The Formal Specification Column

by

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Categorical Frameworks for Graph Transformation and HLR Systems based on the DPO Approach

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Abstract

Several variants of high-level replacement (HLR) and adhesive categories have been introduced in the literature as categorical frameworks for graph transformation and HLR systems based on the double pushout (DPO) approach. In addition to HLR, adhesive, and adhesive HLR categories several weak variants, especially weak adhesive HLR with horizontal and vertical variants, as well as partial variants, including partial map adhesive and partial VK square adhesive categories are reviewed and related to each other. We propose as weakest version the class of vertical weak adhesive HLR categories, short $\mathcal{M}$-adhesive categories, which are still sufficient to obtain most of the main results for graph transformation and HLR systems. The results in this paper are summarized in Fig. 1 showing a hierarchy of all these variants of adhesive, adhesive HLR, and $\mathcal{M}$-adhesive categories, which can be considered as different categorical frameworks for graph transformation and HLR systems.

1   Introduction

The concepts of adhesive [10] and adhesive high-level replacement (HLR) [4] categories have been a break-through for the double pushout approach (DPO) of algebraic graph transformations [6, 5, 11]. Almost all main results in the DPO approach have been formulated and proven in these categorical frameworks and instantiated to a large variety of HLR systems, especially graph transformation systems for different types of graphs. These main results include the Local Church-Rosser, Parallelism, and Concurrency Theorems, the Embedding and Extension Theorem, Completeness of Critical Pairs, and the Local Confluence Theorem [2].
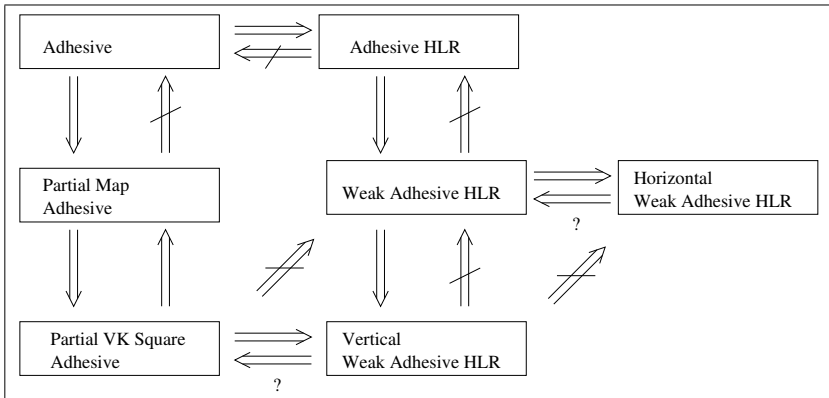

Figure 1: Hierarchy of Adhesive Categories

The notion of HLR systems has been introduced in [3] as categorical framework, based on a variety of axiomatic HLR properties, which have been used to prove the main results mentioned above. The concept of adhesive categories, introduced by Lack and Sobocinski [10] , allows to prove most of these HLR properties from the axioms of adhesive categories, which was an important step towards an elegant categorical framework. Instantiations of this framework include the categories of sets, graphs, typed graphs, and hypergraphs, but unfortunately not the categories of typed attributed graphs, algebraic specifications, and different kinds of Petri nets. In [4], adhesive categories have been generalized to adhesive HLR categories essentially by replacing the class of all monomorphisms in adhesive categories by a suitable subclass M of all monomorphisms. This allows to obtain also typed attributed graphs and algebraic specifications as instantiations of adhesive HLR categories, where the class M is a proper subclass of all monomorphisms. This was another important improvement, because typed attributed graphs are most important to model visual languages and various aspects

of model driven software development.

It turned out, however, that place/transition nets and several other variants of Petri nets do not satisfy the axioms of adhesive HLR categories [2]. For this reason, weak adhesive HLR categories were introduced, which still allow to prove the same kind of HLR properties mentioned above. Weak adhesive HLR means that pushouts along $\mathcal{M}$-morphisms are weak van Kampen (VK) squares. This means that the corresponding VK properties – roughly spoken compatibility of pushouts and pullbacks – hold for VK cubes, where all horizontal or all vertical morphisms are $\mathcal{M}$-morphisms.

These two cases are called in this paper horizontal resp. vertical weak adhesive HLR. In fact, all Petri net categories we considered so far are weak adhesive HLR and all HLR properties mentioned above have been shown also for weak adhesive HLR categories in [2]. Recently, we recognized by inspection of the proofs that it is sufficient to consider vertical weak adhesive HLR categories in order to obtain all these HLR properties. For this reason, we propose in this paper to use the short name $\mathcal{M}$-adhesive category for this important class of vertical weak adhesive HLR categories. Note, that this is in contrast to some other papers like [1], where weak adhesive HLR categories are called $\mathcal{M}$-adhesive.

Another variant of adhesive categories has been introduced by Heindel [8], called partial map adhesive categories. They are based on the requirement that pushouts along $\mathcal{M}$-morphisms are hereditary [9]. This means that such pushouts are preserved by the inclusion functor from the category **C** into the category of partial maps over **C**. A partial map from A to B is a span ($A \xleftarrow{m} A' \xrightarrow{f} B$) of morphisms in **C** with $m \in \mathcal{M}$, where A' is unique up to isomorphism. In [8], it is pointed out that the existence of an $\mathcal{M}$-partial map classifier, which means that the inclusion functor has a right adjoint, is an interesting sufficient condition for pushouts to be hereditary. Hence we have a sufficient condition for partial map adhesiveness, because left adjoint functors preserve colimits and especially pushouts.

On the other hand, it is shown in [8] that hereditary pushouts can be characterized by partial VK squares, which can be seen as a variant of weak VK squares in vertical weak adhesive HLR categories, i.e. in $\mathcal{M}$-adhesive categories. In fact, it is easy to see that partial VK square adhesiveness implies vertical weak adhesiveness, but the inverse implication is open. In [8], it is shown that all the relevant HLR properties are valid in partial map adhesive categories. Actually, this is now a consequence of the fact that partial map adhesive categories are also vertical weak adhesive, which – as mentioned above – is sufficient to show these HLR properties.

Summarizing, we will review in this paper all the variants of HLR and adhesive categories mentioned above leading to the hierarchy shown in Fig. 1.

2   From Adhesive to $\mathcal{M}$-Adhesive Categories

Adhesive categories have been introduced by Lack and Sobocinski [10] as a categorical framework for graph transformation systems, which allows to verify the variety of axiomatic HLR properties required for the theory of high-level replacement (HLR) systems in [3]. Adhesive categories are based on van Kampen (VK) squares as pushouts which are – roughly spoken – stable under pullbacks and vice versa.
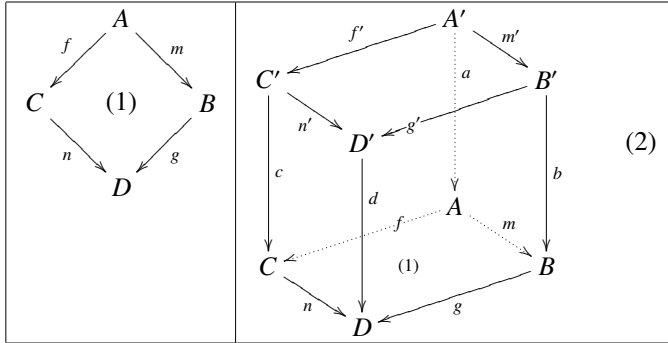
Figure 2: Pushout and VK cube for VK property.

Definition 2.1 (Adhesive Category and VK Square)**.** *A category* **C** *is called an* adhesive category*, if:*

1. **C** *has pushouts along monomorphisms.*

2. **C** *has pullbacks.*

3. *Pushouts in* **C** *along monomorphisms are* VK *squares, i.e. the following VK property holds for all commutative cubes with the given pushout in the bottom and the back squares being pullbacks (see Fig. 2). The* VK *property is the following equivalence:*
   The top square is a pushout if and only if the front squares are pullbacks.

    Important examples of adhesive categories are the categories **Sets** of sets, **Graphs** of graphs, and **Graphs**$_{TG}$ of typed graphs over a type graph TG, while the category **AGraphs**$_{\textbf{ATG}}$ of typed attributed graphs is not adhesive. But in the latter case, pushouts along $\mathcal{M}$-morphisms are VK squares for the class $\mathcal{M}_{mono-iso}$ of all monos, which are isomorphic in the data type part. In fact, **AGraphs**$_{ATG}$ is an adhesive HLR category in the following sense:

Definition 2.2 (Adhesive HLR Category and PO-PB Compatibility)**.** *Given a PO-PB compatible class $\mathcal{M}$ of monomorphisms in* **C** *(see below), then* $(\mathbf{C}, \mathcal{M})$ *is called* adhesive HLR category, *if pushouts along $\mathcal{M}$-morphisms are VK squares (see Def. 2.1).*

   A class $\mathcal{M}$ of monomorphisms in **C** *is called* PO-PB compatible, *if*

1. *Pushouts along $\mathcal{M}$-morphisms exist and $\mathcal{M}$ is stable under pushouts.*

2. *Pullbacks along $\mathcal{M}$-morphisms exist and $\mathcal{M}$ is stable under pullbacks.*

3. *$\mathcal{M}$ contains all identities and is closed under composition.*

Remark. *A PO-PB compatible class $\mathcal{M}$ contains all isomorphisms and is also closed under decomposition, i.e. $g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$. In fact, isomorphisms can be obtained by pushouts or pullbacks along identities, and the pullback of $(g \circ f, g)$ is (id, f) s.t. pullback closure of $\mathcal{M}$ implies $f \in \mathcal{M}$ [8]. It can be shown that the class $\mathcal{M}_{mono}$ of all monomorphisms in an adhesive category* **C** *is PO-PB compatible [10], such that $(\mathbf{C}, \mathcal{M}_{mono})$ is also an adhesive HLR category leading to the implication "adhesive implies adhesive HLR" in Fig. 1. This implication is proper, because* **AGraphs**$_{ATG}$ *is not adhesive, but (* **AGraphs**$_{ATG}$, $\mathcal{M}_{mono-iso}$*) is adhesive HLR [2].*

   However, there are important examples, like the category (**PTNets**, $\mathcal{M}_{mono}$) of place/transition nets with the class $\mathcal{M}_{mono}$ of all monos, which are not adhesive HLR, but only weak adhesive in the following sense:

Definition 2.3 (Weak Adhesive HLR Category)**.** *Given a PO-PB compatible class $\mathcal{M}$ of monomorphisms in* **C** *then (* **C**, $\mathcal{M}$*) is called* weak adhesive HLR category, *if it is horizontal and vertical weak adhesive HLR in the following sense:*

1. *(* **C**, $\mathcal{M}$*) is called* horizontal weak adhesive HLR, *if pushouts in* **C** *along $m \in \mathcal{M}$ exist and they are* horizontal weak VK squares, *i.e. the VK square property (see Def. 2.1) is only required for commutative cubes, where all horizontal morphisms $(f, g, m, n, f', g', m', n')$ are in $\mathcal{M}$.*

2. *(* **C**, $\mathcal{M}$*) is called* vertical weak adhesive HLR, *if pushouts in* **C** *along $m \in \mathcal{M}$ exist and they are* vertical weak VK squares, *i.e. the VK property (see Def. 2.1) is only required for commutative cubes, where all vertical morphisms (a, b, c, d) are in $\mathcal{M}$.*

Remark. *In the horizontal case it is sufficient to assume $f, m \in \mathcal{M}$, because closure under pushouts and pullbacks implies that all horizontal morphisms are in $\mathcal{M}$. Similar in the vertical case it is sufficient to assume in addition to $m \in \mathcal{M}$ also $b, c, d \in \mathcal{M}$, because this implies $a \in \mathcal{M}$ and also $n, m', n' \in \mathcal{M}$ by*

*closure under pushouts and pullbacks. By definition we have the implications on the right hand side of Fig. 1 between the different adhesive HLR properties. The category (**PTNets**, $\mathcal{M}_{mono}$) shows that the implication from "adhesive HLR" to "weak adhesive HLR" is proper. Later we will conclude that also the implication from "weak adhesive HLR" to "vertical weak adhesive HLR" is proper, but it is open, whether also the corresponding one to "horizontal weak adhesive HLR" is proper.*

The following HLR properties, which are essential to prove the main results for graph transformation and HLR systems mentioned in the introduction, are valid not only for weak adhesive HLR categories as shown in [2], but already for vertical weak adhesive HLR categories. For this reason we propose the short notion "$\mathcal{M}$- adhesive category" for this case.

Definition 2.4 (M-Adhesive Category). *An $\mathcal{M}$-adhesive category ($\mathbf{C}$, $\mathcal{M}$) is a short notation for a vertical weak adhesive HLR category. The corresponding vertical weak VK squares are called $\mathcal{M}$-VK squares.*

Since the proof of the HLR properties in [2] Thm. 4.26 – stated for weak adhesive HLR categories – uses only the axioms of vertical weak adhesive categories, which are now called $\mathcal{M}$-adhesive, we can conclude the following main result:

Theorem 2.5 (HLR Properties in $\mathcal{M}$-Adhesive Categories). *Given an $\mathcal{M}$-adhesive category ($\mathbf{C}$, $\mathcal{M}$) the following HLR properties are valid:*

1. *Pushouts along $\mathcal{M}$-Morphisms are Pullbacks.*

2. *$\mathcal{M}$-Pushout-Pullback Decomposition Lemma*

3. *Cube Pushout-Pullback Lemma*

4. *Uniqueness of Pushout Complements*

These HLR properties are essential to prove the main results for graph transformation and HLR systems in [2].

3 Partial Map and Partial VK Square Adhesive Categories

Heindel proposes in [8] to reconsider hereditary pushouts of [9] in connection with adhesive and weak adhesive HLR categories. Hereditary pushouts in a category $\mathbf{C}$ are those pushouts that are preserved by the embedding into the associated category $\mathbf{Par}_{\mathcal{M}}(\mathbf{C})$ of partial maps over $\mathbf{C}$. Heindel has shown that hereditary pushouts

can be characterized by a variant of VK squares, called partial VK squares, which are closely related to weak VK squares in weak adhesive HLR categories. This leads to the new concept of partial map adhesive categories, introduced by Heindel in [8], which are equivalent to partial VK square adhesive categories. Moreover we will discuss the relationship with the concepts of adhesiveness introduced in Section 2.

The concepts in this section are based on an admissable class $\mathcal{M}$ of monos according to [8], which we call PB compatible in anology to PO-PB compatible classes in Def. 2.1.

Definition 3.1 (Partial Map Category and PB Compatibility). *Given a PB compatible class $\mathcal{M}$ of monos in $\mathbf{C}$ (see below) the category $\mathbf{Par}_{\mathcal{M}}(\mathbf{C})$ of $\mathcal{M}$-partial maps over $\mathbf{C}$, called* partial map category, *has the same objects as $\mathbf{C}$ and morphisms from A to B are (isomorphism classes of) spans $(A \overset{m}{\leftarrow} A' \overset{f}{\rightarrow} B)$ of morphisms in $\mathbf{C}$ with $m \in \mathcal{M}$, called* partial maps $(m, f) : A \rightarrow B$ in $\mathbf{Par}_{\mathcal{M}}(\mathbf{C})$. *Identities are identical spans and composition of spans is defined by pullbacks.*

A class $\mathcal{M}$ of monos in $\mathbf{C}$ is called PB compatible, *if*

1. *Pullbacks along $\mathcal{M}$-morphisms exist and $\mathcal{M}$ is stable under pullbacks.*

2. *$\mathcal{M}$ contains all identities and is closed under composition.*

Remark. *A PB compatible class $\mathcal{M}$ is PO-PB compatible, if pushouts along $\mathcal{M}$-morphisms exist and $\mathcal{M}$ is stable under pushouts.*

There is an inclusion functor $I : \mathbf{C} \rightarrow \mathbf{Par}_{\mathcal{M}}(\mathbf{C})$, called graphing functor in [8], which is the identity on objects and maps each morphism $f : A \rightarrow B$ in $\mathbf{C}$ to the partial map $I(f) = (id, f) : A \rightarrow B$ in $\mathbf{Par}_{\mathcal{M}}(\mathbf{C})$.

Now we are able to define hereditary pushouts and partial map adhesive categories.

Definition 3.2 (Hereditary Pushout and Partial Map Adhesive Category). *Given a PB compatible class $\mathcal{M}$ of monos in $\mathbf{C}$, then a pushout in $\mathbf{C}$ is called* hereditary, *if it is preserved by the inclusion functor $I : \mathbf{C} \rightarrow \mathbf{Par}_{\mathcal{M}}(\mathbf{C})$.*

The category $(\mathbf{C}, \mathcal{M})$ is called partial map adhesive, *if pushouts along $\mathcal{M}$-morphisms exist and are hereditary.*

Remark. *If $\mathbf{C}$ has pushouts along $\mathcal{M}$-morphisms a sufficient condition for $(\mathbf{C}, \mathcal{M})$ to be partial map adhesive is the existence of a cofree construction leading to a right adjoint functor R for the inclusion functor $I : \mathbf{C} \rightarrow \mathbf{Par}_{\mathcal{M}}(\mathbf{C})$. In this case I is left adjoint and preserves all colimits, especially pushouts along $\mathcal{M}$-morphisms, such that these pushouts are hereditary. In the case $(\mathbf{C}, \mathcal{M}) = (\mathbf{Sets}, \mathcal{M}_{mono})$ we have that $\mathbf{Par}_{\mathcal{M}}(\mathbf{C})$ is isomorphic to the category of partial functions, where*

R(X) is the extension of a set X by one distinguished ("undefined") element.In several examples like **Sets** *the construction of a right adjoint R is much easier than the explicit verification of VK properties. Moreover, it is shown in [8] that in partial map adhesive categories double pushout transformation in (C, M) can be considered as single pushout transformation in* **Par**$_M$**(C)** *with total match and comatch.*

Hereditary pushouts are important in connection with adhesive categories, because it has been shown in [8] that they are equivalent to partial VK squares in the following sense:

Definition 3.3 (Partial VK Square Adhesive Category)**.** *Given a PB compatible class M of monos in* **C***, then (***C***, M) is called* partial VK square adhesive*, if it has pushouts along m* ∈ *M which are* partial VK squares*, i.e.for any commutative cube with the given pushout in the bottom (see Def. 2.1 and Fig. 2), where the back squares are pullbacks with b, c in M, the following equivalence holds: The top square is a pushout if and only if the front squares are pullbacks with d in M.*

Remark. *Partial VK squares are closely related to vertical weak VK squares in Def. 2.3, but the assumption resp. conclusion concerning d in M is different. Both will be compared below.*

The following theorem shows that partial VK square and partial map adhesive categories are in fact equivalent. The proof is sketched in [8] based on results in [7]. Moreover, the highly non-trivial proof has been verified independently by the authors of this paper. As indicated in [8], the statement and proof remains valid if "pushout along *M*-morphisms" is replaced by arbitrary "pushouts".

Theorem 3.4 (Equivalence of Partial Map and Partial VK Square Adhesive Categories)**.** *Given a PB compatible class M of monos in* **C** *such that* **C** *has pushouts along M-morphisms. Then pushouts along M-morphisms are hereditary if and only if they are partial VK squares.*

Remark. *Based on [7] it is shown in [8] that adhesive categories are also partial map adhesive for M = M$_{mono}$. Moreover, an example of a category* **lSet** *of list sets is given in [8], which is partial map adhesive, but not adhesive. Together with Thm. 3.4 this leads to the implication and equivalence on the left hand side of Fig. 1. It remains to analyse the relationship with the different adhesive HLR notions.*

In [8], it is shown that PB compatibility already implies PO-PB compatibility such that we have the following equivalence:

Theorem 3.5 (Equivalence of PB and PO-PB Compatibility)**.** *In partial VK square adhesive categories (***C***, M) PB compatibility is equivalent to PO-PB compatibility.*

Proof. It suffices to show that in partial VK square adhesive categories M is stable under pushouts. Given a pushout with $m \in M$ in the bottom of the partial VK cube in Fig. 3, which has pullbacks in the back squares with $id, f \in M$ and a pushout in the top. Hence the partial VK property implies that the front squares are pullbacks with $n \in M$. Hence M is stable under pushouts. $\square$
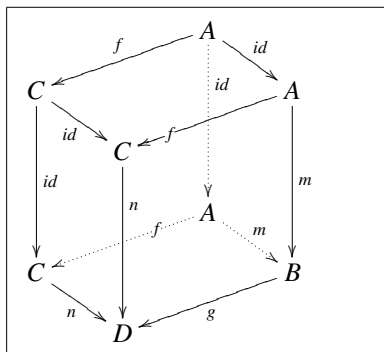

Figure 3: Special partial VK-cube.

Now we study the relationship between partial VK square and M-adhesive categories.

Theorem 3.6 (Partial VK Square Adhesive Categories are M-Adhesive)**.** *Given a partial VK square adhesive category (***C***, M) with PB compatible M then (***C***, M) is an M-adhesive category with PO-PB compatible M.*

Proof. Given a partial VK square adhesive category (**C**, M) then M is already PO-PB compatible by Thm. 3.5. Moreover, pushouts along M-morphisms satisfy the vertical weak VK square property, because we only have to consider cubes with $a, b, c, d \in M$ in contrast to partial VK squares, where the equivalence holds under the assumption $a, b, c \in M$. Hence (**C**, M) is vertical weak adhesive HLR and hence also M-adhesive by definition. $\square$

Remark. *By Thm. 3.6 we have the implication from "partial VK square adhesive" to "M-adhesive" in Fig. 1. But it is open whether the implication is an equivalence. Moreover, Heindel [8] has shown that his example* **lSet** *of list sets is partial map adhesive – and hence partial VK square adhesive – but violates*

the property that pushouts of pairs of M-morphisms are VK squares. Hence this is a counterexample for horizontal weak adhesive HLR, and hence also for weak adhesive HLR categories. This implies that there is no implication from "partial VK square adhesive" to "weak adhesive HLR" in Fig. 1. This explains also that there are no implications from "M-adhesive" to "weak adhesive HLR" and to "horizontal weak adhesive HLR. The remaining implications are open.

4   Conclusion

In this paper we have reviewed several variants of adhesive and adhesive HLR categories and discussed how they are related to each other. The main results are summarized in Fig. 4. For graph transformations and high-level replacement (HLR) systems it is most important that the HLR properties in Thm. 2.5 are valid, which are essential for the main results including the Local Church-Rosser, Parallelism, and Concurrency Theorems, the Embedding and Extension Theorem, Completeness of Critical Pairs, and the Local Confluence Theorem [2]. These HLR properties are valid already in vertical weak adhesive HLR categories, short M-adhesive categories, where M is a PO-PB compatible class of monomorphisms. The latter ones are closely related – but probably slightly weaker – then partial map adhesive categories. They are recommended by Heindel [8], based on heraditary pushouts advocated by Kennaway [9]. A useful sufficient condition for the existence of hereditary pushouts is the existence of a right adjoint for the inclusion functor from **C** into the corresponding partial map category $\mathbf{Par}_M(\mathbf{C})$.
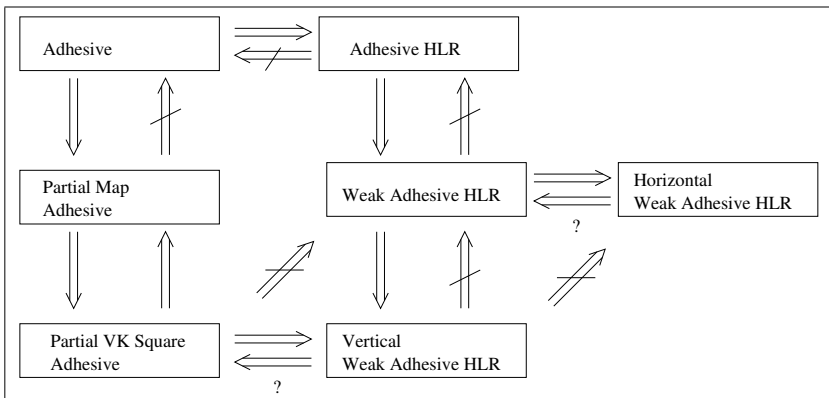

Figure 4: Hierarchy of Adhesive Categories

Altogether we advocate now $\mathcal{M}$-adhesive categories as main categorical framework for graph transformation and HLR systems, while we have used (weak) adhesive HLR categories in our book [2].

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