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CASTELNUOVO–MUMFORD REGULARITY OF SIMPLICIAL SEMIGROUP RINGS WITH ISOLATED SINGULARITY

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ABSTRACT. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n \geq 2$ variables over a field K and $\mathfrak{m}$ its graded maximal ideal. Let $f_1, \ldots, f_m \in S$ be homogeneous polynomials of degree $d - 1 \geq 2$ generating an $\mathfrak{m}$-primary ideal, and let $g_1, \ldots, g_r \in S$ be arbitrary homogeneous polynomials of degree d. In the present paper it will be proved that the Castelnuovo–Mumford regularity of the standard graded K-algebra $A = K[\{f_i x_j\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}, g_1, \ldots, g_r]$ is at most $(d-2)(n-1)$. By virtue of this result, it follows that the regularity of a simplicial semigroup ring $K[C]$ with isolated singularity is at most $e(K[C]) - \operatorname{codim}(K[C])$, where $e(K[C])$ is the multiplicity of $K[C]$ and $\operatorname{codim}(K[C])$ is the codimension of $K[C]$.

INTRODUCTION

Castelnuovo–Mumford regularity of graded rings and ideals is one of the most active research topics in computational commutative algebra and computational algebraic geometry.

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n \geq 2$ variables over a field K, and let M be a finitely generated graded S-module. If

$$\cdots \longrightarrow F_j \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is the graded minimal free S-resolution of M, then the Castelnuovo–Mumford regularity $\operatorname{reg}(M)$ of M is the nonnegative integer $\operatorname{reg}(M) = \max\{b_j - j \colon j = 0, 1, \ldots\}$, where b_j is the maximal degree of the generators of the graded free S-module F_j.

We are especially interested in the Castelnuovo–Mumford regularity of the standard graded K-algebra $A = S/I$, where I is a homogeneous ideal of S. Eisenbud and Goto conjectured in their paper [3] that if A is an integral domain, then $\operatorname{reg}(A)$ satisfies the inequality

$$\operatorname{reg}(A) \leq e(A) - \operatorname{codim}(A), \tag{1}$$

where $e(A)$ is the multiplicity of A and $\operatorname{codim}(A)$ is the codimension of A. The Eisenbud–Goto conjecture turns out to be true in several special cases considered

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in algebraic geometry; see [5] and [6]. However, the conjecture is widely open in general; even in the case that A is an affine semigroup ring.

Let $\mathfrak{m}$ denote the graded maximal ideal of S. In the present paper, we pay attention to the Castelnuovo–Mumford regularity of the standard graded K-algebra $A = K[f_1, \ldots, f_m]$, where $I = (f_1, \ldots, f_m) \subset S$ is a homogeneous ideal generated in degree d such that $I^k = \mathfrak{m}^{dk}$ for some $k > 0$. For such a K-algebra A one has $e(A) = d^{n-1}$ and $\operatorname{codim}(A) \leq \binom{d+n-1}{n-1} - n$.

For a particular class of such K-algebras we can bound the regularity. This is shown in Theorem 1.1. As a consequence we obtain in Corollary 1.3 that for such a K-algebra A, one has $\operatorname{reg}(A) \leq e(A) - \operatorname{codim}(A)$, if $n \geq 3$.

Recently, in Hoa and Stückrad [4] the regularity of simplicial semigroup rings was studied. Their work strongly stimulates the research to find reasonable classes of simplicial semigroup rings satisfying inequality (1). As a conclusion of Corollary 1.3 and a simple counting argument, we show in our final Corollary 2.2 that the Eisenbud–Goto conjecture holds for simplicial semigroup rings with isolated singularity.

1. Regularity of certain graded rings generated by d-forms

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in $n \geq 2$ variables over K with the graded maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$.

Theorem 1.1. *Let $f_1, \ldots, f_m \in S$ be homogeneous polynomials of degree $d-1 \geq 2$ generating an $\mathfrak{m}$-primary ideal, and let $g_1, \ldots, g_r \in S$ be arbitrary homogeneous polynomials of degree d. Then the regularity of the standard graded K-algebra $A = K[\{f_i x_j\}_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}}, g_1, \ldots, g_r]$ is at most $(d-2)(n-1)$.*

Proof. We may assume that K is an infinite field. Let $J = (f_1, \ldots, f_m)$. Then there exists an ideal $L \subset J$ generated by a regular sequence of length n consisting of elements of degree $d-1$. For a finite length graded S-module N we set $s(N) = \max\{i \colon N_i \neq 0\}$. It is known that $\operatorname{reg}(N) = s(N)$. Therefore for k it follows that

$$\operatorname{reg}(J^k) = \operatorname{reg}(S/J^k) + 1 \leq \operatorname{reg}(S/L^k) + 1 = \operatorname{reg}(L^k).$$

Let $L = (\ell_1, \ldots, \ell_n)$. Since L is generated by a regular sequence of length n of elements of degree $d-1$, the resolution of L^k is given by the Eagon–Northcott complex (see, e.g., [2]) attached to the $(d-1) \times (n+d-2)$ matrix

$$\begin{pmatrix} \ell_1 & \cdots & \ell_n & 0 & \cdots & 0 \\ 0 & \ell_1 & \cdots & \ell_n & \cdots & 0 \\ \vdots & & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & \ell_1 & \cdots & \ell_n \end{pmatrix}.$$

It follows immediately from this resolution that $\operatorname{reg}(L^k) = (d-1)k + (d-2)(n-1)$.

For the convenience of the reader we give a direct proof of this fact: The ideal $\mathfrak{m}^k = (x_1, \ldots, x_n)^k$ has a K-linear resolution. In particular, the generators in the last step of the resolution are of degree $k + n - 1$. Consider the flat map $\varphi \colon S \to S$ with $\varphi(x_i) = \ell_i$ for $i = 1, \ldots, n$. Then $\varphi(\mathfrak{m}^k) = L^k$, and so the resolution of L^k is obtained from that of $\mathfrak{m}^k$ by replacing each x_i with ℓ_i. This implies that all the shifts are multiplied by $d-1$. Hence the generators in the last step of the resolution of L^k are of degree $(k+n-1)(d-1)$. From this we conclude that $\operatorname{reg}(L^k) = (k+n-1)(d-1) - (n-1) = (d-1)k + (d-2)(n-1)$.

Let $I = J\mathfrak{m}$. We claim that I^k has a linear resolution if $k \geq (d-2)(n-1)$. In fact, $I^k = J^k\mathfrak{m}^k = (J^k)_{\geq (d-1)k+k}$, where for a graded module M we set $M_{\geq j} = \bigoplus_{i\geq j} M_i$. Recall from [3] (or [1, Theorem 4.3.1]) that

$$\operatorname{reg} M = \min\{\, j\colon M_{\geq j} \quad \text{has a linear resolution}\,\}.$$

It follows that I^k has a linear resolution if and only if $(d-1)k + k \geq \operatorname{reg}(J^k)$. In particular, I^k has a linear resolution if $(d-1)k + k \geq (d-1)k + (d-2)(n-1)$, namely if $k \geq (d-2)(n-1)$.

Next we notice that an $\mathfrak{m}$-primary ideal H generated in one degree, say h, has a linear resolution if and only if it is a power of $\mathfrak{m}$. To see why this is true, we observe that H has a linear resolution if and only if $\operatorname{reg}(H) = h$. But $\operatorname{reg}(H) = s(S/H) + 1 = h$ if and only if $H = \mathfrak{m}^h$.

Applied to our situation we conclude that $I^k = \mathfrak{m}^{dk}$ for $k \geq (d-2)(n-1)$. This implies that $A_k = S_{dk}$ for all $k \geq (d-2)(n-1)$. Let A^* be the integral closure of A. Then $A^* = S^{(d)}$, the dth Veronese subring of S, and A^*/A is of finite length with $s(A^*/A) \leq (d-2)(n-1) - 1$.

Let $\mathfrak{n}$ be the graded maximal ideal of A. Local cohomology applied to the exact sequence

$$0 \longrightarrow A \longrightarrow A^* \longrightarrow A^*/A \longrightarrow 0$$

yields that $H^0_{\mathfrak{n}}(A^*/A) = H^1_{\mathfrak{n}}(A)$ and $H^i_{\mathfrak{n}}(A) = H^i_{\mathfrak{n}}(A^*)$ for $i > 1$. Since A^* is Cohen–Macaulay, one also has $H^i_{\mathfrak{n}}(A^*) = 0$ for $i < d = \dim(A) = \dim(A^*)$. Hence $\operatorname{reg}(A) = \max\{\operatorname{reg}(A^*), \operatorname{reg}(A^*/A) + 1\}$. Since $\operatorname{reg}(A^*/A) = s$, it follows that

$$\operatorname{reg}(A) = \max\{\operatorname{reg}(A^*), s(A^*/A) + 1\} \leq (d-2)(n-1),$$

since the regularity of the Cohen–Macaulay algebra A^* is n plus its a-invariant, and hence at most $n-1$, and since $n-1 \leq (d-2)(n-1)$ because $d \geq 3$. □

Theorem 1.1 suggests the following question: Let $f_1, \ldots, f_m$ be homogeneous polynomials of degree d and suppose that $A_k = S_{dk}$ for some k. Does this imply that $\operatorname{reg}(A) \leq (d-2)(n-1)$?

We shall need the following numerical result.

Lemma 1.2. *If $n \geq 3$ and $d \geq 3$, then $(d-2)(n-1) \leq d^{n-1} - (\binom{n+d-1}{n-1} - n)$.*

Proof. Replace $n-1$ with n in the required inequality, and what we must prove is the inequality

$$(d-2)n \leq d^n - \left(\binom{n+d}{n} - (n+1)\right) \tag{2}$$

for $n \geq 2$ and $d \geq 3$. Inequality (2) is equivalent to the inequality

$$d^n - \binom{n+d}{n} \geq (d-2)n - (n+1).$$

Thus we must prove the inequality

$$d^n - \prod_{i=1}^{n}(1 + \frac{d}{i}) \geq nd - 3n - 1 \tag{3}$$

for $n \geq 2$ and $d \geq 3$.

Fix $d \geq 3$. By using induction on $n \geq 2$ we will prove (3).

If $n = 2$, then the inequality (3) coincides with $(d-3)(d-4) \geq 0$.

Let $n \geq 2$ and suppose that the inequality (3) is true. Inequality (3) for $n+1$ then follows from the computation below:

$$\begin{aligned}
& d^{n+1} - \prod_{i=1}^{n+1}(1+\frac{d}{i}) \\
= \quad & (d^{n+1} - d^n) - (\prod_{i=1}^{n+1}(1+\frac{d}{i}) - \prod_{i=1}^{n}(1+\frac{d}{i})) + d^n - \prod_{i=1}^{n}(1+\frac{d}{i}) \\
\geq \quad & d^n(d-1) - \frac{d}{n+1}\prod_{i=1}^{n}(1+\frac{d}{i}) + (nd - 3n - 1) \\
\geq \quad & d^n(d-1) - \frac{d^n}{n+1}(1+d) + (nd - 3n - 1) \\
= \quad & d^n((d-1) - \frac{d+1}{n+1}) + (nd - 3n - 1) \\
\geq \quad & d^n((d-1) - \frac{d+1}{2}) + (nd - 3n - 1) \\
= \quad & \frac{d^n(d-3)}{2} + (nd - 3n - 1) \\
\geq \quad & (d-3) + (nd - 3n - 1) \\
= \quad & (n+1)d - 3(n+1) - 1.
\end{aligned}$$

□

Corollary 1.3. *Let A be the K-algebra as defined in Theorem* 1.1*, and assume that $n \geq 3$. Then*

$$\operatorname{reg}(A) \leq e(A) - \operatorname{codim}(A).$$

Proof. If $n \geq 3$, then the assertion follows from Theorem 1.1 together with Lemma 1.2 because $e(A) = e(S^{(d)}) = d^{n-1}$ and $\operatorname{codim}(A) \leq \binom{n+d-1}{n-1} - n$. □

2. Simplicial semigroup rings with isolated singularity

Let C be a positive affine semigroup of rank n, i.e., the associated group $\mathbb{Z}C$ is isomorphic to $\mathbb{Z}^n$ and $\{0\}$ is the only subgroup contained in C. Let G be the minimal set of generators of C. We say that C is standard graded if there exists a hyperplane $H \subset \mathbb{Z}C \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $G \subset H$. Let P be the convex hull of G in $\mathbb{Z}C \otimes_{\mathbb{Z}} \mathbb{Q}$. We say that C is *simplicial* if P is a simplex. Let $v_1, \dots, v_n$ be the vertices of P. After the choice of a basis of $\mathbb{Z}C$, the vertices v_i can be identified with integral vectors. Let

$$A = (v_1^t, \dots, v_n^t)$$

be the $n \times n$ matrix whose columns are the transpose of the vertices v_i. We denote by A^* the adjoint matrix of A. Let $\delta = \det(A)$. Then $\delta \neq 0$ and $A^*A = \delta E_n$, where E_n is the unit matrix of size n.

Let $\varphi\colon \mathbb{Z}C \to \mathbb{Z}^n$ be the linear map associated with A^*. It then follows that the simplex $P' = \varphi(P) \subset \mathbb{Z}^n$ has the vertices $\delta\varepsilon_i$, where ε_i denotes the ith standard unit vector of $\mathbb{Q}^n$. Let $C' = \varphi(C)$ and $G' = \varphi(G)$. Note that C' is isomorphic to C and that G' is the minimal set of generators of C' with $\{\delta\varepsilon_1, \dots, \delta\varepsilon_n\} \subset G' \subset P'$. Let t be the greatest common divisor of all the components of all the vectors belonging to G', and set $d = \delta/t$. Denote by $C'' \subset \mathbb{Z}$ the semigroup generated by $G'' = \frac{1}{t}G'$.

Then it is clear that the convex hull of G'' is the simplex with vertices $d\varepsilon_1, \dots, d\varepsilon_n$, that $C'' \cong C$ and that $[\mathbb{Z}^n : \mathbb{Z}C''] = d$. We call d the *index* of C. It is in fact an invariant of C, i.e., does not depend on the particular basis of $\mathbb{Z}C$ which was chosen to define the matrix A. We say that the simplicial semigroup C'' is standard embedded.

Theorem 2.1. *Let C be a simplicial semigroup of rank $n > 1$ with index $d > 2$. Let K be a field and $K[C]$ the semigroup ring associated with C. Suppose that $K[C]$ is a K-algebra with isolated singularity. Then*

$$\operatorname{reg}(K[C]) \leq (d-2)(n-1).$$

Proof. Since $K[C] \cong K[C']$ we may assume that the embedding of C itself is standard. Let $[n] = \{1, \dots, n\}$. Write $\mathbf{x}^{\mathbf{w}} = x_1^{w_1} \cdots x_n^{w_n}$ if $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$.

For each $1 \leq i \neq j \leq d$, we write q_i^j (≥ 1) for the biggest integer satisfying

$$d\varepsilon_i + \frac{1}{q_i^j}(d\varepsilon_j - d\varepsilon_i) \in G.$$

Since the localization

$$K[C]_{x_i^d} = K[x_i^d, \frac{1}{x_i^d}][\{\frac{\mathbf{x}^{\mathbf{w}}}{x_i^d}\}_{\mathbf{w}\in G}] = K[x_i^d, \frac{1}{x_i^d}][\{\mathbf{x}^{\mathbf{w}-d\varepsilon_i}\}_{\mathbf{w}\in G}]$$

is regular if and only if

$$K[C]_{x_i^d} = K[x_i^d, \frac{1}{x_i^d}][\{\mathbf{x}^{\frac{1}{q_i^j}(d\varepsilon_j - d\varepsilon_i)}\}_{j\in[n]\setminus\{i\}}],$$

and since $K[C]$ is a K-algebra with isolated singularity, it follows that, for any $\mathbf{w} \in G$ and for any $1 \leq i \leq n$, there exists $0 \leq p_j \in \mathbb{Z}$, $j \in [n] \setminus \{i\}$, such that

$$\mathbf{w} - d\varepsilon_i = \sum_{j\in[n]\setminus\{i\}} \frac{p_j}{q_i^j}(d\varepsilon_j - d\varepsilon_i).$$

This simple observation yields the crucial result that $q_i^j = q_k^\ell$ for all i, j, k, ℓ with $i \neq j$ and $k \neq \ell$. In fact, in case of $1 \leq i \neq j \leq n$, since

$$(d\varepsilon_j + \frac{1}{q_j^i}(d\varepsilon_i - d\varepsilon_j)) - d\varepsilon_i = \frac{p}{q_i^j}(d\varepsilon_j - d\varepsilon_i), \qquad 0 \leq p \in \mathbb{Z},$$

one has $q_i^j = q_j^i(q_i^j - p)$. Thus q_j^i divides q_i^j. Similarly, q_i^j divides q_j^i. Hence $q_i^j = q_j^i$. Also, in case of $i, k, \ell \in [n]$ with $i \neq k$, $k \neq \ell$ and $i \neq \ell$, since

$$d\varepsilon_k - d\varepsilon_i + \frac{1}{q_k^\ell}(d\varepsilon_\ell - d\varepsilon_k) = \sum_{j\in[n]\setminus\{i\}} \frac{p_j}{q_i^j}(d\varepsilon_j - d\varepsilon_i), \qquad 0 \leq p_j \in \mathbb{Z},$$

one has $\frac{1}{q_k^\ell} = \frac{p_\ell}{q_i^\ell}$. Thus q_k^ℓ divides q_i^ℓ. Similarly, q_i^ℓ divides q_k^ℓ. Hence $q_i^\ell = q_k^\ell$.

Let $q = q_i^j$ for all $1 \leq i \neq j \leq n$. Then $0 < \frac{d}{q} \in \mathbb{Z}$ divides each component of any vector belonging to G. Since the embedding of C is standard, it follows that $q = d$.

We now conclude that

$$K[C] = K[\{x_i^{d-1}x_j\}_{\substack{i=1,\dots,n\\ j=1,\dots,n}}, g_1, \dots, g_r], \tag{4}$$

where $g_1, \dots, g_r$ are monomials of degree d. Hence we are in the situation of Theorem 1.1 with $f_i = x_i^{d-1}$ for $i = 1, \dots, n$. ∎

The following final result follows partly from Corollary 1.3.

Corollary 2.2. *Let $K[C]$ be a simplicial semigroup ring with isolated singularity. Then*

$$\operatorname{reg}(K[C]) \leq e(K[C]) - \operatorname{codim}(K[C]).$$

Proof. Fix a standard embedding of C. Let $\operatorname{rank} C = n$. For $n \geq 3$, the assertion follows from (4) and Corollary 1.3.

Now let $n = 2$. Then

$$K[C] = K[\{x_1^{d-a_i} x_2^{a_i}\}_{i=0,\ldots,r+1}],$$

with $0 = a_0 < 1 = a_1 < a_2 < \cdots < d-1 = a_r < d = a_{r+1}$.

Therefore, $e(K[C]) = d$, and $\operatorname{codim}(K[C]) = r$. Thus we need to show that $\operatorname{reg}(K[C]) \leq d - r$, or equivalently, that

$$K[C]_{d-r} = K[\{x_1^{(d-r)d-j} x_2^j\}_{j=0,\ldots,(d-r)d}].$$

Set $k = d - r$, and let $X = \{j\colon x_1^{kd-j} x_2^j \in K[C]_{d-r}\}$. Since $a_0 = 0$, it follows that

$$X = \{\sum_{i=1}^{r+1} k_i a_i \colon k_i \geq 0, \quad \sum_{i=1}^{r+1} k_i \leq k\},$$

and we have to show that $X = \{0, \ldots, kd\}$.

For any two integers $a \leq b$ we set $[a, b] = \{c \in \mathbb{Z}\colon a \leq c \leq b\}$. Fix a number $j \in \{0, \ldots, k\}$. Then $a_i + jd \in X \cap [jd, (j+1)d]$ for $i = 0, \ldots, r+1$.

Next we notice that $a_i + ja_{r+1} + la_1 = a_i + jd + l \in X$ for $l = 0, \ldots, k-1-j$, and that $a_{i+1} + la_r + (j-l)a_{r+1} = a_{i+1} + jd - l \in X$ for $l = 0, \ldots, j$. Thus we see that

$$[a_i + jd, a_i + jd + (k-1-j)] \cup [a_{i+1} + jd - j, a_{i+1} + jd] \subset X. \tag{5}$$

Since

$$(a_{i+1} + jd - j) - (a_i + jd + (k-1-j)) = (a_{i+1} - a_i) - (k-1) \leq (d-r) - (d-r-1) = 1,$$

it follows that

$$[a_i + jd, a_{i+1} + jd] = [a_i + jd, a_i + jd + (k-1-j)] \cup [a_{i+1} + jd - j, a_{i+1} + jd],$$

so that by (5), $[a_i + jd, a_{i+1} + jd] \in X$ for all $i = 0, \ldots, r$ and all $j = 0, \ldots, k$. Since $[0, kd] = \bigcup_{\substack{i=0,\ldots,r \\ j=0,\ldots,k}} [a_i + jd, a_{i+1} + jd]$, the assertion follows. □

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