

CAT(0) SPACES ON WHICH A CERTAIN TYPE OF SINGULARITY IS BOUNDED

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Abstract

In this paper, we will consider a family \mathcal{Y} of complete CAT(0) spaces such that the tangent cone $TC_p Y$ at each point $p \in Y$ of each $Y \in \mathcal{Y}$ is isometric to a (finite or infinite) product of the Euclidean cones $\text{Cone}(X_\alpha)$ over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces. Each element of such \mathcal{Y} is a space presented by Gromov [4] as an example of a “CAT(0) space with “bounded” singularities”. We will show that the Izeki-Nayatani invariants of spaces in such a family are uniformly bounded from above by a constant strictly less than 1.

1. Introduction

In [4], Gromov introduced the term “CAT(0) space with ‘bounded’ singularities”, and remarked that there exist infinite groups which admit no uniform embeddings into such a space. He used this terminology without providing its precise definition, but as examples of such spaces, he presented CAT(0) spaces Y such that the tangent cone $TC_p Y$ at each point $p \in Y$ is isometric to a (finite or infinite) product of Euclidean cones $\text{Cone}(X_\alpha)$ over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces.

On the other hand, Izeki and Nayatani [5] defined an invariant $\delta(Y) \in [0, 1]$ of a complete CAT(0) space Y . And some general results for CAT(0) spaces whose Izeki-Nayatani invariants are bounded from above were proved by Izeki, Kondo, and Nayatani ([5], [6], [7], [8], [9]). Group Γ is said to have the *fixed-point property* for a metric space Y , if for any group homomorphism $\rho: \Gamma \rightarrow \text{Isom}(Y)$ there exists a point $p \in Y$ such that $\rho(\gamma)p = p$ for all $\gamma \in \Gamma$. Izeki, Kondo and Nayatani [7] proved that a certain random group has the fixed-point property for all elements Y of a family \mathcal{Y} of CAT(0) spaces whose Izeki-Nayatani invariants are uniformly bounded from above by a constant strictly less than 1:

$$\sup\{\delta(Y) \mid Y \in \mathcal{Y}\} < 1.$$

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Moreover, it is straightforward to see that an expander admits no uniform embedding into a complete CAT(0) space Y with $\delta(Y) < 1$ (see [9]). Combining this with Gromov's argument in [4], the existence of infinite groups which admit no uniform embeddings into a space Y with $\delta(Y) < 1$ follows. This seems to suggest that the Izeki-Nayatani invariant measures a certain type of "singularity" similar to Gromov's notion.

Although these general results were proved, the computation of the Izeki-Nayatani invariant is difficult. It is still unclear what kind of CAT(0) spaces Y or families \mathcal{Y} of CAT(0) spaces have the boundedness property as above. It had been even unknown whether there exists a complete CAT(0) space Y with $\delta(Y) = 1$ or not, until Kondo [9] showed the existence of CAT(0) spaces with $\delta = 1$ fairly recently.

In this paper, we prove the following theorem.

THEOREM 1.1. *Let \mathcal{Y} be a family of complete CAT(0) spaces such that the tangent cone $TC_p Y$ at each point $p \in Y$ on each $Y \in \mathcal{Y}$ is isometric to a (finite or infinite) product of the Euclidean cones $\text{Cone}(X_\alpha)$ over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of complete CAT(1) spaces. Then we have*

$$\sup_{Y \in \mathcal{Y}} \delta(Y) < 1.$$

Here, we use the word *product* of Euclidean cones T_1, T_2, \dots in the sense of ℓ^2 -product of the pointed metric spaces $(T_1, O_1), (T_2, O_2), \dots$, where each O_n is the cone point of T_n . That is, the product T of the cones T_1, T_2, \dots consists of all sequences $(x_n)_n$ such that $x_n \in T_n$ and $\sum_n d_n(O_n, x_n)^2 < \infty$, and T is equipped with the metric function d defined by

$$d(x, y)^2 = \sum_{n=1}^{\infty} d_n(x_n, y_n)^2$$

for any $x = (x_1, x_2, \dots) \in T$ and any $y = (y_1, y_2, \dots) \in T$, where d_n is the metric function on T_n for each n . Then, T also has a cone structure with the cone point $O = (O_1, O_2, \dots)$. And completeness and CAT(0) condition are preserved by this construction.

Combining Theorem 1.1 with the general results mentioned above, we have the following corollary.

COROLLARY 1.2. **(i)** *If Y is a complete CAT(0) space such that the tangent cone at each point $y \in Y$ is isometric to a (finite or infinite) product of Euclidean cones $\text{Cone}(X_\alpha)$ over elements X_α of some Gromov-Hausdorff precompact family $\{X_\alpha\}$ of CAT(1) spaces, then there exists infinite groups which admit no uniform embeddings into Y . **(ii)** *There exist infinite groups which has the fixed-point property for all elements Y in such a family \mathcal{Y} as in Theorem 1.1.**

Here, (i) has already been remarked in [4]. And (ii) follows from the general result in [7]. (ii) can be stated in terms of random groups (see [7]).

In the end of this paper, we claim that by the same technique used in the proof of Theorem 1.1, we can prove a more general statement, which includes Theorem 1.1 as a special case (Proposition 5.4).

2. Preliminaries on CAT(0) spaces

In this section we recall some basic definitions and facts concerning CAT(0) spaces. For a detailed exposition, we refer the reader to [1], [2] or [11].

For $\kappa > 0$ let M_κ^2 denote the simply connected, complete 2-dimensional Riemannian manifold of constant Gaussian curvature κ , and let d_κ be its distance function. Let $D_\kappa \in (0, \infty]$ be the diameter of M_κ^2 .

Let (Y, d_Y) be a metric space. A *geodesic* in Y is an isometric embedding γ of a closed interval $[a, b]$ into Y . A *geodesic triangle* in Y is a triple $\triangle = (\gamma_1, \gamma_2, \gamma_3)$ of geodesics $\gamma_i : [a_i, b_i] \rightarrow Y$ such that

$$\gamma_1(b_1) = \gamma_2(a_2), \quad \gamma_2(b_2) = \gamma_3(a_3), \quad \gamma_3(b_3) = \gamma_1(a_1).$$

If \triangle has a perimeter less than $2D_\kappa : \sum_{i=1}^3 |b_i - a_i| < 2D_\kappa$, then there is a geodesic triangle

$$\triangle^\kappa = (\gamma_1^\kappa, \gamma_2^\kappa, \gamma_3^\kappa), \quad \gamma_i : [a_i, b_i] \rightarrow M_\kappa^2$$

in M_κ^2 , which has the same side lengths as \triangle . This triangle \triangle^κ is unique up to isometry of M_κ^2 , and we call it the *comparison triangle* of \triangle in M_κ^2 . Then \triangle is said to be κ -thin if

$$d_Y(\gamma_i(s), \gamma_j(t)) \leq d_\kappa(\gamma_i^\kappa(s), \gamma_j^\kappa(t))$$

whenever $i, j \in \{1, 2, 3\}$ and $s \in [a_i, b_i]$, and $t \in [a_j, b_j]$.

DEFINITION 2.1. A metric space (Y, d) is called a *CAT(κ) space*, if for any pair of points $p, q \in Y$ with $d(p, q) < D_\kappa$ there exists a geodesic from p to q , and any geodesic triangle in Y with perimeter $< 2D_\kappa$ is κ -thin.

Next, we recall the definition of the Euclidean cone. Let (X, d_X) be a metric space. The cone $\text{Cone}(X)$ over X is the quotient of the product $X \times [0, \infty)$ obtained by identifying all points in $X \times \{0\} \subset X \times [0, \infty)$. The point represented by $(x, 0)$ is called the *cone point* of $\text{Cone}(X)$ and we will denote this point by $O_{\text{Cone}(X)}$ in this paper. The cone distance $d_{\text{Cone}(X)}(v, w)$ between two points $v, w \in \text{Cone}(X)$ represented by $(x, t), (y, s) \in X \times [0, \infty)$ respectively, is defined by

$$d_{\text{Cone}(X)}(v, w) = \sqrt{t^2 + s^2 - 2ts \cos(\min\{\pi, d_X(x, y)\})}.$$

Then $(\text{Cone}(X), d_{\text{Cone}(X)})$ is a metric space, and we call it the *Euclidean cone* over (X, d_X) . It is known that a metric space (X, d_X) is a CAT(1) space if and only if $(\text{Cone}(X), d_{\text{Cone}(X)})$ is a CAT(0) space.

Suppose that Y is a CAT(0) space. Then by the definition of CAT(0) space, there is a unique geodesic joining any pair of points in Y . So, for any triple of points (p, q, r) in Y , it makes sense to denote by $\triangle(p, q, r)$ the geodesic triangle consisting of three geodesics joining each pair of the three points.

Let $\gamma : [a, b] \rightarrow Y$, $\gamma' : [a', b'] \rightarrow Y$ be two geodesics in a CAT(0) space Y such that

$$\gamma(a) = \gamma'(a') = p \in Y.$$

We define the *angle* $\angle_p(\gamma, \gamma')$ between γ, γ' as

$$\angle_p(\gamma, \gamma') = \lim_{t \rightarrow a, t' \rightarrow a'} \angle_p^0(\gamma(t), \gamma'(t')),$$

where $\angle_p^0(\gamma(t), \gamma'(t'))$ is the corresponding angle of the comparison triangle of $\triangle(p, \gamma(t), \gamma'(t'))$ in $M_0^2 = \mathbf{R}^2$. The existence of the limit follows from the definition of CAT(0) space.

DEFINITION 2.2. Let (Y, d_Y) be a complete CAT(0) space, and let $p \in Y$. We denote by $(S_p Y)^\circ$ the set of all geodesics $\gamma : [a, b] \rightarrow Y$ such that $\gamma(a) = p$. Then the angle \angle_p defines a pseudometric on $(S_p Y)^\circ$. The *space of directions* $S_p Y$ at p is the metric completion of the quotient space of $(S_p Y)$ where we identify any $x, y \in S_p Y$ with $\angle_p(x, y) = 0$. We define the *tangent cone* $TC_p Y$ of Y at p to be the Euclidean cone $\text{Cone}(S_p Y)$ over the space of directions at p .

If (Y, d_Y) is a complete CAT(0) space and if $p \in Y$, then it can be proved that the space of directions $S_p Y$ at p is a complete CAT(1) space. Hence, the tangent cone $TC_p Y$ at p is a complete CAT(0) space.

Finally, we recall some basic notions and facts about probability measures on a metric space (Y, d_Y) . In this paper, we will treat only finitely supported measures. Measure ν on Y is *finitely supported* if there exists a finite subset $S \subset Y$ such that $\nu(Y \setminus S) = 0$. We call the minimal subset S with such a property the *support* of ν , and denote it by $\text{supp}(\nu)$. We denote by $\mathcal{P}(Y)$ the set of all finitely supported probability measures on Y . If $\text{supp}(\nu) = \{p_1, \dots, p_n\}$, then ν can be represented as

$$(2.1) \quad \nu = \sum_{i=1}^n t_i \text{Dirac}_{p_i}$$

by nonnegative real numbers t_1, \dots, t_n with $\sum_{i=1}^n t_i = 1$, where Dirac_{p_i} stands for the Dirac measure at $p_i \in Y$. We will also use the notation $\mathcal{P}'(Y)$ to denote the subset of $\mathcal{P}(Y)$ consisting of all measures whose supports contain at least two

points. Let Z be a set and let $\phi : Y \rightarrow X$ be a map. Then for any $\nu \in \mathcal{P}(Y)$, we define the *pushforward* measure $\phi_*\mu$ on X as

$$\phi_*\nu(A) = \mu(\phi^{-1}(A)), \quad A \subset X$$

If we write ν as in the form (2.1), we can write $\phi_*\nu$ as

$$\phi_*\nu = \sum_{i=1}^n t_i \operatorname{Dirac}_{\phi(p_i)}$$

If (Y, d_Y) is a complete CAT(0) space, and if $\nu \in \mathcal{P}(Y)$, there exists a unique point $\operatorname{bar}(\nu) \in Y$ which minimizes the function

$$y \mapsto \int_Y d(y, z)^2 \nu(dz)$$

defined on Y . This point is called the *barycenter* of ν . We refer the reader to [11] for the existence and uniqueness of barycenter.

3. Hilbert sphere valued maps and an invariant of a CAT(1) space

In this section, we define a certain invariant of complete CAT(1) spaces. First we set up some notations for Hilbert sphere valued maps on CAT(1) spaces. Let \mathcal{H} be a real Hilbert space, and let $\phi : X \rightarrow \mathcal{H}$ be a map whose image is contained in the unit sphere in \mathcal{H} . Thus $\|\phi(x)\| = 1$ for all $x \in X$. Let $\mu \in \mathcal{P}(X)$ be a finitely supported probability measure on X . We define the vector $\mathbf{E}_\mu[\phi] \in \mathcal{H}$ as

$$\mathbf{E}_\mu[\phi] = \int_X \phi(x) \mu(dx).$$

And if the vector $\mathbf{E}_\mu[\phi]$ is not the zero vector, we denote by $\tilde{\mathbf{E}}_\mu[\phi]$ the unit vector parallel to $\mathbf{E}_\mu[\phi]$:

$$\tilde{\mathbf{E}}_\mu[\phi] = \frac{1}{\|\mathbf{E}_\mu[\phi]\|} \mathbf{E}_\mu[\phi].$$

Then the value $\|\mathbf{E}_\mu[\phi]\| \in [0, 1]$ amounts to a sort of concentration of the pushforward measure $\phi_*\mu$ around $\tilde{\mathbf{E}}_\mu[\phi]$ on the unit sphere. By simple calculation, we have

$$(3.1) \quad \|\mathbf{E}_\mu[\phi]\| = \int_X \langle \tilde{\mathbf{E}}_\mu[\phi], \phi(x) \rangle \mu(dx)$$

whenever $\|\mathbf{E}_\mu[\phi]\| \neq 0$.

Now we define an invariant of a complete CAT(1) space by using the notations introduced above. This invariant is designed for estimating the Izeki-Nayatani invariant of a CAT(0) space, whose definition will be recalled in the next section.

DEFINITION 3.1. Let (X, d_X) be a metric space, and let $\mu \in \mathcal{P}(X)$. We define $\tilde{\delta}(\mu) \in [0, 1]$ to be

$$\tilde{\delta}(\mu) = \inf_{\phi} \|\mathbf{E}_{\mu}[\phi]\|^2,$$

where the infimum is taken over all maps $\phi : X \rightarrow \mathcal{H}$ to some Hilbert space \mathcal{H} such that

$$(3.2) \quad \|\phi(x)\| = 1, \quad \angle(\phi(x), \phi(y)) \leq d_X(x, y)$$

for any $x, y \in X$. Here and henceforth, we denote the angle between two vectors v, w in any Hilbert space by $\angle(v, w)$.

Suppose (X, d_X) is a complete CAT(1) space and $\iota : X \rightarrow \text{Cone}(X)$ is the canonical inclusion of X into its Euclidean cone. Then, we define $\tilde{\delta}(X)$ to be

$$\tilde{\delta}(X) = \sup\{\tilde{\delta}(\mu) \mid \mu \in \mathcal{P}(X), \text{bar}(\iota_*\mu) = O_{\text{Cone}(X)}\}.$$

When there is no measure satisfying such a condition, we define $\tilde{\delta}(X) = -\infty$.

To estimate this invariant in the proceeding sections, we will use the following fact:

LEMMA 3.2. Let (X, d_X) be a complete CAT(1) space. For $v, w \in \text{Cone}(X)$ represented by $(x, t), (y, s) \in X \times \mathbf{R}$ respectively, we set

$$\langle v, w \rangle = ts \cos(\min\{\pi, d_X(x, y)\}).$$

Then for any $\nu \in \mathcal{P}(\text{Cone}(X))$ the following two conditions are equivalent:

- (i) $\text{bar}(\nu) = O_{\text{Cone}(X)}$.
- (ii) $\int_{\text{Cone}(X)} \langle E_x, v \rangle \nu(dw) \leq 0$, whenever $x \in X$ and E_x is an element of $\text{Cone}(X)$ represented by $(x, 1)$.

Proof. For $w \in \text{Cone}(X)$ represented by $w = (y, s) \in X \times \mathbf{R}$, we write $\|w\| = s$. Fix $x \in X$ and let v_t be an element of $\text{Cone}(X)$ represented by $(x, t) \in X \times \mathbf{R}$. Suppose that $\text{bar}(\nu) = O_{\text{Cone}(X)}$. Then the function

$$(3.3) \quad \begin{aligned} F_x(t) &= \int_{\text{Cone}(X)} d_{\text{Cone}(X)}(v_t, w)^2 \nu(dw) \\ &= \int_{\text{Cone}(X)} \{t^2 + \|w\|^2 - 2t\langle E_x, w \rangle\} \nu(dw), \end{aligned}$$

defined on $[0, \infty)$ must attain its minimum at $t = 0$. This happens if and only if

$$F'_x(t) = 2 \left(t - \int_{\text{Cone}(X)} \langle E_x, w \rangle \nu(dw) \right) \geq 0.$$

for all $t \in \mathbf{R}$. So (ii) follows.

Conversely, if (ii) holds, then the function F_x on $[0, \infty)$ as (3.3) attains its minimum at $t = 0$ for each $x \in X$. And it is easily seen that $\text{bar}(v) = O_{\text{Cone}(X)}$. \square

In the final section, we will use this lemma in the following form.

COROLLARY 3.3. *Let (X, d_X) be a complete CAT(1) space, and let $\iota : X \rightarrow \text{Cone}(X)$ be the canonical inclusion. If $\mu \in \mathcal{P}(X)$ satisfies $\text{bar}(\iota_*\mu) = O_{\text{Cone}(X)}$, then we have*

$$\mu(\{y \in X \mid d_X(x, y) \leq \theta\}) \leq \frac{1}{1 + \cos \theta}$$

for any $x \in X$ and any $0 \leq \theta < \frac{\pi}{2}$. In particular, we have

$$\mu\left(\left\{y \in X \mid d_X(x, y) \leq \frac{\pi}{3}\right\}\right) \leq \frac{2}{3}$$

for all $x \in X$.

Proof. Suppose there is $x_0 \in X$ such that

$$\mu(\{y \in X \mid d_X(x_0, y) \leq \theta\}) > \frac{1}{1 + \cos \theta}.$$

Then we would have

$$\begin{aligned} & \int_X \cos(\min\{\pi, d_X(x_0, x)\})\mu(dx) \\ &= \int_{\{x \in X \mid d_X(x, x_0) \leq \theta\}} \cos(\min\{\pi, d_X(x_0, x)\})\mu(dx) \\ & \quad + \int_{X \setminus \{x \in X \mid d_X(x, x_0) \leq \theta\}} \cos(\min\{\pi, d_X(x_0, x)\})\mu(dx) \\ &> \cos \theta \times \frac{1}{1 + \cos \theta} + (-1) \times \left(1 - \frac{1}{1 + \cos \theta}\right) \\ &= 0. \end{aligned}$$

This implies $\text{bar}(\iota_*\mu) \neq O_{\text{Cone}(X)}$ by Lemma 3.2, which is a contradiction. \square

4. Izeki-Nayatani invariant

In this section, we recall the definition of the invariant δ of a complete CAT(0) space introduced by Izeki and Nayatani [5]. We will then derive a relation between δ and the invariant $\tilde{\delta}$ of a complete CAT(1) space defined in the previous section. More information about the Izeki-Nayatani invariant δ can be found in [5], [6], [7], [8] and [10].

DEFINITION 4.1 ([5]). Let (Y, d_Y) be a complete CAT(0) space. Recall that $\mathcal{P}'(Y)$ is the subset of $\mathcal{P}(Y)$ consisting of all measures whose supports contain at least two points. For any $\nu \in \mathcal{P}'(Y)$, we define $\delta(\nu)$ to be

$$\delta(\nu) = \inf_{\phi} \frac{\left\| \int_Y \phi(p) \nu(dp) \right\|^2}{\int_Y \|\phi(p)\|^2 \nu(dp)},$$

where the infimum is taken over all maps $\phi : \text{supp}(\nu) \rightarrow \mathcal{H}$ from the support of ν to some Hilbert space \mathcal{H} such that

$$(4.1) \quad \|\phi(p)\| = d(\text{bar}(\nu), p),$$

$$(4.2) \quad \|\phi(p) - \phi(q)\| \leq d(p, q)$$

for all $p, q \in \text{supp}(\nu)$. Then the Izeki-Nayatani invariant $\delta(Y)$ of Y is defined by

$$\delta(Y) = \sup\{\delta(\nu) \mid \nu \in \mathcal{P}'(Y)\}.$$

By definition, we have $0 \leq \delta(\nu) \leq 1$ and $0 \leq \delta(Y) \leq 1$. When Y is a Euclidean cone, we define $\delta(Y, O_Y) \in [0, 1]$ to be

$$\delta(Y, O_Y) = \sup\{\delta(\nu) \mid \nu \in \mathcal{P}'(Y), \text{bar}(\nu) = O_Y\},$$

where O_Y is the cone point of Y . When there is no measure satisfying such a condition, we define $\delta(Y, O_Y) = -\infty$. The following lemma is shown in [5].

LEMMA 4.2 ([5]). Suppose that Y is a complete CAT(0) space, and $\nu \in \mathcal{P}'(Y)$. Then we have

$$\delta(\nu) \leq \delta(TC_{\text{bar}(\nu)} Y, O_{TC_{\text{bar}(\nu)} Y}).$$

In particular, we have

$$\delta(Y) \leq \sup\{\delta(TC_p Y, O_{TC_p Y}) \mid p \in Y\}.$$

The following lemma is a slight generalization of Proposition 6.5 in [5].

LEMMA 4.3. Let $(T_1, d_1), (T_2, d_2), (T_3, d_3), \dots$ be complete CAT(0) spaces which are isometric to Euclidean cones, and let O_1, O_2, \dots be their cone points respectively. Let T be the cone obtained as the product of T_1, T_2, \dots with the cone point $O = (O_1, O_2, \dots)$. Then we have

$$\delta(T, O) = \sup_n \delta(T_n, O_n).$$

Proof. The following proof is almost the same argument as in the proof of Proposition 6.5 in [5]. We however include it for the sake of completeness.

First, the inequality $\delta(T, O) \geq \sup_n \delta(T_n, O_n)$ is obvious. Because we have the canonical isometric embedding $\mathcal{J}_n : T_n \rightarrow T$ for each n , and for each $\mu \in \mathcal{P}'(T_n)$ with $\text{bar}(\mu) = O_n$, it is easy to see that $\text{bar}(\mathcal{J}_{n*}\mu) = O$ and $\delta(\mu) = \delta(\mathcal{J}_{n*}\mu)$.

Let

$$\mu = \sum_{i=1}^m t_i \operatorname{Dirac}_{v_i} \in \mathcal{P}'(T)$$

be an arbitrary measure in $\mathcal{P}'(T)$ with $\operatorname{bar}(\mu) = O$, where $v_1, \dots, v_m \in T$ and $t_1, \dots, t_m > 0$ with $\sum_{i=1}^m t_i = 1$. Write $v_i = (v_i^{(1)}, v_i^{(2)}, \dots)$ and let

$$\mu_n = \sum_{i=1}^m t_i \operatorname{Dirac}_{v_i^{(n)}} \in \mathcal{P}'(T_n), \quad n = 1, 2, \dots$$

Then $\operatorname{bar}(\mu_n) = O_n$ for each n . Because if we have $\operatorname{bar}(\mu_n) \neq O_n$ for some n , it is easy to show that

$$\int_T d(w, B)^2 \mu(dw) < \int_T d(w, O)^2 \mu(dw),$$

where $B \in T$ is a point in T such that all of its components are the cone points but $\operatorname{bar}(\mu_n)$ for the n -th component, and it contradicts the assumption that $\operatorname{bar}(\mu) = O$.

Let $\varepsilon > 0$ be an arbitrary positive number. By the definition of $\delta(T_n, O_n)$, there exists a map $\phi_n : \operatorname{supp}(\mu_n) \rightarrow \mathcal{H}_n$ from the support of μ_n to some Hilbert space \mathcal{H}_n with the properties (4.1) and (4.2) with respect to μ_n , satisfying

$$\frac{\|\int_{T_n} \phi_n(v) \mu_n(dv)\|^2}{\int_{T_n} \|\phi_n(v)\|^2 \mu_n(dv)} \leq \delta(T_n, O_n) + \varepsilon.$$

We define a map $\phi : \operatorname{supp}(\mu) \rightarrow \mathcal{H}$ from the support of μ to the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ to be

$$\phi(v_i) = (\phi_1(v_i^{(1)}), \phi_2(v_i^{(2)}), \dots), \quad i = 1, \dots, m.$$

Then it is straightforward to see that ϕ satisfies the properties (4.1) and (4.2) with respect to μ . And we have

$$\begin{aligned} \delta(\mu) &\leq \frac{\|\int_T \phi(v) \mu(dv)\|^2}{\int_T \|\phi(v)\|^2 \mu(dv)} = \frac{\sum_{n=1}^{\infty} \|\sum_{i=1}^m t_i \phi_n(v_i^{(n)})\|^2}{\sum_{n=1}^{\infty} \sum_{i=1}^m t_i \|\phi_n(v_i^{(n)})\|^2} \\ &\leq \sup_n \frac{\|\sum_{i=1}^m t_i \phi_n(v_i^{(n)})\|^2}{\sum_{i=1}^m t_i \|\phi_n(v_i^{(n)})\|^2} \leq \sup_n (\delta(T_n, O_n) + \varepsilon). \end{aligned}$$

Since this holds for an arbitrary $\varepsilon > 0$ and an arbitrary $\mu \in \mathcal{P}'(T)$ with $\operatorname{bar}(\mu) = O$, we have $\delta(T, O) \leq \sup_n \delta(T_n, O_n)$. \square

For a CAT(1) space X , we prove the following relation between $\delta(\text{Cone}(X), O_{\text{Cone}(X)})$ and $\tilde{\delta}(X)$

PROPOSITION 4.4. *Let (X, d_X) be a complete CAT(1) space. Then we have*

$$\delta(\text{Cone}(X), O_{\text{Cone}(X)}) \leq \tilde{\delta}(X).$$

Before proving Proposition 4.4, we establish the following two lemmas.

LEMMA 4.5. *Let (X, d_X) be a complete CAT(1) space. Let*

$$v = \sum_{i=1}^m t_i \text{Dirac}_{v_i} \in \mathcal{P}'(\text{Cone}(X)),$$

where $v_i \in \text{Cone}(X)$ for $i = 1, \dots, m$ and $t_1, \dots, t_m > 0$ with $\sum_{i=1}^m t_i = 1$. Suppose that $\text{bar}(v) = O_{\text{Cone}(X)}$. If $v_1 = O_{\text{Cone}(X)}$ and if

$$v' = \sum_{i=2}^m \frac{t_i}{1 - t_1} \text{Dirac}_{v_i},$$

then $\text{bar}(v') = O_{\text{Cone}(X)}$ and $\delta(v) \leq \delta(v')$.

Proof. The former assertion follows immediately from Lemma 3.2. Let $\phi' : \text{supp}(v') \rightarrow \mathcal{H}$ be a map from the support of v' to some Hilbert space \mathcal{H} satisfying (4.1) and (4.2) with respect to v' . Define $\phi : \text{supp}(v) \rightarrow \mathcal{H}$ by

$$\begin{aligned} \phi(v_1) &= 0, \\ \phi(v_i) &= \phi'(v_i), \quad i = 2, \dots, m. \end{aligned}$$

Then ϕ satisfies (4.1) and (4.2) with respect to v . Moreover, an easy computation shows that

$$\frac{\|\int_{\text{Cone}(X)} \phi(v) v(dv)\|^2}{\int_{\text{Cone}(X)} \|\phi(v)\|^2 v(dv)} \leq \frac{\|\int_{\text{Cone}(X)} \phi'(v) v'(dv)\|^2}{\int_{\text{Cone}(X)} \|\phi'(v)\|^2 v'(dv)}.$$

Hence, by the definition of δ , the latter assertion follows. \square

LEMMA 4.6. *Let (X, d_X) be a complete CAT(1) space and let*

$$v = \sum_{i=1}^m t_i \text{Dirac}_{[x_i, r_i]} \in \mathcal{P}'(\text{Cone}(X)),$$

where $[x_i, r_i]$ is the point on $\text{Cone}(X)$ represented by $(x_i, r_i) \in X \times [0, \infty)$. Suppose that $\alpha > 0$, $l \in \{1, 2, \dots, m-1\}$, and

$$v' = \frac{1}{\sum_{i=1}^l \frac{t_i}{\alpha} + \sum_{i=l+1}^m t_i} \left(\sum_{i=1}^l \frac{t_i}{\alpha} \text{Dirac}_{[x_i, \alpha r_i]} + \sum_{i=l+1}^m t_i \text{Dirac}_{[x_i, r_i]} \right).$$

Then $\text{bar}(v') = O_{\text{Cone}(X)}$ if and only if $\text{bar}(v) = O_{\text{Cone}(X)}$. Moreover, if $\text{bar}(v) = \text{bar}(v') = O_{\text{Cone}(X)}$ and if $\alpha > 1$ (resp. $0 < \alpha < 1$), then the inequality $\delta(v) \leq \delta(v')$ holds if and only if

$$(4.3) \quad \alpha \frac{\sum_{i=1}^l t_i r_i^2}{\sum_{i=l+1}^m t_i r_i^2} \leq \frac{\sum_{i=1}^l t_i}{\sum_{i=l+1}^m t_i} \quad \left(\text{resp. } \alpha \frac{\sum_{i=1}^l t_i r_i^2}{\sum_{i=l+1}^m t_i r_i^2} \geq \frac{\sum_{i=1}^l t_i}{\sum_{i=l+1}^m t_i} \right).$$

Proof. The equivalence between $\text{bar}(v) = O_{\text{Cone}(X)}$ and $\text{bar}(v') = O_{\text{Cone}(X)}$ is an immediate consequence of Lemma 3.2. Assume that $\text{bar}(v) = \text{bar}(v') = O_{\text{Cone}(X)}$, and fix some real Hilbert space \mathcal{H} of dimension $\geq m$. Then there is a natural bijection $\phi \mapsto \phi'$ between the set of all maps from $\text{supp}(v)$ to \mathcal{H} satisfying (4.1) and (4.2) with respect to v , and the set of all maps from $\text{supp}(v')$ to \mathcal{H} satisfying (4.1) and (4.2) with respect to v' : it is given by

$$\begin{aligned} \phi'[x_i, \alpha r_i] &= \alpha \phi[x_i, r_i], \quad i = 1, \dots, l, \\ \phi'[x_i, r_i] &= \phi[x_i, r_i], \quad i = l+1, \dots, m. \end{aligned}$$

Let $\phi : \text{supp}(v) \rightarrow \mathcal{H}$ and $\phi' : \text{supp}(v') \rightarrow \mathcal{H}$ be the maps satisfying (4.1) and (4.2) with respect to v and v' respectively, and corresponding to each other under this bijection. Let

$$T = \frac{1}{\frac{1}{\alpha} \sum_{i=1}^l t_i + \sum_{i=l+1}^m t_i}.$$

Then we have

$$\frac{\| \int_{\text{Cone}(X)} \phi'(p) v'(dp) \|^2}{\int_{\text{Cone}(X)} \|\phi'(p)\|^2 v'(dp)} = T \frac{\| \sum_{i=1}^m t_i \phi[x_i, r_i] \|^2}{\alpha \sum_{i=1}^l t_i \|\phi[x_i, r_i]\|^2 + \sum_{i=l+1}^m t_i \|\phi[x_i, r_i]\|^2}.$$

Hence,

$$\begin{aligned} (4.4) \quad & \frac{\| \int_{\text{Cone}(X)} \phi'(p) v'(dp) \|^2}{\int_{\text{Cone}(X)} \|\phi'(p)\|^2 v'(dp)} - \frac{\| \int_{\text{Cone}(X)} \phi(p) v(dp) \|^2}{\int_{\text{Cone}(X)} \|\phi(p)\|^2 v(dp)} \\ &= \left\| \sum_{i=1}^m t_i \phi[x_i, r_i] \right\|^2 \frac{T \sum_{i=1}^m t_i r_i^2 - \alpha \sum_{i=1}^l t_i \|\phi[x_i, r_i]\|^2 - \sum_{i=l+1}^m t_i \|\phi[x_i, r_i]\|^2}{(\alpha \sum_{i=1}^l t_i \|\phi[x_i, r_i]\|^2 + \sum_{i=l+1}^m t_i \|\phi[x_i, r_i]\|^2) (\sum_{i=1}^m t_i r_i^2)}. \end{aligned}$$

We also have

$$(4.5) \quad \begin{aligned} & T \sum_{i=1}^m t_i r_i^2 - \alpha \sum_{i=1}^l t_i \|\phi[x_i, r_i]\|^2 - \sum_{i=l+1}^m t_i \|\phi[x_i, r_i]\|^2 \\ &= \frac{1-\alpha}{(1-\alpha)(\sum_{i=1}^l t_i) + \alpha} \left\{ \alpha \left(\sum_{i=l+1}^m t_i \right) \left(\sum_{i=1}^l t_i r_i^2 \right) - \left(\sum_{i=1}^l t_i \right) \left(\sum_{i=l+1}^m t_i r_i^2 \right) \right\}. \end{aligned}$$

By (4.4) and (4.5), the inequality

$$\frac{\|\int_{\text{Cone}(X)} \phi'(p) v'(dp)\|^2}{\int_{\text{Cone}(X)} \|\phi'(p)\|^2 v'(dp)} \geq \frac{\|\int_{\text{Cone}(X)} \phi(p) v(dp)\|^2}{\int_{\text{Cone}(X)} \|\phi(p)\|^2 v(dp)}$$

holds if and only if

$$\alpha \geq 1, \quad \alpha \left(\sum_{i=l+1}^m t_i \right) \left(\sum_{i=1}^l t_i r_i^2 \right) - \left(\sum_{i=1}^l t_i \right) \left(\sum_{i=l+1}^m t_i r_i^2 \right) \leq 0$$

or

$$0 < \alpha \leq 1, \quad \alpha \left(\sum_{i=l+1}^m t_i \right) \left(\sum_{i=1}^l t_i r_i^2 \right) - \left(\sum_{i=1}^l t_i \right) \left(\sum_{i=l+1}^m t_i r_i^2 \right) \geq 0.$$

The lemma follows easily from this equivalence and the bijectivity of the correspondence $\phi \leftrightarrow \phi'$. \square

Proof of Proposition 4.4. First suppose that $\mu \in \mathcal{P}(\text{Cone}(X))$, $\text{bar}(\mu) = O_{\text{Cone}(X)}$, and $\text{supp}(\mu) \subset \iota(X)$. Let $\iota : X \rightarrow \text{Cone}(X)$ be the canonical inclusion, and let $\iota^{-1} : \iota(X) \rightarrow X$ be the inverse map. Let $\tilde{\phi} : X \rightarrow \mathcal{H}$ be a map from X to some Hilbert space \mathcal{H} satisfying (3.2). Then the restriction $\phi = [\tilde{\phi} \circ \iota^{-1}]|_{\text{supp}(\mu)}$ of $\tilde{\phi} \circ \iota^{-1} : \iota(X) \rightarrow \mathcal{H}$ to $\text{supp}(\mu)$ satisfies (4.1) and (4.2). Moreover we have

$$\|\mathbf{E}_{\iota_*^{-1}\mu}[\tilde{\phi}]\|^2 = \frac{\|\int_{\text{Cone}(X)} \phi(v) \mu(dv)\|^2}{\int_{\text{Cone}(X)} \|\phi(v)\|^2 \mu(dv)}.$$

Hence by the definitions of $\tilde{\delta}(\iota_*^{-1}\mu)$ and $\delta(\mu)$, we have

$$\delta(\mu) \leq \tilde{\delta}(\iota_*^{-1}\mu).$$

Thus, if we prove the existence of $v' \in \mathcal{P}(\text{Cone}(X))$ such that

$$(4.6) \quad \delta(v) \leq \delta(v'), \quad \text{supp}(v') \subset \iota(X)$$

for any

$$v = \sum_{i=1}^m t_i \operatorname{Dirac}_{[x_i, r_i]} \in \mathcal{P}'(\operatorname{Cone}(X))$$

with $\bar{v} = O_{\operatorname{Cone}(X)}$, then the desired assertion follows. Here, we can assume $r_i > 0$ for all $i \in \{1, \dots, m\}$ by Lemma 4.5. And, if $r_1 = r_2 = \dots = r_m$, we can take

$$v' = \sum_{i=1}^m t_i \operatorname{Dirac}_{[x_i, 1]},$$

and v' satisfies (4.6) because it is straightforward that $\delta(v) = \delta(v')$. So we can assume $r_1 = \dots = r_l < r_{l+1} \leq \dots \leq r_m$ without loss of generality. Then we have

$$\left(\frac{\sum_{i=1}^l t_i}{\sum_{i=l+1}^m t_i} \right) / \left(\frac{\sum_{i=1}^l t_i r_i^2}{\sum_{i=l+1}^m t_i r_i^2} \right) \geq \frac{r_{l+1}^2}{r_1^2} \geq \frac{r_{l+1}}{r_1}.$$

Hence, if we set

$$v_0 = \frac{1}{\frac{r_1}{r_{l+1}} \sum_{i=1}^l t_i + \sum_{i=l+1}^m t_i} \left(\sum_{i=1}^l \frac{r_1 t_i}{r_{l+1}} \operatorname{Dirac}_{[x_i, r_{l+1}]} + \sum_{i=l+1}^m t_i \operatorname{Dirac}_{[x_i, r_i]} \right),$$

then we have

$$\delta(v_0) \geq \delta(v)$$

by Lemma 4.6. Repeating this procedure, we finally get

$$v_1 = \sum_{i=1}^m s_i \operatorname{Dirac}_{[x_i, r_m]},$$

which satisfies $\delta(v_1) \geq \delta(v)$. If we set $v' = \sum_{i=1}^m s_i \operatorname{Dirac}_{[x_i, 1]}$, it is easily seen that $\delta(v') = \delta(v_1)$, and the assertion follows. \square

5. Proof of the theorem

Recall that the Gromov-Hausdorff precompactness is known to be equivalent to the uniformly total boundedness. We call the family \mathcal{X} of metric spaces *uniformly totally bounded* if the following two conditions are satisfied:

- There is a constant D such that $\operatorname{diam}(X) \leq D$ for all $X \in \mathcal{X}$.
- For any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that each $X \in \mathcal{X}$ contains a subset $S_{X, \varepsilon}$ with the following property: the cardinality of $S_{X, \varepsilon}$ is no greater than $N(\varepsilon)$ and X is covered by the union of all ε -balls whose centers are in $S_{X, \varepsilon}$.

By Lemma 4.2, Lemma 4.3 and Proposition 4.4, to prove Theorem 1.1 it suffices to prove the following proposition.

PROPOSITION 5.1. *Let (X, d_X) be a complete CAT(1) space. Assume that there exist $N \in \mathbf{N}$ and a subset $S = \{x_i\}_{i=1}^N \subset X$ such that X is covered by the union of all $\frac{\pi}{12}$ -balls whose centers are in S . Then there exists a constant $C(N) < 1$, depending only on N , such that*

$$\tilde{\delta}(X) < C(N).$$

Remark 5.2. It follows from the argument in the proof of Proposition 5.1, we can take

$$C(N) = \left(\frac{2}{3} + \frac{1}{3} \sqrt{\frac{e^{-\pi^2/36N} + 1}{2}} \right)^2.$$

as a constant $C(N)$ in the proposition.

Before proving Proposition 5.1, we will recall a well-known construction of a map from a Hilbert space to the unit sphere in another Hilbert space, and derive some necessary estimates for them. We follow Dadarlat and Guentner [3] to explain this construction. Let \mathcal{H} be a Hilbert space. Let

$$\text{Exp}(\mathcal{H}) = \mathbf{R} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots,$$

and define $\text{Exp} : \mathcal{H} \rightarrow \text{Exp}(\mathcal{H})$ by

$$\text{Exp}(\zeta) = 1 \oplus \zeta \oplus \left(\frac{1}{\sqrt{2!}} \zeta \otimes \zeta \right) \oplus \left(\frac{1}{\sqrt{3!}} \zeta \otimes \zeta \otimes \zeta \right) \oplus \cdots.$$

For $t > 0$, define a map G_t from \mathcal{H} to $\text{Exp}(\mathcal{H})$ to be

$$G_t(\zeta) = e^{-t\|\zeta\|^2} \text{Exp}(\sqrt{2t}\zeta).$$

Then simple computation shows that

$$(5.1) \quad \cos \angle(G_t(\zeta), G_t(\zeta')) = \langle G_t(\zeta), G_t(\zeta') \rangle = e^{-t\|\zeta - \zeta'\|^2}$$

for all $\zeta, \zeta' \in \mathcal{H}$. In particular, $\|G_t(\zeta)\| = 1$ for all $\zeta \in \mathcal{H}$. Hence we can regard G_t as a map from \mathcal{H} to the unit sphere in $\text{Exp}(\mathcal{H})$.

We need the following estimate to prove Proposition 5.1.

LEMMA 5.3. *Let (X, d_X) be a metric space, and let $F : X \rightarrow \mathcal{H}$ be an L -Lipschitz map ($L > 0$) to some Hilbert space. Suppose that $0 < tL^2 \leq \frac{1}{2}$. Then the map $\phi = G_t \circ F : X \rightarrow \text{Exp}(\mathcal{H})$ satisfies*

$$\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x, y)\}$$

for all $x, y \in X$.

Proof. By (5.1) and L -Lipschitz continuity of F , it is sufficient to show that

$$(5.2) \quad e^{-tL^2 d_X(x, y)^2} \geq \cos(\min\{\pi, d_X(x, y)\})$$

for all $x, y \in X$ and all $t \in \left(0, \frac{1}{2L^2}\right)$. When $d_X(x, y) \geq \frac{\pi}{2}$, (5.2) is obvious. So, if we put $a = tL^2$ and $d = d_X(x, y)$, then what we have to show is that

$$(5.3) \quad a \leq \frac{-\log(\cos d)}{d^2}$$

holds for any $a \in \left(0, \frac{1}{2}\right]$ and any $d \in \left[0, \frac{\pi}{2}\right)$. But this is obvious because the right-hand side of (5.3) is non-decreasing with respect to d . \square

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. First we define a map F_S from X to \mathbf{R}^N by

$$F_S(x) = (d_X(x, x_1), d_X(x, x_2), \dots, d_X(x, x_N))$$

for $x \in X$. Then F_S is \sqrt{N} -Lipschitz since

$$\|F_S(x) - F_S(y)\| = \left\{ \sum_{i=1}^N (d_X(x, x_i) - d_X(y, x_i))^2 \right\}^{1/2} \leq \sqrt{N} \cdot d_X(x, y).$$

On the other hand, by the definition of the subset S , for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$, there exist $i_0, i_1 \in \{1, \dots, N\}$ such that

$$\begin{aligned} d_X(x_{i_0}, x) &\geq \frac{\pi}{4}, & d_X(x_{i_0}, y) &\leq \frac{\pi}{12}, \\ d_X(x_{i_1}, y) &\geq \frac{\pi}{4}, & d_X(x_{i_1}, x) &\leq \frac{\pi}{12}. \end{aligned}$$

Hence

$$(5.4) \quad \begin{aligned} \|F_S(x) - F_S(y)\| &\geq \sqrt{(d_X(x_{i_0}, x) - d_X(x_{i_0}, y))^2 + (d_X(x_{i_1}, x) - d_X(x_{i_1}, y))^2} \\ &\geq \frac{\pi}{3\sqrt{2}} \end{aligned}$$

for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$.

We now set $\phi = G_{1/2N} \circ F_S : X \rightarrow \text{Exp}(\mathbf{R}^N)$. Then the all values of ϕ are contained in the unit sphere of $\text{Exp}(\mathbf{R}^N)$, and ϕ satisfies

$$\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x, y)\}$$

for all $x, y \in X$ by Lemma 5.3. Moreover (5.1) and (5.4) imply that

$$(5.5) \quad \angle(\phi(x), \phi(y)) \geq \arccos(e^{-\pi^2/36N})$$

for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$.

Set $\eta = \arccos(e^{-\pi^2/36N})$, and let μ be an arbitrary measure in $\mathcal{P}(X)$ with $\text{bar}(\iota_*\mu) = O_{\text{Cone}(X)}$, where $\iota: X \rightarrow \text{Cone}(X)$ is the canonical inclusion and $O_{\text{Cone}(X)}$ is the cone point of $\text{Cone}(X)$. Then we have

$$(5.6) \quad \phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right) \leq \frac{2}{3}$$

for any point v on the unit sphere in $\text{Exp}(\mathbf{R}^N)$, where

$$B\left(v, \frac{\eta}{2}\right) = \left\{u \in \text{Exp}(\mathbf{R}^N) \mid \|u\| = 1, \angle(v, u) < \frac{\eta}{2}\right\}.$$

This is because if there exists some vector $\phi(x_0)$ contained in $B\left(v, \frac{\eta}{2}\right) \cap \phi(X)$, then by (5.5) and Corollary 3.3 we have

$$\begin{aligned} \phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right) &\leq \phi_*\mu(B(\phi(x_0), \eta)) \\ &= \mu(\phi^{-1}(B(\phi(x_0), \eta))) \\ &\leq \mu\left(B\left(x_0, \frac{\pi}{3}\right)\right) \leq \frac{2}{3}, \end{aligned}$$

where $B\left(x_0, \frac{\pi}{3}\right)$ is the open ball in X centered at x_0 with radius $\frac{\pi}{3}$. In the case $B\left(v, \frac{\eta}{2}\right) \cap \phi(X) = \emptyset$, (5.6) obviously holds.

By (5.6), we have

$$\begin{aligned} \int_X \langle v, \phi(x) \rangle \mu(dx) &= \int_{\mathcal{S}} \langle v, u \rangle \phi_*\mu(du) \\ &= \int_{B(v, \eta/2)} \langle v, u \rangle \phi_*\mu(du) + \int_{\mathcal{S} \setminus B(v, \eta/2)} \langle v, u \rangle \phi_*\mu(du) \\ &\leq 1 \times \phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right) + \cos \frac{\eta}{2} \times \left\{1 - \phi_*\mu\left(B\left(v, \frac{\eta}{2}\right)\right)\right\} \\ &\leq 1 \times \frac{2}{3} + \left(\cos \frac{\eta}{2}\right) \times \frac{1}{3}, \end{aligned}$$

where \mathcal{S} is the unit sphere in $\text{Exp}(\mathbf{R}^N)$.

Setting $v = \tilde{\mathbf{E}}_\mu[\phi]$ in the above inequality and using (3.1), we have

$$\|\mathbf{E}_\mu[\phi]\| = \left\| \int_X \langle \tilde{\mathbf{E}}_\mu[\phi], \phi(x) \rangle \mu(dx) \right\| \leq c_N,$$

where

$$c_N = 1 \times \frac{2}{3} + \left(\cos \frac{\eta}{2}\right) \times \frac{1}{3} = \frac{2}{3} + \frac{1}{3} \sqrt{\frac{e^{-\pi^2/36N} + 1}{2}}$$

Thus, by the definition of $\tilde{\delta}(X)$,

$$\tilde{\delta}(X) \leq c_N^2 < 1$$

which proves the proposition. \square

Finally, we remark that the proof of Proposition 5.1 works for the following more general statement.

PROPOSITION 5.4. *Let $0 < \theta < \frac{\pi}{2}$, $0 < \alpha < 1$ and $\varepsilon > 0$. Let (X, d_X) be a complete CAT(1) space. Assume that there exists a finite subset $S \subset X$ such that*

$$\#\{s \in S \mid \|d_X(x, s) - d_X(y, s)\| \geq \varepsilon\} \geq \alpha \#S$$

whenever $x, y \in X$ and $d(x, y) \geq \theta$. Here, $\#S$ stands for the cardinality of S . Then there exists a constant $C = C(\theta, \alpha, \varepsilon) < 1$ such that

$$\tilde{\delta}(X) \leq C.$$

Proof. We denote the cardinality of S by N . Let F_S be the map from X to \mathbf{R}^N as in the proof of Proposition 5.1 with respect to our set S . Then F_S is \sqrt{N} -Lipschitz and we have

$$(5.7) \quad \|F_S(x) - F_S(y)\| \geq \sqrt{\alpha N} \varepsilon$$

for any $x, y \in X$ with $d_X(x, y) \geq \theta$. If we set $\phi = G_{1/2N} \circ F_S : X \rightarrow \text{Exp}(\mathbf{R}^N)$, then all the values of ϕ are contained in the unit sphere of $\text{Exp}(\mathbf{R}^N)$, and ϕ satisfies

$$\angle(\phi(x), \phi(y)) \leq \min\{\pi, d_X(x, y)\}$$

for all $x, y \in X$ by Lemma 5.3. Moreover (5.1) and (5.7) imply that

$$(5.8) \quad \angle(\phi(x), \phi(y)) \geq \arccos(e^{-\alpha\varepsilon^2/2})$$

for any $x, y \in X$ with $d_X(x, y) \geq \theta$.

Now the rest of the proof is done exactly in the same manner as in the proof of Proposition 5.1, and we have

$$\tilde{\delta}(X) \leq (c_{\theta, \alpha, \varepsilon})^2,$$

where

$$\begin{aligned} c_{\theta, \alpha, \varepsilon} &= 1 \times \frac{1}{1 + \cos \theta} + \left(\cos \frac{\arccos(e^{-\alpha\varepsilon^2/2})}{2} \right) \times \left(1 - \frac{1}{1 + \cos \theta} \right) \\ &= \frac{1}{1 + \cos \theta} + \sqrt{\frac{e^{-\alpha\varepsilon^2/2} + 1}{2}} \times \frac{\cos \theta}{1 + \cos \theta} < 1. \end{aligned} \quad \square$$

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