). Here we consider cash sub-additive risk measures generated by BSDEs.

8.1 Some results on BSDEs

Let $X_T \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ and $g(t, y, z)$ be a $\mathcal{P}_1 \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$-measurable coefficient, where $\mathcal{P}_1$ is the set of real-valued progressively measurable processes. Consider the pair of squared-integrable progressively measurable processes $(Y, Z) := (Y_t, Z_t)_{t\in[0,T]}$ solution of the following BSDE associated to (g, X_T),

$$-dY_t = g(t, Y_t, Z_t)dt - \langle Z_t, dW_t\rangle, \quad Y_T = X_T.$$

The existence and the uniqueness of the solution $(Y_t, Z_t)_{t\in[0,T]}$ depend on the properties of the coefficient g. Pardoux and Peng (1990) prove that the solution exists and is unique when g is uniformly Lipschitz continuous with respect to (y, z). In this case g is called standard coefficient.

When, for any given $t \in [0,T]$, g is continuous with respect to (y,z) $\mathbb{P}$-a.s. and $|g(t,y,z)| \leq C(1+z^2+y)$, $\forall (t,y,z)$ $\mathbb{P}$-a.s., (g with linear-quadratic growth, in the sequel), Kobylanski (2000) and Lepeltier and San Martin (1998) show that the BSDE associated with (g, X_T) has a maximal and minimal solution. Uniqueness holds under some additional assumptions.

The following theorem, called *Comparison Theorem*, is a crucial tool in the study of one-dimensional BSDEs and corresponding dynamic measures of risk.

Theorem 8.1 *Let X_T^1 and $X_T^2 \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ and g^1 and g^2 both standard (or both with linear-quadratic growth) coefficients. Let (Y^1, Z^1) and (Y^2, Z^2) be the (maximal) solutions associated to (g^1, X_T^1) and (g^2, X_T^2), respectively. If $X_T^1 \geq X_T^2$, $\mathbb{P}$-a.s., and $g^1(t, Y_t^2, Z_t^2) \geq g^2(t, Y_t^2, Z_t^2)$ $d\mathbb{P} \times dt$-a.s., then $Y_t^1 \geq Y_t^2$ a.s. $\forall\, t \in [0,T]$. In particular, the maximal solution is still monotone decreasing with respect to the terminal condition.*

The comparison theorem and the existence of the maximal solution ensure that, if the coefficient g is convex, the solution Y_t of the BSDE $(g, -X_T)$ is also convex when Y_t is considered as a functional of its terminal condition $-X_T$. Moreover, the existence of the maximal solution ensures the *time consistency* of $(Y_t)_{[0,T]}$, that is: $\forall\ 0 \leq t_1 \leq t_2 \leq T,\quad Y_{t_1}(X_T) = Y_{t_1}(-Y_{t_2}(X_T))$. For the derivations of this result see, for instance, El Karoui, Peng, and Quenez (1997), Peng (2004), Barrieu and El Karoui (2006) and Rosazza Gianin (2006).

8.2 BSDEs and cash sub-additive risk measures

The link between measures of risk and BSDEs is particularly interesting because it enhances interpretation and tractability of risk measures. Barrieu and El Karoui (2006) point out that the coefficient g of BSDEs can be interpreted as infinitesimal risk measure over a time interval $[t, t+dt]$ as $\mathbb{E}_{\mathbb{P}}[dY_t|\mathcal{F}_t] = -g(t, Y_t, Z_t)dt$ where Z_t is the local volatility of the conditional risk measure, $\mathbb{V}(Y_t|\mathcal{F}_t) = |Z_t|^2 dt$. Choosing carefully the coefficient g enables to generate g-conditional risk measures that are locally compatible with the different agent beliefs.

Example 8.2 Ambiguous interest rates. *Assume that locally $\mathbb{E}_{\mathbb{P}}[-dY_t|\mathcal{F}_t]$ is driven by the worst case scenario generated by an ambiguous discount rate $\beta = (\beta_t)_{t \in [0,T]}$, where β is an adapted* *process ranging between two adapted and bounded processes* $(r_t)_{t\in[0,T]}$ *and* $(R_t)_{t\in[0,T]}$*, that is*

$$\mathbb{E}_{\mathbb{P}}[-dY_t^{r,R}|\mathcal{F}_t] = \sup_{0\le r_t\le\beta_t\le R_t}(-\beta_t Y_t^{r,R})dt.$$

$Y^{r,R}$ *is the first component solution of the BSDE*

$$-dY_t = -\big(rY_t^+ - R_t Y_t^-\big)dt - \langle Z_t, dW_t\rangle, \quad Y_T = -X_T,$$

where $y^+ = \sup(y,0)$ *and* $y^- = \sup(-y,0)$*. More precisely, since* $(r_t)_{t\in[0,T]}$ *and* $(R_t)_{t\in[0,T]}$ *are assumed to be bounded,* $(Y^{r,R}, Z^{r,R})$ *is the unique solution of the standard BSDE with convex Lipschitz coefficient*

$$g^{r,R}(t,y) = R_t y^- - r_t y^+ = \sup_{r_t\le\beta_t\le R_t}(-\beta_t y). \tag{8.1}$$

Notice that $y \mapsto g^{r,R}(t,y)$ *is a monotone non increasing function. To provide the intuition on this risk measure, we apply the comparison theorem to the coefficients* $g^{r,R}(t,y)$ *and* $g(t,y) = (-\beta_t y)$*,* $\beta_t \in [r_t, R_t]$*, with the same terminal condition* $-X_T$*. Since* $g^{r,R}(t,y) \ge (-\beta_t y)$*,* $Y_t^{r,R} \ge Y_t^\beta$ *where* Y^β *is the solution of the linear BSDE*

$$-dY_t = -\beta_t Y_t dt - \langle Z_t, dW_t\rangle, \quad Y_T = -X_T,$$

and it can be represented as $Y_t^\beta = \mathbb{E}_{\mathbb{P}}[e^{-\int_t^T \beta_s ds}(-X_T)|\mathcal{F}_t]$*,* $\forall t \in [0,T]$*. Then it follows that* $Y_t^{r,R} \ge \operatorname{ess\,sup}_{0\le r_t\le\beta_t\le R_t} Y_t^\beta$*. As the process* $\overline{\beta}_t = R_t 1_{Y_t^{r,R}\le 0} + r_t 1_{Y_t^{r,R}>0}$ *achieves the maximum of* $\sup_{r_t\le\beta_t\le R_t}(-\beta_t Y_t^{r,R}) = -\overline{\beta}_t Y_t^{r,R}$*, then the equality* $Y_t^{r,R} = Y_t^{\overline{\beta}_t}$ *holds. Thus, the dual representation of* $Y_t^{r,R}$ *follows*

$$Y_t^{r,R} = Y_t^{\overline{\beta}_t} = \operatorname{ess}\sup_{0\le r_t\le\beta_t\le R_t}\mathbb{E}_{\mathbb{P}}[e^{-\int_t^T\beta_s ds}(-X_T)|\mathcal{F}_t].$$

Notice that, for any $t \in [0,T]$*,* $Y_t^{r,R}$ *is dominated, but in general not equal to the conditional risk measure* $\mathcal{R}_t^{D^R,D^r}$ *associated with the worst case discounted factors* $D_{t,T}^R \le D_{t,T} \le D_{t,T}^r$*, where* $D_{t,T}^R = \exp\{-\int_t^T R_s ds\}$ *and* $D_{t,T}^r = \exp\{-\int_t^T r_s ds\}$*,*

$$Y_t^{r,R}(-X_T) \le \mathcal{R}_t^{D^R,D^r}(X_T) = \mathbb{E}_{\mathbb{P}}[D_{t,T}^R(-X_T)^- + D_{t,T}^r(-X_T)^+|\mathcal{F}_t]. \tag{8.2}$$

$\mathcal{R}^{D^R,D^r} := (\mathcal{R}_t^{D^R,D^r})_{t\in[0,T]}$ *is a cash sub-additive risk measure which is not time consistent in contrast to* $Y^{r,R} = (Y_t^{r,R})_{t\in[0,T]}$.

In the sequel we consider risk measures generated by BSDEs which generalize Example 8.2. For the remain part of the paper $g(t,y,z)$ denotes a convex generator in (y,z), standard or with linear growth with respect to y and quadratic growth in z. The comparison theorem ensures that the (maximal) solution (Y,Z) associated with a $(g,-X_T)$ exists and, for any $t\in[0,t]$, Y_t is convex and decreasing with respect to the final condition $-X_T$.

The coefficient $g^{r,R}(t,y)$ in equation (8.1) depends on y in a convex decreasing way. As observed by Peng (2004) and Barrieu and El Karoui (2006), this is never the case for conditional cash additive risk measures generated by BSDEs. Under some mild additional assumptions, Peng (2004) shows that, for any $t\in[0,T]$, the (maximal) solution Y_t associated with $(g,-X_T)$ is cash additive as functional of its terminal condition if and only if g does not depend on y for any $t\in[0,T]$. Barrieu and El Karoui (2006) study these cash additive solutions as a dynamic risk measure $(\rho_t(X_T))_{t\in[0,T]}$, $\rho_t(X_T) = Y_t(-X_T)$, that they call g-conditional risk measures[4].

In the following proposition we show that conditional risk measures generated by BSDEs are cash sub-additive when the convex coefficient $g(t,y,z)$ depends on both y and z and is decreasing with respect to y.

Proposition 8.3 *If the convex $g(t,y,z)$ is decreasing with respect to y then the (maximal) solution Y_t of the BSDE associated with $(g,-X_T)$ is a conditional cash sub-additive risk measure, $\mathcal{R}_t^g(X_T) = Y_t$ and $\mathcal{R}^g = (\mathcal{R}_t^g(X_T))_{t\in[0,T]}$ is a time consistent cash sub-additive risk measure. We call $\mathcal{R}^g = (\mathcal{R}_t^g(X_T))_{t\in[0,T]}$* g-conditional cash sub-additive risk measure.

Proof. For the convexity and the decreasing monotonicity of Y_t with respect to the terminal condition see, for instance, El Karoui and Quenez (1996) and Peng (1997).
Cash sub-additivity: Consider the BSDE satisfied by $\mathcal{R}_t^g(X_T + m1_T) + m = Y_t^m$,

$$-dY_t^m = g^m\big(t, Y_t^m, Z_t^m\big)dt - \langle Z_t^m, dW_t\rangle, \qquad Y_T^m = -X_T.$$

Since $g^m(t,y,z) = g(t,y-m,z)$, then $g^m(t,y,z)$ is increasing in y (as g is decreasing in y). From the comparison theorem it follows that $\mathcal{R}_t^g(X_T + m1_T) + m = Y_t^m$ is increasing in m. □

[4]If $g(t,0) = 0$ for any $t\in[0,T]$, the g-conditional risk measures coincide with the non linear expectation originally studied by Peng (2004); see also Rosazza Gianin (2006).

8.3 Dual Representation

In this section we focus on a dual representation for g-conditional cash sub-additive risk measures $\mathcal{R}^g$ as in the static case. For the cash additive g-conditional risk measures such a representation has been derived in Barrieu and El Karoui (2006). The next result is a straightforward generalization of their results.

The key tool to obtain dual representations is the Legendre transform of the generator g defined by

$$G(t,\beta,\mu) := \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \{-\beta y - \langle \mu, z\rangle - g(t,y,z)\}.$$

The following lemma summarizes the properties of G and g.

Lemma 8.4 *Let g be a continuous convex function on $\mathbb{R}\times\mathbb{R}^d$ satisfying the growth control: there exist two positive constants $C>0$ and $k>0$ such that $|g(t,y,z)| \le |g(t,0,0)| + C|y| + \frac{k}{2}|z|^2$.*

i) Then the Legendre transform of g, $G(t,\beta,\mu)$, takes infinite values if $\beta \notin [0,C]$. Moreover,

$$G(t,\beta,\mu) \ge -|g(t,0,0)| + \frac{1}{2k}|\mu|^2. \tag{8.3}$$

ii) Since g is continuous, for any $t\in[0,T]$, $g(t,Y_t,Z_t) = \sup_{\beta,\mu}\{-\beta_t Y_t - \langle \mu_t, Z_t\rangle - G(t,\beta_t,\mu_t)\}$. The maximum is achieved at $(\overline{\beta}_t,\overline{\mu}_t)$ with $0\le\overline{\beta}_t\le C$ and $|\overline{\mu}_t|^2 \le A\big(|g(t,0,0)| + C|Y_t|\big) + B|Z_t|^2$, for some A and B positive constants.

Proof. i) $G(t,\beta,\mu) \ge -\beta y - g(t,y,0) \ge -\beta y - |g(t,0,0)| - C|y|$. Then, if $|\beta|>C$, $\sup_{y\in\mathbb{R}}\{-\beta y - C|y|\} = +\infty$. Moreover, since $g(t,y,z)$ is monotone decreasing with respect to y, $-g(t,y,0) \ge -g(t,0,0)$, $\forall y>0$ and $G(t,\beta,\mu) \ge -\beta y - g(t,0,0)$, $\forall y>0$. Then $G(t,\beta,\mu) = +\infty$ if $\beta<0$. To prove the inequality (8.3), we observe that $G(t,\beta,\mu) \ge \langle \mu, -z\rangle - g(t,0,z) \ge \langle\mu,-z\rangle - |g(t,0,0)| - \frac{k}{2}|z|^2$. As $\max_{z\in\mathbb{R}}\{\langle\mu,-z\rangle - \frac{k}{2}|z|^2\} = \frac{1}{2k}|\mu|^2$ the result follows.
ii) Standard results in convex analysis show that, since g is continuous, the duality between g and G holds true and the maximum is achieved.

To show the inequality in *ii)*, we choose a constant ε such that $0<\varepsilon<\frac{1}{2k}$ and we use the inequality (8.3),

$$\begin{aligned}\big(\frac{1}{2k}-\varepsilon\big)|\overline{\mu}_t|^2 &\le |g(t,0,0)|+G(t,\overline{\beta}_t,\overline{\mu}_t)-\varepsilon|\overline{\mu}_t|^2\\ &\le |g(t,0,0)|-\overline{\beta}_tY_t+\langle\overline{\mu}_t,-Z_t\rangle-g(t,Y_t,Z_t)-\varepsilon|\overline{\mu}_t|^2\\ &\le 2|g(t,0,0)|+2C|Y_t|+\frac{k}{2}|Z_t|^2+\sup_{\mu_t}\{\langle\mu_t,-Z_t\rangle-\varepsilon|\mu_t|^2\}.\end{aligned}$$

As $\max_{\mu_t\in\mathbb{R}}\{\langle\mu_t,-Z_t\rangle-\varepsilon|\mu_t|^2\}=\frac{|Z_t|^2}{4\varepsilon}$, then $\big(\frac{1}{2k}-\varepsilon\big)|\overline{\mu}_t|^2\le 2|g(t,0,0)|+2C|Y_t|+\big(\frac{k}{2}+\frac{1}{4\varepsilon}\big)|Z_t|^2$, which proves the inequality. □

Now we introduce the class of probability measures that appears in the dual representation. As in Barrieu and El Karoui (2006) the reference is the Girsanov theorem for the BMO-exponential martingales such as defined in Kazamaki (1994),

$$\Gamma_t^\mu=\mathcal{E}(M_t^\mu)=\exp\big(-\int_0^t\mu_s dW_s-\frac{1}{2}\int_0^t|\mu_s|^2ds\big),$$

where $M_t^\mu=\int_0^t\mu_s dW_s$ is a BMO($\mathbb{P}$)-martingale, that is μ belongs to BMO($\mathbb{P}$),

$$\text{BMO}(\mathbb{P}):=\{\psi\in\mathcal{H}^2\text{ such that }\exists C>0:\mathbb{E}_\mathbb{P}[\int_t^T\psi_s^2ds|\mathcal{F}_t]\le C\ a.s.,\forall t\in[0,T]\}.$$

Using Kazamaki (1994, Section 3.3), Γ_T^μ is the likelihood of an equivalent probability measure on $\mathcal{F}_T$ with respect to $\mathbb{P}$ defined by $d\mathbb{Q}^\mu=\Gamma_T^\mu d\mathbb{P}$. Moreover, if $v\in$ BMO($\mathbb{P}$) then $v\in$ BMO($\mathbb{Q}^\mu$). Recall that Γ_t^μ is the solution of the forward stochastic differential equation

$$d\Gamma_t^\mu=\Gamma_t^\mu\langle-\mu_t,dW_t\rangle,\quad \Gamma_0^\mu=1.$$

Now we establish the duality theorem.

Theorem 8.5 *Let g be a convex coefficient, decreasing with respect to y and with growth $|g(t,y,z)|\le|g(t,0,0)|+C|y|+\frac{k}{2}|z|^2$. Moreover, assume that there exists a constant $K>0$ such that $\mathbb{E}\big[\int_t^T|g(s,0,0)|ds|\mathcal{F}_t\big]\le K$, $\forall t\in[0,T]$. Then the (maximal) solution (Y,Z) of the BSDE*

$$-dY_t=g(t,Y_t,Z_t)-\langle Z_t,dW_t\rangle,\quad Y_T=-X_T,\quad X_T\in L^\infty(\mathbb{P}),$$

is bounded and Z is in BMO($\mathbb{P}$). Let $G(t,y,z)$ be the Fenchel transform of g and

$$\mathcal{A}:=\big\{(\beta_t,\mu_t)_{t\in[0,T]}|\,G(t,\beta_t,\mu_t)<+\infty,\ 0\le\beta_t\le C,\ \forall t\in[0,T]\text{ and }\mu\in\text{ BMO}(\mathbb{P})\big\}.$$

Then, the g-conditional cash sub-additive risk measure $\mathcal{R}^g = (\mathcal{R}^g_t(X_T))_{t\in[0,T]}$, $\mathcal{R}^g_t(X_T) = Y_t$, *has the following dual representation*

$$\mathcal{R}^g_t(X_T) = \operatorname{ess\,sup}_{(\beta,\,\mu)\in\mathcal{A}} \mathbb{E}_{\mathbb{Q}^\mu}\Big[e^{-\int_t^T \beta_s ds}\,(-X_T) - \int_t^T e^{-\int_t^s \beta_u du} G(s,\beta_s,\mu_s)ds\big|\mathcal{F}_t\Big]. \tag{8.4}$$

Remark 8.6 *The dual representation of* $\mathcal{R}^g$ *in equation (8.4) is similar to the dual representation of static cash sub-additive risk measures. Here, the sub-probability measures are replaced by the* $\mathcal{F}_t$*-conditional sub-probability measures* $R^{\beta,\mu}$

$$\frac{dR^{\beta,\mu}}{d\mathbb{P}}|\mathcal{F}_t := \exp\big(-\int_t^T \mu_s dW_s - \frac{1}{2}\int_t^T |\mu_s|^2 ds - \int_t^T \beta_s ds\big)$$

and the penalty function becomes

$$\alpha_t(R^{\beta,\mu}) := R^{\beta,\mu}\left(\int_t^T e^{-\int_t^s \beta_u du} G(s,\beta_s,\mu_s)ds\big|\mathcal{F}_t\right).$$

Proof. *i)* To show that $Z \in$ BMO($\mathbb{P}$) we refer the reader to the proof in Barrieu and El Karoui (2006).

ii) From the Girsanov theorem for BMO-martingales we known that for any $0 \le \beta_t \le C$, $\mu \in BMO(\mathbb{P})$, $dW^\mu_t = dW_t + \mu_t dt$ is a $\mathbb{Q}^\mu$-Brownian motion and

$$\begin{aligned} -dY_t &= g(t,Y_t,Z_t) - \langle Z_t, dW_t\rangle \\ &= \big[g(t,Y_t,Z_t) + \beta_t Y_t + \langle \mu_t, Z_t\rangle\big]dt - \beta_t Y_t dt - \langle Z_t, dW^\mu_t\rangle. \end{aligned}$$

Then it follows

$$\begin{aligned} Y_t(-X_T) &= \mathbb{E}_{\mathbb{Q}^\mu}\Big[e^{-\int_t^T \beta_s ds}\,(-X_T) + \int_t^T e^{-\int_t^s \beta_u du}\big[g(s,Y_s,Z_s) + \beta_s Y_s + \langle \mu_s, Z_s\rangle\big]ds|\mathcal{F}_t\Big] \\ &\ge \mathbb{E}_{\mathbb{Q}^\mu}\Big[e^{-\int_t^T \beta_s ds}\,(-X_T) - \int_t^T e^{-\int_t^s \beta_u du} G(s,\beta_s,\mu_s)ds|\mathcal{F}_t\Big]. \end{aligned} \tag{8.5}$$

To prove the last equality in (8.5) at the optimal control $(\overline{\beta},\overline{\mu})$,

$$G(t,\overline{\beta},\overline{\mu}) = -\overline{\beta}_t Y_t - \langle \overline{\mu}_t, Z_t\rangle - g(t,Y_t,Z_t),\ \forall t \in [0,T],$$

we need to verify that $(\overline{\beta},\overline{\mu})$ is admissible. Since $0 \le \overline{\beta}_t \le C$, we only need to verify that $\overline{\mu}$ is in BMO($\mathbb{P}$). We use the inequality in Lemma 8.4, $|\overline{\mu}_t|^2 \le A\big(|g(t,0,0)| + c|Y_t|\big) + B|Z_t|^2$. Since $|g(t,0,0)|^{1/2}$ belongs to BMO($\mathbb{P}$), Y is bounded and $Z \in$ BMO($\mathbb{P}$), then $\overline{\mu} \in$ BMO($\mathbb{P}$),

$$Y_t(-X_T) = \mathcal{R}^g(X_T) = \mathbb{E}_{\mathbb{Q}^{\overline{\mu}}}\Big[e^{-\int_t^T \overline{\beta}_s ds}\,(-X_T) - \int_t^T e^{-\int_t^s \overline{\beta}_u du} G(s,\overline{\beta}_s,\overline{\mu}_s)ds|\mathcal{F}_t\Big]$$

and this establishes the dual representation. □

9 Conclusion

We propose a new class of risk measures called cash sub-additive risk measures which account for the risk/ambiguity on interest rates when assessing the risk of future financial, nonfinancial and insurance positions. This goal is achieved by relaxing the debated cash additive axiom into the cash sub-additive axiom. We provide several examples of the new risk measures in the static and the dynamic frameworks, such as the put options premium and the robust expected utility. In the dynamic framework cash sub-additive risk measures are generated by BSDEs enhancing their tractability and interpretability. Cash sub-additive risk measures represent a promising research area as these risk measures overcome the issues arising from the cash additive axiom.

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