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Causal Construction of Yang-Mills Theories I

SW 9445

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Abstract. - Pure quantized Yang-Mills theories are constructed by causal perturbation theory. We study operator gauge transformations which lead in a natural way to the introduction of ghost fields. Considering fermionic as well as bosonic ghosts, we find that only the former save gauge invariance in second order. We work with free quantum fields throughout, so that all expressions are mathematically well-defined.

PACS. 11.10 - Field theory, 12.35C - General properties of quantum chromodynamics.

*) Work supported by Schweizerischer Nationalfonds

Zürich University Preprint

1. Introduction

For quantization of non-abelian gauge theories there exist two popular methods: the path integral formalism [1] and the canonical quantization method [2]. Both, however, suffer from lack of mathematical control: the basic objects (functional integrals or Heisenberg field operators) are ill-defined. For this reason we follow here a third alternative, that is causal perturbation theory. In this approach only well defined quantities, namely free fields, appear: *all quantum field operators in this paper are understood to be free fields.*

The method goes back to Epstein and Glaser [3]. In recent years it has been worked out for abelian gauge theories (QED) [4]. In contrast to the usual (Feynman) perturbation theory, the ultraviolet and infrared problems are fully under control: the former by careful splitting of causal distributions and the latter by adiabatic switching of the interaction. The S-matrix is constructed inductively order by order in perturbation theory in the form

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n), \tag{1.1}$$

where $g(x)$ is a tempered test function that switches the interaction. The first order $T_1(x)$ must be given and defines the theory. For pure Yang-Mills theories (without matter fields), that we are going to consider, $T_1(x)$ contains the following gluon part

$$T_1^A(x) = igf_{abc} : A_{\mu a}(x) A_{\nu b}(x) \partial^\nu A_c^\mu(x) : \tag{1.2}$$

$$= i\frac{g}{2} f_{abc} : A_{\mu a} A_{\nu b} F_c^{\nu\mu} : \tag{1.3}$$

where

$$F_c^{\mu\nu}(x) = \partial^\mu A_c^\nu - \partial^\nu A_c^\mu \tag{1.4}$$

are the *free* asymptotic Yang-Mills fields. f_{abc} are the usual antisymmetric structure constants of the gauge group, say $SU(N)$, and $A_{\mu a}(x)$ are the gauge potentials, satisfying free field commutation rules

$$[A_a^\mu(x),\, A_b^\nu(y)] = i\delta_{ab} g^{\mu\nu} D(x-y) \tag{1.5}$$

$$[A_a^{(-)\mu}(x),\, A_b^{(+)\nu}] = i\delta_{ab} g^{\mu\nu} D^{(+)}(x-y), \tag{1.6}$$

where $A^{(\pm)}$ are the emission and absorption parts (negative and positive frequency parts) of A and $D^{(\pm)}$ the usual (mass zero) Pauli-Jordan distributions. The time dependence of A is given by the wave equation

$$\square A_a^\mu(x) = 0. \tag{1.7}$$

One may wonder why (1.2) does not contain a quadrilinear term $\sim g^2$. As we shall see in Sect.3 this term is automatically generated in second order by gauge invariance, similarly as in scalar QED [5]. Gauge invariance also requires the coupling of the gauge potentials to ghost fields. This will be discussed in the next section by investigating operator gauge transformations. We study the usual fermionic as well as bosonic ghost fields. In sect.3 we prove that $T_2(x, y)$ with fermionic ghosts and appropriate normalisation is gauge invariant. On the other hand, we show explicitly in sect.4 that gauge invariance cannot be satisfied with a bosonic ghost field. This was first observed by Feynman and by De Witt [6]. In the last section higher orders are briefly discussed.

2. Operator Gauge Transformations

In QED one considers the simple gauge transformation

$$A'^{\mu}(x)=A^{\mu}+\lambda\partial^{\mu}u(x) \tag{2.1}$$

where $u(x)$ is a C-number field satisfying the wave equation

$$\Box u(x)=0. \tag{2.2}$$

This transformation can be pseudo-unitarily implemented in the form

$$A'^{\mu}(x)=e^{-i\lambda Q}A^{\mu}e^{i\lambda Q}=A^{\mu}-\frac{i\lambda}{1!}[Q,\,A^{\mu}]-\frac{\lambda^2}{2!}\big[Q,\,[Q,\,A^{\mu}]\big]+\dots \tag{2.3}$$

by means of the charge

$$Q=\int d^3x\,(\partial_\nu A^\nu\overleftrightarrow{\partial}_0 u). \tag{2.4}$$

This is a consequence of the commutation rule

$$[Q,\,A^{\mu}(x)]=i\partial^{\mu}u(x), \tag{2.5}$$

and all higher commutators vanish. Perturbative QED is invariant under this transformation [4].

We now try to apply the same gauge transformation in the Yang-Mills theory (1.2), where, of course, we must sum over the colour index a

$$Q=\int d^3x\,(\partial_\nu A_a^\nu\overleftrightarrow{\partial}_0 u_a). \tag{2.6}$$

For the sake of completeness we calculate the commutators

$$[Q,\,A_a^{(\pm)\mu}(x)]=\int d^3y\,[\partial_\nu A_b^{(\mp)\nu}\partial_0 u_b-\partial_0\partial_\nu A_b^{(\mp)\nu}u_b,\,A_a^{(\pm)\mu}(x)]$$

$$=\int d^3y\,i\Big(\partial_y^\mu D^{(\pm)}(y-x)\partial_0 u_a(y)-\partial_0^y\partial_y^\mu D^{(\pm)}(y-x)u_a(y)\Big)$$

$$=i\partial^\mu u_a^{(\pm)}(x), \tag{2.7}$$

because $u_a(x)$ are solutions of the wave equation. For the commutator of Q with $T_1^A(x)$ (1.2) we now obtain

$$[Q,\,T_1^A(x)]=igf_{abc}:\Big\{[Q,\,A_{\mu a}]A_{\nu b}\partial^\nu A_c^\mu$$

$$+A_{\mu a}[Q,\,A_{\nu b}]\partial^\nu A_c^\mu+A_{\mu a}A_{\nu b}[Q,\,\partial^\nu A_c^\mu]\Big\}:$$

$$= -g f_{abc} : \Big\{ \partial_\mu u_a A_{\nu b} \partial^\nu A_c^\mu + \partial_\nu (A_{\mu a} u_b \partial^\nu A_c^\mu) \Big\} : . \tag{2.8}$$

Here the last term is a divergence, but the first term spoils gauge invariance. To restore it, this term must be compensated. With a C-number gauge function $u_a(x)$ this is impossible. We therefore consider $u_a(x)$ to be a *free* quantum field that has additional couplings to the gauge potentials. In this way we are led to operator gauge transformations. The fields $u(x)$ occurring in the gauge transformations are called ghost fields.

According to the path-integral method (Fadeev-Popov procedure [1]) one exclusively uses fermionic ghost fields. Therefore we study this case first. Let $u_a(x)$ and $\tilde{u}_a(x)$ be free massless Fermi fields satisfying the anti-commutation relations

$$\{u_a^{(\pm)}(x),\, \tilde{u}_b^{(\mp)}(y)\} = -i\delta_{ab} D^{\mp}(x-y), \tag{2.9}$$

all other anti-commutators vanish. We take the same charge operator Q as before (2.6) and find by a similar calculation as above (2.7)

$$\{Q,\, \tilde{u}_a^{(\pm)}(x)\} = -i\partial_\nu A_a^{(\pm)\nu}(x), \tag{2.10}$$

$$\{Q,\, u_b^{(\pm)}\} = 0, \quad \{Q,\, \partial^\mu \tilde{u}_a(x)\} = -i\partial_\nu F_a^{\mu\nu}(x).$$

The ghost fields must be coupled to the gauge potentials by derivative couplings

$$T_1^u = igf_{abc} : A_{\mu a} u_b \partial^\mu \tilde{u}_c : . \tag{2.11}$$

For the commutator with Q we get

$$[Q,\, T_1^u] = -gf_{abc} : \{\partial_\mu u_a u_b \partial^\mu \tilde{u}_c + A_{\mu a} u_b \partial^\mu \partial^\nu A_{\nu c}\} : . \tag{2.12}$$

Taking the antisymmetry of f_{abc} into account, we see that the first term is a divergence

$$f_{abc} \partial_\mu u_a u_b \partial^\mu \tilde{u}_c = \frac{1}{2} f_{abc} \partial_\mu (u_a u_b \partial^\mu \tilde{u}_c),$$

because $\tilde{u}_a(x)$ fulfills the wave equation. The second term can be transformed as follows

$$\begin{aligned}[Q,\, T_1^u] = -gf_{abc} : \{&\tfrac{1}{2}\partial^\mu (u_a u_b \partial_\mu \tilde{u}_c) + \partial^\nu (A_{\mu a} u_b \partial^\mu A_{\nu c}) \\ &-\partial^\nu u_b A_{\mu a} \partial^\mu A_{\nu c}\} : . \end{aligned} \tag{2.13}$$

Interchanging $a \leftrightarrow b$ and $\mu \leftrightarrow \nu$ in the last term, it becomes equal to the first term in (2.8). Hence, the desired compensation can be achieved by taking

$$\begin{aligned} T_1(x) &= T_1^A(x) - T_1^u(x) \\ &= igf_{abc} : \{A_{\mu a} A_{\nu b} \partial^\nu A_c^\mu - A_{\mu a} u_b \partial^\mu \tilde{u}_c\} : . \end{aligned} \tag{2.14}$$

Then the commutator

$$[Q,\, T_1(x)] = gf_{abc} : \Big\{ -\partial_\nu \Big(A_{\mu a} u_b (\partial^\nu A_c^\mu - \partial^\mu A_c^\nu) \Big) + \frac{1}{2} \partial_\nu (u_a u_b \partial^\nu \tilde{u}_c) \Big\} : \tag{2.15}$$

is a sum of divergences of normal products. If this is integrated with a test function $g(x)$, the result vanishes in the adiabatic limit $g \to 1$. We therefore call (2.15) gauge invariance of $T_1(x)$.

To investigate the finite gauge transformation generated by Q as in (2.3), we must compute the higher commutators in the Lie series. One easily finds

$$[Q,\, u(x)] = -2u(x)Q, \tag{2.16}$$

$$[Q,\, \tilde{u}(x)] = -2\tilde{u}(x)Q - i\partial_\nu A^\nu(x), \tag{2.17}$$

$$[Q,\, u(x)Q] = -2u(x)Q^2. \tag{2.18}$$

The last commutator and all higher ones vanish because

$$Q^2 = \tfrac{1}{2}\{Q,\, Q\} = 0 \tag{2.19}$$

in case of fermionic ghost fields. Then the gauge transformation of the gauge potentials is given by

$$A'_{\mu a} = A_{\mu a} + \lambda\partial_\mu u_a + i\lambda^2\partial_\mu u_a Q, \tag{2.20}$$

$$u'_a = u_a + 2i\lambda u_a Q, \tag{2.21}$$

$$\tilde{u}'_a = \tilde{u}_a - \lambda(\partial_\nu A^\nu_a - 2i\tilde{u}_a Q) - i\lambda^2\partial_\nu A^\nu_a Q. \tag{2.22}$$

These operator gauge transformations have some resemblance with the BRS transformations. In fact, they can be obtained by restricting the latter to lowest order in the coupling constant g. We prefer the notion operator gauge transformation because, in contrast to the BRS transformations that refer to interacting fields, we consider only gauge transformations of free fields. Moreover, such transformations can also be generated by bosonic ghost fields which we are now going to discuss.

As bosonic ghost fields charged or neutral Bose fields are possible. To have the biggest difference to the case of fermionic ghosts just considered, we assume a neutral free Bose field $v_a(x)$ with commutation rules

$$[v_a^{(\mp)}(x),\, v_b^{(\pm)}(y)] = -i\delta_{ab}D^{(\pm)}(x-y), \tag{2.23}$$

and all other commutators vanishing. Using the same charge operator as before

$$Q = \int d^3x\, (\partial_\nu A^\nu_a \overleftrightarrow{\partial}_0 v_a) \tag{2.24}$$

we find

$$[Q,\, v_a^{(\pm)}] = -i\partial_\nu A_a^{(\pm)\nu}(x). \tag{2.25}$$

Again, this scalar field must be coupled to the gauge potential by derivative coupling

$$T_1^v = igf_{abc} : A_{\mu a} v_b \partial^\mu v_c : . \tag{2.26}$$

Then we get the commutator

$$[Q,\, T_1^v(x)] = gf_{abc} : A_{\mu a}(\partial_\nu A^\nu_b \partial^\mu v_c + v_b\partial^\mu\partial_\nu A^\nu_c) : .$$

The way to transform this into a divergence form is not unique. In the following we shall always take out the derivatives that originate from the commutation with Q; this is ∂_ν here:

$$[Q,\, T_1^v(x)] = gf_{abc} : \Big\{\partial_\nu(A_{\mu a}A^\nu_b\partial^\mu v_c + A_{\mu a}v_b\partial^\mu A^\nu_c)$$
$$+2\partial_\nu v_a A_{\mu b}\partial^\mu A^\nu_c\Big\} : . \tag{2.27}$$

The last term again coincides with the first term in (2.8) that must be compensated. Hence, choosing

$$T_1(x) = T_1^A(x) + \frac{1}{2}T_1^v(x)$$
$$= igf_{abc} : \{A_{\mu a}A_{\nu b}\partial^\nu A_c^\mu + \frac{1}{2}A_{\mu a}v_b\partial^\mu v_c\} :, \qquad (2.28)$$

we arrive again at a gauge invariant first order distribution:

$$[Q,\, T_1(x)] = gf_{abc}\partial_\nu : \{\frac{1}{2}A_{\mu a}v_b\partial^\mu A_c^\nu - A_{\mu a}v_b\partial^\nu A_c^\mu$$
$$+\frac{1}{2}A_{\mu a}A_b^\nu\partial^\mu v_c)\} : . \qquad (2.29)$$

It remains to be seen whether and how gauge invariance can be maintained in higher orders.

3. Second Order: Fermionic Ghosts

The second order T-distribution $T_2(x,y)$ is obtained from $T_1(x)$ [4] by first calculating

$$R_2'(x,y) = T_1(x)T_1(y), \qquad (3.1)$$
$$A_2'(x,y) = T_1(y)T_1(x), \qquad (3.2)$$
$$D_2(x,y) = R_2'(x,y) - A_2'(x,y) = [T_1(x),\, T_1(y)]. \qquad (3.3)$$

D_2 has a causal support which is evident from the commutator (3.3). Decomposing it into a retarded (R_2) and advanced part (A_2)

$$D_2(x,y) = R_2(x,y) - A_2(x,y), \qquad (3.4)$$

T_2 is given by

$$T_2(x,y) = R_2(x,y) - R_2'(x,y). \qquad (3.5)$$

To split the operator-valued distribution (3.4), we have first to perform normal ordering and then split the numerical distributions in front of the normal products [3, 4]. Without the normal ordering the operation is ill-defined.

We are interested in the commutator $[Q, D_2]$. The commutation does not affect the numerical distributions in D_2, it only changes the field operators without disturbing normal ordering. Consequently, in the splitting of $[Q,\, D_2]$ we have to split only such numerical distributions that appear also in D_2. With the same convention of normalisation in the splitting of these numerical distributions, we can calculate $[Q, R_2]$ directly by splitting $[Q, D_2]$. This procedure has the advantage that it preserves the divergence structure and shows immediately where gauge invariance may break down. We start from

$$[Q,\, D_2(x,y)] = [Q,\, [T_1(x),\, T_1(y)]]$$
$$= [[Q,\, T_1(x)],\, T_1(y)] - [[Q,\, T_1(y)],\, T_1(x)]. \qquad (3.6)$$

The first term is a divergence with respect to x and the second with respect to y. In fact, the second term is obtained from the first by interchanging x and y and multiplying by -1. The question is, whether the same (divergence form) is true for the commutator $[Q,\, R_2(x,y)]$ obtained by causal splitting of (3.6). Since this commutator agrees with (3.6) on $V^+ \setminus \{0\} = \{(x-y)^2 \geq 0, x^0 - y^0 > 0\}$, gauge invariance can only be spoiled by local terms with support $x = y$. But such terms do arrise in the process of distribution splitting of (3.6). First we must normally oder the commutator (3.6). In this process we get tree and loop graphs, owing to the relation

$$[:ABC:,\, :DEF:] =: [ABC,\, DEF]: + \text{loops},$$

which is true in our situation.

We must only examine the first term on the right of (3.6). Substituting (2.15) we get a first tree contribution

$$K_1(x,y) = -i f_{abc} f_{a'b'c'} \partial_\nu^x : [A_{\mu a} u_b \partial^\nu A_c^\mu,\, A_{\mu' a'} A_{\nu' b'} \partial^{\nu'} A_{c'}^{\mu'}] : \stackrel{\text{def}}{=} \partial_\nu^x K_1^\nu, \tag{3.7}$$

where, after calculating the commutator,

$$K_1^\nu = f_{abc} u_b(x) : A_{\mu a}(x) \{ f_{cb'c'} \partial_x^\nu D(x-y) A_{\nu' b'} \partial_y^{\nu'} A_{c'}^\mu \tag{3.8a}$$

$$+ f_{a'cc'} \partial_x^\nu D(x-y) A_{\mu' a'} \partial^\mu A_{c'}^{\mu'} \tag{3.8b}$$

$$+ f_{a'b'c} \partial_x^\nu \partial_y^{\nu'} D(x-y) A_{a'}^\mu A_{\nu' b'} \} : \tag{3.8c}$$

$$+ f_{abc} u_b(x) : \{ f_{a'b'a} \partial_y^{\nu'} D(x-y) A_{\mu a'} A_{\nu' b'} + \ldots \} \partial^\nu A_c^\mu(x) : . \tag{3.8d}$$

The splitting of (3.7) must be performed as follows: We carry out the derivative ∂_ν^x and, then, we split the causal D-distributions in each term

$$D(x-y) = D^{\text{ret}}(x-y) - D^{\text{av}}(x-y). \tag{3.9}$$

We look whether the resulting retarded distribution R_1 is again a divergence, that means, whether the derivative ∂_ν^x can again be taken out after the splitting. This is not the case because

$$\partial_\nu \partial^\nu D^{\text{ret}}(x-y) = \delta(x-y) \tag{3.10}$$

in contrast to $\square D(x-y) = 0$. Here is the origin of local terms:

$$R_1(x,y) = \partial_\nu^x \Big\{ f_{abc} u_b(x) : A_{\mu a}(x) f_{cb'c'} \partial_x^\nu D^{\text{ret}}(x-y) A_{\nu' b'} \partial_x^{\nu'} A_{c'}^\mu + \ldots \Big\}$$

$$- f_{abc} u_b(x) : A_{\mu a}(x) \Big\{ f_{cb'c'} \delta(x-y) A_{\nu' b'} \partial^{\nu'} A_{c'}^\mu$$

$$+ f_{a'cc'} \delta(x-y) A_{\mu' a'} \partial_y^\mu A_{c'}^{\mu'} + f_{a'b'c} \partial_y^{\nu'} \delta(x-y) A_{a'}^\mu A_{\nu' b'} \Big\} : . \tag{3.11}$$

The local terms compensate those coming from the divergence, if ∂_ν^x operates on $\partial_x^\nu D^{\text{ret}}$; (3.8d) does not give rise to such terms.

For gauge invariance these local terms must drop out. In the splitting of the second commutator in (3.6) with x and y interchanged, the retarded part contains advanced Pauli-Jordan distributions (3.9) $-D^{\text{av}}(y-x)$. These give rise to local terms with the same sign as in (3.11), so that there is no compensation. But the local terms coming from the vertex x cancel out separately. We write (3.11) as follows

$$R_1(x,y) = \partial_\nu^x R_1^\nu + f_{abc}f_{a'b'c}u_b(x) : \Big\{\partial_x^{\nu'} A_a^\mu(x)\delta(x-y)A_{\mu a'}(y)A_{\nu' b'}(y)$$

$$+A_a^\mu(x)\partial_x^{\nu'}\delta(x-y)A_{\mu a'}(y)A_{\nu' b'}(y)\Big\}:,$$

by means of the Jacobi identity for the f_{abc}, or

$$= \partial_\nu^x R_1^\nu + f_{abc}f_{a'b'c} : \Big\{\partial_\nu^x\Big(u_b A_a^\mu \delta(x-y)A_{\mu a'}A_{b'}^\nu\Big) \tag{3.12}$$

$$-\partial_\nu^x u_b A_a^\mu \delta(x-y)A_{\mu a'}A_{b'}^\nu\Big\}: . \tag{3.12a}$$

The second commutator K_2 contributing to (3.6) on the tree level

$$K_2(x,y) = if_{abc}f_{a'b'c'}\partial_\nu^x : [A_{\mu a}u_b\partial^\mu A_c^\nu,\ A_{\mu' a'}A_{\nu' b'}\partial^{\nu'}A_{c'}^{\mu'}] :$$

gives no local term because there is no second derivative ∂_x^ν. But the third commutator

$$K_3(x,y) = if_{abc}f_{a'b'c'}\partial_\nu^x : [A_{\mu a}u_b\partial_x^\nu A_c^\mu,\ A_{\mu' a'}u_{b'}\partial_y^{\mu'}\tilde u_{c'}] : \stackrel{\text{def}}{=} \partial_\nu^x K_3^\nu \tag{3.13}$$

has such a derivative and produces a δ-term:

$$K_3^\nu = f_{abc} : \Big\{f_{cb'c'}u_b A_{\mu a}\partial_x^\nu D(x-y)\partial^\mu \tilde u_{c'}u_{b'}$$

$$+f_{ab'c'}u_b\partial^\nu A_c^\mu D(x-y)\partial_\mu^y \tilde u_{c'}u_{b'} - f_{a'b'b}A_{\mu' a'}u_{b'}\partial_y^{\mu'}D(x-y)A_{\mu a}\partial^\nu A_c^\mu\Big\}:, \tag{3.14}$$

namely,

$$R_3 = \partial_\nu^x R_3^\nu - f_{abc}f_{cb'c'} : u_b A_{\mu a}\delta(x-y)\partial^\mu \tilde u_{c'}u_{b'} : . \tag{3.15}$$

The last local term is again transformed by means of the Jacobi identity

$$L_3 \stackrel{\text{def}}{=} f_{abc}f_{cb'c'} : u_b u_{b'}A_{\mu a}\partial^\mu \tilde u_{c'} : \delta(x-y) \tag{3.16}$$

$$= -f_{b'ac}f_{cbc'} : u_b u_{b'}A_{\mu a}\partial^\mu \tilde u_{c'} : \delta - f_{bb'c}f_{cac'}\ldots \tag{3.16a}$$

Interchanging $b \leftrightarrow b'$ in the first term, it agrees with (3.16) up to the minus sign. Hence,

$$L_3 = -\frac{1}{2}f_{bb'c}f_{cac'} : u_b u_{b'}A_{\mu a}\partial^\mu \tilde u_{c'} : \delta(x-y). \tag{3.17}$$

The commutator

$$K_4 = -if_{abc}f_{a'b'c'}\partial_\nu^x : [A_{\mu a}u_b\partial^\mu A_c^\nu,\ A_{\mu' a'}u_{b'}\partial^{\mu'}\tilde u_{c'}] :$$

gives no local term, one further commutator (K_5) vanishes, but the final one

$$K_6(x,y) = -\frac{i}{2}f_{abc}f_{a'b'c'}\partial_\nu^x : [u_a u_b\partial^\nu \tilde u_c,\ A_{\mu' a'}u_{b'}\partial^{\mu'}\tilde u_{c'}] : \stackrel{\text{def}}{=} \partial_\nu^x K_6^\nu$$

$$K_6^\nu = -\frac{i}{2}f_{abc}f_{a'b'c'} : \Big\{u_a u_b\{\partial^\nu \tilde u_c,\ u_{b'}\}\partial^{\mu'}\tilde u_{c'}$$

$$-u_a u_{b'}\{u_b,\ \partial^{\mu'}\tilde u_{c'}\}\partial^\nu \tilde u_c - u_{b'}\{u_a,\ \partial^{\mu'}\tilde u_{c'}\}u_b\partial^\nu \tilde u_c\Big\}: A_{\mu' a'}$$

$$= \frac{1}{2} f_{abc} f_{a'cc'} : u_a u_b \partial_x^\nu D(x-y) \partial^{\mu'} \tilde{u}_{c'} : A_{\mu' a'} + \ldots , \tag{3.18}$$

produces another local term

$$R_6 = \partial_\nu^x R_6^\nu - \frac{1}{2} f_{abc} f_{a'cc'} : u_a u_b A_{\mu a'} \partial^\mu \tilde{u}_{c'} : \delta(x-y).$$

Changing $a' \to a$, $a \to b'$, this cancels against L_3 (3.17).

Summing up, the only breakdown of gauge invariance so far, is the local term (3.12a). This term is just the commutator of the usual four-gluon interaction with Q, therefore, adding this coupling by hand restores gauge invariance. However, before employing such an ad hoc procedure, we recall that distribution splitting is not unique if a C-number distribution has singular order $\omega \geq 0$ [4]. As in scalar QED [5], there is one tree graph with $\omega = 0$ in D_2 (3.3), namely

$$D_2(x,y) = -f_{abc} f_{a'b'c'} : [A_{\mu a} A_{\nu b} \partial^\nu A_c^\mu , \, A_{\mu' a'} A_{\nu' b'} \partial^{\nu'} A_{c'}^{\mu'}] :$$

$$= -i f_{abc} f_{a'b'c} : A_{\mu a} A_{\nu b} \partial_x^\nu \partial_y^{\nu'} D(x-y) A_{a'}^\mu A_{\nu' b'} : + \ldots$$

The general splitting solution contains an undetermined normalisation term

$$L_1 = -iC f_{abc} f_{a'b'c} : A_{\mu a} A_{\nu b} \delta(x-y) A_{a'}^\mu A_{b'}^\nu : . \tag{3.19}$$

Its commutator with Q

$$[Q, \, L_1] = 4C f_{abc} f_{a'b'c} : \partial_\mu u_a A_{\nu b} \delta(x-y) A_{a'}^\mu A_{b'}^\nu : \tag{3.20}$$

has the same form as (3.12a). Consequently, choosing $C = \frac{1}{2}$, the unwanted local term in (3.12) and the corresponding one from the second term in (3.6) drop out. With this normalisation the tree graphs are gauge invariant.

Let us finally turn to the loop graphs where we need essentially no explicit computation. The "vacuum polarisation" graphs (gluon and ghost loops added up)

$$D_2^7(x,y) =: A_{\mu a}(x) d^{\mu\nu}(x-y) A_{\nu a}(y) : \tag{3.21}$$

give rise to the following commutator

$$[Q, \, D_2^7] = i\partial_\mu^x u_a(x) d^{\mu\nu}(x-y) A_{\nu a}(y) + i A_{\mu a}(x) d^{\mu\nu}(x-y) \partial_\nu^y u_a(y). \tag{3.22}$$

On the other hand, we know from (3.6) that the derivatives coming from the commutator with Q can be taken out, that means (3.21) can be written as follows

$$[Q, \, D_2^7] = \partial_\mu^x \Big\{ i u_a(x) d^{\mu\nu}(x-y) A_{\nu a}(y) \Big\}$$

$$- \partial_\mu^y \Big\{ i u_a(y) d^{\mu\nu}(y-x) A_{\nu a}(x) \Big\},$$

hence

$$\partial_\mu d^{\mu\nu} = 0 = \partial_\nu d^{\mu\nu}. \tag{3.23}$$

The question is whether the retarded part $r^{\mu\nu}$ of $d^{\mu\nu}$ obeys the same divergence relations. If follows from (3.23) that

$$d^{\mu\nu}(x) = (\partial^\mu \partial^\nu - g^{\mu\nu} \Box) d_7(x), \tag{3.24}$$

where $d_7(x)$ has causal support (the logically simplest way to prove causality of d_7 is explicit calculation). We get a splitting solution of (3.24) if we split the scalar distribution $d_7(x)$:

$$r^{\mu\nu}(x)=(\partial^\mu\partial^\nu-g^{\mu\nu}\square)r_7(x). \tag{3.25}$$

Then

$$\partial_\mu r^{\mu\nu}=0=\partial_\nu r^{\mu\nu} \tag{3.26}$$

and gauge invariance is preserved

$$[Q,\,R_2^7]=\partial_\mu^x\Big\{iu_a(x)r^{\mu\nu}(x-y)A_{\nu a}(y)\Big\}$$
$$-\partial_\mu^y\Big\{iu_a(y)r^{\mu\nu}(y-x)A_{\nu a}(x)\Big\}. \tag{3.27}$$

Concerning external field operators $\partial^\mu A^\nu$, we note that these occur in D_n always in the form $F^{\mu\nu}$ (1.4), due to (1.3) and (2.11). It immediately follows from (2.7) that

$$[Q,\,F^{\mu\nu}]=0, \tag{3.28}$$

such that gauge invariance cannot break down here. There is a mixed loop of the following form

$$D_2^8(x,y)=:A_{\mu a}(x)d_\nu(x-y)F_a^{\mu\nu}(y):, \tag{3.29}$$

where

$$d_\nu(x)=\partial_\nu d_8(x). \tag{3.30}$$

This can be easily verified by direct computation. Now

$$\partial_\mu^x d_\nu(x-y)F_a^{\mu\nu}(y)=\partial_\mu^x\partial_\nu^x d_8(x-y)F_a^{\mu\nu}(y)=0, \tag{3.31}$$

and this property remains true after splitting if we again split the scalar causal distribution $d_8(x)$. Finally there is a loop with external ghost fields

$$D_2^9(x,y)=:u_a(x)d_9^\mu(x-y)\partial_\mu^y\tilde{u}_a(y):, \tag{3.32}$$

with

$$d_9^\mu(x)=\partial^\mu d_9(x). \tag{3.33}$$

We have

$$[Q,\,D_2^9]=u_a(x)\partial_\mu^y d_9(x-y)i\partial_\nu^y F_a^{\mu\nu}(y), \tag{3.34}$$

because

$$[Q,\,u\partial\tilde{u}]=\{Q,\,u\}\partial\tilde{u}-u\{Q,\,\partial\tilde{u}\} \tag{3.35}$$

as a consequence of (2.10). Now

$$[Q,\,D_2^9]=i\partial_\nu^y\Big\{u_a(x)\partial_\mu^y d_9(x-y)F_a^{\mu\nu}(y)\Big\}, \tag{3.36}$$

as it must be, and this remains valid after splitting if the scalar distribution d_9 is splitted. This finishes the construction of a gauge invariant retarded distribution R_2. Since R_2' in (3.5) is trivially gauge invariant, we arrive at a gauge invariant two-point distribution $T_2(x,y)$.

4. Second Order: Bosonic Ghost

After the discussion of fermionic ghosts the bosonic case can be treated concisely. The local terms in $[[Q, T_1(x)], T_2(y)]$ come only from the second term in (2.29)

$$[Q, T_1(x)] = -g f_{abc} \partial_\nu : A_{\mu a} v_b \partial^\nu A_c^\mu : + \dots \tag{4.1}$$

The first tree contribution

$$K_1(x,y) = -i f_{abc} f_{a'b'c'} \partial_\nu^x : [A_{\mu a} v_b \partial^\nu A_c^\mu,\ A_{\mu' a'} A_{\nu' b'} \partial^{\nu'} A_{c'}^{\mu'}] : \tag{4.2}$$

agrees with (3.7). Therefore the result (3.12) for the retarded distribution can immediately be taken over

$$R_1(x,y) = \partial_\nu^x R_1^\nu + f_{abc} f_{a'b'c} : \Big\{ \partial_\nu^x (v_b A_a^\mu \delta(x-y) A_{\mu a'} A_{b'}^\nu \tag{4.3}$$

$$-\partial_\nu^x v_b A_a^\mu \delta(x-y) A_{\mu a'} A_{b'}^\nu \Big\} : . \tag{4.3a}$$

The only other local term comes from the commutator

$$K_3 = -\frac{i}{2} f_{abc} f_{a'b'c'} \partial_\nu^x : [A_{\mu a} v_b \partial^\nu A_c^\mu,\ A_{\mu' a'} v_{b'} \partial^{\mu'} v_{c'}] :, \tag{4.4}$$

if we commute $\partial^\nu A_c^\mu$ with $A_{\mu' a'}$:

$$K_3^\nu = \frac{1}{2} f_{abc} f_{cb'c'} : A_{\mu a} v_b \partial_x^\nu D(x-y) v_{b'} \partial^\mu v_{c'} : + \dots \tag{4.5}$$

It leads to the local term

$$L_3 = -\frac{1}{2} f_{abc} f_{cb'c'} : A_{\mu a} v_b \delta(x-y) v_{b'} \partial^\mu v_{c'} :$$

$$= \frac{1}{2} f_{a'b'c} f_{abc} : A_{\mu a'} v_{b'} \delta(x-y) v_b \partial^\mu v_a :, \tag{4.6}$$

where we have generated the same colour tensor as in (4.3). There is no cancellation between (4.3a) and (4.6) and the only possibility to save gauge invariance is that these local terms can be compensated by normalisation terms.

The first normalisation term coming from the pure gluon coupling is the same as before (3.19). It leads to a commutator (3.20) that compensates the local term (4.3a). In contrast to the fermionic case there is a second normalisation term coming from pure ghost coupling

$$D_2^2(x,y) = -\frac{1}{4} f_{abc} f_{a'b'c'} : [A_{\mu a} v_b \partial^\mu v_c,\ A_{\mu' a'} v_{b'} \partial^{\mu'} v_{c'}] :$$

$$= -\frac{i}{4} f_{abc} f_{a'b'c} : A_{\mu a} v_b \partial_x^\mu \partial_y^{\mu'} D(x-y) A_{\mu' a'} v_{b'} : + \dots \tag{4.7}$$

The normalisation term corresponding to this $\omega = 0$ distribution is equal to

$$L_2 = -C \frac{i}{4} f_{abc} f_{a'b'c} : A_{\mu a} v_b \delta(x-y) A_{a'}^\mu v_{b'}. \tag{4.8}$$

It gives rise to the following commutator

$$[Q,\, L_2] = \frac{C}{2} f_{abc} f_{a'b'c} : \Big(\partial_\mu v_a v_b A^\mu_{a'} v_{b'} - A_{\mu a} \partial_\nu A^\nu_b A^\mu_{a'} v_{b'}\Big) \delta(x-y). \tag{4.9}$$

Here the first term allows compensation of L_3 (4.6), but the non-vanishing second term in (4.8) remains unbalanced. It is therefore impossible to save gauge invariance by suitable normalisation. This example shows that gauge invariance of the first order is not sufficient for a gauge invariant theory.

We have also investigated the case of a pair of charged bosonic ghosts. Here gauge invariance can be established in first order in a similar way as above. But it breaks down in second order because the local terms generated by the tree graphs cannot be completely compensated by normalisation.

5. Higher Orders

Gauge invariance can be violated in n-th order by local terms with point support $x_1 = x_2 = \ldots = x_n$, only. This follows from the fact that $[Q,\, D_n]$ and $[Q,\, R_n]$ coincide in the retarded region with respect to x_n. The possibilities to generate such a singular distribution are dying out in higher orders. The reason is that a contraction can only give rise to a 4-dimensional δ-distribution, as we have seen before, so that a $(4n-8)$-dimensional δ-distribution must already be present in T_{n-1}. There are only a few possibilities for it.

Let us discuss this in third order

$$D_3(x,y,z) = [T_2(x,z),\, \tilde{T}_1(y)] + [T_2(y,z),\, \tilde{T}_1(x)] + [T_1(z),\, \tilde{T}_2(x,y)]. \tag{5.1}$$

Since

$$\begin{aligned} \tilde{T}_1 &= -T_1, \\ \tilde{T}_2(x,y) &= -T_2(x,y) + T_1(x)T_1(y) + T_1(y)T_1(x), \end{aligned} \tag{5.2}$$

we have

$$\begin{aligned} D_3 = [T_1(y),\, T_2(x,z)] &+ [T_1(x),\, T_2(y,z)] \\ &- [T_1(z),\, T_2(x,y)] + \ldots \end{aligned} \tag{5.3}$$

where only those terms that can generate a local term on the tree level have been written down; we will always do so in the following. We consider the commutator of the first term in (5.3) with Q

$$[Q,\, D_{31}] = [[Q,\, T_1(y)],\, T_2(x,z)] + [T_1(y),\, [Q,\, T_2(x,z)]]. \tag{5.4}$$

Here we look first at the second term, say K_2. $[Q, T_2]$ must contain a $\delta(x-z)$ in order to produce a local term that violates gauge invariance. We know from sect.3 that $[Q, T_2]$ is a divergence and the only local tree contribution from the vertex x is the second term in (3.12):

$$K_2 =: \Big[A_{\alpha d} A_{\beta e} \partial^\beta_y A^\alpha_g,\, \partial^x_\nu \Big(u_b A^\mu_a \delta(x-z) A_{\mu a'} A^\nu_{b'}\Big)\Big] : \stackrel{\text{def}}{=} \partial^x_\nu K^\nu_2, \tag{5.5}$$

omitting all uninteresting factors. This remains a divergence also after splitting, because the second subscript ν is not on a derivative. Hence, we must only concentrate on the first term K_1 in (5.4).

Again $T_2(x,z)$ must contain $\delta(x-z)$, consequently only the 4-gluon interaction (3.19) must be taken into account as a tree graph. In $[Q,T_1]$ (2.15) only the first term is able to produce a local term:

$$K_1 = \frac{i}{2} f_{abc} f_{a'b'c} f_{deg} \partial^y_\alpha : [A_{\beta d} u_e \partial^\alpha_y A^\beta_g,\ A_{\mu a} A_{\nu b} A^\mu_{a'} A^\nu_{b'}(z)] : \delta(x-z). \tag{5.6}$$

Commuting $\partial^\alpha A^\beta$ with the 4-gluon term, we do get local terms after splitting by the previous mechanism:

$$L_1 = \tfrac{1}{2} f_{abc} f_{a'b'c} \delta(x-z)\delta(y-z) : \Big(f_{dea} A_{\mu d} u_e A_{\nu b} A^\mu_{a'} A^\nu_{b'}$$
$$+ f_{deb} A_{\nu d} u_e A_{\mu a} A^\mu_{a'} A^\nu_{b'} + f_{dea'} A_{\mu d} u_e A^\mu_a A_{\nu b} A^\nu_{b'}$$
$$+ f_{deb'} A_{\nu d} u_e A^\mu_a A^\nu_b A_{\mu a'} \Big) : . \tag{5.7}$$

Interchanging the colour indices of summation, this can be written as follows

$$L_1 = \tfrac{1}{2} \delta(x-z)\delta(y-z) : A_{\mu a} A^\mu_{a'} A_{\nu b} A^\nu_{b'} u_e :$$
$$\Big(f_{dbc} f_{a'b'c} f_{aed} + f_{adc} f_{a'b'c} f_{bed}$$
$$+ f_{abc} f_{db'c} f_{a'ed} + f_{abc} f_{a'dc} f_{b'ed} \Big). \tag{5.8}$$

The bracket is equal to

$$\Big(\dots \Big) = -f_{dec} (f_{abc} f_{b'a'd} + f_{a'b'c} f_{bad}), \tag{5.9}$$

due to the Jacobi identity. Interchanging $d \leftrightarrow c$, the last expression is transformed into its negative. Hence, (5.9) and L_1 vanish.

Summing up, we have found that the third order is gauge invariant on the tree level. Moreover, there is no local divergence term in $[Q,T_3]$ on the tree level, nor a local tree term in T_3. Then tree graphs cannot cause violation of gauge invariance in forth order. There remain loops to be discussed. We cannot exclude that a loop graph contains a contribution with point support which could generate a local term in higher orders, breaking gauge invariance. But we have learnt in sect.3 (3.10) that the violation of gauge invariance is due to the retarded solution of $\Box r = \delta$. This solution is ***unique***, $r = D^{\rm ret}$. $D^{\rm ret}$ occurs in simple tree-like contractions only.
A detailed investigation will be given in a later paper.

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