CATEGORICITY IN POWER(1)

BY
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Introduction. A theory, Σ, (formalized in the first order predicate calculus) is *categorical in power* κ if it has exactly one isomorphism type of models of power κ. This notion was introduced by Łoś [9] and Vaught [16] in 1954. At that time they pointed out that a theory (e.g., the theory of dense linearly ordered sets without end points) may be categorical in power $\aleph_0$ and fail to be categorical in any higher power. Conversely, a theory may be categorical in every uncountable power and fail to be categorical in power $\aleph_0$ (e.g., the theory of algebraically closed fields of characteristic 0). Łoś then raised the following question.

Is a theory categorical in one uncountable power necessarily categorical in every uncountable power?

The principal result of this paper is an affirmative answer to that question. We actually prove a stronger result, namely: If a theory is categorical in some uncountable power then every uncountable model of that theory is *saturated.* (Terminology used in the Introduction will be defined in the body of the paper; roughly speaking, a model is saturated, or *universal-homogeneous,* if it contains an element of every possible elementary type relative to its subsystems of strictly smaller power.) It is known(2) that a theory can have (up to isomorphism) at most one saturated model in each power. It is interesting to note that our results depend essentially on an analogue of the usual analysis of topological spaces in terms of their derived spaces and the Cantor-Bendixson theorem.

The paper is divided into five sections.

In §1 terminology and some meta-mathematical results are summarized. In particular, for each theory, Σ, there is described a theory, Σ^*, which has essentially the same models as Σ but is "neater" to work with.

In §2 is defined a topological space, $S(A)$, corresponding to each subsystem, A, of a model of a theory, Σ; the points of $S(A)$ being the "isomorphism types" of elements with respect to A. With each monomorphism (= isomorphic imbedding), $f: A \to B$, is associated a "dual" continuous map, $f^*: S(B) \to S(A)$. Then there is defined for each $S(A)$ a decreasing sequence $\{S^\alpha(A)\}$ of subspaces which is analogous to (but different from)

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(1) Except for minor emendations this paper is identical with the author's doctoral dissertation submitted to the University of Chicago in August 1962.

(2) Cf. [10] where the result was shown to follow from the more general result of [5].

the usual sequence of derived spaces in a topological space(3). The basic difference is that for us the definition of "derived space" will involve not only $S(A)$ but all of its inverse images under maps of the type, $f^*: S(B) \to S(A)$; that is, not only A but every system A can be imbedded into. It is well known that those topological spaces whose αth derived space vanishes at some ordinal α have particularly simple properties. Similarly, those theories, Σ, such that for some ordinal α, $S^\alpha(A)$ vanishes for every A which is a subsystem of a model of Σ have particularly simple properties. We have chosen to call such theories totally transcendental. Theorem 2.8, which is an analogue of the Cantor-Bendixson theorem, states that totally transcendental theories are characterized by a certain countability condition.

§3 gathers together some results depending on Ramsey's theorem. In particular, Theorem 3.8 states that any theory categorical in an uncountable power is totally transcendental. Much of §3 is related to the results of Ehrenfeucht and Mostowski [3] and Ehrenfeucht [1] and [2].

Some properties of models of totally transcendental theories are established in §4. These have to do with the existence of prime models and the existence of sets of *indiscernible* elements.

Finally §5 applies the results of the preceding sections to solve the problem of Łoś.

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1. **Preliminaries.** Ordinals are defined so that each ordinal is equal to the set of smaller ordinals. Cardinals are those ordinals not set-theoretically equivalent to any preceding ordinal. We use the Greek letters $\alpha, \beta, \gamma, \cdots$ to denote ordinals, reserving δ for limit ordinals; λ and κ will always denote cardinals and m and n non-negative integers. κ^+ denotes the least cardinal $> \kappa$. The cardinality of a set X is denoted by $\kappa(X)$. An infinite cardinal κ is *regular* if for every $\beta < \kappa$ and every well-ordered set $[\lambda_\alpha; \alpha < \beta]$ of cardinals with each $\lambda_\alpha < \kappa, \sum_{\alpha<\beta} \lambda_\alpha < \kappa$. In much that follows finite cardinals will present anomalous cases; therefore, we shall use the notation $\kappa = \kappa'$ (modulo $\aleph_0$) to mean $\kappa + \aleph_0 = \kappa' + \aleph_0$.

A relational system, $A = \langle |A|, R_i^A \rangle_{i \in I}$ is a set $|A|$ together with an indexed set $\{R_i^A\}_{i \in I}$ of finitary relations on $|A|$. Then $|A|$ is the *universe* of $A, \kappa(A) = \kappa(|A|)$, the power of A, R_i^A the ith relation of A, and I the *index* set of A. If $\tau \in \omega^I$ and each R_i^A is a $\tau(i)$-ary relation, then τ is the *similarity type* of A. Suppose A and B are systems of similarity type τ. Then a map $f: |A| \to |B|$ is a *monomorphism* if f is one-one, and, for each

(3) As defined, for example, in [7, pp. 126-134].

$i \in I$ and $a_1, \cdots, a_{\tau(i)} \in A$, $R_i^A a_1, \cdots, a_{\tau(i)}$ if and only if $R_i^B f(a_1), \cdots, f(a_{\tau(i)})$. If a monomorphism maps A *onto* B it is an *isomorphism* and A is isomorphic to $B(A \cong B)$. If $|A| \subseteq |B|$ and the identity map is a monomorphism of A into B then A is a *subsystem* of $B(A \subseteq B)$. Corresponding to each $X \subseteq |A|$, there is a unique subsystem of A with universe X, denoted by $A|X$.

In certain auxiliary constructions it is convenient to consider *generalized relation systems* which have in addition to finitary relations, a set of *distinguished elements* and a set of finitary *operations*. The preceding concepts may be extended to generalized relation systems in an obvious fashion. In particular, a subsystem will always contain all the distinguished elements and be closed under all the operations.

Corresponding to each similarity type τ is a first order (with identity) language, L_τ. The symbols of L_τ are the usual logical connectives: $\sim$, $\vee, \wedge, \rightarrow, \leftrightarrow$; quantifiers: $\exists, \forall$; an equality sign: $=$; a denumerable set of variables: $v_0, v_1, \cdots$; and a $\tau(i)$-ary relation symbol, R_i, for each $i \in I$. (Corresponding to generalized relation systems we have *generalized languages* which have, in addition to the preceding symbols, individual constants and operation symbols.) The language, L_τ is *countable* if it has only a countable number of symbols. The reader is assumed familiar with the notion of *term* and *formula* in such a language. An *open* formula is a formula containing no quantifiers. A *sentence* is a formula with no free variables. A *universal* sentence is a sentence in prenex form containing no existential quantifiers. If ψ is a formula of L_τ with no free variables other than $v_0, \cdots, v_{n-1}$, A is a system of type τ, and $a_0, \cdots, a_{n-1} \in A$; then $\vdash_A \psi(a_0, \cdots, a_{n-1})$ means that $a_0, \cdots, a_{n-1}$ satisfies ψ in A (in the usual sense) when v_m denotes a_m. If $t(v_1, \cdots, v_n)$ is a term of L_τ and $\vdash_A a_0 = t(a_1, \cdots, a_n)$, then we say a_0 is the value of the term t when v_m denotes $a_m (m \leq n)$ and write $a_0 = t^A(a_1, \cdots, a_n)$. A consistent set, Σ, of sentences of L is a theory of L_τ. A system, A (of similarity type τ), is a *model* of Σ if for every $\sigma \in \Sigma$, $\vdash_A \sigma$. If ψ is a sentence of L_τ, $\vdash_\Sigma \psi$ means that for every model A of Σ, $\vdash_A \psi$. The theory Σ is *complete* if for every sentence ψ of L, either $\vdash_\Sigma \psi$ or $\vdash_\Sigma \sim \psi$. If Σ is a theory having an infinite model and κ is an infinite cardinal then Σ is *categorical* in power κ (κ-categorical) if all models of Σ of power κ are isomorphic. By a result of Vaught [16] and Łoś [9], if Σ is κ-categorical and has no finite models then Σ is complete.

If A is a system of type τ and $X \subseteq |A|$ we may form a new system $(A, a)_{a \in X}$ by taking each element of X as a distinguished element. We denote by $L(A)$ the language corresponding to the similarity type of $(A, a)_{a \in |A|}$. (The symbols of $L(A)$ are the symbols of L together with a new individual constant $\bar{a}$ for each $a \in A$ (4).) The *diagram* of $A, \mathscr{D}(A)$, is the

(4) To avoid all ambiguities one should write $\bar{a}^A$ rather than $\bar{a}$; however, in our uses the A will always be clear from context.

set of all open sentences (i.e., formulas without variables) of $L(A)$ which are valid in $(A,a)_{a\in|A|}$. If A and B are systems of type τ then A is *elementary equivalent* to B if A and B are models of the same complete theory of L_τ $(A\equiv B)$. If $X\subseteq|A|$ and f is a mapping of X into $|B|$ then f is an *elementary monomorphism* $((A,x)_{x\in X}\equiv(B,f(x))_{x\in X})$ if for every $x_0,\cdots,x_n\in X$ and every formula, ψ, of L_τ, $\vdash_A\psi(x_0,\cdots,x_n)$ implies $\vdash_B\psi(f(x_0),\cdots,f(x_n))$.

Suppose $A=\langle A,R_i^A\rangle_{i\in I}$ is a relation system of type τ. For each formula ψ of L_τ, if m is the smallest number such that the free variables of ψ are among $v_0,\cdots,v_{m-1}$, then we denote by ψ^A the m-ary relation on $|A|$ such that $\psi^Aa_0,\cdots,a_{m-1}$ if and only if $\vdash_A\psi(a_0,\cdots,a_{m-1})$. Then define

$$A^*=(A,\psi^A)_{\psi\in}\qquad\text{formulas of } L_\tau.$$

Let τ^* be similarity type of A^*. If Σ is a theory in L_τ define Σ^* as those sentences ψ of L_{ρ^*} such that $\vdash_{A^*}\psi$ for every model A of Σ. The next lemma follows easily from these definitions.

Lemma 1.1. (a) *A' is a model of Σ^* if and only if there is a model A of Σ such that $A^*=A'$.*

(b) *$A\cong B$ if and only if $A^*\cong B^*$.*

(c) *If A and B are models of Σ, $X\subseteq|A|$, and f a map of X into B, then $(A,x)_{x\in X}\equiv(B,f(x))_{x\in X}$ if and only if the map $f:A^*|X\to B^*$ is a monomorphism.*

(d) *Σ is κ-categorical if and only if Σ^* is κ-categorical.*

(e) *If Σ is a theory in L_τ and ψ is a formula in L_{τ^*} having no free variables other than $v_0,\cdots,v_{n-1}$, then there is a relation symbol R of degree n in L_{τ^*} such that $\vdash_{\Sigma^*}\psi(v_0,\cdots,v_{n-1})\leftrightarrow R(v_0,\cdots,v_{n-1})$.*

For the case that Σ is a complete theory the following results were established in [10].

Lemma 1.2. *Suppose Σ is a complete theory in L_τ. Denote by $\mathscr{N}(\Sigma^*)$ the class of subsystems of models of Σ^*.*

(a) *Σ^* is a complete theory in L_{τ^*}.*

(b) *If $\{A_\alpha;\alpha<\delta\}$ is an increasing chain of members of $\mathscr{N}(\Sigma^*)$ then $\bigcup_{\alpha<\delta}A_\alpha\in\mathscr{N}(\Sigma^*)$. If each A_α is a model of Σ^* then the union is a model of Σ^*.*

(c) *If $A_1,A_2\in\mathscr{N}(\Sigma^*)$ then there is an $A_3\in\mathscr{N}(\Sigma^*)$ and monomorphisms $f_1:A_1\to A_3$ and $f_2:A_2\to A_3$.*

(d) *If $A_0,A_1,A_2\in\mathscr{N}(\Sigma^*)$ and $g_1:A_0\to A_1$ and $g_2:A_0\to A_2$ are monomorphisms, then there is an $A_3\in\mathscr{N}(\Sigma^*)$ and monomorphisms $f_1:A_1\to A_3$ and $f_2:A_2\to A_3$ such that $f_1g_1=f_2g_2$*(5).

(5) In [5] Jónsson considered classes of relation systems satisfying certain condition which he numbered I-VI. In order to apply Jónsson's result to an arbitrary complete theory, [10] devised the Σ^* theory and showed that $\mathscr{N}(\Sigma^*)$ satisfied Jósson's conditions. In Theorem 1.2, (b), (c) and (d) assert respectively that $\mathscr{N}(\Sigma^*)$ satisfies Jónsson's conditions V, III, and IV.

A notion that we shall find convenient is that of a category of maps. If $\mathcal{S}$ is a class of mathematical objects(6) then a *category* which object class $\mathcal{S}$ is a class, $\mathcal{C}$, of triples called maps (denoted by $f: A \to B$) where $A, B \in \mathcal{S}$, f is a function of $|A|$ into $|B|$, and such that (i) (identity: $A \to A) \in \mathcal{C}$ for each $A \in \mathcal{S}$ and (ii) if $(f: A \to B)$ and $(g: B \to C) \in \mathcal{C}$ then $(gf: A \to C) \in \mathcal{C}$. A is the *domain* and B the *co-domain* of $f: A \to B$(7).

2. **Transcendence in rank.** We shall be interested in elementary monomorphisms among subsystems of a complete theory Σ. By 1.1(c) it is therefore convenient to consider Σ^* instead of Σ. *Throughout the rest of this paper we shall adopt the following convention*: T will always denote a complete theory in a countable language, L, T has an infinite model, and there is a theory Σ such that $T = \Sigma^*$. We will denote the class of subsystems of models of T by $\mathcal{M}(T)$.

If $A \in \mathcal{M}(T)$ it follows from 1.1(e) that $T(A) = T \cup \mathcal{D}(A)$ is a complete theory in $L(A)$(8). We denote by $F(A)$ the set of formulas of $L(A)$ which have no free variable other than v_0. If the formulas of $F(A)$ which are equivalent in the theory $T(A)$ are identified (i.e., ψ is identified with ψ' if $\vdash_{T(A)} (\forall v_0)\psi \leftrightarrow \psi'$)(9) then $F(A)$ may be considered as a Boolean algebra with $\wedge$, $\vee$, and $\sim$ as $\cap$, $\cup$, and complementation respectively (10). A maximally consistent set of formulas in $F(A)$ will be a dual prime ideal (ultrafilter) in $F(A)$ considered as a Boolean algebra. The set of such dual prime ideals is the Stone space of $F(A)$ and will be denoted by $S(A)$. $S(A)$ is a Boolean space with a basis consisting of the sets.

$$U_\psi = \{p \in S(A); \psi \in p\} \qquad (\psi \in F(A)).$$

It follows that $S(A)$ has a basis of power $= \kappa(A)$ (modulo $\aleph_0$).

The space $S(A)$ may be thought of as the ways of extending $T(A)$ to a complete theory in a language having one more individual constant than $L(A)$ has. Suppose $A, B \in \mathcal{M}(T), B \supseteq A, b \in B$, and $\bar{b}$ is the constant in the $L(B)$ corresponding to b. We denote by $p_{b,B}$ the unique $p \in S(B)$ con-

(6) A "mathematical object," A, is a set $|A|$ with some associated structure. In every case in this paper an object is either a relational system or a topological space.

(7) It is more usual to abstract the composition properties of the maps and define a category as a class of elements with a binary operation defined for some pairs of elements and which satisfies certain axioms. Since we are interested not in categories, *per se*, but in certain instances of them, the definition we have given is more convenient.

(8) We could have chosen to present this entire section "syntactically" by considering, instead of the class $\mathcal{M}(T)$, the class of all complete extensions of T in languages which are extensions of L by the addition of new individual constants.

(9) It follows from 1.1(e) that we would get the same Boolean algebra if we assumed that $F(A)$ contained only open formulas. Notice that for open formulas $\vdash_A$ is equivalent to $\vdash_{T(\bar{A})}$, but for formulas in general the two are not equivalent unless A is a model of T.

(10) The close relationship between the properties of the various Boolean algebras of formulas of the language L and the model-theoretic properties of T has been observed by several authors. See especially [13] and [18].

taining the formula: $v_0 = \overline{b}$. Let $p_{b,B,A} = p_{b,B} \cap F(A)$. If $q \in S(A)$ we say b *realizes* q in B if $q = p_{b,B,A}$. Clearly, every $b \in B$ realizes some point of $S(A)$. By the Completeness Theorem every $p \in S(A)$ is realized in some extension of A. Suppose $B_1, B_2 \in \mathscr{N}(T), B_1 B_2 \supseteq A$ and $b_1 \in B_1, b_2 \in B_2$; then the map: $A \cup \{b_1\} \to A \cup \{b_2\}$ which is the identity on A and maps b_1 to b_2 is a monomorphism if and only if b_1 and b_2 realize the same point in $S(A)$. Thus, $S(A)$ is the set of "isomorphism types of elements with respect to A."

LEMMA 2.1. *If $A \in \mathscr{N}(T)$ then there is a model of $T, B, B \supseteq A$ such that each $p \in S(A)$ is realized in B.*

Proof. Let $\{p_\alpha; \alpha < \gamma\}$ be a well-ordered list of the points of $S(A)$. We assert there exists an increasing chain $\{B_\alpha; \alpha < \gamma\}$ of models of T such that each $B_\alpha \supseteq A$ and each p_β with $\beta < \alpha$ is realized in B_α. The proof is by induction on α. Assume the sequence exists for all $\beta < \alpha$. If α is a limit ordinal let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and the result follows from 1.2(b). Suppose $\alpha = \beta + 1$. By the Completeness Theorem there is a model of $T, C, C \supseteq A$ such that p_β is realized in C. By 1.2(d) there is a model of T, D, and monomorphisms $f_1: C \to D$ and $f_2: B_\beta \to D$ such that $f_1 = f_2$ on A. If we identify B_β with $f_2(B_\beta)$ then D may be taken as B_α. B_γ is the B satisfying the theorem.

Suppose that $A, B \in \mathscr{N}(T)$ and $f: A \to B$ is a monomorphism. Then f induces a monomorphism $\tilde{f}: F(A) \to F(B)$ defined by: $\tilde{f}(\psi)$ is the formula obtained by substituting (for each $a \in A$) $\overline{f(a)}$ for each occurrence of $\overline{a}$ in ψ. In turn, $\tilde{f}$ induces a map $f^*: S(B) \to S(A)$ defined by $f^*(p) = \tilde{f}^{-1}(p)$. The map f^* is continuous (cf. [14]), indeed $f^{*-1}(U_\psi) = U_{\tilde{f}(\psi)}$; the map f^* is onto $S(A)$, for if $q \in S(A)$ there is some $p \in S(B)$ with $p \supseteq f(q)$. If, in particular, $B \supseteq A$ and $i_{AB}: A \to B$ is the identity map[11] and $p \in S(B)$ then $i^*_{AB}(p) = p \cap F(A)$.

Let $\mathscr{S}(T) = \{S(A); A \in \mathscr{N}(T)\}$ and $\mathscr{C}(T) = \{(f^*: S(B) \to S(A)); A, B \in \mathscr{N}(T)$ and $f: A \to B$ a monomorphism$\}$. Then $\mathscr{C}(T)$ is a category of continuous onto maps with object class $\mathscr{S}(T)$. It is "dual" to the category of monomorphisms between members of $\mathscr{N}(T)$. Therefore, corresponding to each of 1.2(b), (c) and (d) there is a dual statement which holds in the category $\mathscr{C}(T)$. It should be especially noted that since a formula, ψ, involves only a finite number of individual constants, for each U_ψ in the basis of $S(A)$ there is some finite $B \subseteq A$ such that U_ψ is the inverse image under i^*_{BA} of a member of the basis of $S(B)$.

The next definition is a generalization of the usual definition of derived spaces to a definition involving a class of spaces and a category of maps between them. Though we shall deal explicitly only with the category

(11) Henceforth, whenever $A \subseteq B$ the identity map of A into B will be denoted by i_{AB}.

$\mathscr{C}(T)$, it will be obvious that Definition 2.2 and many of the following results and proofs remain valid in many other categories of continuous onto maps betweem compact spaces.

DEFINITION 2.2. For each ordinal α and each $S(A) \in \mathscr{S}(T)$, subspaces $S^\alpha(A)$ and $\mathrm{Tr}^\alpha(A)$ are defined inductively by:

(1) $S^\alpha(A) = S(A) - \bigcup_{\beta<\alpha} \mathrm{Tr}^\beta(A)$

(2) $p \in \mathrm{Tr}^\alpha(A)$ if (i) $p \in S^\alpha(A)$ and (ii) for every map $(f^*: S(B) \to S(A)) \in \mathscr{C}(T), f^{*-1}(p) \cap S^\alpha(B)$ is a set of isolated points in $S^\alpha(B)$.

$p \in S(A)$ is *algebraic* if $p \in \mathrm{Tr}^0(A)$; *p is transcendental in rank* α if $p \in \mathrm{Tr}^\alpha(A)$ (12).

THEOREM 2.3. (a) *$S^\alpha(A)$ is a closed and hence compact subspace of $S(A)$.*

(b) *If $(f^*: S(B) \to S(A)) \in \mathscr{C}(T)$ then* (i) *$f^*(S^\alpha(B)) = S^\alpha(A)$, and* (ii) *if $p \in S^\alpha(A)$ then $p \in \mathrm{Tr}^\alpha(A)$ if and only if $f^{*-1}(p) \cap S^\alpha(B) \subseteq \mathrm{Tr}^\alpha(B)$.*

Proof. (a) The proof is by induction on α. Suppose $\alpha = \beta + 1$. Then $S^\alpha(A) = S^\beta(A) - \mathrm{Tr}^\beta(A)$. $\mathrm{Tr}^\beta(A)$ is a set of isolated points in $S^\beta(A)$ and is therefore open in $S^\beta(A)$. So $S^\alpha(A)$ is closed.

Suppose $\alpha = \delta$. Then $S^\delta(A) = \bigcap_{\beta<\delta} S^\beta(A)$ and is closed since it is the intersection of closed sets.

(b) Notice first that, since $\mathrm{Tr}^\alpha(A) = S^\alpha(A) - S^{\alpha+1}(A)$, (b)(ii) will follow immediately from (b)(i). We shall use the following topological result.

PROPOSITION. *Suppose G is a compact space, H a Hausdorff space, f: $G \to H$ a continuous onto map, and $p \in H$, a limit point of H; then $f^{-1}(p)$ contains a limit point of G.*

Proof of proposition. If $f^{-1}(p)$ contained only isolated points then $G - f^{-1}(p)$ would be closed and hence compact. Then $f(G - f^{-1}(p)) = H - \{p\}$ would be compact and hence closed, so p would not be a limit point of H.

The proof of (b)(i) is by induction on α. Assume result for all $\beta < \alpha$. We first show that $f^*(S^\alpha(B)) \subseteq S^\alpha(A)$; that is, we show for each $\beta < \alpha$ that if $q \in S^\alpha(B)$ then $f^*(q) = p \notin \mathrm{Tr}^\beta(A)$. Since $q \notin \mathrm{Tr}^\beta(B)$ there is some $(g^*: S(C) \to S(B)) \in \mathscr{C}(T)$ such that $g^{*-1}(q) \cap S^\beta(C)$ contains a limit point, say r. Then $(f^*g^*: S(C) \to S(A)) \in \mathscr{C}(T)$ and $r \in (f^*g^*)^{-1}(p)$ so $p \notin \mathrm{Tr}^\beta(A)$.

Finally, to prove that $f^*(S^\alpha(\beta)) \supseteq S^\alpha(A)$ we must show for each $p \in S^\alpha(A)$ that $f^{*-1}(p) \cap S^\alpha(B) \neq \emptyset$. Suppose the contrary for some $p \in S^\alpha(A)$. Since f^* is onto, $f^{*-1}(p)$ is closed and compact and therefore there is a largest β (necessarily $< \alpha$) such that $f^{*-1}(p) \cap S^\beta(B) \neq \emptyset$. Then $f^{*-1}(p)$

(12) The terminology *algebraic* and *transcendental* are suggested by the theory of algebraically closed fields of characteristic 0, see Example I below. Our notion of algebraic is also related to a generalized notion of algebraic extension considered by Jónsson [6].

$\cap S^\beta(B) \subseteq \mathrm{Tr}^\beta(B)$. Since $p \notin \mathrm{Tr}^\beta(A)$ there is some $(g^*: S(C) \to S(A)) \in \mathscr{L}(T)$ such that $g^{*-1}(p) \cap S^\beta(C)$ contains a limit point of $S^\beta(C)$, say r. By 1.2(d) there is a $D \in \mathscr{N}(T)$ and monomorphisms $h_1: B \to D$ and $h_2: C \to D$ such that $h_1 f = h_2 g$. By the induction assumption, h_2^* maps $S^\beta(D)$ onto $S^\beta(C)$. By the proposition above, $h_2^{*-1}(r)$ contains a limit point of $S^\beta(D)$ say s. Then $h_1^*(s) \in S^\beta(B) \cap f^{*-1}(p)$ but $h_1^*(s) \notin \mathrm{Tr}^\beta(B)$ from 2.2. This contradicts $f^{*-1}(p) \cap S^\beta(B) \subseteq \mathrm{Tr}^\beta(B)$ and the result is established.

COROLLARY 2.4. *If* $p \notin \mathrm{Tr}^\alpha(A)$ *then there is a finite* $F \subseteq A$ *such that* $i_{FA}^*(p) \in \mathrm{Tr}^\alpha(F)$.

Proof. $S(A)$ has a neighborhood U such that $S^\alpha(A) \cap U = \{p\}$. As remarked earlier, since U is determined by some formula there is some finite $F \subseteq A$ such that $S(F)$ has a neighborhood V with $U = i_{FA}^{*-1}(V)$. By 2.3 (b)(i), $i_{FA}^*(p) = i_{FA}^*(U \cap S^\alpha(A)) = V \cap S^\alpha(F)$. Therefore, $i_{FA}^*(p) \in \mathrm{Tr}^\alpha(F)$.

THEOREM 2.5. (a) *If* $p \in \mathrm{Tr}^\alpha(A)$ *there is an integer* n *such that for every* $(f^*: S(B) \to S(A)) \in \mathscr{L}(T)$ *the set* $f^{*-1}(p) \cap S^\alpha(B)$ *has power* $\leqq n$. *The least such integer will be called the degree of* p[13]

(b) *If* $p \in \mathrm{Tr}^\alpha(A)$ *and* $(f^*: S(B) \to S(A)) \in \mathscr{L}(T)$ *then degree* $p = \sum_q$ *degree* $q(q \in f^{*-1}(p) \cap \mathrm{Tr}^\alpha(B))$.

Proof. (a) Suppose the opposite for some $p \in \mathrm{Tr}^\alpha(A)$. Then there would be, for each $n \in \omega$, a $B_n \in \mathscr{N}(T)$ and monomorphisms f_n: $A \to B_n$ such that $f_n^{*-1}(p) \cap S^\alpha(B_n)$ has power $> n$. By iterative applications of 1.2(d) to these B_n's there is a sequence $A \subseteq A_1 \subseteq A_2 \cdots$ such that $i_{AA_n}^{-1}(p) \cap S^\alpha(A_n)$ has power greater than n. Let $A' = \bigcup_{n \in \omega} A_n$. Then $i_{AA'}^{*-1}(p) \cap S^\alpha(A')$ is infinite and since it is compact, has a limit point. So $p \notin \mathrm{Tr}^\alpha(A)$ contradicting the assumption.

(b) For each $q \in \mathrm{Tr}^\alpha(B) \cap f^{*-1}(p)$ there is some $C_q \in \mathscr{N}(T)$ and a monomorphism $g_q: B \to C_q$ such that $g_q^{*-1}(q) \cap S^\alpha(C_q)$ has power degree q. Similarly there is some $C_p \in \mathscr{N}(T)$ and a monomorphism $g_p: A \to C_p$ such that $g_p^{*-1}(p) \cap S^\alpha(C_p)$ has power degree p. By repeated applications of 1.2(d) there is a $C \in \mathscr{N}(T)$ and a monomorphism $g: B \to C$ such that $g^{*-1}(q) \cap S^\alpha(C)$ has power degree q (for each $q \in f^{*-1}(p) \cap S^\alpha(B)$) and $(f^*g^*)^{-1}(p) \cap S^\alpha(C)$ has power degree p. But

$$(f^*g^*)^{-1}(p) \cap S^\alpha(C) = \bigcup_q g^{*-1}(q) \cap S^\alpha(C) \quad (q \in f^{*-1}(p) \cap S^\alpha(B))$$

and the result follows.

Lemma 2.6. (a) *There is an ordinal* $\alpha_T < (2^{\aleph_0})^+$ *which is the least ordinal*

[13] It is possible to combine the rank and degree into a single new rank by varying the Definition 2.2 slightly. To do so replace in 2.2(2) the words "set of isolated points" by "a single point."

such that for all $A \in \mathcal{N}(T)$ *and all* $\beta > \alpha_T$, $S^{\alpha_T}(A) = S^{\beta}(A)$.

(b) *If* $S^{\alpha_T}(A) = \emptyset$ *for some* $A \in \mathcal{N}(T)$, *then* α_T *is not a limit ordinal and for every* $B \in \mathcal{N}(T)$, $S^{\alpha_T}(B) = \emptyset$ *and* $S^{\beta}(B) = \emptyset$ *for any* $\beta < \alpha_T$.

Proof. (a) From 2.4 it follows that $\mathrm{Tr}^{\alpha}(A)$ is empty for every $A \in \mathcal{N}(T)$ if it is empty for every finite $A \in \mathcal{N}(T)$. There are at most $2^{\aleph_0}$ isomorphism types of finite systems $\in \mathcal{N}(T)$ and for each such finite system $\kappa(S(A)) \leqq 2^{\aleph_0}$.

(b) Suppose $A, B \in \mathcal{N}(T)$ and $S^{\beta}(A) = \emptyset$ Then by 1.2(c) and 2.3(b) $S^{\beta}(B) = \emptyset$. That the least ordinal at which this occurs cannot be a limit ordinal follows from 2.3(a) and the compactness of $S(A)$.

We say T is *totally transcendental* if $S^{\alpha_T}(A) = \emptyset$ for some (and hence every) $A \in \mathcal{N}(T)$.

Theorem 2.7. *If T is totally transcendental then* $\kappa(S(A)) = \kappa(A)$ (*modulo* $\aleph_0$) *for every* $A \in \mathcal{N}(T)$.

Proof. For every $p \in \mathrm{Tr}^{\alpha}(A)$ we may choose a member $U(p)$ of the basis of $S(A)$ such that $U(p) \cap S^{\alpha}(A) = \{p\}$. Clearly if $p \neq p'$ then $U(p) \neq U(p')$. Since T is totally transcendental, every $p \in S(A)$ is transcendental in some rank. Thus the correspondence of p to $U(p)$ is a one-one correspondence between $S(A)$ and a subset of the basis of $S(A)$. So $\kappa(S(A)) \leqq \kappa(A) + \aleph_0$. On the other hand, the formula: $v_0 = \bar{a}$, determines for each $a \in A$ a unique element of $S(A)$; so $\kappa(S(A)) \geqq \kappa(A)$.

The next theorem is an analogue of the Cantor-Bendixson theorem and the proof is similar to proofs of that theorem.

Theorem 2.8. *T is totally transcendental if and only if $S(A)$ is countable for every countable* $A \in \mathcal{N}(T)$.

Proof. If T is totally transcendental then $S(A)$ is countable for countable A by Theorem 2.7.

Conversely, suppose T is not totally transcendental. Then for every $A \in \mathcal{N}(T), S^{\alpha_T}(A) \neq \emptyset$. There is some $A \in \mathcal{N}(T)$ such that $S^{\alpha_T}(A)$ has more than one point; for otherwise, every $p \in S^{\alpha_T}(A)$ would be transcendental in rank α_T, and by definition, there are no points transcendental in rank α_T. Thus, there is some $A_1 \in \mathcal{N}(T)$ such that $S^{\alpha_T}(A_1)$ may be divided into two disjoint nonempty components (closed-open sets), say U_0 and U_1. As remarked earlier, U_0 and U_1 are determined by finite subsets of A_1. Hence, without loss of generality we take A_1 to be finite. There must be some $B \in \mathcal{N}(T), B \supseteq A_1$, such that $i^{*-1}_{A_1B}(U_0) \cap S^{\alpha_T}(B)$ has more than one point; for otherwise, each $p \in U_0$ would be transcendental in rank α_T Similarly for U_1. By 1.2(d) we may find an $A_2 \supseteq A_1$ such that $i^{*-1}_{A_1A_2}(U_0) \cap S^{\alpha_T}(A_2)$ and $i^{*-1}_{A_1A_2}(U_1) \cap S^{\alpha_T}(A_2)$ both have more than one point. Thus we may decompose $S^{\alpha_T}(A_2)$ into four disjoint nonempty components, U_{00},

U_{01}, U_{10}, U_{11} such that $i^{*-1}_{A_1A_2}(U_j) \cap S^{\alpha T}(A_2) = U_{j0} \cup U_{j1}$ $(j = 0, 1)$. As before we may take A_2 to be finite. We proceed inductively to find an increasing chain of systems $\{A_n; n < \omega\}$ such that each $A_n \in \mathscr{N}(T)$, is finite, each $S^{\alpha T}(A_n)$ may be decomposed into 2^n disjoint nonempty components $U_{j_0 \cdots j_{n-1}}$ $(j_k = 0, 1)$ and

$$i^{*-1}_{A_nA_{n+1}}(U_{j_0 \cdots j_{n-1}}) \cap S^{\alpha T}(A_{n+1}) = U_{j_0 \cdots j_{n-1}0} \cup U_{j_0 \cdots j_{n-1}1}.$$

Let $A = \bigcup_n A_n$. For each $t \in 2^w$, Let $V_t = \bigcap_n i^{*-1}_{A_nA}(U_{t(0) \cdots t(n-1)}) \cap S^{\alpha T}(A)$. Then $V_t \neq \emptyset$ since it is the intersection of closed nonempty sets. Obviously, $t_1 \neq t_2$ implies $V_{t_1} \cap V_{t_2} = \emptyset$. Thus $S^{\alpha T}(A)$ has power $2^{\aleph_0}$ though A is countable.

We shall conclude this section with three examples. In each case we shall describe the theory Σ such that $T = \Sigma^*$. We shall then describe $S(A)$ for each $A \in \mathscr{N}(T)$. To do this it is convenient to know when a consistent set of formulas of $F(A)$ is contained in a unique $p \in S(A)$. We give the following sufficient condition:

A consistent set of formulas, $Q \subseteq F(A)$, is contained in a unique $p \in S(A)$ if whenever B is a model of $T, B \supseteq A$, and $b, b' \in B$ satisfy every formula of Q, then there is an automorphism of B carrying b to b' and leaving each element of A fixed(14).

For suppose p and p' were points of $S(A)$ which contain Q. By 2.1 there is a model of $T, B, B \supseteq A$, and with $b, b' \in B$ realizing p and p' respectively. Our condition then asserts that there is an automorphism of B having A fixed and carrying b to b'. Therefore b and b' realize the same point of $S(A)$, that is $p = p'$.

Example I. Let Σ be the theory of algebraically closed fields of characteristic 0(15). As mentioned earlier this theory is categorical in very uncountable power but not in power $\aleph_0$. Suppose $A \in \mathscr{N}(T)$, let $\Delta(A)$ be the field generated by A. Suppose $Q(v_0)$ is a polynomial with coefficients in $\Delta(A)$ and irreducible over $\Delta(A)$. By the condition above the formula: $Q(v_0) = 0$ determines a unique point of $S(A)$. Since this point is determined by a single formula it is an isolated point of $S(A)$. Let P be the set of all formulas: $Q(v_0) \neq 0$ where $Q(v_0)$ is a polynomial with coefficients in $\Delta(A)$. Then all the formulas of P are satisfied precisely by those elements transcendental (in the usual field-theoretical sense) over $\Delta(A)$. Therefore, by our condition and the Steinitz theorems P is included in a unique $p \in S(A)$. Obviously, the above are all the points of $S(A)$. Since $S(A)$ is infinite and compact it must have a limit point which can only be the point deter-

(14) If we weaken this condition to assert that there is a model of $T, C \supseteq B$, such that C has an automorphism carrying b to b' and leaving each element of A fixed, then this condition is also necessary; cf. [10].

(15) For a more detailed discussion of this case see Abraham Robinson, *Complete theories*, North-Holland, Amsterdam, 1956.

mined by P. Thus $S(A)$ consists of: (1) isolated points corresponding to the distinct elements of $\Delta(A)$ and to the algebraic extensions of $\Delta(A)$, and (2) a single limit point corresponding to the transcendental extensions of $\Delta(A)$. If $B \supseteq A$ then any element algebraic over $\Delta(A)$ is *a fortiori* algebraic over $\Delta(B)$. So if $p \in S(A)$ is an isolated point, then $i_{AB}^{*-1}(p)$ is a set of isolated points; hence $p \in \mathrm{Tr}^0(A)$. For each $A \in \mathscr{N}(T), S^1(A)$ is then a single point so $S^1(A) = \mathrm{Tr}^1(A)$. Thus T is totally transcendental and $\alpha_T = 2$.

Example II. Suppose there are two relation symbols: R_0, a one-ary relation symbol, and R_1, an $(n+1)$-ary relation symbol and let the formulas of Σ assert that in any model of Σ $A = \langle |A|, R_0^A, R_1^A \rangle$:

(1) $|A|$ is infinite, and

(2) the set of pairs $(a_0, (a_1, \cdots, a_n))$ such that $R_1^A a_0, a_1, \cdots, a_n$ is a one-one correspondence between $|A| - R_0^A$ and the n-tuples of distinct elements of R_0^A.

This theory is obviously categorical in every infinite power.

For each n-tuple of distinct elements of $R_0^A, a_1, \cdots, a_n$, let $\langle a_1, \cdots, a_n\rangle$ denote the unique a_0 such that $R_1^A, a_0, a_1, \cdots, a_n$. Suppose $B \in \mathscr{N}(T)$. By 2.1 there is a model of $T, A \supseteq B$, such that every $p \in S(B)$ is realized in A. Denote by $\hat{B}$ the closure of B in A; more precisely, $\hat{B}$ is the smallest subsystem such that $B \subseteq \hat{B} \subseteq A$, and $\langle a_1, \cdots, a_n\rangle \in \hat{B}$ if and only if $a_1, \cdots, a_n \in \hat{B}$. It is easy to see that every $a \in \hat{B}$ is characterized by a unique formula of $F(B)$, and so each $a \in \hat{B}$ realizes an isolated point of $S(B)$.

Notice that every one-one map of $R_0^A - \hat{B}$ onto itself induces an automorphism of A which leaves $\hat{B}$ fixed. So every element of $R_0^A - \hat{B}$ realizes the same point of $S(B)$. Similarly, two elements $\langle a_1, \cdots, a_n\rangle$ and $\langle a_1', \cdots, a_n'\rangle$ realizes the same point of $S(B)$ if and only if for all $1 \leqq i \leqq n, a_i = a_i'$ whenever a_i or $a_i' \in \hat{B}$. Call a point $p \in S(B)$ of type m if it is realized by an element $\langle a_1, \cdots, a_n\rangle$ and exactly m of the a_i's $\in \hat{B}$;

Suppose $C \in \mathscr{N}(T)$, $A \supseteq C \supseteq B$, $\langle a_1, \cdots, a_n\rangle \in A, m$ of the a_i's $\in \hat{B}$, and $m+1$ of the a_i's $\in \hat{C}$. Then $\langle a_1, \cdots, a_n\rangle$ realizes a point of type m in $S(B)$ and of type $m+1$ in $S(C)$. Thus, for every $B \in \mathscr{N}(T)$ we can find a $C \supseteq B$ such that for every $p \in S(B)$ of type $m < n, i_{BC}^{*-1}(p)$ contains an infinite set of points of type $m+1$.

From the above considerations it is easy to show that: (1) the points of $S(B)$ realized by elements of $\hat{B} \in \mathrm{Tr}^0(B)$, (2) the point of $S(B)$ realized by the elements of $R_0^A - \hat{B}$ is transcendental in rank 1, and (3) the points of $S(B)$ of type m are transcendental in rank $n - m$. Therefore, T is totally transcendental and $\alpha_T = n + 1$.

Example III. Consider the Cantor set, i.e., 2^ω with the product topology. Let Y be a closed nonempty subset of 2^ω. There will be a denumerable set R_n $(n \in \omega)$ of singulary relation symbols and the theory Σ will assert that for any model of Σ, A, and any two finite sets $K_0, K_1 \subseteq \omega, \bigcap_{n \in K_1} R_n^A \cap \bigcap_{n \in K_0} (|A| - R_n^A)$ is empty or infinite depending on whether

$$\left\{ t \in Y;\ \bigwedge_{n \in K_1} t(n) = 1 \wedge \bigwedge_{n \in K_0} t(n) = 0 \right\}$$

is empty or nonempty. Thus the points of Y correspond to the isomorphism types of single element subsystems of models of Σ. If $A \in \mathscr{N}(T)$, then the points of $S(A)$ realized by elements of A are isolated, indeed algebraic points; while the points realized by elements not in A form a space homeomorphic to Y. None of the latter points can be algebraic since each one could be realized by an infinite set of elements in some $B \supseteq A$. So $S^1(A)$ is homeomorphic to Y, and, if $B \supseteq A, i^*_{AB}$ maps $S^1(B)$ homeomorphically onto $S^1(A)$. A point $p \in S^1(A)$ will be in $S^{1+\alpha}(A)$ if and only if the corresponding point of Y is in $Y^{(\alpha)}$, the αth derived set of Y. The theory is totally transcendental if and only if Y has a vanishing perfect kernel, that is, if Y is countable. If α_Y is the least ordinal such that $Y^{(\alpha)} = Y^{(\alpha+1)}$ then $\alpha_T = 1 + \alpha_Y$.

3. **Results depending on Ramsey's theorem.** In this section we have gathered together some results depending on the following theorem of Ramsey [12].

THEOREM 3.1 (RAMSEY). *Suppose Y is an infinite set and $Y^{(n)}$ the set of subsets of Y having exactly n elements. If $Y^{(n)} = C_1 \cup \cdots \cup C_m$ is a partition of $Y^{(n)}$ into a finite number of mutually disjoint sets, then there is a $j \leqq m$ and an infinite set $Y_1 \subseteq Y$ such that $Y_1^{(n)} \subseteq C_j$.*

Much of this section is related to results of Ehrenfeucht and Mostowski [3] and Ehrenfeucht [1], [2]. In particular, Theorems 3.2, 3.4 and 3.5 below are only slight variants of the results of [3].

THEOREM 3.2. *Suppose Σ is a (generalized) theory in a language L, Σ has an infinite model, and $(X, <)$ is an arbitary linearly ordered set. Then there is a model of Σ, A, such that $|A| \supseteq X$ and whenever $n \in \omega, x_0 < \cdots < x_{n-1}$ and $x'_0 < \cdots < x'_{n-1}$ are contained in X and ψ is a formula of L with no free variables other than $v_0, \cdots, v_{n-1}$, then $\vdash_A \psi(x_0, \cdots, x_{n-1}) \leftrightarrow \psi(x'_0, \cdots, x'_{n-1})$.*

Proof. Suppose there is added to L a new constant, $\bar{x}$, for each $x \in X$, and there is added to Σ the sentence

(I) $$\text{“}\overline{x_1} \neq \overline{x_2}\text{”}$$

for each pair x_1, x_2 of distinct elements of X. Suppose further that whenever $n \in \omega, x_0 < \cdots < x_{n-1}$ and $x'_0 < \cdots < x'_{n-1}$ and ψ is a formula of L with free variables among $v_0, \cdots, v_{n-1}$, there is added to Σ the formula

(II) $$\text{“}\psi(\overline{x_0}, \cdots, \overline{x_{n-1}}) \leftrightarrow \psi(\overline{x'_0}, \cdots, \overline{x'_{n-1}})\text{.”}$$

The result is a set of sentences, say $\bar{\Sigma}$, extending Σ.

To prove the theorem it is sufficient to prove $\bar{\Sigma}$ consistent. Suppose $\bar{\Sigma}$ is inconsistent. Then there is an inconsistent $\Sigma_1 \subseteq \bar{\Sigma}$ such that $\Sigma_1 = \Sigma$ together with a finite number of sentences of type (I) and (II). Let the sentences of type (II) appearing in Σ_1 be:

$$\text{“}\psi_1[\bar{x}]_1 \leftrightarrow \psi_1[\bar{x}]_1',\text{”}, \cdots, \text{“}\psi_m[\bar{x}]_m \leftrightarrow \psi_m[\bar{x}]_m',\text{”}$$

where the notation $[\bar{x}]$ is an abbreviation for a sequence of constants $\bar{x}_0, \cdots, \bar{x}_{n-1}$.

Consider first the case where each $[\bar{x}]_j$ has the same number of elements, say n. Let A be an infinite model of Σ and "$<^*$" a linear ordering of $|A|$ (in general, having nothing to do with any of the original relations of A). If $[a]$ and $[a]'$ are n-tuples of $|A|$ which are properly ordered by $<^*$ then we say

$$[a] \approx [a]' \quad \text{if} \quad \vdash_A \bigwedge_{j \leq m} \psi_j[a] \leftrightarrow \psi_j[a]'.$$

This equivalence $\rightarrow$ partitions $|A|^{(n)}$ into (at most) 2^m equivalence classes. Applying Ramsey's theorem, we may find some infinite subset $Y \subseteq |A|$ such that $Y^{(n)}$ lies entirely within one equivalence class. That is, if $[a]$ and $[a]'$ are properly ordered n-tuples of Y then $[a] \approx [a]'$. Since Σ contains only a finite number of sentences of type (I) and (II), it contains of the new constants added to L, only those corresponding to some finite subset of X, say X_1. We may now pick in Y a finite subset, Y_1, which is order-isomorphic to X_1. Then $(A, a)_{a \in Y_1}$ is a model of Σ_1, contradicting its inconsistency.

Consider the general case where all the $[x]_j$'s do not necessarily have the same number of elements. Notice that it is sufficient to prove the theorem for X, a linearly ordered set without maximal elements, since any linearly ordered set can be imbedded in such a one. Now, let N be the maximum number of elements in any $[x]_j$ $(j \leq m)$. Then a properly ordered $[x] = (x_0, \cdots, x_{n-1})$ may be imbedded in a properly ordered set $(x_0, \cdots, x_{n-1}, x_n, \cdots, x_{N-1})$. The general result then follows from the first considered case.

The next theorem expresses the well-known fact that one can eliminate existential quantifiers by the use of operation symbols. A proof may be found in the first chapter of [4].

THEOREM 3.3. *Suppose Σ is a theory in a countable language, L; then there is a countable generalized language, $L^\# \supseteq L$ and a theory $\Sigma^\#$ of $L^\#$ such that*:

(i) *every $\sigma \in \Sigma^\#$ is a universal sentence, and*

(ii) *for every sentence, ψ of L, $\vdash_{\Sigma^\#} \psi$ if and only if $\vdash_\Sigma \psi$.*

Thus, if A is a model of $\Sigma^\#$, then $A \upharpoonright L$ (A restricted to the relations corresponding to symbols of L) is a model of Σ.

Suppose A is a model of $\Sigma^\#$ and $X \subseteq |A|$. The set of elements $a \in A$ such that there is a term, $t(v_1, \cdots, v_n)$ in $L^\#$ and $x_1, \cdots, x_n \in X$ with $a = t^A(x_1, \cdots, x_n)$ is by 3.3(i) the universe of a model of $\Sigma^\#$, denoted by $M(X, A)$.

THEOREM 3.4. *If Σ is a theory of L, has an infinite model, and $(X, <)$ is an arbitrary linearly ordered set; then there is a model of $\Sigma^\#, A$, with $X \subseteq |A|$ such that if:* (i) $t_0(v_0, \cdots, v_{n_0}), \cdots, t_m(v_0, \cdots, v_{n_m})$ *are terms in $L^\#$,* (ii) x_{jk} *and x'_{jk} $(j \leqq m, k \leqq n_j)$ are elements of X and the mapping of x_{jk} to x'_{jk} is an order isomorphism between them, and* (iii) ψ *is a formula of $L^\#$ with free variables among $v_0, \cdots, v_m$; then*

$$\vdash_A \psi(t_0^A(x_{00}, \cdots, x_{0n_0}), \cdots, t_m^A(x_{m0}, \cdots, x_{mn_m}))$$
$$\leftrightarrow \psi(t_0^A(x'_{00}, \cdots, x'_{0n_0}), \cdots, t_0^A(x'_{m0}, \cdots, x'_{mn_m})).$$

Proof. Apply Theorem 3.2 to $\Sigma^\#$.

THEOREM 3.5. *Suppose Σ is a theory with an infinite model and $(X, <)$ is an arbitrary linearly ordered set. Then there is a model of $\Sigma, B, |B| \supseteq X$, such that any order endomorphism (automorphism) of X may be extended to an endomorphism (automorphism) of B.*

Proof. Extend Σ to $\Sigma^\#$ and apply Theorem 3.4 to get a model of $\Sigma^\#$ containing X. Take $B = M(X, A) \upharpoonright L$. If $f: X \rightarrow X$ is an order endomorphism, define $\tilde{f}: M(X, A) \rightarrow M(X, A)$ by $\tilde{f}(t^A(x_0, \cdots, x_n)) = t^A(f(x_0), \cdots, f(x_n))$. By 3.4 $\tilde{f}$ is well defined and is a monomorphism; it is obviously onto if f is onto.

The preceding two theorems may be strengthened by extending Σ to $\Sigma^{*\#}$ rather than $\Sigma^\#$. Using 1.1(c) this will then prove:

THEOREM 3.6. (a) *For formulas, ψ, of L, Theorem 3.4 remains valid if in the last line $\vdash_A$ is replaced by $\vdash_{M(X,A)}$.*

(b) *Under the hypothesis of Theorem 3.5 there is a model of $\Sigma, B, |B| \supseteq X$, such that any order endomorphism (automorphism) of X may be extended to an elementary endomorphism (automorphism) of B.*

Suppose A is a model of $\Sigma, X \subseteq |A|$, and $a, a' \in A$. We *say a is elementarily equivalent over X with respect of A* to a' if the map $: X \cup \{a\} \rightarrow X \cup \{a'\}$ which is the identity on X and maps a to a' is an elementary monomorphism.

THEOREM 3.7. *Suppose Σ is a theory in a countable language, L, and Σ has an infinite model. Then for every infinite κ there is a model of $\Sigma, A, \kappa(A) = \kappa$, such that for every countable $X \subseteq |A|, A$ contains only a countable number of elementary equivalence classes over X.*

Proof[16]. Let $(X_\kappa, <)$ be a linearly ordered set having the order type of initial ordinal κ. Apply Theorem 3.4 to $\Sigma^{*\#}$ to get B and let $A = M(X_\kappa, B) \upharpoonright L$. Suppose Y is a countable subset of $|A|$. Then $Y \subseteq M(X_0, A)$ for some countable subset $X_0 \subseteq X_\kappa$. For each $a \in A$ there is some term $t(v_0, \cdots, v_n)$ in $L^{*\#}$ and elements $x_0, \cdots, x_n \in X_\kappa$ such that $a = t^A(x_0, \cdots, x_n)$. By 3.6(a) the elementary equivalence class of a over Y is determined by $t(v_0, \cdots, v_n)$ and the ordering relations between $x_0, \cdots, x_n$ and X_0. $L^{*\#}$ is a countable language and has only a countable number of distinct terms. X_0 is a countable well-ordered set and so there are only a countable number of ways of interpolating a finite set into it. Therefore A has only a countable number of equivalence classes over Y.

THEOREM 3.8. *If T is categorical in some power $\kappa > \aleph_0$ then T is totally transcendental.*

Proof[17]. Suppose T were not totally transcendental. Then by 2.8 there would be a countable $C \in \mathscr{M}(T)$ with $\kappa(S(C)) > \aleph_0$. So we could certainly have a model of T, B, such that $\kappa(B) = \kappa, B \supseteq C$, and an uncountable number of points of $S(C)$ are realized in B. This B is clearly not isomorphic to the model of power κ proven to exist in Theorem 3.7.

A theory T may be categorical in power $\aleph_0$ and not be totally transcendental. For example, consider the theory of dense linearly ordered sets without end points. Let A be a linearly ordered set having the order type of the rationals. It can be shown that distinct Dedekind cuts in A correspond to distinct points in $S(A)$ so $\kappa(S(A)) = 2^{\aleph_0}$. By 2.8 the theory cannot be totally transcendental. Theorem 3.9, below, is proved by a generalization of this argument.

Suppose A is a model of T, R a relation of degree n of $A, X \subseteq |A|$, and S_n the permutation group on $(0, \cdots, n-1)$. Following Ehrenfeucht [1] we define R to be *connected over* X if for every sequence of n distinct elements $x_0, \cdots, x_{n-1}$ of X there is an $s \in S_n$ such that $\vdash_A R(x_{s(0)}, \cdots, x_{s(n-1)})$. R is *anti-symmetric over* X if for every sequence of n distinct elements $x_0, \cdots, x_{n-1}$ of X there is an $s \in S_n$ such that $\vdash_A \sim R(x_{s(0)}, \cdots, x_{s(n-1)})$.

THEOREM 3.9. *If T is totally transcendental and A a model of T, then no relation of A is connected and anti-symmetric over any infinite $X \subseteq |A|$* [18].

[16] For the case $X = \emptyset$, this result was obtained by Ehrenfeucht [2]. Indeed, he showed that if equivalence of two elements of A is defined to mean that there is an automorphism of A mapping one to the other, there is still a model of Σ of power κ which has only a countable number of equivalence classes. The proof is similar to that of 3.7 but X must be taken as a somewhat more complicated linear ordering.

[17] The crux of this proof, that $S(C)$ is countable for every countable $C \in \mathscr{M}(T)$, was established by Vaught [10] for the case where T is categorical in power $\kappa = \kappa^{\aleph_0}$.

[18] For the case where T is categorical in power 2^κ, this result was obtained by Ehrenfeucht [1]. Dana Scott (unpublished), by a different and simpler proof, extended the result to theories categorical in power $\kappa^{\aleph_0}$.

Proof. Suppose some relation of degree n of A, say R, were connected and anti-symmetric over an infinite set $X \subseteq |A|$. Impose an arbitrary linear order on X and say that two properly ordered n-tuples of X are equivalent, $(x_0, \cdots, x_{n-1}) \approx (x'_0, \cdots, x'_{n-1})$ if

$$\vdash_A \bigwedge_{s \in S_n} R(x_{s(0)}, \cdots, x_{s(n-1)}) \leftrightarrow R(x'_{s(0)}, \cdots, x'_{s(n-1)}).$$

Then "$\approx$" partitions the properly ordered n-tuples of X into a finite number of equivalence classes. By Ramsey's theorem we may find an infinite $Y \subseteq X$ such that every properly ordered n-tuple of Y is in the same equivalence class. That is, S_n may be decomposed into two sets S_n^+ and S_n^- such that for any $y_0 < \cdots < y_{n-1} \in Y$,

$$\text{(I)} \qquad \vdash_A \bigwedge_{s \in S_n^+} R(y_{s(0)}, \cdots, y_{s(n-1)}) \wedge \bigwedge_{s \in S_n^-} \sim R(y_{s(0)}, \cdots, y_{s(n-1)}).$$

R is connected and anti-symmetric on Y so neither S_n^+ nor S_n^- is empty. Hence there exists an $s_1 \in S_n^+$, $s_2 \in S_n^-$, and a cycle $(m-1, m)$ such that $s_1 = s_2 \cdot (m-1, m)$.

Using the Completeness Theorem one easily shows that the existence of Y implies that for any arbitrary order type, γ, there is a model of T, B, containing an ordered set, Y, of type γ and such that any $y_0 < \cdots < y_{n-1} \in Y$ satisfies (I). In particular, let Y have the order type of the real numbers and let $Z \subseteq Y$ be a countable dense subset. We assert that distinct elements in Y realize distinct points in $S(Z)$. For suppose $y < y' \in Y$. Pick $n-1$ elements of $Z, z_0, \cdots, z_{m-1}, z_{m+1}, \cdots, z_{n-1}$ such that

$$z_0 < \cdots < z_{m-2} < y < z_{m-1} < y' < z_{m+1} < \cdots < z_{n-1}.$$

Then $(z_0, \cdots, z_{m-1}, y', z_{m+1}, \cdots, z_{n-1})$ will, after permutation by s_1, satisfy R. But $(z_0, \cdots, z_{m-1}, y, z_{m+1}, \cdots, z_{n-1})$ will, after permutation by s_1, not satisfy R, since its proper order will now be permuted by s_2. So $\kappa(S(Z)) = 2^{\aleph_0}$ and T cannot be totally transcendental.

4. Models of totally transcendental theories.

A neat characterization of models of a theory T is given by the following lemma.

LEMMA 4.1. *$A \in \mathscr{N}(T)$ is a model of T if and only if the points of $S(A)$ which are realized in A form a dense subset of $S(A)$.*

Proof. By 2.1 there is a $B \supseteq A$ such that B is a model of T which realizes every point in $S(A)$. By 1.1(c) a necessary condition for A to be a model of T is that i_{AB} be an elementary monomorphism. Trivially, this is also a sufficient condition. By a theorem of Tarski[19] a necessary and sufficient condition that i_{AB} be an elementary monomorphism is that every formula of $F(A)$ which is satisfied by some $b \in B$ be also satisfied by some $a \in A$.

[19] Theorem 1.10 of [15].

But every $\psi \in F(A)$ and consistent with $T(A)$ (i.e., $\neq 0$ in the Boolean algebra $F(A)$) is satisfied in B. Hence, a necessary and sufficient condition that A be a model of T is that every $\psi \neq 0$ in $F(A)$ be satisfied in A, which is equivalent to the condition that some point in each of the sets $U_\psi = \{p \in S(A); \psi \in p\}$ is realized in A. But the sets U_ψ ($\psi \in F(A)$) form a basis for $S(A)$, and the lemma is proved.

LEMMA 4.2. *If T is totally transcendental then for every $A \in \mathscr{N}(T)$ the isolated points are dense in $S(A)$; indeed if U is an open set of $S(A)$ and $p \in U$ is a point of the minimal transcendental rank of the points of U, then p is an isolated point in $S(A)$.*

Proof. Suppose $p \in U$ is of the minimal transcendental rank, say α, of the points of U. By definition there is a neighborhood V of p such that $V \cap S^\alpha(A) = \{p\}$. But $U \cap S^\alpha(A) = U$. So $V \cap S^\alpha(A) \cap U = V \cap U = \{p\}$, and p is isolated.

Suppose $A, B \in \mathscr{N}(T), B \supseteq A$, and B is a model of T. B is *prime* over A if for every model of T, B', and monomorphism $f: A \to B'$, there is a monomorphism $g: B \to B'$ with $f = g$ on A.

THEOREM 4.3. *Suppose T is such that for every $A \in \mathscr{N}(T)$ the isolated points are dense in $S(A)$, then every $A \in \mathscr{N}(T)$ has a model of T prime over it*[20].

Proof. Let $A \in \mathscr{N}(T)$ and $\kappa = \kappa(A) + \aleph_0$. Then $S(A)$ has at most κ isolated points. Let $\{p_\alpha; \alpha < \kappa\}$ be a listing (possibly with repetitions) of the isolated points of $S(A)$. Choose some increasing chain $\{A_\alpha; \alpha < \kappa\}$ of members of $\mathscr{N}(T)$ such that: (1) $A_0 = A$, (2) $A_\delta = \bigcup_{\beta<\delta} A_\beta$, (3) $A_{\alpha+1} = A_\alpha$ if p_α is realized in A_α, and (4) if p_α is not realized in A_α, then $A_{\alpha+1} - A_\alpha$ has a single element, a_α, which realizes some isolated point q in $S(A_\alpha)$ such that $q \supseteq p_\alpha$.

If C is a model of T and $f_0: A \to C$ is a monomorphism then there is a sequence of monomorphisms $\{(f_\alpha: A_\alpha \to C); \alpha < \kappa\}$ such that for $\alpha' > \alpha$, $f_{\alpha'}$ extends f_α. This is proved by induction on α. The induction is trivial in cases (1), (2), and (3) above. In case (4) suppose $f_\alpha: A_\alpha \to C$ is a monomorphism and $a_\alpha \in A_{\alpha+1} - A_\alpha$ satisfies the isolated point q in $S(A_\alpha)$. Then $f_\alpha^{*-1}(q)$ is an open set in $S(C)$ and by hypothesis contains an isolated point, say q'. By 4.1 there is a $c \in C$ realizing q'. Let $f_{\alpha+1}(a_\alpha) = c$ and the monomorphism is extended.

$A_\kappa = \bigcup_{\alpha<\kappa} A_\alpha$ then realizes every isolated point in $S(A)$ and every monomorphism of A into a model of T can be extended to a monomorphism of A_κ. We may now list the isolated points of $S(A_\kappa)$ and repeat the above process to get an $A_{\kappa\cdot 2}$ realizing every isolated point in $S(A_\kappa)$ and

(20) For the case where A is countable the existence of a prime model over A was proved under a somewhat weaker hypothesis in [18].

such that every monomorphism of A into a model of T may be extended to a monomorphism of $A_{\kappa\cdot 2}$. Iterating ω times we obtain

$$A_{\kappa\cdot\omega} = \bigcup_{n\in\omega} A_{\kappa\cdot n}$$

such that any monomorphism of A into a model of T can be extended to a monomorphism of $A_{\kappa\cdot\omega}$, and $A_{\kappa\cdot\omega}$ realizes every isolated point in $S(A_{\kappa\cdot n})$ for each $n \in \omega$. But the topology on $S(A_{\kappa\cdot\omega})$ is that induced by the $S(A_{\kappa\cdot n})$'s; for each formula $\psi \in F(A_{\kappa\cdot\omega})$ must be already in some $F(A_{\kappa\cdot n})$, hence the neighborhood U_ψ of $S(A_{\kappa\cdot\omega})$ is the inverse image of the corresponding neighborhood in $S(A_{\kappa\cdot n})$. So, $A_{\kappa\cdot\omega}$ realizes every isolated point in $S(A_{\kappa\cdot\omega})$ and is by 4.1 a model of T.

For the next theorem we shall need some results about increasing sequences of systems and the corresponding sequence of Boolean spaces. We summarize these in the next lemma.

Lemma 4.4. *Suppose T is totally transcendental.* (a) *If $\{A_\alpha; \alpha < \gamma\}$ is an increasing sequence of members of $\mathscr{N}(T)$, $A = \bigcup_{\alpha<\gamma} A_\alpha$, and $\{p_\alpha; \alpha < \gamma\}$ a sequence such that $p_\alpha \in S(A_\alpha)$ and $i^*_{A_\alpha A_\beta}(p_\beta) = p_\alpha$ $(\alpha \leqq \beta < \gamma)$ then*:

(i) *there is an $\alpha_0 < \gamma$ such that for all α, if $\alpha_0 \leqq \alpha < \gamma$ then transcendental rank and degree of p_α equal the transcendental rank and degree of p_{α_0}, and*

(ii) *there is a unique $p \in S(A)$ such that*

$$p \in \bigcap_{\alpha<\gamma} i^{*-1}_{A_\alpha A}(p_\alpha).$$

This point will have transcendental rank and degree equal to that of the p_{α_0} defined in (i).

(b) *If $\{A_\alpha; \alpha < \gamma\}$ is an increasing sequence of members of $\mathscr{N}(T)$ and p is an isolated point in $S(A_0)$, then there is a sequence $\{p_\alpha; \alpha < \gamma\}$ of points such that $p_\alpha \in S(A_\alpha)$ $(\alpha < \gamma)$, $p_0 = p$, $i_{A_\alpha A_\beta}(p_\beta) = p_\alpha$ $(\alpha < \beta < \gamma)$ and each p_α is isolated in $S(A_\alpha)$.*

Proof. (a) If $\gamma = \beta + 1$ then $A = A_\beta$ and the result is trivial. Suppose $\gamma =$ a limit ordinal δ. By 2.3, $\beta \geqq \alpha$ implies transcendental rank $p_\beta \leqq$ transcendental rank p_α. Since there can be no infinite decreasing sequence of ordinal numbers, the transcendental rank must remain constant from some α on. By a similar argument (now using 2.5), the transcendental rank and degree must remain constant from some α_0 on. Let p_{α_0} have transcendental rank ν and degree n. By 2.3, $\bigcap_{\alpha<\delta} i^{*-1}_{A_\alpha A}(p_\alpha)$ can have no point of rank $> \nu$, and $i_{A_0 A}(p_{\alpha_0}) \cap S^\nu(A)$ is not empty; but by 2.5(b) $i^{*-1}_{A_{\alpha_0} A_\alpha}(p_{\alpha_0}) \cap S^\nu(A_\alpha) = \{p_\alpha\}$ (for $\alpha \geqq \alpha_0$) and so

$$\bigcap_{\alpha<\delta} i^{*-1}_{A_\alpha A}(p_\alpha) \cap S^\nu(A) = i^{*-1}_{A_{\alpha_0} A}(p_{\alpha_0}) \cap S^\nu(A).$$

We assert that $\bigcap_{\alpha<\delta} i^{*-1}_{A_\alpha A}(p_\alpha)$ can contain only one point. For suppose it contained distinct points p_1 and p_2. Then there would be a formula $\psi \in F(A)$ such that $\psi \in p_1$ and $\tilde{}\psi \in p_2$. There is some $\alpha < \delta$ such that $\psi \in F(A_\alpha)$ and so $\psi \in p_\alpha$ and $\tilde{\psi} \in p_\alpha$ which is impossible. (Topologically, this argument amounts to the statement that $S(A)$ is a Hausdorff space with its topology determined by that of the $S(A_\alpha)$'s.) By 2.5 this unique $p \in S(A)$ must have degree equal to the degree of p_{α_0} and (a) is proved.

(b) There is no loss of generality in assuming that for limit ordinals $\delta, A_\delta = \bigcup_{\alpha<\delta} A_\alpha$; for whenever it is not so for some δ we may interpolate $\bigcup_{\alpha<\delta} A_\alpha$ into the sequence. Consider a sequence $\{p_\alpha; \alpha < \gamma\}$ such that for each $\alpha, p_\alpha \in S(A_\alpha)$ and p_α is a point of minimal transcendental rank in $\bigcap_{\beta<\alpha} i^{*-1}_{A_\beta A_\alpha}(p_\beta)$. We show inductively that such a sequence exists and that it satisfies (b). Assume a sequence defined and satisfying (b) for $\beta < \alpha$. If $\alpha = \beta + 1$, then by 4.2 any point of minimal transcendental rank in $i^{*-1}_{A_\beta A_\alpha}(p_\beta)$ is isolated.

If $\alpha = \delta$ then by (a) and its proof

$$\bigcap_{\beta<\delta} i^{*-1}_{A_\beta A_\delta}(p_\beta) = i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0}) \cap S^\nu(A_\delta)$$

and is a single point, say p_δ. The point p_{α_0} is isolated in $S(A_{\alpha_0})$ so $i_{A_{\alpha_0} A_\delta}(p_{\alpha_0})$ is an open set in $S(A_\delta)$. Thus to prove p_δ isolated in $S(A_\delta)$ it will suffice to show that $i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0}) \cap S^\nu(A_\delta) = i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0})$. Suppose this equality did not hold. Then there would be a $p' \in i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0})$ with transcendental rank of $p' < \nu$. By the argument used in the proof of (a) there would be a $\beta, \alpha_0 \leqq \beta < \delta$, such that $i^*_{A_\beta A_\delta}(p') \neq i_{A_\beta A_\delta}(p_\delta) = p_\beta$. Since p_β has transcendental rank and degree the same as p_{α_0} and $i^*_{A_\beta A_\delta}(p') \in i^{*-1}_{A_{\alpha_0} A_\beta}(p_{\alpha_0})$, by 2.5 transcendental rank of $i^*_{A_\beta A_\delta}(p') <$ transcendental rank $(p_{\alpha_0}) =$ transcendental rank (p_β). This contradicts the assumption that p_β is of minimal rank in $\bigcap_{\beta'<\beta} i^{*-1}_{A_{\beta'} A_\beta}(p_{\beta'})$

Theorem 4.5. *Suppose T is totally transcendental and $\{A_\alpha; \alpha < \gamma\}$ is an increasing chain of members of $\mathscr{N}(T)$ such that for each limit ordinal $\delta < \gamma$, $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$. Then there is an increasing chain $\{B_\alpha; \alpha < \gamma\}$ of models of T such that B_α is prime over A_α (for each $\alpha < \gamma$) and for each limit ordinal $\delta < \gamma, B_\delta = \bigcup_{\alpha<\delta} B_\alpha$*(21).

Proof. Let $A = \bigcup_{\alpha<\gamma} A_\alpha$. We shall show inductively that there exists an increasing sequence of systems $\{C_\alpha; \alpha < \gamma\}$ and of models of $T, \{B_\alpha; \alpha < \gamma\}$ such that (i) $C_\alpha = A \cup B_\alpha$, (ii) $B_\alpha \supseteq A_\alpha$, (iii) $B_\delta = \bigcup_{\beta<\delta} B_\beta$ (for limit ordinals $\delta < \gamma$), and (iv) if D is a model of T, $\alpha = \beta + 1$, $\alpha' \geqq \alpha$ and $f: A_{\alpha'} \cup B_\beta \to D$ is a monomorphism then there is a monomorphism $g: A_{\alpha'} \cup B_\alpha \to D$ with $g \supseteq f$. The sequence $\{B_\alpha; \alpha < \gamma\}$ will then satisfy the theorem.

(21) It may be shown by example that the assumption that T is totally transcendental is stronger than the assumption that the isolated points are dense in $S(A)$ for every $A \in \mathscr{N}(T)$. Theorem 4.3 was proved under the weaker assumption but we have been unable to do the same for Theorem 4.5.

Assume the sequence $\{C_\beta; \beta < \alpha\}$ satisfying (i)-(iv). If $\alpha = \delta$ let $C_\delta = \bigcup_{\beta<\delta} C_\beta$ and $B_\delta = \bigcup_{\beta<\delta} B_\beta$.

If $\alpha = \beta + 1$ we proceed as in the proof of Theorem 4.3. Let $\{p_\nu; \nu < \kappa\}$ be a list of the isolated points of $S(A_\alpha \cup B_\beta)$. By 4.4 we may find a sequence of points $\{p_{0,\eta}; \alpha \leqq \eta < \gamma\}$ such that $p_{0,\alpha} = p_0, p_{0,\eta}$ is an isolated point of $S(A_\eta \cup B_\beta)$ and $\eta' > \eta$ implies $p_{0,\eta'} \supseteq p_{0,\eta}$. Let $q_0 = \bigcup_{\eta<\gamma} p_{0,\eta}$. If there is an element of C_β realizing q_0 denote it by a_0; otherwise add an element satisfying q_0 to C_β and denote it by a_0. By the method of the proof of 4.3 we may iterate this process $\kappa \cdot \omega$ times and find a sequence $\{a_\nu; \nu < \kappa \cdot \omega\}$ such that $A_\alpha \cup B_\beta \cup \{a_\nu; \nu < \kappa \cdot \omega\}$ is a model of T, and for each α' $(\alpha \leqq \alpha' < \gamma)$ a_ν realizes an isolated point in $S(A_{\alpha'} \cup B_\beta \cup \{a_{\nu'}; \nu' < \nu\})$. This latter condition implies (by the same argument used in the proof of 4.3) that (iv) holds for α.

Let $C_\alpha = C_\beta \cup \{a_\nu; \nu < \kappa \cdot \omega\}$ and $B_\alpha = B_\beta \cup \{a_\nu; \nu < \kappa \cdot \omega\}$.

Using condition (iv), above, a simple induction shows that any monomorphism of A_α into a model of T may be extended to a monomorphism of B_α into the same model, i.e., B_α is prime over A_α. Theorem 4.5 is proved.

Suppose $A, B \in \mathscr{N}(T), A \subseteq B$, and $X \subseteq |B| - |A|$. X is a set of elements *indiscernible* over A if every one-one map of

$$|A| \cup X \rightarrow |A| \cup X$$

which is the identity on $|A|$ is a monomorphism. That is, for any open formula, ψ, of L, any $a_1, \cdots, a_m \in A$, and any two sets of distinct elements $x_1, \cdots, x_n$ and $x'_1, \cdots, x'_n \in X$; $\psi(a_1, \cdots, a_m, x_1, \cdots, x_n)$ if and only if $\psi(a_1, \cdots, a_m, x'_1, \cdots, x'_n)$.

THEOREM 4.6. *Suppose T is totally transcendental, $A, B \in \mathscr{N}(T), A \subseteq B$, and $\kappa(A) < \kappa(B) = \kappa$. Then* (i) *if κ is a regular uncountable cardinal, there is an $X \subseteq |B| - |A|$ such that $\kappa(X) = \kappa$ and X is a set of elements indiscernible over A*; (ii) *if κ is uncountable but not regular there is still for each $\lambda < \kappa$ a set $X \subseteq |B| - |A|$ such that $\kappa(X) > \lambda$ and X is a set of elements indiscernible over A.*

Proof. Since for every infinite λ, λ^+ is regular, (ii) will follow immediately from (i) by choosing some $C, A \subseteq C \subseteq B$ and $\kappa(C)$ regular.

So assume κ regular. Suppose $C \in \mathscr{N}(T), \kappa(C) < \kappa$ and $A \subseteq C \subseteq B$. By 2.7 $\kappa(S(C)) < \kappa$, and from the regularity of κ it follows that there is some $p \in S(C)$ which is realized by κ distinct elements of B. From the set of all pairs (C, p) satisfying the above conditions we pick one, say (C_0, p_0), such that transcendental rank of p_0 is the minimum, say ν, and the degree of p_0 is the minimum, say n, among those having rank ν.

Suppose $C' \in \mathscr{N}(T)$, $\kappa(C') < \kappa$ and $C_0 \subseteq C' \subseteq B$. Then $i^{*-1}_{C_0C'}(p_0)$ has power $< \kappa$ and hence must contain some point, p', which is realized by κ elements of B. Since ν is the minimal transcendental rank of such points, transcendental rank $(p') \geqq \nu$. But by 2.3 transcendental rank $(p') \leqq$ transcendental rank $(p_0) = \nu$. A similar argument may be made for degree, so transcendental rank $(p_0) =$ transcendental rank (p') and degree $(p_0) = \text{degree}(p')$. By 2.5 there is only one such point in $i^{*-1}_{C_0C'}(p_0)$. Thus, $i^{*-1}_{C_0C'}(p_0)$ has exactly one point realized by κ elements and that point has transcendental rank ν and degree n.

We show that inductively that there exists a set of κ distinct elements $\{x_\alpha; \alpha < \kappa\} \subseteq |B| - |C_0|$ such that, letting $C_\alpha = C_0 \cup \{x_\beta; \beta < \alpha\}$ and $p_\alpha \in S(C_\alpha)$ the point realized by x_α, p_α is the unique point of rank ν and degree n in $i^{*-1}_{C_0C_\alpha}(p_0)$. For if $\{x_\beta; \beta < \alpha\}$ is defined, then by the discussion of the preceding paragraph there are κ elements of B which realize p_α and we pick x_α to be one of these. Notice that $\beta < \alpha$ implies $i^*_{C_\beta C_\alpha}(p_\alpha) = p_\beta$ and hence x_α realizes p_β for all $\beta \leqq \alpha$.

Suppose $\beta_1 < \cdots < \beta_m$ and $\beta'_1 < \cdots < \beta'_m$. Denote by D_m and D'_m the systems having universe $|C_0| \cup \{x_{\beta_1}, \cdots, x_{\beta_m}\}$ and $|C_0| \cup \{x_{\beta'_1}, \cdots, x_{\beta'_m}\}$ respectively. We assert that the map $f_m: D_m \to D'_m$ which is the identity on C_0 and carries x_{β_i} to $x_{\beta'_i}$ $(i \leqq m)$ is an isomorphism. Then proof is by induction on m. Assume $f_{m-1}: D_{m-1} \to D'_{m-1}$ is an isomorphism. Let q be the point of $S(D_{m-1})$ realized by x_{β_m} and q' the point of $S(D'_{m-1})$ realized by $x_{\beta'_m}$. To prove f_m to be an isomorphism it is sufficient to show that $f^*_{m-1}(q') = q$. Since x_{β_m} realizes a point (namely p_0) of transcendental rank ν and degree n in $S(C_0)$ and a point (namely p_{β_m}) of transcendental rank ν and degree n in $S(C_{\beta_m})$ and $C_0 \subseteq D_{m-1} \subseteq C_{\beta_m}$, it follows from 2.3 and 2.5 that q is of transcendental rank ν and degree n. As proved above, there is a unique point of transcendental rank ν and degree n in $i^{*-1}_{C_0D_{m-1}}(p_0)$, and q must be this point. Similary, q' must be the unique point of rank ν and degree n in $i^{*-1}_{C_0D'_{m-1}}(p_0)$. Since f_{m-1} is the identity on C_0,

$$f^*_{m-1}(i^*_{C_0D'_{m-1}}(p_0)) = i^{*-1}_{C_0D_{m-1}}(p_0).$$

Therefore $f^*_{m-1}(q') = q$ and f_m is an isomorphism.

Finally, we assert that X is indiscernible over A, indeed over C_0. Consider an open formula ψ of $L, a_1, \cdots, a_m \in C_0$, and sequences of distinct elements $(x_{\beta_1}, \cdots, x_{\beta_r})$ and $(x_{\beta'_1}, \cdots, x_{\beta'_r})$ in X. We must show that $\psi(a_1, \cdots, a_m, x_{\beta_1}, \cdots, x_{\beta_r})$ if and only if $\psi(a_1, \cdots, a_m, x_{\beta'_1}, \cdots, x_{\beta'_r})$. We have already shown this in the case when $\beta_1 < \cdots < \beta_r$ and $\beta'_1 < \cdots < \beta'_r$. But by 3.9, $\psi(a_1, \cdots, a_m, x_{\beta_1}, \cdots, x_{\beta_r})$ cannot depend on the order of the β_i's. (We actually apply 3.9 to the theory $T(\{a_1, \cdots, a_m\})$ which extends T by adding $a_1, \cdots, a_m$ as "distinguished elements" but by 2.8, $T(\{a_1, \cdots, a_m\})$ is totally transcendental if T is.) Theorem 4.6 is now proved.

5. **Saturated models and categoricity in power.** Suppose B is an infinite system $\in \mathscr{N}(T)$. B is *saturated* if for every $A \subseteq B$ with $\kappa(A) < \kappa(B)$, every point of $S(A)$ is realized in B.

From 4.1 we see that if $B \in \mathcal{N}(T)$ is saturated, then B is a model of T. Saturated systems were considered in [10], and the following result was established(22).

Theorem 5.1. *If A and B are saturated models of T of the same power, then A is isomorphic to B.*

Thus a sufficient condition for T to be categorical in power κ is that every model of T of power κ be saturated(23).

Suppose $B \in \mathcal{N}(T)$ is an uncountable system. B is *saturated over countable subsystems* if for every countable $A \subseteq B$, B realizes every point of $S(A)$. By 4.1, every $B \in \mathcal{N}(T)$ which is saturated over countable subsystems is a model of T.

Theorem 5.2. *If T is totally transcendental and $\kappa > \aleph_0$, then there is a model of T of power κ which is saturated over countable subsystems*(24).

Proof. Let B_0 be an arbitrary model of T of pwer κ. Then $S(B_0) = \kappa$ by 2.7. Therefore, there is a model of T, $B_1 \supseteq B_0$ such that $\kappa(B_1) = \kappa$ and every point of $S(B_0)$ is realized in B_1. Proceeding inductively, we see that there is an increasing chain of models of T of power κ, $\{B_\alpha; \alpha < \omega_1\}$ such that every point of $S(B_\alpha)$ is realized in $B_{\alpha+1}$ (for all $\alpha < \omega_1$). Then $B = \bigcup_{\alpha<\omega_1} B_\alpha$ is a model of T of power κ which is saturated over countable subsystems. For if A is a countable subsystem of B, then there is an $\alpha < \omega_1$ such that $A \subseteq B_\alpha$; then every $p \in S(A)$ is realized in $B_{\alpha+1}$ and, *a fortiori*, in B.

Lemma 5.3. *Suppose T is totally transcendental and B is an uncountable model of T which is not saturated. Then there is a countable model of T, $A \subseteq B$, with a subsystem $A' \subseteq A$ such that* (i) *there is an infinite set $Y \subseteq |A| - |A'|$ of elements indiscernible over A', and* (ii) *there is a $q \in S(A')$ which is not realized in A.*

(22) In [10] *universal homogeneous* systems are considered. This is a terminology of Jónsson [5]. If K is a class of similar relational systems and $A \in K$ then: (1) A is universal for K if A contains an isomorphic image of every $B \in K$ with $\kappa(B) \leq \kappa(A)$, (2) A is homogeneous in K if whenever $B_1, B_2 \in K, B_1, B_2 \subseteq A$. $\kappa(B_i) < \kappa(A)$, and $f: B_1 \to B_2$ is an isomorphism, then f may be extended to an automorphism of A. Jónsson showed that under certain simple conditions on K that any two universal homogenous systems of the same power are isomorphic. In the case that $K = \mathcal{N}(T)$, universal-homogeneous is equivalent to saturated. This was shown in the countable case by Vaught [18] and in the uncountable case by Keisler (Theorem A2 of [8]).

(23) That the problem of categoricity in power could be approached this way was noticed by Vaught. He proved [10;17] (assuming the generalized continuum hypothesis) that if T is categorical in an increasing sequence of powers then it is categorical in the limit power.

(24) In the case $\kappa = \kappa^{\aleph_0}$, this result was proved in [10] without the assumption that T is totally transcendental. However, it is possible to give an example of a theory T which is not totally transcendental and a cardinal $\kappa > \aleph_0$ with $\kappa^{\aleph_0} \neq \kappa$ such that no model of T of power κ is countably saturated.

Proof. Since B is not saturated there is some $C \subseteq B, \kappa(C) < \kappa(B)$, and a $p \in S(C)$ which is not realized in B. By 4.6 there is a countable infinite set, Y, of elements indiscernible over C contained in $|B| - |C|$. By the Löwenheim-Skolem theorem there is a countable submodel of B, A_0, such that $A_0 \supseteq Y$. For each $a \in A_0$ let p_a be the point of $S(C)$ realized by a. Then no $p_a = p$ since no element of B realizes p. Hence, there is for each $a \in A_0$ a formula $\psi_a \in F(C)$ such that $\psi_a \in p_a$ and $\sim \psi_a \in p$. Since ψ_a involves only a finite number of sysmbols we may find for each some finite $C_a \subseteq C$ such that $\psi_a \in F(C_a)$. Let $A_1' = \bigcup_{a \in A_0} C_a$. Then no $a \in A_0$ realizes $i^*_{A_1'C}(p)$ in $S(A_1')$. Let A_1 be a countable submodel of B such that $A_1 \supseteq A_0 \cup A_1'$. By iteration we may find a sequence of countable models, $A_0 \subseteq \cdots A_n \subseteq \cdots$, and a sequence of systems, $A_1' \subseteq \cdots A_n' \subseteq \cdots$, such that $A_n' \subseteq A_n \cap C$ and no $a \in A_n$ realizes $i^*_{A'_{n+1}C}(p)$ in $S(A'_{n+1})$. Let $A = \bigcup_{n \in \omega} A_n$ and $A' = \bigcup_{n \in \omega} A_n'$. Then $Y \subseteq |A| - |A'|$ is a set of elements indiscernible over A' and no $a \in A$ realizes $i^*_{A'C}(p)$ in $S(A')$.

THEOREM 5.4. *Suppose T is totally transcendental and has an uncountable model which is not saturated. Then for each $\kappa > \aleph_0$, T has a model of power κ which is not saturated over countable subsystems.*

Proof. Let A, A', and Y be as in Lemma 5.3 and $q \in S(A')$ be not realized in A. By the completeness theorem there is an $A_\kappa \in \mathscr{N}(T)$ such that $A_\kappa \supseteq A' \cup Y$ and $A_\kappa - A'$ is a set of κ elements indiscernible over A'. (For we can assert the existence of such an A_κ by a set Σ (of power κ) of sentences, and the existence of $A' \cup Y$ shows that every finite subset of Σ, and therefore Σ, is consistent.) Let $\{y_\alpha; \alpha < \kappa\}$ be a well-ordering of $A_\kappa - A'$, and $A_\alpha = A' \cup \{y_\beta; \beta < \alpha\}$. Apply Theorem 4.5 to get an increasing chain of models of T, $\{\beta_\alpha; \alpha < \kappa\}$, with B_α prime over A_α and for each limit ordinal $\delta < \kappa$, $B_\delta = \bigcup_{\alpha < \delta} B_\alpha$.

We assert that q is not realized in any B_α. The proof is by induction on α. For $\alpha < \omega$, the existence of the model $A \supseteq A' \cup Y$ and not realizing q, implies B_α does not realize q. If $\alpha = \delta$, the induction hypothesis implies no B_β $(\beta < \delta)$ realizes q and, hence, $B_\delta = \bigcup_{\beta < \delta} B_\beta$ does not realize q. Finally, if $\alpha = \beta + 1 > \omega$, then by the indiscernibility of $A_\kappa - A'$ over A', there is an isomorphism of A_α onto A_β which is the identity on A'. So there is a monomorphism of B_α into B_β which is the identity on A'. By the induction hypothesis B_β does not realize q, therefore B_α does not realize q.

$B_\kappa = \bigcup_{\alpha < \kappa} B_\alpha$ is of power κ and does not realize q.

THEOREM 5.5. *If T is categorical in some power $\kappa > \aleph_0$, then every uncountable model of T is saturated.*

Proof. By 3.8, T is totally transcendental. By 5.2, there is a model of T of power κ which is saturated over countable subsystems. If T had an uncountable model which was not saturated, then by 5.4 it would have a model of power κ which was not saturated over countable subsystems, and T would not be categorical in power κ.

THEOREM 5.6. *If T is categorical in one uncountable power then T is categorical in every uncountable power.*

Proof. The proof is immediate from 5.1 and 5.5.

We shall conclude by mentioning some open questions([25]). The first two questions are about theories categorical in uncountable powers but not in power $\aleph_0$.

(1) Does every such theory have exactly $\aleph_0$ isomorphism types of countable models?

(2) Is any such theory finitely axiomatizable?

The next two questions concern theories in languages with an uncountable number of symbols.

(3) If $\kappa > \aleph_0$, Σ is a theory in a language having $\leqq \kappa$ symbols, and Σ is categorical in some power $> \kappa$, is Σ necessarily categorical in every power $> \kappa$?

(4) If $\kappa > \aleph_0$ and every model of Σ has power $\geqq \kappa$ can Σ be categorical in power κ?

We return to theories in countable languages. From 4.3 and 4.6 it follows that if T is totally transcendental and $\kappa > \aleph_0$ we may find a model of T, A, and a set $X \subseteq |A|$ with $\kappa(X) = \kappa(A) = \kappa$ such that any one-one map of X into itself may be extended to an endomorphism of A. This raises the following question.

(5) If T is totally transcendental and $\kappa \geqq \aleph_0$, is there always a model of T, A, with a set $X \subseteq |A|$ such that $\kappa(A) = \kappa(X) = \kappa$ and any one-one map of X onto itself may be extended to an automorphism of A?

Notice this would follow from 3.5 and 3.9 if whenever T were totally transcendental we could find a $T^\#$ which was totally transcendental. In [1], Theorem 2 asserts the affirmative of this question for theories categorical in power 2^κ, but Vaught has pointed out a fallacy in the proof given.

Finally, we consider some questions about the ordinal α_T defined in 2.6. In 2.6 we showed that $a_T < (2^{\aleph_0})^+$. The first question is:

(6) Is α_T ever uncountable?

We can answer this question in one case.

THEOREM 5.7. *If T is totally transcendental, $\alpha_T < \omega_1$.*

Proof. By 2.4 if $p \in \mathrm{Tr}^\alpha(A)$ there is a finite $B \subseteq A$ such that $i^*_{BA}(p) \in \mathrm{Tr}^\alpha(B)$. By 2.7 $S(B)$ is countable for every finite $B \in \mathscr{N}(T)$. Thus we

([25]) Problems (1) through (4) below are not due to the author; they seem to have been considered by several people. Problem (5) has recently been answered affirmatively by Jack Silver.

need only to show that there are only a countable number of isomorphism types of finite members of $\mathscr{N}(T)$. We prove inductively for each $n \in \omega$ that there are only a countable number of isomorphism types of members of $\mathscr{N}(T)$ of power n. For $n = 0$ there is obviously only one. (Strictly, the empty set is not a subsystem. But since we can define $F(\emptyset)$, there is no harm in treating it as a member of $\mathscr{N}(T)$.) Assume only a countable number of isomorphism types of systems of power m. By 2.7 there are only a countable number of ways of adding an element to each system of power m, so there are only a countable number of isomorphism types of members of $\mathscr{N}(T)$ of power $m + 1$.

Another question is:

(7) What model-theoretical conditions on T imply that α_T is finite? Plausible possiblilities are T being categorical in some power, or $T = \Sigma^*$ with Σ finitely axiomatizable.

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