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Causal Independence and the Energy-Level Density of States in Local Quantum Field Theory

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Dedicated to H. J. Borchers on the occasion of his 60th birthday

Abstract. Within the general framework of local quantum field theory a physically motivated condition on the energy-level density of well-localized states is proposed and discussed. It is shown that any model satisfying this condition obeys a strong form of the principle of causal (statistical) independence, which manifests itself in a specific algebraic structure of the local algebras ("split property"). It is also shown that the proposed condition holds in a free field theory.

1. Introduction

It is well known that the general postulates in quantum field theory, concerning locality, Poincaré covariance, and the spectrum condition, do not exclude models with manifestly unphysical properties. Examples are the generalized free field with continuous mass spectrum, which does not describe particles, or models with an infinite number of particles in the same mass multiplet, for which the familiar relation between spin and statistics need not hold. It has been a long-standing problem in the theory of local algebras [1, 2] and in the standard version of quantum field theory [3], to find conditions of a "local" character which would guarantee an interpretation of the theory in terms of asymptotic particle states.

A first step towards the solution of this problem was taken by Haag and Swieca [4], who pointed out that in any theory with a reasonable particle interpretation the number of states occupying a finite volume of phase space should be limited due to the uncertainty principle. Based on this physical input Haag and Swieca proposed a "compactness criterion" which every quantum field theory ought to satisfy if it is to describe particles. They also showed that their criterion excludes the above mentioned examples of physically unreasonable models. But the difficult problem of whether the compactness criterion ensures a particle interpretation remains open to date. (For some partial results cf. [5].) One may surmise that the compactness criterion is still too general and does not fully reflect the specific phase space properties of a particle theory.

In the present article we therefore propose a sharpened version of this criterion. The precise statement requires a discussion of some technical points, but the underlying physical idea can roughly be stated as follows: the number of states which one can accommodate within some bounded region of configuration space should grow with the total energy available in a specific manner, suggested by the energy-level density of an arbitrary number of indistinguishable particles confined to a container ("box") of finite volume. In Sect. 2 we give the proper formulation of our level density condition (the "nuclearity condition"), and we will show in the Appendix that our condition is satisfied in the free field theory of a spinless massive particle.

In the second part of this study (Sect. 3) we will demonstrate that our condition has implications for the structure of the physical states which resemble the tensor product structure (Fock structure) of collision states in theories with a particle interpretation. To be more specific: let $\mathfrak{A}(\mathcal{O}_1)$ be the algebra of local observables (respectively fields) associated with any bounded region $\mathcal{O}_1$ of Minkowski space. Then there exists another bounded region $\mathcal{O}_2 \supset \mathcal{O}_1$ such that the physical Hilbert space $\mathcal{H}$ can be represented as a tensor product, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, on which the operators[1] $A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)'$ act according to $A = A_1 \otimes 1_2$ and $B = 1_1 \otimes B_2$, respectively. In particular, there exists a total set of vectors $\Phi \in \mathcal{H}$ such that

$$(\Phi, AB\Phi) = (\Phi, A\Phi) \cdot (\Phi, B\Phi) \tag{1.1}$$

for all $A \in \mathfrak{A}(\mathcal{O}_1)$ and $B \in \mathfrak{A}(\mathcal{O}_2)'$. Thus the vectors Φ describe states for which all measurements in the regions $\mathcal{O}_1$ and $\mathcal{O}_2'$, respectively, are uncorrelated. For further discussions on this strong form of causal (statistical) independence see the publications [6–8].

It is a remarkable fact that the presence of this specific property of the theory manifests itself in a clearcut way also in the algebraic structure of the local algebras [8, 9]. Namely, for each pair of regions $\mathcal{O}_1$, $\mathcal{O}_2$ related as above there exists a factor $\mathfrak{M}$ of type I (i.e. a von Neumann algebra which is algebraically isomorphic to the algebra of all bounded operators on some Hilbert space) such that

$$\mathfrak{A}(\mathcal{O}_1)'' \subset \mathfrak{M} \subset \mathfrak{A}(\mathcal{O}_2)''. \tag{1.2}$$

The question of whether the nets of local algebras do have this "split property" [9] in general was originally raised by Borchers. The split property was later established for theories of non-interacting particles in [8] and [10] (cf. also [11]), and by these means also for interacting theories which are locally Fock, such as the $P(\varphi)_2$ models [12] and the Yukawa theory in two dimensions [10]. But there exists also an abundance of models which do not have the split property. Examples are all theories with a non-compact global symmetry group and models of an infinite number of free particles such that the number of species grows very rapidly with the mass [9]. Note that the Haag-Swieca compactness criterion still holds in the latter case.

It is a common feature of these counter-examples that they describe systems with a large number of local degrees of freedom. (This is discussed in a more

[1] If $\mathfrak{B}$ is any algebra acting on $\mathcal{H}$ we denote by $\mathfrak{B}'$ its commutant in $\mathcal{B}(\mathcal{H})$

quantitative manner in Sect. 2.) Our present results indicate that it is precisely this number which is of decisive importance for the question of whether the split property holds in a model. They also suggest that the split property is a quite general characteristic of theories with a sensible particle interpretation. In view of the "local" nature of the feature in question this is a very useful piece of information. For applications of the split property to the construction of local current algebras and a quantum version of Noether's theorem, see the publications [13–15]. Some implications relating to the superselection structure of models are discussed in [10, 16, 17].

It follows from these investigations and the present results that our nuclearity condition distinguishes a class of models exhibiting many physically desirable properties. There is also evidence that this strengthened version of the Haag-Swieca compactness criterion is relevant to the problem of asymptotic completeness [18]. Moreover, as was demonstrated in [19], the nuclearity condition is connected with thermodynamical properties of field theoretic models. Thus it seems that this condition provides a natural basis for the investigation of problems involving considerations of phase space.

The discussion in this paper is within the algebraic framework of quantum field theory [1, 2]. We shall be concerned with a net of local algebras on a separable Hilbert space $\mathscr{H}$, containing the (up to a phase unique) vacuum vector Ω. Specifically we shall assume that to every open double cone $\mathcal{O}$ in Minkowski space there corresponds a von Neumann algebra $\mathfrak{A}(\mathcal{O})$ for which the vector Ω is cyclic. The standard conditions concerning Poincaré covariance, locality and isotony are assumed, and we also assume the usual spectrum condition for the translation subgroup of the Poincaré group.

2. The Nuclearity Condition

The intuitive idea underlying the paper by Haag and Swieca quoted before [4] can be stated roughly as follows: let $\mathscr{L}_r \subset \mathscr{H}$ be the set of vector states describing in some local field theory all excitations of the vacuum which are localized at time $t=0$ in the ball $\mathbf{B}_r = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < r\}$. If one applies to $\mathscr{L}_r$ the orthogonal projection P_E onto the states with total energy less than E one obtains a set $\mathscr{L}_{r,E}$ of states of limited extension in configuration and momentum space. In theories with an asymptotically complete particle interpretation in terms of a finite number of species of particles, these states should describe systems evolving within a characteristic time interval τ into configurations of freely moving particles which, because of the maximal propagation speed $c=1$, are localized in the region $\mathbf{B}_{r+\tau}$. So the number of linearly independent states in $\mathscr{L}_{r,E}$ should not exceed the number of different configurations of non-interacting particles which can be placed in the region $\mathbf{B}_{r+\tau}$ and have total energy less than E. Applying now the rule that the number of quantum states of a particle which can be associated with a finite volume Γ of phase space is equal to $\Gamma/(2\pi)^3$ (setting $\hbar=1$) there should be only a finite number of such configurations. Hence by this heuristic argument one is led to the conclusion that in theories with a reasonable particle interpretation the number of independent states in $\mathscr{L}_{r,E}$ ought to be "finite."

It was pointed out by Haag and Swieca that the above considerations give only a rough idea of the actual situation in local field theory. In particular, there does not exist a notion of localization of states which has all the properties familiar from non-relativistic quantum mechanics. If one adopts, for example, the concept of strict localization introduced by Knight [20] (cf. also the discussion below) one is faced with the problem that the linear span of the set $\mathscr{L}_r$ is dense in the physical Hilbert space. Consequently the sets $\mathscr{L}_{r,E}$ cannot be finite dimensional. But a careful estimation of the long range correlations of the states in $\mathscr{L}_{r,E}$ led Haag and Swieca to the conclusion that, at least in massive particle theories, these sets should be (strongly) compact. Moreover, using the concept of approximate dimension, they were able to estimate the "size" of these sets as a function of the cutoff energy E.

If one follows the reasoning of Haag and Swieca in detail one finds that their conclusions are in two respects unnecessarily conservative. Firstly, it follows from their arguments that the sets $\mathscr{L}_{r,E}$ should not only be compact, but even nuclear (cf. the definition given below). And secondly, the size of the sets $\mathscr{L}_{r,E}$ was overestimated, since the indistinguishability of particles was not taken into account in the discussion.

Instead of directly modifying the reasoning of Haag and Swieca, we will present here an alternative, likewise heuristic argument, shedding some light on the properties of the sets $\mathscr{L}_{r,E}$. For the subsequent analysis it is actually more appropriate to consider the sets $e^{-\beta H}\mathscr{L}_r$, $\beta > 0$, where the energy H has been cut off smoothly.

Our considerations are based on a rough analogy between the sets $e^{-\beta H}\mathscr{L}_r$ and the grand canonical ensembles appearing in statistical mechanics. Using this analogy we will motivate a condition on the size of these sets which rests upon the assumption that the theory in question has decent thermodynamical properties, as one may expect in models with a realistic particle spectrum. Our heuristic input consists of the following three premises.

1. Boundary effects should play a secondary rôle for the problem at hand, so it seems reasonable to assume that the size of the sets $e^{-\beta H}\mathscr{L}_r$ remains essentially unchanged if one proceeds from the given (infinite volume) theory to the corresponding theory for finite volume V, provided V is sufficiently large compared to r^3. A weakened version of this hypothesis can be stated as follows: let $\mathscr{H}_V$ and H_V be the Hilbert space and the Hamiltonian of the finite volume theory, respectively. Then it should be possible to identify the set $e^{-\beta H}\mathscr{L}_r$ with a subset of $e^{-\beta H_V}\mathscr{H}_{V,1}$, where $\mathscr{H}_{V,1}$ is the unit ball in $\mathscr{H}_V$. Namely, for each r and β there should exist a similarity transformation S (i.e. a bounded, invertible operator) mapping $\mathscr{H}_V$ onto $\mathscr{H}$, such that

$$e^{-\beta H}\mathscr{L}_r \subset S \cdot e^{-\beta H_V}\mathscr{H}_{V,1}. \tag{2.1}$$

The norm of S should, for fixed β, converge to 1 in the limit of large V/r^3.

2. In statistical mechanics the operators $e^{-\beta H_V}$ describe, for any given volume $V < \infty$ and temperature $\beta^{-1} > 0$, the Gibbs equilibrium states, and in most theories of physical interest these operators have a finite trace. The few exceptions to this rule are theories with a "maximal temperature" such as the so-called string-theories (cf. for example [21]). But disregarding these models, the sets $e^{-\beta H_V}\mathscr{H}_{V,1}$

appearing in the right-hand side of (2.1) are the images of the unit ball in $\mathscr{H}_V$ under the actions of some trace class operators. Such sets are the simplest examples of nuclear sets, as defined by Grothendieck [22].

Definition. A subset $\mathscr{N}$ of a Hilbert space $\mathscr{H}$ is called a *nuclear set* if there exists a sequence of linear functionals l_n, $n \in \mathbb{N}$ defined on the linear span of $\mathscr{N}$, and a sequence of unit vectors Φ_n, $n \in \mathbb{N}$ such that

i) $$\sum_n \lambda_n < \infty\,, \quad \text{where} \quad \lambda_n = \sup\{|l_n(\Psi)|:\ \Psi \in \mathscr{N}\}\,,$$

ii) $$\sum_n l_n(\Psi) \cdot \Phi_n = \Psi \quad \text{for all} \quad \Psi \in \mathscr{N}\,.$$

The *nuclearity index* of $\mathscr{N}$ can then be defined, setting $\nu(\mathscr{N}) = \inf \sum_n \lambda_n$, where the infimum is to be taken with respect to all functionals l_n, $n \in \mathbb{N}$ and vectors Φ_n, $n \in \mathbb{N}$ complying with the above conditions.

It is easy to verify that similarity transformations S map nuclear sets $\mathscr{N}$ onto nuclear sets, and that $\nu(S \cdot \mathscr{N}) \leqq \|S\| \cdot \nu(\mathscr{N})$. Therefore it follows from the previous assumptions [cf. in particular relation (2.1)] that the sets $e^{-\beta H}\mathscr{L}_r \subset \mathscr{H}$ are nuclear. Moreover, since clearly $\nu(e^{-\beta H_V}\mathscr{H}_{V,1}) \leqq \mathrm{Tr}\, e^{-\beta H_V}$, we obtain

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq \|S\| \cdot \mathrm{Tr}\, e^{-\beta H_V}\,. \tag{2.2}$$

Bearing in mind that $\|S\|$ should be close to 1 if V/r^3 is sufficiently large, one can derive from this estimate bounds on the nuclearity index $\nu(e^{-\beta H}\mathscr{L}_r)$ if one has sufficient information on the level density of H_V.

3. It is also obvious from relation (2.2) that the dependence of the nuclearity index $\nu(e^{-\beta H}\mathscr{L}_r)$ on r and β is linked to the thermodynamical properties of the theory in question: since $\mathrm{Tr}\, e^{-\beta H_V}$ is the grand partition function (for zero "chemical potential") the quantity

$$p = (\beta V)^{-1} \cdot \ln(\mathrm{Tr}\, e^{-\beta H_V}) \tag{2.3}$$

is to be interpreted as the pressure of the grand canonical ensemble occupying the volume V at temperature β^{-1}. Now in theories of physical interest the pressure should stay bounded (for fixed β) in the thermodynamic limit $V \to \infty$. In this generic case it then follows from (2.2) and the remark following it that for all $r \geqq r_0$, where r_0 is some arbitrarily fixed length,

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq e^{r^3 \phi(\beta)}\,. \tag{2.4}$$

Here ϕ is some model-dependent function which tends to infinity as β approaches 0.

If both, $\|S\|$ and p would converge uniformly for small β in the limit of large V/r^3 and V, respectively, one could replace $\phi(\beta)$ in (2.4) by $\beta \cdot p_\infty(\beta)$, where p_∞ is the pressure in the thermodynamic limit. But in general it might be necessary to modify this expression by some additional β-dependent factor, which subsumes the boundary effects.

The dominant contribution to ϕ should, however, be due to the pressure p_∞. For non-interacting particles one obtains in the limit of small β (neglecting the particle masses and using Stefan-Boltzmann's law) $p_\infty = c \cdot \beta^{-4}$. We note that this relation holds irrespective of the particle statistics. A similar behaviour of the pressure is expected in theories which are asymptotically free [23]. So in these cases ϕ should have an at most power-like singularity at $\beta = 0$. On the other hand, there exist models where p_∞, and therefore also ϕ, has an essential singularity at $\beta = 0$. (An artificial example is the theory of an infinite number of non-interacting particles, the number of which grows sufficiently rapidly with mass.) But the bound (2.4) with $\phi(\beta) = c \cdot \beta^{-n}$ for some $n > 0$ is expected to hold in most theories of physical interest. We will therefore restrict our attention to these cases.

Having thus explained the heuristic basis of our nuclearity condition we can proceed now to its precise formulation. To this end we must merely specify the sets $\mathscr{L}_r$ of well localized states. We distinguish these states by the following conditions: firstly, it should be possible to generate these states from the vacuum by some operation inside the region $\mathbf{B}_r$ at time $t = 0$. Hence the vectors Φ representing these states should be of the form $\Phi = W\Omega$ for suitable operators W from the algebra $\mathfrak{A}(\mathcal{O}_r)$; here $\mathcal{O}_r$ denotes the "double cone" with base $\mathbf{B}_r$ at $t = 0$. And secondly, it should be impossible to distinguish the well-localized states from the vacuum by measurements in the spacelike complement $\mathcal{O}_r'$ of $\mathcal{O}_r$. Thus the vectors Φ must satisfy

$$(\Phi, A\Phi) = (\Omega, A\Omega) \tag{2.5}$$

for all observables A which are localized in $\mathcal{O}_r'$. As was shown by Knight [20] and Licht [24], this condition implies that the operators W generating the vectors Φ from the vacuum must be isometries. We therefore identify the well-localized states with the set of vectors

$$\mathscr{L}_r = \{W\Omega\colon\ W \in \mathfrak{A}(\mathcal{O}_r),\ W^*W = 1\}\,. \tag{2.6}$$

This is essentially the class of states considered by Haag and Swieca [4].

We now formulate our nuclearity condition, which we expect to be satisfied by any local quantum field theory admitting a particle interpretation and having regular thermodynamical properties.

Condition of Nuclearity. The sets $e^{-\beta H}\mathscr{L}_r$ must be nuclear for all $\beta > 0$ and $r > 0$. Moreover, there must exist positive constants c, n and r_0, β_0 such that

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq e^{cr^3\beta^{-n}} \tag{2.7}$$

for all $r \geqq r_0$ and $0 < \beta \leqq \beta_0$.

We show in the Appendix by a direct computation that this condition is satisfied in the field-theory of a non-interacting spinless, massive particle. This result provides evidence to the effect that the boundary effects alluded to in our heuristic discussion play indeed the secondary rôle which we anticipated.

Our condition is clearly stronger than the criterion proposed by Haag and Swieca. In particular, it restricts the admissible particle spectrum at high energies. Yet since a physically acceptable theory should not only allow a particle interpretation, but also exhibit a realistic thermodynamical behaviour, we believe that our condition characterizes within the general setting of quantum field theory a relevant class of models.

3. The Split Property as a Consequence of Nuclearity

We will now show that the local algebras have the split property outlined in the Introduction if they satisfy the nuclearity condition. Our goal is the following

Theorem 3.1. *Let $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ be a local net of von Neumann algebras subject to the standard conditions mentioned in the Introduction and the condition of nuclearity. Then there exists for any bounded region $\mathcal{O}_1$ another bounded region $\mathcal{O}_2 \supset \mathcal{O}_1$ such that:*

i) *(Existence of product states) There is an isometry V mapping $\mathcal{H}$ onto $\mathcal{H} \otimes \mathcal{H}$ such that*

$$VA_1A_2'V^{-1} = A_1 \otimes A_2'$$

for all $A_1 \in \mathfrak{A}(\mathcal{O}_1)$ and $A_2' \in \mathfrak{A}(\mathcal{O}_2)'$.

ii) *(Split property) There exists a factor $\mathfrak{M}$ of type I such that*

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{M} \subset \mathfrak{A}(\mathcal{O}_2).$$

For the derivation of this result one actually does not need all the properties of the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ mentioned in the premises of the theorem. In particular, the specific dependence of the nuclearity index on the radius r of the localization region given in relation (2.7) is unessential; all that matters are the limitations on the level density of localized states at high energies. In order to reveal which properties of the net are relevant for our argument as well as for later reference we give below a list of the specific assumptions on which our proof of the theorem is based.

1. We will make use of the fact that the von Neumann algebras $\mathfrak{A}(\mathcal{O})$ constitute a net with respect to the open, bounded regions $\mathcal{O} \subset \mathbb{R}^4$; thus $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ whenever $\mathcal{O}_1 \subset \mathcal{O}_2$. We remark that the spacelike commutation properties of the local operators do not enter into our argument. But it will be essential that the continuous, unitary representation $x \to U(x)$ of the translations $\mathbb{R}^4$ acts covariantly on the net,

$$U(x)\mathfrak{A}(\mathcal{O})U(x)^{-1} \subset \mathfrak{A}(\mathcal{O}+x), \tag{3.1}$$

and that the joint spectrum of the generators of $x \to U(x)$ is contained in the closed forward lightcone $\bar{V}_+$. Moreover, we will rely on the fact that there is an (up to a phase unique) unit vector $\Omega \in \mathcal{H}$ which is invariant under the action of $U(x)$ and which is cyclic for each $\mathfrak{A}(\mathcal{O})$.

2. We require in our argument the existence of some vector $\Phi \in \mathcal{H}$ which is cyclic for the algebras $\mathfrak{A}(\mathcal{O}_1)' \cap \mathfrak{A}(\mathcal{O}_2)$ whenever the closure of $\mathcal{O}_1$ is contained in the interiour of $\mathcal{O}_2$, i.e. $\bar{\mathcal{O}}_1 \subset \mathcal{O}_2$. In the present case of a *local* net it is obvious from the preceding assumption that Ω satisfies this condition. But the existence of vectors Φ with the desired property can also be established for non-local nets, provided the algebras $\mathfrak{A}(\mathcal{O}_1)' \cap \mathfrak{A}(\mathcal{O}_2)$ are properly infinite [26].

3. We also rely on the following property of algebraic independence of the algebras $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)'$, whenever $\bar{\mathcal{O}}_1 \subset \mathcal{O}_2$: if the product $A_1 \cdot A_2'$ of operators $A_1 \in \mathfrak{A}(\mathcal{O}_1)$ and $A_2' \in \mathfrak{A}(\mathcal{O}_2)'$ is zero, then either $A_1 = 0$ or $A_2' = 0$. This property was established for local nets by Schlieder [6], and by a slight modification of his argument it can be established for any net satisfying the preceding two conditions.

4. Besides these standard properties we will use in our argument the following weakened version of our nuclearity condition: let $\mathfrak{U}(\mathcal{O})$ be the group of all unitaries in $\mathfrak{A}(\mathcal{O})$ and let H be the positive generator of the time translations $t \to U(t \cdot e)$, where $e \in \mathbb{R}^4$ is a timelike vector fixing the time direction. Then we assume that the set of vectors $e^{-\beta H}\mathfrak{U}(\mathcal{O})\Omega$ is nuclear for any $\beta > 0$. Moreover, we require that for small β

$$\nu(e^{-\beta H}\mathfrak{U}(\mathcal{O})\Omega) \leqq e^{c\beta^{-n}}. \tag{3.2}$$

where c and n are positive constants (independent of β, but dependent on $\mathcal{O}$).

We emphasize that the statement of the theorem holds true for any net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ satisfying these four conditions.

We turn now to the proof of Theorem 3.1 which will be given in several steps. Let $\mathcal{O}_1, \mathcal{O}_2$ be regions such that $\bar{\mathcal{O}}_1 \subset \mathcal{O}_2$, and let $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$ be the *-algebra which is generated from $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)'$ by taking all finite sums of products of operators in these algebras. We will consider two representations of the algebra $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$. The first one, denoted by π, acts on $\mathcal{H}$ and is given by

$$\pi\left(\sum_k A_k A_k'\right) = \sum_k A_k A_k', \tag{3.3}$$

where $A_k \in \mathfrak{A}(\mathcal{O}_1)$ and $A_k' \in \mathfrak{A}(\mathcal{O}_2)'$. Thus π is the identical representation. Note, however, that π depends on the choice of the regions $\mathcal{O}_1$ and $\mathcal{O}_2$. Yet in order not to overburden the notation we do not indicate this dependence explicitly. The second representation of $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$ to be considered is denoted by π_p and acts on $\mathcal{H} \otimes \mathcal{H}$. It is defined by

$$\pi_p\left(\sum_k A_k A_k'\right) = \sum_k A_k \otimes A_k' \tag{3.4}$$

with $A_k \in \mathfrak{A}(\mathcal{O}_1)$ and $A_k' \in \mathfrak{A}(\mathcal{O}_2)'$. That this definition is consistent, i.e. that $\sum_k A_k A_k' = 0$ implies $\sum_k A_k \otimes A_k' = 0$, is a consequence of the algebraic independence of $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)'$, as was shown by Roos [7].

It is our aim to show that for any bounded region $\mathcal{O}_1$ there exists another bounded region $\mathcal{O}_2 \supset \bar{\mathcal{O}}_1$ such that the representations π and π_p of $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$ are unitarily equivalent. This is exactly the statement in the first part of the theorem. The second part is then an immediate consequence: namely, let V be any isometry of $\mathcal{H}$ onto $\mathcal{H} \otimes \mathcal{H}$ implementing the equivalence of π and π_p. Then $\mathfrak{M} = V^{-1}(\mathcal{B}(\mathcal{H}) \otimes 1)V$ is clearly a type I factor. From the trivial inclusion

$$V^{-1}(\mathfrak{A}(\mathcal{O}_1) \otimes 1)V \subset V^{-1}(\mathcal{B}(\mathcal{H}) \otimes 1)V \subset (V^{-1}(1 \otimes \mathfrak{A}(\mathcal{O}_2)')V)' \tag{3.5}$$

it then follows that the split property holds.

It was pointed out in [8] that the problem of establishing the equivalence of the representations π and π_p can be reduced to an estimate of the norm-difference of two specific functionals ω and ω_p on the algebra $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$. These functionals are, for $C \in \mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$, given by

$$\omega(C) = (\Omega, \pi(C)\Omega), \qquad \omega_p(C) = (\Omega \otimes \Omega, \pi_p(C)\Omega \otimes \Omega). \tag{3.6}$$

We denote the norm difference of these functionals by

$$\|\omega - \omega_p\|_{\mathcal{O}_1; \mathcal{O}_2} = \sup\{|\omega(C) - \omega_p(C)|: C \in \mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2), \|C\| \leqq 1\}. \tag{3.7}$$

The following proposition provides the basis for the subsequent investigations.

Proposition 3.2.[2] *Let $\mathcal{O}_a, \mathcal{O}_b$ be open, bounded regions such that $\bar{\mathcal{O}}_a \subset \mathcal{O}_b$, and let*

$$\|\omega - \omega_p\|_{\mathcal{O}_a;\, \mathcal{O}_b} < 2. \tag{3.8}$$

Then the representations π and π_p of $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$ are unitarily equivalent for any pair of regions $\mathcal{O}_1, \mathcal{O}_2$ satisfying $\bar{\mathcal{O}}_1 \subset \mathcal{O}_a \subset \bar{\mathcal{O}}_b \subset \mathcal{O}_2$.

Proof. The proof of this statement is based on standard arguments and may be omitted in a first reading. It consists of the following three steps.

i) Let $\tilde{\pi}$ and $\tilde{\pi}_p$ be the representations of $\tilde{\mathfrak{C}} := \mathfrak{C}(\mathcal{O}_a; \mathcal{O}_b)$ defined according to relations (3.3) and (3.4), respectively. It then follows from the definition (3.6) of ω and ω_p, and from the assumption (3.8) that these representations are non-disjoint (cf. the Appendix of [25]). This means that there exist non-trivial subrepresentations of $\tilde{\pi}$ and $\tilde{\pi}_p$, respectively, which are unitarily equivalent.

ii) Let $\tilde{\pi}_s$ be any non-trivial subrepresentation of $\tilde{\pi}$. We want to show next that if $\mathcal{O}_1, \mathcal{O}_2$ satisfy the premises of the proposition, then the restrictions of $\tilde{\pi}_s$, respectively $\tilde{\pi}$, to the algebra $\mathfrak{C} := \mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2) \subset \tilde{\mathfrak{C}}$ are unitarily equivalent. To that end we must verify that the projection E_s onto the relevant subspace $\mathcal{H}_s \subset \mathcal{H}$ reducing $\tilde{\pi}$ can be represented in the form $E_s = V_s V_s^*$, where $V_s \in \tilde{\pi}(\mathfrak{C})'$ is an isometry, i.e. $V_s^* V_s = 1$.

It is well known that such an isometry exists if there is some vector $\Phi \in \mathcal{H}$ which is cyclic for $\tilde{\pi}(\mathfrak{C})$ and which has the property that $E_s\Phi$ is separating for $\tilde{\pi}(\mathfrak{C})''$. Because then it follows from a Radon-Nikodym type of theorem [26, Theorem 2.7.9] that there exists another vector $\Psi \in \mathcal{H}$ which is cyclic for $\tilde{\pi}(\mathfrak{C})$, and

$$(\Psi, \tilde{\pi}(C)\Psi) = (E_s\Phi, \tilde{\pi}(C)E_s\Phi)$$

for all $C \in \mathfrak{C}$. Taking into account these features of Ψ and Φ as well as the fact that $E_s \in \tilde{\pi}(\tilde{\mathfrak{C}})' \subset \tilde{\pi}(\mathfrak{C})'$, it is then easy to verify that the operator V_s defined by

$$V_s \cdot \tilde{\pi}(C)\Psi = \tilde{\pi}(C)E_s\Phi \quad \text{for} \quad C \in \mathfrak{C}$$

is an isometry with the desired properties. So it remains to establish the existence of Φ.

Now according to our assumptions on the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ there exists a vector $\Phi \in \mathcal{H}$ which is cyclic for $\mathfrak{A}(\mathcal{O}_\alpha)' \cap \mathfrak{A}(\mathcal{O}_\beta)$, whenever $\bar{\mathcal{O}}_\alpha \subset \mathcal{O}_\beta$. In particular Φ is cyclic for $\tilde{\pi}(\mathfrak{C}) = \mathfrak{C}$. That $E_s\Phi$ is separating for $\tilde{\pi}(\mathfrak{C})''$ can be seen as follows: since $\bar{\mathcal{O}}_1 \subset \mathcal{O}_a \subset \mathcal{O}_2$, it is clear that $\mathfrak{C}' \cap \tilde{\mathfrak{C}} \supset \mathfrak{A}(\mathcal{O}_1)' \cap \mathfrak{A}(\mathcal{O}_a)$, so Φ is separating for $\tilde{\pi}(\mathfrak{C})'' \vee \tilde{\pi}(\tilde{\mathfrak{C}})'$ $= \mathfrak{C}'' \vee \tilde{\mathfrak{C}}'$. Hence if $Z \cdot E_s\Phi = 0$ for some projection $Z \in \tilde{\pi}(\mathfrak{C})'' = \mathfrak{C}''$, we obtain $Z \cdot E_s = 0$. Moreover, with the notation $Z(x) = U(x)ZU(x)^{-1}$, we have $[Z(x), E_s] = 0$ for x in some open neighbourhood of the origin in $\mathbb{R}^4$; the latter assertion follows from $E_s \in \tilde{\pi}(\tilde{\mathfrak{C}})' = \tilde{\mathfrak{C}}'$ and the fact that $U(x)\mathfrak{C}U(x)^{-1} \subset \tilde{\mathfrak{C}}$ for all x satisfying $\bar{\mathcal{O}}_1 + x \subset \mathcal{O}_a \subset \bar{\mathcal{O}}_b \subset \mathcal{O}_2 + x$.

It is a well-known consequence of the above relations between Z and E_s and the assumed spectral properties of $U(x)$ that $ZU(x)E_s = 0$ for all $x \in \mathbb{R}^4$ [27]. Since E_s commutes with the operators $A_k(x_k)$, $k = 1, \ldots, n$ if $A_k \in \mathfrak{A}(\mathcal{O}_1)$ and x_k is contained in

[2] The nuclearity condition is not needed here

some sufficiently small neighbourhood of the origin in $\mathbb{R}^4$, it is also clear that

$$ZU(x)A_1(x_1)\dots A_n(x_n)E_s=0\,.$$

But this equation extends, by the edge of the wedge theorem, to arbitrary $x, x_1, \dots, x_n$ because of the spectral properties of $U(x)$. Now the von Neumann algebra generated by $\mathfrak{A}(\mathcal{O}_1)$ and $U(x)$ is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. [Recall that Ω is cyclic for $\mathfrak{A}(\mathcal{O}_1)$ and the only state in $\mathcal{H}$ which is invariant under translations.] So we conclude from the above discussion that $Z\cdot\mathcal{B}(\mathcal{H})\cdot E_s=0$. Since $E_s\neq 0$, this is only possible if $Z=0$, which proves that the vector $E_s\Phi$ is separating for $\pi(\mathfrak{C})''$.

iii) Finally, let $\bar{\pi}_{ps}$ be any non-trivial subrepresentation of $\bar{\pi}_p$. Arguing as in the previous step one can show that the restrictions of these representations to the algebra $\mathfrak{C}$ are equivalent: the counterpart to the vector Φ is, in the present case, the vector $\Phi\otimes\Phi$, which is cyclic for $\pi_p(\mathfrak{C})$ and separating for $\pi_p(\mathfrak{C})''\vee\pi_p(\bar{\mathfrak{C}})'$; the rôle of the translations $x\to U(x)$ is now played by $x\to U_p(x):=U(x)\otimes U(x)$ which also satisfies the spectrum condition. Making use of the facts that $\Omega\otimes\Omega$ is a cyclic vector for $\mathfrak{A}(\mathcal{O}_1)\otimes\mathfrak{A}(\mathcal{O}_1)$ and the only state in $\mathcal{H}\otimes\mathcal{H}$ which is invariant under $U_p(x)$, the previous argument can be taken over almost literally.

Summing up, we have seen in i) that there exist subrepresentations $\bar{\pi}_s$ and $\bar{\pi}_{ps}$ of $\bar{\pi}$ and $\bar{\pi}_p$, respectively, which are unitarily equivalent: $\bar{\pi}_s\sim\bar{\pi}_{ps}$. From ii) we know that $\pi=\bar{\pi}\restriction\mathfrak{C}\sim\bar{\pi}_s\restriction\mathfrak{C}$, and from iii) that $\pi_p=\bar{\pi}_p\restriction\mathfrak{C}\sim\bar{\pi}_{ps}\restriction\mathfrak{C}$. Hence we conclude that $\pi\sim\pi_p$. □

By the above proposition the problem of establishing the equivalence of π and π_p has, so to say, been reduced to a "computational" problem. Making use of the nuclearity condition (3.2) we shall now carry out this computation and show that for any given bounded region $\mathcal{O}_a$ there exists another bounded region $\mathcal{O}_b\supset\bar{\mathcal{O}}_a$ such that $\|\omega-\omega_p\|_{\mathcal{O}_a;\mathcal{O}_b}<2$.

Let $\mathcal{O}$ be a fixed bounded region and let, for any $r>0$, $\mathcal{O}_r$ be another bounded region which is sufficiently large such that $\bar{\mathcal{O}}+t\cdot e\subset\mathcal{O}_r$ for all $|t|<r$; here $e\in\mathbb{R}^4$ is the timelike vector appearing in the formulation of the nuclearity condition (3.2). Picking arbitrarily a finite number of operators $A_k\in\mathfrak{A}(\mathcal{O})$ and $A'_k\in\mathfrak{A}(\mathcal{O}_r)'$ for which $\left\|\sum_k A_kA'_k\right\|\leqq 1$, we define two functions $f_+(z)$ and $f_-(z)$ as follows.

$$\begin{aligned} f_+(z)&=\sum_k(\Omega, A'_k(1-P_\Omega)e^{izH}A_k\Omega)\quad\text{for}\quad z\in\mathbb{C},\quad \operatorname{Im}z\geqq 0,\\ f_-(z)&=\sum_k(\Omega, A_k(1-P_\Omega)e^{-izH}A'_k\Omega)\quad\text{for}\quad z\in\mathbb{C},\quad \operatorname{Im}z\leqq 0. \end{aligned}\tag{3.9}$$

Here H is the positive generator of the time translations $e^{itH}=U(t\cdot e)$, $t\in\mathbb{R}$, and P_Ω is the projection onto Ω.

The function $f_+(z)$ is continuous on the closed upper half of the complex z-plane, and analytic for $\operatorname{Im}z>0$. Similarly $f_-(z)$ is continuous on the closed lower half of the complex z-plane, and analytic for $\operatorname{Im}z<0$. Since Ω is invariant under all time translations, and since $A_k(t\cdot e)$ commutes with A'_k for $|t|<r$, it follows that $f_+(t)=f_-(t)$ for $|t|<r$. Let $\mathcal{P}_r$ be the cut plane,

$$\mathcal{P}_r=\mathbb{C}\backslash\{z:\operatorname{Im}z=0,\ |\operatorname{Re}z|\geqq r\}\,.\tag{3.10}$$

From what was said above we can conclude that there exists a function $f_r(z)$, defined and analytic on $\mathscr{P}_r$, which coincides with $f_+(z)$ when $z \in \mathscr{P}_r$ and $\operatorname{Im} z \geqq 0$, and similarly coincides with $f_-(z)$ when $z \in \mathscr{P}_r$ and $\operatorname{Im} z \leqq 0$. Notice that

$$f_r(0) = \omega\Big(\sum_k A_k A_k'\Big) - \omega_p\Big(\sum_k A_k A_k'\Big). \tag{3.11}$$

Hence if we can show that there exist constants r and $\delta < 2$ such that $|f_r(0)| \leqq \delta$ for any choice of the operators $A_k \in \mathfrak{A}(\mathcal{O})$ and $A_k' \in \mathfrak{A}(\mathcal{O}_r)'$ with $\Big\|\sum_k A_k A_k'\Big\| \leqq 1$, the desired bound $\|\omega - \omega_p\|_{\mathcal{O};\mathcal{O}_r} < 2$ follows. The pivotal point in the proof of this is expressed in the following lemma, which depends in an essential way on the nuclearity condition (3.2).

Lemma 3.3. *The analytic functions f_r defined above satisfy the condition*

$$|f_r(z)| \leqq h(|\operatorname{Im} z|) \cdot e^{c|\operatorname{Im} z|^{-n}} \quad \textit{for} \quad \operatorname{Im} z \neq 0.$$

Here c and n are positive constants and $h(s)$, $s \geqq 0$ is a continuous function which tends monotonically to 0 as s tends to infinity. The quantities c, n, and h depend only on $\mathcal{O}$. They are in particular independent of r and the specific choice of the operators $A_k \in \mathfrak{A}(\mathcal{O})$ and $A_k' \in \mathfrak{A}(\mathcal{O}_r)'$, provided that $\Big\|\sum_k A_k A_k'\Big\| \leqq 1$.

Proof. We begin with some preliminary remarks about two simple algebraic facts. Firstly, let $\Phi, \Psi \in \mathscr{H}$ and let $\|\cdot\|_p$ denote the operator norm on $\mathscr{B}(\mathscr{H}) \otimes \mathscr{B}(\mathscr{H})$. Then it is obvious that

$$\Big\|\sum_k (\Psi, A_k'\Phi) \cdot A_k\Big\| \leqq \|\Psi\| \cdot \|\Phi\| \cdot \Big\|\sum_k A_k \otimes A_k'\Big\|_p.$$

Making use of the existence of the representation π_p of $\mathfrak{C}(\mathcal{O}; \mathcal{O}_r)$ defined in (3.4), it is also clear that

$$\Big\|\sum_k A_k \otimes A_k'\Big\|_p = \Big\|\pi_p\Big(\sum_k A_k A_k'\Big)\Big\|_p \leqq \Big\|\sum_k A_k A_k'\Big\|.$$

Hence, taking into account that $\Big\|\sum_k A_k A_k'\Big\| \leqq 1$, we arrive at the useful estimate

$$\Big\|\sum_k (\Psi, A_k'\Phi) \cdot A_k\Big\| \leqq \|\Psi\| \cdot \|\Phi\|.$$

Secondly, we note that any operator $A \in \mathfrak{A}(\mathcal{O})$ with $\|A\| \leqq 2$ can be represented as a sum of four unitaries in $\mathfrak{A}(\mathcal{O})$, since $\mathfrak{A}(\mathcal{O})$ is norm closed. Hence if for some linear functional l on $\mathfrak{A}(\mathcal{O})$ and a fixed $\lambda \geqq 0$ the bound $|l(U)| \leqq \lambda$ holds for all unitary $U \in \mathfrak{A}(\mathcal{O})$, it follows that $|l(A)| \leqq 2\lambda \cdot \|A\|$ for $A \in \mathfrak{A}(\mathcal{O})$.

After these preliminaries we can now discuss the consequence of the nuclearity condition for the problem at hand. By the condition (3.2) and by the above remark there exist, for any $\sigma > 0$, a sequence of linear functionals l_m on $e^{-\sigma H}\mathfrak{A}(\mathcal{O})\Omega$ and a sequence of unit vectors Φ_m in $\mathscr{H}$ such that for any $A \in \mathfrak{A}(\mathcal{O})$, we have

$$e^{-\sigma H} A\Omega = \sum_{m=1}^{\infty} l_m(e^{-\sigma H} A\Omega) \cdot \Phi_m,$$

where the sum is absolutely convergent. In fact, setting

$$\lambda_m(\sigma) = \sup\{|l_m(e^{-\sigma H} A\Omega)| : A \in \mathfrak{A}(\mathcal{O}),\ \|A\| \leqq 1\},$$

we have for sufficiently small $\sigma > 0$ the estimate

$$\sum_{m=1}^{\infty} \lambda_m(\sigma) \leqq 4 e^{c\sigma^{-n}},$$

where c and n are the constants appearing in the nuclearity condition (3.2).

Now let $z = t + is$, where $t \in \mathbb{R}$ and $s > 0$. Then it follows from the definition of f_r that $f_r(t+is) = f_+(t+is)$, and with the arbitrary decomposition $s = \sigma + \tau$, where $\sigma > 0$, $\tau \geqq 0$ we obtain from (3.9) the equality

$$f_r(t+is) = \sum_k (\Omega, A_k' e^{(it-\tau)H}(1-P_\Omega) e^{-\sigma H} A_k \Omega).$$

Making use of the above mentioned properties of the functionals l_m and the vectors Φ_m, we can convert this expression into the equalities

$$\begin{aligned} f_r(t+is) &= \sum_k \sum_{m=1}^{\infty} (\Omega, A_k' e^{(it-\tau)H}(1-P_\Omega)\Phi_m) \cdot l_m(e^{-\sigma H} A_k \Omega) \\ &= \sum_{m=1}^{\infty} l_m(e^{-\sigma H} A^{(m)} \Omega), \end{aligned}$$

where the operators

$$A^{(m)} = \sum_k (\Omega, A_k' e^{(it-\tau)H}(1-P_\Omega)\Phi_m) \cdot A_k$$

are elements of the algebra $\mathfrak{A}(\mathcal{O})$. From the above preliminary remarks it follows that

$$\|A^{(m)}\| \leqq \|e^{-\tau H}(1-P_\Omega)\Phi_m\|,$$

and hence we arrive at the estimate

$$|f_r(t+is)| \leqq \sum_{m=1}^{\infty} \lambda_m(\sigma) \cdot \|e^{-\tau H}(1-P_\Omega)\Phi_m\|.$$

Setting first $\sigma = s$, $\tau = 0$ and taking into account that $\|\Phi_m\| = 1$ as well as the bound on $\sum_{m=1}^{\infty} \lambda_m(\sigma)$, we obtain

$$|f_r(t+is)| \leqq 4 \cdot e^{cs^{-n}}$$

for $0 < s \leqq s_0$, provided s_0 is sufficiently small. If $s \geqq s_0$, we set $\sigma = s_0$, $\tau = s - s_0$, and we then have

$$|f_r(t+is)| \leqq \sum_{m=1}^{\infty} \lambda_m(s_0) \cdot \|e^{-(s-s_0)H}(1-P_\Omega)\Phi_m\|.$$

The right-hand side of this inequality is continuous for $s \geqq s_0$ and tends monotonically to 0 as s tends to infinity, because each individual term $\|e^{-(s-s_0)H}(1-P_\Omega)\Phi_m\|$ has these properties (recall that Ω is the only invariant state under translations) and the sum over all coefficients $\lambda_m(s_0)$ is finite. By a combination of these bounds on $|f_r(t+is)|$ the assertion of the lemma follows for

$s>0$. To see that this assertion also holds for $s<0$ we note, with reference to (3.9), that for $s<0$

$$f_r(t+is)=f_-(t+is)=\overline{g_+(t-is)},$$

where g_+ is obtained from f_+ by replacing the operators A_k, A'_k in (3.9) by their adjoints A^*_k and A'^*_k, respectively. Since

$$\left\|\sum_k A^*_k A'^*_k\right\| = \left\|\left(\sum_k A'_k A_k\right)^*\right\| = \left\|\sum_k A_k A'_k\right\| \leqq 1,$$

the conclusion then follows from what has already been proved. □

Let c, n be the positive constants and h be the function appearing in the formulation of the previous lemma. We consider for any $r>0$ the family $\mathscr{F}_r$ of *all* functions f which are analytic on the cut plane $\mathscr{P}_r$ and satisfy the inequality

$$|f(z)| \leqq h(|\mathrm{Im}\, z|) \cdot e^{c|\mathrm{Im}\, z|^{-n}} \tag{3.12}$$

for all $z \in \mathbb{C}$, $\mathrm{Im}\, z \neq 0$. The functions f_r considered earlier are clearly elements of $\mathscr{F}_r$. The desired bound on $|f_r(0)|$ now follows from the next lemma, which is based on a Phragmén-Lindelöf type of argument.

Lemma 3.4. *For any $\delta>0$ there exists an $(\mathscr{F}_{r(\delta)})>0$ such that $|f(0)|<\delta$ for all $f\in\mathscr{F}_{r(\delta)}$.*

Proof. Let $r>0$ be fixed in the following. We will show that the family $\mathscr{F}_r$ of analytic functions on $\mathscr{P}_r$ is normal (i.e. uniformly bounded on each compact subset of $\mathscr{P}_r$). Anticipating this result, the proof of the lemma can be completed as follows: if the statement were wrong, there would exist a sequence of functions $g_n \in \mathscr{F}_{n\cdot r}$, $n\in\mathbb{N}$ such that $|g_n(0)|\geqq\delta$. We then consider the scaled functions $\hat{g}_n(z) := g_n(n\cdot z)$, $z\in\mathscr{P}_r$ which are all elements of $\mathscr{F}_r$, since the bound on the right-hand side of (3.12) decreases monotonically if $|\mathrm{Im}\, z|$ increases. Moreover, since this bound converges to 0 as $|\mathrm{Im}\, z|$ approaches infinity, it follows that $\lim_n \hat{g}_n(z)=0$ if $|\mathrm{Im}\, z|\neq 0$. Making use now of the fact that $\mathscr{F}_r$ is a normal family of analytic functions, it is then an immediate consequence that $\lim_n \hat{g}_n(z)=0$ for all $z\in\mathscr{P}_r$. But since $0\in\mathscr{P}_r$ and $\hat{g}_n(0)=g_n(0)$, this result is incompatible with the assumption that $|g_n(0)|\geqq\delta$.

For the proof that $\mathscr{F}_r$ is normal it suffices to show that the functions in $\mathscr{F}_r$ are uniformly bounded on some neighbourhood of each real point $t\in\mathscr{P}_r$. To this end we introduce for any given ε with $0<\varepsilon<r$ the auxiliary function

$$a_1(z)=a_0(r-\varepsilon+z)\cdot a_0(r-\varepsilon-z), \qquad z\neq\pm(r-\varepsilon),$$

where

$$a_0(z)=\exp(-4e^{1/z}), \qquad z\neq 0.$$

It is obvious that the restriction of a_1 to the open set

$$\mathscr{D}=\{z\colon |\mathrm{Re}\, z|+|\mathrm{Im}\, z|^{1/2}<r-\varepsilon\}$$

is analytic; moreover, this restriction extends in a continuous fashion to the closure $\bar{\mathscr{D}}$ of $\mathscr{D}$. We denote this extension of $a_1 \restriction \mathscr{D}$ by a and note that $a(r-\varepsilon)=a(-r+\varepsilon)=0$.

Now let $f \in \mathscr{F}_r$ and let $g(z) = a(z) \cdot f(z)$ for $z \in \bar{\mathscr{D}}$. Since $\bar{\mathscr{D}} \subset \mathscr{P}_r$ it is clear that g is analytic on $\mathscr{D}$, continuous on $\bar{\mathscr{D}}$, and $g(r-\varepsilon) = g(-r+\varepsilon) = 0$. The maximum modulus principle is then applicable, and hence $|g(z)| \leqq M(g)$, where $M(g)$ is the maximum of $|g(z)|$ on the boundary $\partial\bar{\mathscr{D}}$ of $\bar{\mathscr{D}}$; this maximum is assumed at some point different from $z = \pm(r-\varepsilon)$. On the basis of the bound (3.12) on the functions $f \in \mathscr{F}_r$ and the specific properties of the auxiliary function a we obtain by a straightforward calculation the estimate

$$M(g) \leqq \sup_{t>0} \{\exp(-e^{1/t}) \cdot h(t^2) e^{ct^{-2n}}\} = M\,,$$

where M is a constant independent of the particular choice of $f \in \mathscr{F}_r$. Since a does not vanish on the open set $\mathscr{D}$, we thus arrive at the bound

$$|f(z)| \leqq M \cdot |a(z)|^{-1}\,, \qquad z \in \mathscr{D}$$

which holds for all $f \in \mathscr{F}_r$. This result implies in particular that the functions in $\mathscr{F}_r$ are uniformly bounded in some neighbourhood of any real point t with $|t| < r - \varepsilon$. But $\varepsilon > 0$ was arbitrary, and taking also into account the a priori bound (3.12) it is then clear that the functions in $\mathscr{F}_r$ are locally bounded on $\mathscr{P}_r$. Hence $\mathscr{F}_r$ is a normal family of analytic functions. □

Summarizing the results of the previous discussion we have seen [cf. Eq. (3.11), Lemma 3.3, and the definition of the families $\mathscr{F}_r$] that

$$\|\omega - \omega_p\|_{\mathcal{O};\mathcal{O}_r} \leqq \sup\{|f(0)| : f \in \mathscr{F}_r\}\,. \tag{3.13}$$

From Lemma 3.4 it then follows that $\|\omega - \omega_p\|_{\mathcal{O};\mathcal{O}_r} < 2$, provided r (and consequently $\mathcal{O}_r$) is sufficiently large. We have thus established

Proposition 3.5. *Let $\mathcal{O}_a$ be any open, bounded region. Then there exists another open bounded region $\mathcal{O}_b \supset \bar{\mathcal{O}}_a$ such that $\|\omega - \omega_p\|_{\mathcal{O}_a;\mathcal{O}_b} < 2$.*

The proof of Theorem 3.1 is now accomplished by combining Propositions 3.5 and 3.2: given $\mathcal{O}_1$, we can choose two open, bounded regions $\mathcal{O}_a, \mathcal{O}_b$ such that $\mathcal{O}_a \supset \bar{\mathcal{O}}_1$, $\mathcal{O}_b \supset \bar{\mathcal{O}}_a$, and $\|\omega - \omega_p\|_{\mathcal{O}_a;\mathcal{O}_b} < 2$. It then follows that the representations π and π_p of $\mathfrak{C}(\mathcal{O}_1; \mathcal{O}_2)$, where $\mathcal{O}_2$ is any region containing $\bar{\mathcal{O}}_b$ in its interiour, are unitarily equivalent. As already discussed, this implies the theorem.

It is obvious from our arguments that these results can be generalized in various ways. For example, we never depended on $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ being a net on $\mathbb{R}^4$, and accordingly the theorem holds in any number of space-time dimensions.

It is also clear that the bound given in the nuclearity condition (3.2) could be relaxed. The actual borderline, where the theorem breaks down is not known to us. One can show, however, that the conclusion in Lemma 3.4 is false if the bound (3.12) entering into the definition of the families $\mathscr{F}_r$ is replaced by $\exp(e^{1/|\mathrm{Im}\, z|})$ in a neighbourhood of the real axis [30]. Our particular approach is thus not applicable in this case.

Finally, we would like to point out that, by a refinement of the argument in Lemma 3.4, one can derive an estimate on the size of the region $\mathcal{O}_2$ in the theorem in terms of the parameters c, n, and h in (3.12). We have refrained from doing so since the size of $\mathcal{O}_2$ obtained in this way is far from optimal. In fact, on the basis of a slightly more restrictive version of our nuclearity condition, it has recently been shown [31] that the statement of the theorem holds true for any region $\mathcal{O}_2$ containing $\bar{\mathcal{O}}_1$ in its interiour.

4. Concluding Remarks

In the present investigation we have focussed attention on certain configuration space aspects of our nuclearity condition. We have seen that the asymptotic energy level density of the well localized states, which can be read off the nuclearity index $\nu(e^{-\beta H}\mathcal{L}_r)$ for small β and fixed r, governs the nature of the correlations between observables in spacelike separated regions of Minkowski space. In models where the nuclear index does not grow too rapidly as β approaches 0, one can represent all physical states as superpositions of "product states" which do not exhibit any correlations between certain pairs of spacelike separated regions.

By a straightforward generalization of the present methods one can extend this result to any number of spacelike separated regions $\mathcal{O}_1, \ldots, \mathcal{O}_n$ provided these regions are bounded (with the possible exception of one member), and their mutual distances are sufficiently large. Namely, given any such collection of regions, which may be regarded as lacunary paving of some spacelike surface, there exists an orthonormal basis of product states $\Phi \in \mathcal{H}$ for which

$$(\Phi, A_1 \ldots A_n\Phi) = \prod_{k=1}^{n} (\Phi, A_k\Phi) \tag{4.1}$$

for any choice of the operators $A_k \in \mathfrak{A}(\mathcal{O}_k)$, $k=1, \ldots, n$. In all models satisfying our nuclearity condition the physical states can thus be interpreted in terms of completely uncorrelated subsystems which are localized in the panels of such lacunary pavings of spacelike planes.

In collision theory one is interested in the properties of the subsystems appearing in these resolutions of physical states at asymptotic times. In this connection it is of particular interest under which circumstances these subsystems can be interpreted as particles. We expect that the volume dependence of the energy level density of the well-localized states, which likewise can be read off the nuclearity index, is of decisive importance in this context.

Because of the additivity of the energy of multiply localized states (i.e. states consisting of several localization centers) [33], it is clear that the nuclearity index $\nu(e^{-\beta H}\mathcal{L}_r)$ increases in all models at least as rapidly as e^{cr^3} in the limit of large r and fixed β. If, on the other hand, the nuclearity index would grow considerably faster, this would mean that there exist excitations of fixed energy E in the model, occupying arbitrarily large regions of space, which are not composed of several localization centers whose total energy adds up to E. It is evident that such states cannot have a particle interpretation. The r-dependence of the nuclearity index stated in our nuclearity condition therefore seems to be a necessary condition for a theory to have a particle interpretation. In fact, there are indications [18] that this condition is also sufficient, if one defines a single particle state as a state singly localized at all times [5]. This more general concept of a particle can also be extended to infra-particles [18], (i.e. particle like excitations which do not correspond to discrete points in the mass spectrum [34]).

5. Appendix

In this Appendix we show that our nuclearity criterion is satisfied in the theory of a free hermitian scalar field associated with a single spinless particle of mass $m>0$. This is the simplest example of a large class of free field theories which can be treated by similar methods.

We begin with a brief review, partly to establish our notation. The Hilbert space $\mathscr{H}$ is the Fock space over the Hilbert space of all single particle states, which we identify with the space $L^2(\mathbb{R}^3)$ of momentum space wave functions $f(\mathbf{p})$ equipped with the scalar product

$$\langle f|g\rangle = \int d^3p\,\overline{f(\mathbf{p})}g(\mathbf{p}).$$

The generator ω of the time translations on the single particle subspace is given (on its natural domain) by

$$(\omega f)(\mathbf{p}) = (|\mathbf{p}|^2+m^2)^{1/2}\cdot f(\mathbf{p}).$$

The global space-time translations $U(x)$ are then fixed by the Fock-space structure of $\mathscr{H}$.

To each $f\in L^2(\mathbb{R}^3)$ corresponds a creation operator $a^*(f)$ and a destruction operator $a(f)$ as well as a unitary Weyl operator

$$W(f) = e^{i(a^*(f)+a(f))^-}.$$

The creation and destruction operators satisfy the usual canonical commutation relations, which are equivalent to the well-known law of composition

$$W(f)\cdot W(g) = W(f+g)\cdot e^{\frac{1}{2}(\langle g|f\rangle - \langle f|g\rangle)}$$

for the Weyl-operators.

To each double cone $\mathcal{O}_r=\{x\in\mathbb{R}^4\colon |x_0|+|\mathbf{x}|<r\}$ there corresponds a *real* subspace $K(\mathcal{O}_r)$ of $L^2(\mathbb{R}^3)$ consisting of all wave functions of the form

$$(|\mathbf{p}|^2+m^2)^{-1/2}f(\mathbf{p}) + i(|\mathbf{p}|^2+m^2)^{1/2}g(\mathbf{p}),$$

where f, g are test functions whose Fourier transforms are real and have support in the ball $|\mathbf{x}|<r$. The local algebras $\mathfrak{A}(\mathcal{O}_r)$ are then defined as the von Neumann algebras generated by all Weyl-operators $W(h)$ with $h\in K(\mathcal{O}_r)$, i.e.

$$\mathfrak{A}(\mathcal{O}_r) = \{W(h)\colon h\in K(\mathcal{O}_r)\}''.$$

With the aid of these special algebras and the translations we can define the algebras corresponding to arbitrary bounded regions $\mathcal{O}$ by additivity. The resulting net $\mathcal{O}\to\mathfrak{A}(\mathcal{O})$ has all the standard properties mentioned in the Introduction.

Our goal is the following theorem which shows that our nuclearity condition is satisfied for the free-field theory under consideration.

Theorem. *In the free field theory of a single spinless particle of mass $m>0$ the sets*

$$e^{-\beta H}\mathscr{L}_r = \{e^{-\beta H}U\Omega\colon U\in\mathfrak{A}(\mathcal{O}_r),\ U^*U=1\}$$

are nuclear for any r and any $\beta>0$. Moreover, if $r\geqq m^{-1}$ and $\beta\leqq r$, then

$$\nu(e^{-\beta H}\mathscr{L}_r)\leqq e^{C(r/\beta)^3\cdot|\ln(1-e^{-\beta m/2})|},$$

where C is some constant (independent of r, β, and m).

Remark. Our estimate for the nuclearity index is rather crude and can be improved at the price of added complications.

For the proof that the set of vectors $e^{-\beta H}\mathcal{L}_r$ is nuclear we must exhibit a sequence of unit vectors Φ_n and a sequence of linear functionals l_n with the properties given in Sect. 2. Our construction of these quantities depends on a proper choice of an orthonormal basis e_k, $k \in \mathbb{N}$ in $L^2(\mathbb{R}^3)$. Given such a basis we define for all *finite* sequences $(n_1, \ldots, n_p)$ of non-negative integers ("occupation numbers") the vectors

$$\Phi_n = \prod_{k=1}^{p} \frac{a^*(e_k)^{n_k}}{(n_k!)^{1/2}} \Omega,$$

where n is the label of the sequences $(n_1, \ldots, n_p)$ in an appropriate enumeration. These vectors form an orthonormal basis in $\mathcal{H}$. The associated linear functionals l_n are defined on all of $\mathcal{H}$ by

$$l_n(\Psi) = (\Phi_n, \Psi) \quad \text{for} \quad \Psi \in \mathcal{H}.$$

It is then obvious that

$$\Psi = \sum_{n=1}^{\infty} l_n(\Psi) \cdot \Phi_n,$$

the sum being defined in the sense of strong convergence. It will be shown in the remainder of this section that for any given r and $\beta > 0$ one can find an orthonormal basis e_k, $k \in \mathbb{N}$ in $L^2(\mathbb{R}^3)$ such that the corresponding sequence

$$\lambda_n = \sup\{|l_n(e^{-\beta H} U \Omega)| : U \in \mathfrak{A}(\mathcal{O}_r),\ U^*U = 1\}$$

is summable. This means that $e^{-\beta H}\mathcal{L}_r$ is a nuclear set. Moreover, the sum over all λ_n will provide the upper bound on the nuclearity index of $e^{-\beta H}\mathcal{L}_r$ stated in the theorem.

The proof of this assertion involves mildly cumbersome combinatorial problems which we will handle by generating function methods of the kind which are standard in Fock space theory. To begin with we define for any $\Psi \in \mathcal{H}$ the functional

$$G(f;\ \Psi) = (e^{a^*(f)}\Omega, \Psi), \qquad f \in L^2(\mathbb{R}^3),$$

where the exponential function is understood in terms of its power series expansion. In view of the relation

$$e^{a^*(f)}\Omega = e^{\frac{1}{2}\|f\|^2} \cdot W(-if)\Omega,$$

it is clear that

$$|G(f;\ \Psi)| \leqq e^{\frac{1}{2}\|f\|^2} \cdot \|\Psi\|,$$

where we have put $\|f\|^2 = \langle f | f \rangle$, and this bound is optimal if Ψ is arbitrary. The following observation is now important: if, for some set of vectors $\mathcal{N} \subset \mathcal{H}$, one can establish bounds on the functionals $G(f;\ \Psi)$, $\Psi \in \mathcal{N}$ as stated in the next lemma, then it follows that $\mathcal{N}$ is a nuclear set.

Lemma 1. *Let $\mathcal{N}$ be a subset of vectors in the unit ball of $\mathcal{H}$. If there exists a (selfadjoint) trace class operator $T \geqq 0$ on $L^2(\mathbb{R}^3)$ with norm $\|T\| < 1$ such that*

$$|G(f;\Psi)| \leqq e^{\frac{1}{2}\|Tf\|^2}, \quad f \in L^2(\mathbb{R}^3)$$

for all $\Psi \in \mathcal{N}$, then $\mathcal{N}$ is a nuclear set. Moreover,

$$\nu(\mathcal{N}) \leqq \det(1-T)^{-2}.$$

Proof. Since T is a positive trace class operator in $L^2(\mathbb{R}^3)$ there exists an orthonormal basis of eigenvectors e_k, $k \in \mathbb{N}$ corresponding to the eigenvalues t_k of T. From the fact that $\|T\| < 1$ and $T \geqq 0$ it is also clear that $0 \leqq t_k < 1$. Now let Φ_n and l_n be the unit vectors and functionals, respectively, constructed from e_k, $k \in \mathbb{N}$ as described above. We desire a good estimate on the quantities $l_n(\Psi)$, $\Psi \in \mathcal{N}$.

Let $(n_1, \ldots, n_p)$ be the sequence of integers corresponding to the label n. Denoting by $\mathbf{w} = (w_1, \ldots, w_p)$ and by $\mathbf{z} = (z_1, \ldots, z_p)$ arbitrary p-tuplets of positive and complex numbers, respectively, we introduce the auxiliary $L^2(\mathbb{R}^3)$-valued function

$$f(\mathbf{z}, \mathbf{w}) = \sum_{k=1}^{p} z_k w_k \cdot e_k .$$

Using the notation $z_k = x_k + i y_k$ with $x_k, y_k \in \mathbb{R}$, $d^p z = dx_1 \ldots dx_p dy_1 \ldots dy_p$, and $|\mathbf{z}|^2 = |z_1|^2 + \ldots + |z_p|^2$, we obtain, through a straightforward calculation, the equality

$$(n_1! \ldots n_p!)^{1/2} w_1^{n_1} \ldots w_p^{n_p} l_n(\Psi) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi^p} \int d^p z z_1^{n_1} \ldots z_p^{n_p} G(f(\mathbf{z}, \mathbf{w}); \Psi) e^{-|\mathbf{z}|^2 - \varepsilon |\mathbf{z}|^4}$$

for any $\Psi \in \mathcal{H}$. [Note that due to the presence of the term $e^{-\varepsilon|\mathbf{z}|^4}$, one may interchange the integration with the summation, involved in the definition of $G(f(\mathbf{z}, \mathbf{w}); \Psi)$, for any choice of $w_1, \ldots, w_p$.] Applying the Cauchy-Schwarz inequality to the integral we thus arrive at

$$w_1^{2n_1} \ldots w_p^{2n_p} |l_n(\Psi)|^2 \leqq \frac{1}{\pi^p} \int d^p z |G(f(\mathbf{z}, \mathbf{w}); \Psi)|^2 e^{-|\mathbf{z}|^2},$$

provided that the right-hand side of this inequality exists. Now for vectors $\Psi \in \mathcal{N}$ it follows from the premises in the lemma that

$$|G(f(\mathbf{z}, \mathbf{w}); \Psi)|^2 \leqq e^{\sum_{k=1}^{p} |z_k w_k t_k|^2} .$$

This leads, after integration, to the estimate

$$|l_n(\Psi)| \leqq \prod_{k=1}^{p} w_k^{-n_k} \cdot (1 - (w_k t_k)^2)^{-1/2},$$

which is valid for all w_k satisfying $w_k > 0$, $w_k t_k < 1$. Calculating the minimum of the right-hand side of this inequality with respect to $w_1, \ldots, w_p$ we finally arrive at the bound

$$|l_n(\Psi)| \leqq \prod_{k=1}^{p} (n_k + 1) \cdot t_k^{n_k}$$

which holds for all $\Psi \in \mathcal{N}$. We note that this bound is not optimal, but we selected this form in the interest of simplicity. Setting

$$\lambda_n = \sup\{|l_n(\Psi)|: \Psi \in \mathcal{N}\},$$

and taking into account that the sequence t_k is summable as well as the fact that $0 \leqq t_k \leqq \|T\| < 1$, it follows that

$$\sum_{n=1}^{\infty} \lambda_n \leqq \prod_{k=1}^{\infty} (1-t_k)^{-2} = \det(1-T)^{-2} < \infty.$$

This estimate shows that $\mathcal{N}$ is a nuclear set and it provides the bound on the nuclearity index of $\mathcal{N}$ given in the lemma. □

We will now exploit the specific properties of the sets $e^{-\beta H}\mathcal{L}_r$ and show that they are of the type considered in the previous lemma. We first note the trivial identity

$$G(f;\, e^{-\beta H} U\Omega) = G(e^{-\beta\omega} f;\, U\Omega)$$

which holds for any bounded operator U. The analysis of the consequences of the assumption that $U \in \mathfrak{A}(\mathcal{O}_r)$ is somewhat more complicated. Let $K(\mathcal{O}_r)'$ be the closed *real* subspace of $L^2(\mathbb{R}^3)$ which is conjugate to $K(\mathcal{O}_r)$ in the sense that

$$K(\mathcal{O}_r)' = \{g \in L^2(\mathbb{R}^3):\, \langle g|f\rangle = \langle f|g\rangle \text{ for all } f \in K(\mathcal{O}_r)\}.$$

In view of the composition law for the Weyl operators we then have $W(g) \in \mathfrak{A}(\mathcal{O}_r)'$ for any $g \in K(\mathcal{O}_r)'$.

On the basis of this composition law it is also easy to establish the equality of the various representations of the function $F(z)$, $z \in \mathbb{C}$ given by

$$\begin{aligned} &F(z) \cdot e^{z^2\|g\|^2 + iz\langle h|g\rangle} \\ &\quad = (e^{a^*(h + i\bar{z}g)}\Omega,\, U \cdot e^{a^*(izg)}\Omega) \\ &\quad = e^{\frac{1}{2}\|h\|^2 + |z|^2\|g\|^2 + i\bar{z}\langle h|g\rangle} \cdot (W(\bar{z}g)W(-ih)\Omega,\, U \cdot W(zg)\Omega) \\ &\quad = e^{|z|^2\|g\|^2 + i\bar{z}\langle h|g\rangle}(W(\bar{z}g)e^{a^*(h)}\Omega,\, U \cdot W(zg)\Omega). \end{aligned}$$

These equations hold for any $g, h \in L^2(\mathbb{R}^3)$ and any bounded operator U. By inspection of the first (defining) equality we see that $F(z)$ is an entire analytic function. If $z = x$ is real, $g \in K(\mathcal{O}_r)'$, and $U \in \mathfrak{A}(\mathcal{O}_r)$, then $W(xg)$ commutes with U, and thus it follows from the second equality that $F(x)$ is constant. But this implies that $F(z)$, $z \in \mathbb{C}$ is constant, and setting $z = 0$ in the third member we find that $F(z) = G(h;\, U\Omega)$. Setting now $z = -i/2$, $h = e^{-\beta\omega} f$, and using again the second equality we obtain the estimate

$$|G(f;\, e^{-\beta H} U\Omega)| \leqq e^{\frac{1}{2}\|e^{-\beta\omega} f - g\|^2}$$

which holds for all $f \in L^2(\mathbb{R}^3)$, all $g \in K(\mathcal{O}_r)'$, and all operators $U \in \mathfrak{A}(\mathcal{O}_r)$ of norm less than 1 (i.e. in particular for all isometries).

In the next step we select for every $f \in L^2(\mathbb{R}^3)$ a $g \in K(\mathcal{O}_r)'$ such that $\|e^{-\beta\omega} f - g\|$ is "as small as possible." To this end we define an antiunitary involution J on $L^2(\mathbb{R}^3)$ by

$$(Jh)(\mathbf{p}) = \overline{h(-\mathbf{p})},$$

i.e. J induces complex conjugation of the wave functions in configuration space. We also consider the closed complex-linear subspace $K_r^{(+)}$ of $L^2(\mathbb{R}^3)$ which is the closure of the set of functions $\omega^{1/2}f$, where f runs through all test functions which, in configuration space, have support in the ball $|\mathbf{x}|<r$. Similarly, $K_r^{(-)}$ is defined as the closure of the set of functions $\omega^{-1/2}f$ with f as above.

Now let $E_r^{(+)}$ and $E_r^{(-)}$ be the orthogonal projections onto $K_r^{(+)}$ and $K_r^{(-)}$, respectively. Since $K_r^{(+)}$ and $K_r^{(-)}$ are invariant under J it follows that $E_r^{(+)}$ and $E_r^{(-)}$ commute with J. It is also easy to see that

$$g=\tfrac{1}{2}(1-E_r^{(+)})(1+J)h+\tfrac{1}{2}(1-E_r^{(-)})(1-J)h$$

is an element of $K(\mathcal{O}_r)'$ for any choice of the function $h\in L^2(\mathbb{R}^3)$. Hence if we put $h=e^{-\beta\omega}f$ and choose g as above, we obtain from the earlier estimate on $G(f;\, e^{-\beta H}U\Omega)$ the bound

$$|G(f;\, e^{-\beta H}U\Omega)|\leqq e^{\frac{1}{2}\|T_+\frac{1}{2}(1+J)f\|^2+\frac{1}{2}\|T_-\frac{1}{2}(1-J)f\|^2},$$

where T_+, T_- denote the bounded operators

$$T_+=E_r^{(+)}e^{-\beta\omega}, \qquad T_-=E_r^{(-)}e^{-\beta\omega}.$$

For further progress we need estimates on the norms and trace-norms of these operators, and we thus consider

Lemma 2. *If* $r\geqq m^{-1}$ *and* $0<\beta\leqq r$, *then*

$$\|T_\pm\|\leqq e^{-\beta m} \quad \textit{and} \quad \|T_\pm\|_1\leqq c\cdot(r/\beta)^3e^{-\beta m/2},$$

where c is some constant which does not depend on r, β, or m. Here $\|\cdot\|_1$ *denotes the trace-norm for the operators on* $L^2(\mathbb{R}^3)$.

Proof. (In the following m is set equal to 1, i.e. the quantities r, β are given in units of m^{-1}.) The bounds on the operator norms of T_+ and T_- follow from the trivial inequality $\|T_\pm\|\leqq\|e^{-\beta\omega}\|$ and the fact that $\omega\geqq 1$. For the proof that T_+ and T_- are trace class operators we proceed as follows.

Let $\chi(\mathbf{x})$ be any real test function on configuration space with the property that $\chi(\mathbf{x})=1$ for $|\mathbf{x}|\leqq 1$ and $\chi(\mathbf{x})=0$ for $|\mathbf{x}|\geqq 2$. We then consider the scaled function $\chi_r(\mathbf{x}):=\chi(r^{-1}\mathbf{x})$ and define the bounded, selfadjoint operator χ_r on the momentum space wave functions $h\in L^2(\mathbb{R}^3)$, setting

$$\widetilde{(\chi_r\cdot h)}(\mathbf{x})=\chi_r(\mathbf{x})\cdot\tilde{h}(\mathbf{x}).$$

Here the tilde denotes the Fourier transforms of the respective functions.

It follows from the definition of the space $K_r^{(+)}$ introduced above that $E_r^{(+)}=E_r^{(+)}\cdot\omega^{-1/2}\chi_r\omega^{1/2}$, so we have the trivial identity

$$T_+=E_r^{(+)}\cdot\{\omega^{-1/2}\chi_r\omega^{1/2}(1+\beta^2|\mathbf{P}|^2)^{-1}\}$$
$$\times\{(1+\beta^2|\mathbf{P}|^2)\omega^{-1/2}\chi_r\omega^{1/2}e^{-\beta\omega}\},$$

where $\mathbf{P}$ denotes the momentum operator on $L^2(\mathbb{R}^3)$. It is also easy to verify that the two operators in the curly brackets are in the Hilbert-Schmidt class, and hence T_+ is in the trace class. Moreover, on the basis of the simple inequality

$$(|\mathbf{p}|^2+1)^{1/2}(|\mathbf{q}|^2+1)^{-1/2}\leqq 1+|\mathbf{p}-\mathbf{q}|$$

for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, we obtain after a straightforward calculation the following bound on the Hilbert-Schmidt norm of the first operator:

$$\|\omega^{-1/2}\chi_r\omega^{1/2}(1+\beta^2|\mathbf{P}|^2)^{-1}\|_2 \leqq c_1 \cdot (r/\beta)^{3/2}$$

for $r \geqq 1$ and $0 < \beta \leqq r$. Here c_1 is a constant which only depends on the specific properties of the function $\chi(\mathbf{x})$ selected above. Making also use of the estimate

$$e^{-2\beta(1+|\mathbf{p}|^2)^{1/2}} \leqq e^{-\beta} \cdot e^{-\beta|\mathbf{p}|}$$

for $\mathbf{p} \in \mathbb{R}^3$, we likewise obtain for the second operator

$$\|(1+\beta^2|\mathbf{P}|^2)\omega^{-1/2}\chi_r\omega^{1/2}e^{-\beta\omega}\|_2 \leqq c_2 \cdot (r/\beta)^{3/2}e^{-\beta/2}$$

for $r \geqq 1$, $0 < \beta \leqq r$, and another constant c_2. Combining the above identity for T_+ with these estimates of the Hilbert-Schmidt norms, we now have

$$\|T_+\|_1 \leqq c_1 c_2 \cdot (r/\beta)^3 e^{-\beta/2}$$

for $r \geqq 1$ and $0 < \beta \leqq r$. This is the desired bound on the trace-norm of T_+. The same considerations apply to T_-, so the proof of the lemma is complete. □

With this information at hand we can now continue our analysis of the functionals $G(f;\, e^{-\beta H}U\Omega)$. Let $r \geqq m^{-1}$, $0 < \beta \leqq r$, and let T_+, T_- be the operators considered in the preceding lemma. We then define the positive, bounded operator T by

$$T = ((T_+^* T_+)^n + (T_-^* T_-)^n)^{1/2n},$$

where n is some positive integer. It is obvious that

$$\|T\| \leqq 2^{1/2n} \cdot \mathrm{Max}(\|T_+\|, \|T_-\|) \leqq 2^{1/2n} e^{-m\beta},$$

and hence if we choose n sufficiently large we have $\|T\| \leqq e^{-m\beta/2}$. We next note that for positive operators A, B, and $0 < \alpha \leqq 1$

$$\|(A+B)^\alpha\|_1 \leqq \|A^\alpha\|_1 + \|B^\alpha\|_1,$$

provided A^α and B^α are in the trace class (cf. for example [35]). We thus conclude that T is in the trace class, and we have

$$\|T\|_1 \leqq \|T_+\|_1 + \|T_-\|_1 \leqq 2c \cdot (r/\beta)^3 e^{-m\beta/2}.$$

From the fact that $x \to x^{1/n}$, $x \geqq 0$ is an operator-monotone function for $n \geqq 1$, it finally follows that $T_+^* T_+ \leqq T^2$ and $T_-^* T_- \leqq T^2$, and consequently

$$\|T_+ \tfrac{1}{2}(1+J)f\|^2 + \|T_- \tfrac{1}{2}(1-J)f\|^2 \leqq \|T\tfrac{1}{2}(1+J)f\|^2 + \|T\tfrac{1}{2}(1-J)f\|^2 = \|Tf\|^2,$$

where in the last equality we made use of the fact that T commutes with J. On the basis of the previous estimate of $G(f;\, e^{-\beta H}U\Omega)$, we thus arrive at the inequality

$$|G(f;\, e^{-\beta H}U\Omega)| \leqq e^{\frac{1}{2}\|Tf\|^2}.$$

We are now in a position to complete the proof of the theorem. Since T is a trace class operator and $\|T\| < 1$, it follows from the above bound on $G(f;\, e^{-\beta H}U\Omega)$ and Lemma 1 that the sets $e^{-\beta H}\mathscr{L}_r$ are nuclear for small $\beta > 0$ and large r. But it is then clear that these sets are nuclear for all r and $\beta>0$ because $\mathscr{L}_r \subset \mathscr{L}_{r'}$ if $r \leqq r'$, and $e^{-(\beta-\beta')H}$ is a bounded, invertible operator if $\beta \geqq \beta'$.

According to Lemma 1 we also have

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq \det(1-T)^{-2} = e^{-2\mathrm{Tr}\ln(1-T)},$$

and making use of the fact that the function $x \to -x^{-1} \cdot \ln(1-x)$, $0 \leqq x < 1$ is monotonically increasing, we can obtain the more convenient estimate

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq e^{-2\|T\|^{-1}\ln(1-\|T\|)\cdot\|T\|_1}.$$

Taking into account the explicit bounds on $\|T\|$ and $\|T\|_1$ given above we conclude that, for $r \geqq m^{-1}$ and $0<\beta \leqq r$,

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq e^{c(r/\beta)^3|\ln(1-e^{-\beta m/2})|},$$

where c is some number. This completes our proof of the theorem.

In conclusion we would like to point out that our methods also apply to "many-particle theories" with a countable number of free spinless particles with arbitrary masses $0<m_1 \leqq m_2 \leqq \ldots \leqq m_i \leqq \ldots$. They are obtained from the present theory by a standard tensor-product construction (cf. for example [9]). This class of models is of some theoretical interest because it allows a simple study of the effects of a more complicated particle spectrum on the properties of the sets $e^{-\beta H}\mathscr{L}_r$.

By a slight generalization of the previous arguments one can show that the sets $e^{-\beta H}\mathscr{L}_r$ are nuclear for any r and $\beta>0$, whenever the particle spectrum of the theory is such that

$$\sum_{i=1}^{\infty} e^{-\beta m_i} < \infty \quad \text{for all} \quad \beta > 0.$$

Moreover, for the nuclearity index of these sets one has

$$\nu(e^{-\beta H}\mathscr{L}_r) \leqq e^{c(r/\beta)^3 \sum_{i=1}^{\infty} |\ln(1-e^{-\beta m_i/2})|}$$

for all $r \geqq m_1^{-1}$, $0<\beta \leqq r$ and some constant c. (This result can also directly be deduced from the present theorem if one uses the following general fact [36]: the nuclearity index $\nu(e^{-\beta H}\mathscr{L}_r)$ is, in the "tensor-product theory" constructed from a given set of models, bounded from above by the product of the corresponding nuclearity indexes in the respective underlying models.) It follows from the above estimate that a many particle theory satisfies our nuclearity condition, provided the number of particles in the theory of mass less than m does not grow faster than some power of m.

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Note added in proof. It has recently been shown that our nuclearity condition is also satisfied by field-theories of non-interacting massless particles, such as free quantum electrodynamics [37].