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Cells in Affine Weyl Groups

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Let W be an affine Weyl group and let H be the corresponding Hecke algebra, as defined by Iwahori and Matsumoto [IM]. This paper arose from an attempt to find a procedure which associates a representation of H to an irreducible representation of W. Such a procedure is known for finite Weyl groups [L_2, L_3] and we generalize it to the case of affine Weyl groups. Since the representations of W are relatively well understood, it may be hoped that this will help us understand better the representations of H. The main tool we use is a function $\boldsymbol{a}$: $W \to \mathrm{N}$ which is constant on two-sided cells and is an analogue of the function on a finite Weyl group which essentially measures the Gelfand-Kirillov dimension of $\mathrm{U}(g)/\mathrm{I}_w$, where $\mathrm{U}(g)$ is the corresponding enveloping algebra and I_w is a primitive ideal corresponding to the Weyl group element w. The function $\boldsymbol{a}$ is constructed in a purely combinatorial way in terms of multiplication of elements in the C_w-basis ([KL_1]) of the Hecke algebra. To establish its properties we need, however, some positivity properties which follow from deep results on perverse sheaves [BBD].

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In another direction, we describe explicitly the left cells and two-sided cells of affine Weyl groups of type $\tilde{A}_2$, $\tilde{B}_2$, $\tilde{G}_2$.

§ 1. The basis C_w of the Hecke algebra

1.1. Let $q^{1/2}$ be an indeterminate and let $\mathscr{A}=\mathbf{Z}[q^{1/2}, q^{-1/2}]$ be the ring of Laurent polynomials in $q^{1/2}$. We shall set $\mathscr{A}^+=\mathbf{Z}[q^{1/2}]$.

Let W be a Coxeter group and let S be the corresponding set of simple reflections. We shall denote by H the Hecke algebra (over $\mathscr{A}$) corresponding to W. As an $\mathscr{A}$-module, H is free with basis T_w, $(w \in W)$. The multiplication is defined by

$$T_w T_{w'} = T_{ww'}, \quad \text{if } l(ww')=l(w)+l(w')$$
$$(T_s+1)(T_s-q)=0, \quad \text{if } s \in S;$$

here $l(w)$ is the length of w.

It will be convenient to set

$$\tilde{T}_w = q^{-l(w)/2} T_w.$$

We then have

$$\begin{cases} \tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'}, & \text{if } l(ww')=l(w)+l(w') \\ \tilde{T}_s^2 = 1+(q^{1/2}-q^{-1/2})\tilde{T}_s, & \text{if } s \in S. \end{cases} \tag{1.1.1}$$

1.2. Let $\leq$ be the standard partial order on W. In [KL$_1$], Kazhdan and the author showed that for any $w \in W$, there is a unique element $C_w \in H$ such that

$$\begin{aligned} C_w &= \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{(l(w)-l(y))/2} P_{y,w}(q^{-1}) T_y \\ &= \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{(l(w)+l(y))/2} P_{y,w}(q) T_{y^{-1}}^{-1}, \end{aligned}$$

where $P_{y,w}(q)$ is a polynomial of degree $\leq \frac{1}{2}(l(w)-l(y)-1)$ if $y<w$ and $P_{w,w}(q)=1$.

Note that

$$C_w \in \tilde{T}_w + q^{1/2} \sum_{y<w} \mathscr{A}^+ \cdot \tilde{T}_y \tag{1.2.1}$$

from which by induction on $l(w)$, it follows that

$$\tilde{T}_w \in C_w + q^{1/2} \sum_{y<w} \mathscr{A}^+ \cdot C_y. \tag{1.2.2}$$

In particular, the elements C_w form a basis (called the C-basis) of H as an $\mathscr{A}$-module.

1.3. Following [KL_2] we define polynomials $Q_{y,w}(q)$ for $y \le w$ by the identities

(1.3.1) $$\sum_{y \le z \le w} (-1)^{l(z)-l(y)} Q_{y,z}(q) P_{z,w}(q) = \begin{cases} 1 & \text{if } y=w \\ 0 & \text{if } y<w. \end{cases}$$

It is clear that $Q_{y,w}(q)$ is a polynomial of degree $\le \frac{1}{2}(l(w)-l(y)-1)$ if $y<w$ and $Q_{w,w}(q)=1$. For any $y \in W$, we define

(1.3.2) $$D_y = \sum_{\substack{w \\ y \le w}} Q_{y,w}(q^{-1}) q^{(l(w)-l(y))/2} \tilde{T}_w;$$

this is an element in the set $\hat{H}$ of formal (possibly infinite) $\mathscr{A}$-linear combinations of the elements $\tilde{T}_w$, $(w \in W)$. We have

(1.3.3) $$D_y \in \tilde{T}_y + q^{1/2} \sum_{\substack{w \\ y<w}} \mathscr{A}^+ \cdot \tilde{T}_w$$

hence

(1.3.4) $$\tilde{T}_y \in D_y + q^{1/2} \sum_{\substack{w \\ y<w}} \mathscr{A}^+ \cdot D_w.$$

(Both sums are, in general, infinite sums). Note that $H \subset \hat{H}$ in an obvious way and that the left H-module structure on H extends naturally to a left H-module structure on $\hat{H}$. For example, we have

$$\tilde{T}_s\Big(\sum_w \alpha_w \tilde{T}_w\Big) = \sum_{\substack{w \\ sw>w}} \alpha_{sw} \tilde{T}_w + \sum_{\substack{w \\ sw<w}} (\alpha_{sw} + (q^{1/2} - q^{-1/2})\alpha_w)\tilde{T}_w$$

($\alpha_w \in \mathscr{A}$, $s \in S$, and the sums are infinite, in general). Similarly, $\hat{H}$ is in a natural way a right H-module; the left and right H-module structures on $\hat{H}$ commute with each other: $(h_1\hat{h})h_2 = h_1(\hat{h}h_2)$ for $h_1 \in H$, $\hat{h} \in \hat{H}$, $h_2 \in H$.

1.4. Let $\tau: \hat{H} \to \mathscr{A}$ be the $\mathscr{A}$-linear map defined by $\tau(\sum_w \alpha_w \tilde{T}_w) = \alpha_e$ where e is the neutral element of W. It is easy to check that

(1.4.1) $$\tau(\tilde{T}_x \cdot \tilde{T}_y) = \begin{cases} 1, & \text{if } x=y^{-1} \\ 0, & \text{if } x \ne y^{-1}. \end{cases}$$

It follows that

(1.4.2) $$\tau(h_1\hat{h}) = \tau(\hat{h}h_1) \quad \text{for all } h_1 \in H,\ \hat{h} \in \hat{H}$$

and

(1.4.3) $$\tau(C_x D_y) = \tau(D_y C_x) = \begin{cases} 1, & \text{if } x=y^{-1} \\ 0, & \text{if } x \ne y^{-1}. \end{cases}$$

§ 2. The function $\boldsymbol{a}$

2.1. Given $w \in W$, consider the set

$$\text{(2.1.2)} \quad \begin{aligned} \mathscr{S}_w &= \{i \in \mathbf{N} \mid q^{i/2}\tau(\tilde{T}_x\tilde{T}_y D_w) \in \mathscr{A}^+ \quad \text{for all } x, y \in W\} \\ &= \{i \in \mathbf{N} \mid q^{i/2}\tau(C_x C_y D_w) \in \mathscr{A}^+ \quad \text{for all } x, y \in W\}. \end{aligned}$$

(The last equality follows immediately from (1.2.1), (1.2.2.). If $\mathscr{S}_w$ is non-empty, we denote by $\boldsymbol{a}(w)$ the smallest number in $\mathscr{S}_w$. If $\mathscr{S}_w$ is empty, we set $\boldsymbol{a}(w)=\infty$. We have thus defined a function

$$\boldsymbol{a}\colon W \longrightarrow \mathbf{N} \cup \{\infty\}.$$

An equivalent definition is the following one. We consider the coefficient with which $C_{w^{-1}}$ appears in the product $\tilde{T}_x\tilde{T}_y$ (expressed in the C-basis of H). We consider the order of the pole at 0 of this coefficient (in the parameter $q^{1/2}$). When x, y vary, the order of this pole may be bounded above and then $\boldsymbol{a}(w)$ is the largest such order, or it may be unbounded and then $\boldsymbol{a}(w)=\infty$. We have

Proposition 2.2. $\boldsymbol{a}(w)=\boldsymbol{a}(w^{-1})$

Proof. Consider the antiautomorphism of the algebra H defined by $T_w \to T_{w^{-1}}$ for all w. Applying it to the equality $\tilde{T}_x\tilde{T}_y=\sum_w \alpha_w C_w$, ($\alpha_w \in \mathscr{A}$), we find $\tilde{T}_{y^{-1}}\tilde{T}_{x^{-1}}=\sum_w \alpha_w C_{w^{-1}}$. From this, the proposition follows immediately.

We have

Proposition 2.3. $\boldsymbol{a}(w)=0$ *if and only if* $w=e$.

Proof. First we show that $Q_{e,w}=1$ for all $w \in W$. In view of the definition (1.3.1) this is equivalent to the identity

$$\sum_{y \le w} (-1)^{l(y)} P_{y,w}=0 \quad \text{for all } w \neq e,$$

which follows from the fact that $P_{y,w}=P_{sy,w}$ where s is any element of S such that $sw<w$. See [KL$_1$, (2.3. g)]. It follows that

$$D_e=\sum_w q^{l(w)/2}\tilde{T}_w.$$

For any $s \in S$, we have $\tilde{T}_s D_e=q^{1/2}D_e$. By induction on $l(x)$ it follows that $\tilde{T}_x D_e=q^{l(x)/2}D_e$, ($x \in W$), and therefore

$$\tau(\tilde{T}_x\tilde{T}_y D_e)=q^{(l(x)+l(y))/2}\tau(D_e)=q^{(l(x)+l(y))/2} \in \mathscr{A}^+$$

for all $x, y \in W$. It follows that $\boldsymbol{a}(e)=0$.

Assume now that $w \neq e$ and let $s \in S$ be such that $sw < w$. Then

$$\tilde{T}_s \tilde{T}_w = \tilde{T}_{sw} + (q^{1/2} - q^{-1/2})\tilde{T}_w,$$

and by (1.2.2) this is of the form $(q^{1/2} - q^{-1/2})C_w + \mathscr{A}$-linear combination of elements $C_{w'}$, $w' < w$. As $\tau(C_{w'} D_{w^{-1}}) = 0$ for $w' < w$, we have

$$\tau(\tilde{T}_s \tilde{T}_w D_{w^{-1}}) = \tau((q^{1/2} - q^{-1/2})C_w D_{w^{-1}}) = q^{1/2} - q^{-1/2}$$

so that $0 \in \mathscr{S}_{w^{-1}}$. Thus, $\boldsymbol{a}(w^{-1}) \geq 1$, and the proposition is proved.

The last part of the previous proof can be generalized as follows.

Proposition 2.4. *Let J be a subset of S which generates a finite subgroup of W and let w_J be the longest element in this subgroup. Let $w, w', w'' \in W$ be such that $w = w' w_J w''$, $l(w) = l(w') + l(w_J) + l(w'')$. Then $\boldsymbol{a}(w) \geq l(w_J)$.*

Proof. Note that $\tilde{T}_{w_J} \cdot \tilde{T}_{w_J} = (q^{-l(w_J)/2} + \text{higher powers of } q^{1/2})\tilde{T}_{w_J} + \mathscr{A}$-linear combination of elements $\tilde{T}_y$, $y < w_J$. It follows that

$$\begin{aligned} \tilde{T}_{w'w_J} \tilde{T}_{w_J w''} &= \tilde{T}_{w'} \tilde{T}_{w_J} \tilde{T}_{w_J} \tilde{T}_{w''} \\ &= (q^{-l(w_J)/2} + \text{higher powers of } q^{1/2})\tilde{T}_w + \mathscr{A}\text{-linear} \\ &\quad \text{combination of elements } \tilde{T}_z,\ z < w \\ &= (q^{-l(w_J)/2} + \text{higher powers of } q^{1/2})C_w + \mathscr{A}\text{-linear} \\ &\quad \text{combination of elements } C_z,\ z < w,\ (\text{cf. } (1.2.2)). \end{aligned}$$

As $\tau(C_z D_{w^{-1}}) = 0$ for $z < w$, and $\tau(C_w D_{w^{-1}}) = 1$, we have $\tau(\tilde{T}_{w'w_J} \tilde{T}_{w_J w''} D_{w^{-1}}) = q^{-l(w_J)/2} + \text{higher powers of } q^{1/2}$. This shows that $\boldsymbol{a}(w^{-1}) \geq l(w_J)$. The proposition follows.

§ 3. Positivity

3.1. The Coxeter group (W, S) is said to be *crystallographic* if for any $s \neq s'$ in S, the product ss' has order 2, 3, 4, 6 or ∞. We shall need the following result.

(3.1.1) *Assume that (W, S) is crystallographic and let $x, y \in W$. Then $C_x \cdot C_y = \sum_{z \in W} \alpha_{x,y,z} C_z$ where, for any $z \in W$, $\alpha_{x,y,z} \in \mathscr{A}$ is of the form $\sum_{i \in \mathbf{Z}} c_i (-1)^i q^{i/2}$ with $c_i \in \mathbf{N}$.*

3.2. Let $\Phi: H \to H$ be the ring homomorphism defined by $\Phi(q^{1/2}) = -q^{1/2}$, $\Phi(T_x) = (-q)^{l(x)} T_{x^{-1}}^{-1}$, $(x \in W)$. Then $\Phi^2 = 1$ and $C_x = \Phi(C'_x)$ where $C'_x = q^{-l(x)/2} \sum_{y \leq x} P_{y,x}(q) \cdot T_y$. Hence (3.1.1) is equivalent to the following statement.

(3.2.1) *If (W, S) is crystallographic, then for any $x, y \in W$ we have $C'_x C'_y = \sum_{z \in W} \beta_{x,y,z} C'_z$ where $\beta_{x,y,z} \in \mathscr{A}$ has ≥ 0 coefficients for all $z \in W$.*

In the case where (W, S) is a (finite) Weyl group, this statement is proved in [S, 2.12]. The ingredients of the proof are:

(a) interpreting $P_{y,w}$ in terms of local intersection cohomology of a Schubert variety corresponding to w, cf. [KL$_2$].

(b) interpreting multiplication in the Hecke algebra in terms of operations with complexes of sheaves (inverse image, tensor product, direct image).

(c) applying the powerful decomposition theorem in the theory of perverse sheaves, due to Beilinson-Bernstein-Deligne-Gabber [BBD].

In the general case, the assumption that W is crystallographic, means that it arises from a Kac-Moody Lie algebra (or group); to each $w \in W$ one can again associate a "Schubert variety". (See [KL$_2$, § 5], [L$_4$, 11] for the case of affine Weyl groups and Tits [T] in the general case; see also Kac-Peterson [KP].) The proof of (3.2.1) can then be carried out essentially as in the finite case. For the proof of (a) it is simpler to use instead of [KL$_2$] the arguments in [L$_5$, Ch. 1]. This avoids using the dual Schubert varieties (of finite codimension).

§ 4. Left cells and two-sided cells

4.1. We shall review some definitions and results from [KL$_1$]. Given $y, w \in W$, we say that $y \prec w$ if the following conditions are satisfied: $y < w$, $l(w) - l(y)$ is odd and $P_{y,w}(q) = \mu(y, w) q^{(l(w)-l(y)-1)/2} +$ lower powers of q, where $\mu(y, w)$ is a non-zero integer.

Given $y, w \in W$, we say that y, w are joined ($y - w$) if we have $y \prec w$ or $w \prec y$; we then set $\tilde{\mu}(y, w) = \mu(y, w)$ if $y \prec w$ and $\tilde{\mu}(y, w) = \mu(w, y)$ if $w \prec y$. For any $x \in W$, we set $\mathscr{L}(x) = \{s \in S \mid sx < x\}$, $\mathscr{R}(x) = \{s \in S \mid xs < x\}$.

4.2. Given $x, x' \in W$, we say that $x \underset{L}{\leq} x'$ if there exists a sequence of elements of W: $x = x_0, x_1, \cdots, x_n$ such that for each i, $1 \leq i \leq n$, we have $x_{i-1} - x_i$, $\mathscr{L}(x_{i-1}) \not\subset \mathscr{L}(x_i)$. We say that $x \underset{LR}{\leq} x'$ if there exists a sequence $x = x_0, x_1, \cdots, x_n = x'$ of elements of W such that for each i, $1 \leq i \leq n$, we have either $x_{i-1} \underset{L}{\leq} x_i$ or $x_{i-1}^{-1} \underset{L}{\leq} x_i^{-1}$. Let $\underset{L}{\sim}$ be the equivalence relation associated to the preorder $\underset{L}{\leq}$; thus $x \underset{L}{\sim} x'$ means that $x \underset{L}{\leq} x'$, $x' \underset{L}{\leq} x$. The corresponding equivalence classes are called the left cells of W. A right cell of W is a set of form $\{w \in W \mid w^{-1} \in \Gamma\}$ where Γ is a left cell. Let $\underset{LR}{\sim}$ be the equivalence relation associated to the preorder $\underset{LR}{\leq}$; thus $x \underset{LR}{\sim} x'$

means that $x \underset{LR}{\leq} x'$, $x' \underset{LR}{\leq} x$. The corresponding equivalence classes are called the two-sided cells of W.

4.3. For any $x \in W$ and $s \in S$, we have (cf. [KL₁, (2.3a), (2.3b)]):

$$C_s C_x = \begin{cases} -(q^{1/2}+q^{-1/2})C_x, & \text{if } s \in \mathscr{L}(x) \\ \sum_{\substack{y-x \\ sy<y}} \tilde{\mu}(y, x)C_y, & \text{if } s \notin \mathscr{L}(x) \end{cases} \tag{4.3.1}$$

and

$$C_x C_s = \begin{cases} -(q^{1/2}+q^{-1/2})C_x, & \text{if } s \in \mathscr{R}(x) \\ \sum_{\substack{y-x \\ ys<y}} \tilde{\mu}(y, x)C_y, & \text{if } s \notin \mathscr{R}(x). \end{cases} \tag{4.3.2}$$

It follows that for any $x \in W$ we have

$$H \cdot C_x \subset \sum_{y \underset{L}{\leq} x} \mathscr{A} \cdot C_y \tag{4.3.3}$$

$$C_x \cdot H \subset \sum_{\substack{y \\ y^{-1} \underset{L}{\leq} x^{-1}}} \mathscr{A} \cdot C_y \tag{4.3.4}$$

$$H \cdot C_x \cdot H \subset \sum_{y \underset{LR}{\leq} x} \mathscr{A} \cdot C_y. \tag{4.3.5}$$

4.4. We shall need the following property, see [KL₁, 2.4(i)]:

(4.4.1) *If $x \underset{L}{\leq} y$, then $\mathscr{R}(x) \supset \mathscr{R}(y)$. Hence, if $x \underset{L}{\sim} y$ then $\mathscr{R}(x) = \mathscr{R}(y)$.*

Lemma 4.5. *Let $x, y \in W$. If $C_x D_y \neq 0$, then $y^{-1} \underset{L}{\leq} x$. If $D_y C_x \neq 0$, then $y \underset{L}{\leq} x^{-1}$.*

Proof. Assume first that $C_x D_y \neq 0$. Then $C_x D_y$ can be written as a (possibly infinite) sum $\sum_z \alpha_z D_z$, $\alpha_z \in \mathscr{A}$, with $\alpha_z \neq 0$ for some z. For such z, we have $\tau(C_{z^{-1}} C_x D_y) = \alpha_z \neq 0$. Let us expand $C_{z^{-1}} C_x$ in the C-basis of H. The coefficient of $C_{y^{-1}}$ in this expansion is equal to α_z hence it is non-zero. Using now (4.3.3), it follows that $y^{-1} \underset{L}{\leq} x$.

The proof of the second assertion of the lemma is entirely similar.

§ 5. Cells and the function $\boldsymbol{a}$

5.1. Given $w \in W$ such that $\boldsymbol{a}(w) < \infty$, and two elements $x, y \in W$, we define the integer $c_{x,y,w}$ to be the constant term ($=$coefficient of q^0)

of $q^{a(w)/2}\tau(\tilde{T}_x\tilde{T}_y D_w) \in \mathscr{A}^+$.

Lemma 5.2. *Assume that* $\boldsymbol{a}(w)<\infty$.

(a) $c_{x,y,w}$ *is equal to the constant term of* $q^{a(w)/2}\tau(C_xC_yD_w) \in \mathscr{A}^+$.

(b) *There exist* $x', y' \in W$ *such that* $c_{x',y',w}\neq 0$.

(c) *If* $c_{x,y,w}\neq 0$, *then* $w \underset{L}{\leq} x^{-1}$ *and* $w^{-1}\underset{L}{\leq} y$.

(d) *If W is crystallographic, then for any x, y we have* $(-1)^{a(w)}c_{x,y,w} \geq 0$.

Proof. We have $C_x=\sum_{x'\leq x}\alpha_{x'}\tilde{T}_{x'}$, $C_y=\sum_{y'\leq y}\beta_{y'}\tilde{T}_{y'}$ where $\alpha_x=\beta_y=1$, $\alpha_{x'} \in q^{1/2}\mathscr{A}^+$, $(x'<x)$, $\beta_{y'} \in q^{1/2}\mathscr{A}^+$, $(y'<y)$. Hence

$$q^{a(w)/1}\tau(C_xC_yD_w)=\sum_{\substack{x'\leq x\\ y'\leq y}}\alpha_{x'}\beta_{y'}q^{a(w)/2}\tau(\tilde{T}_{x'}\tilde{T}_{y'}D_w)$$
$$=q^{a(w)/2}\tau(\tilde{T}_x\tilde{T}_yD_w)+\text{ an element of } q^{1/2}\mathscr{A}^+$$

and (a) follows.

(b) is clear from the definition of $\boldsymbol{a}(w)$.

Assume that $c_{x,y,w}\neq 0$. Then $\tau(C_xC_yD_w)\neq 0$ and, by (1.4.2), we have also $\tau(C_yD_wC_x)\neq 0$. In particular, $C_yD_w\neq 0$ and $D_wC_x\neq 0$. Hence (c) follows from Lemma 4.5.

Let $\pi_{x,y,w} \in \mathscr{A}$ be the coefficient of $C_{w^{-1}}$ in $C_x\cdot C_y$ (expressed in the C-basis of H). By (a), $c_{x,y,w}$ is the coefficient of $q^{-a(w)/2}$ in $\pi_{x,y,w}$. Hence, (d) is a special case of (3.1.1).

Lemma 5.3. *Assume that W is crystallographic. Let* $x, y \in W$.

(a) *Let* $z, z' \in W$ *be such that* $z - z'$ *and let* $s \in S$ *be such that* $s \in \mathscr{R}(z')-\mathscr{R}(z)$. *If* $i\geq 0$ *is an integer such that* $q^{i/2}\tau(C_xC_yD_z) \in \mathscr{A}$ *has non-zero constant term, then there exists* $x' \in W$ *such that* $q^{i/2}\tau(C_{x'}C_yD_{z'}) \in \mathscr{A}$ *has non-zero constant term. Moreover, we have* $\boldsymbol{a}(z')\geq \boldsymbol{a}(z)$.

(b) *Let* $z \in W$ *be such that* $\boldsymbol{a}(z)<\infty$. *Assume that* $c_{x,y,z}\neq 0$. *Then* $\mathscr{R}(y)=\mathscr{L}(z)$ *and* $\mathscr{L}(x)=\mathscr{R}(z)$.

Proof. Let z, z', s, i be as in (a). We have $\tau(C_xC_yD_z)=\tau(C_yD_zC_x)\neq 0$ hence $D_zC_x\neq 0$ so that $z\underset{L}{\leq} x^{-1}$, (Lemma 4.5). By (4.4.1), $z\underset{L}{\leq} x^{-1}$ implies $\mathscr{R}(z)\supset\mathscr{R}(x^{-1})$. Since $s \notin \mathscr{R}(z)$, it follows that $s\notin\mathscr{R}(x^{-1})$, hence $sx>x$.

Write $C_xC_y=\sum_w \alpha_wC_w$, $\alpha_w \in \mathscr{A}$. Using (4.3.1), we get

$$C_sC_xC_y=\alpha_{z^{-1}}C_sC_{z^{-1}}+\sum_{w\neq z^{-1}}\alpha_wC_sC_w=\sum_{w'}\beta_{w'}C_{w'},\quad (\beta_{w'} \in A),$$

where

$$\beta_{z'^{-1}}=\alpha_{z^{-1}}\tilde{\mu}(z'^{-1}, z^{-1})+\delta$$

$$\delta=\sum_{\substack{w\neq z^{-1}\\ sw>w\\ z'^{-1}-w}}\alpha_w\cdot\tilde\mu(z'^{-1},w)-\alpha_{z'^{-1}}(q^{1/2}+q^{-1/2}).$$

Let a_i, b_i, d_i be the coefficient of $q^{-i/2}$ in $\alpha_{z^{-1}}$, $\beta_{z'^{-1}}$, δ, respectively. Then $b_i=\tilde\mu a_i+d_i$ where $\tilde\mu=\tilde\mu(z'^{-1},z^{-1})=\tilde\mu(z',z)$. From (3.1.1) it follows that $(-1)^i a_i\geq 0$, $(-1)^i d_i\geq 0$, $\tilde\mu\geq 0$. Our assumptions are that $a_i\neq 0$, $\tilde\mu\neq 0$. It follows that $(-1)^i a_i>0$, $\tilde\mu>0$, $(-1)^i b_i\geq\tilde\mu(-1)^i a_i>0$ so that $b_i\neq 0$. Thus, the coefficient of $q^{-i/2}$ in $\tau(C_sC_xC_yD_{z'})\in\mathscr{A}$ is non-zero. By (4.3.1) we have

$$\tau(C_sC_xC_yD_{z'})=\sum_{\substack{x'-x\\ sx'<x'}}\tilde\mu(x',x)\tau(C_{x'}C_yD_{z'})$$

with $\tilde\mu(x',x)\neq 0$ for all x' in the sum. It follows that there is at least one x' in the sum such that the coefficient of $q^{-i/2}$ in $\tau(C_{x'}C_yD_{z'})$ is non-zero. Hence the first assertion of (a) is proved. We now show that we have an inclusion

$$\mathscr{S}_z\supset\mathscr{S}_{z'}$$

(see (2.1.2)). If this were not true we could find $j\in\mathscr{S}_{z'}$ such that $j\notin\mathscr{S}_z$. Since $j\notin\mathscr{S}_z$, there exists $x_1,y_1\in W$ such that $q^{j/2}\tau(\tilde T_{x_1}\tilde T_{y_1}D_z)\notin\mathscr{A}^+$; hence there exists $j'>0$ such that $q^{(j+j')/2}\tau(\tilde T_{x_1}\tilde T_{y_1}D_z)$ has non-zero constant term. By the first assertion of (a) it follows that $q^{(j+j')/2}\tau(\tilde T_{x_2}\tilde T_{y_1}D_{z'})$ has non-zero constant term for some $x_2\in W$, so that $q^{j/2}\tau(\tilde T_{x_2}\tilde T_{y_1}D_{z'})\notin\mathscr{A}^+$. Thus $j\notin\mathscr{S}_{z'}$, a contradiction.

From $\mathscr{S}_z\supset\mathscr{S}_{z'}$ and the definition of the function $\boldsymbol{a}$, it follows that $\boldsymbol{a}(z')\geq\boldsymbol{a}(z)$.

We now prove (b). With the assumption of (b), we have $z\leq_L x^{-1}$ and $z^{-1}\leq_L y$. Using (4.4.1), it follows that $\mathscr{R}(z)\supset\mathscr{R}(x^{-1})=\mathscr{L}(x)$, $\mathscr{L}(z)=\mathscr{R}(z^{-1})\supset\mathscr{R}(y)$. Assume that there exists $t\in S$ such that $t\in\mathscr{L}(z)$, $t\notin\mathscr{R}(y)$. Write again $C_xC_y=\sum_w\alpha_wC_w$, $(\alpha_w\in\mathscr{A})$. From this we have

$$\begin{aligned}\sum_{\substack{y'-y\\ y't<y'}}C_xC_{y'}\tilde\mu(y',y)=C_xC_yC_t&=\alpha_{z^{-1}}C_{z^{-1}}C_t+\sum_{w\neq z^{-1}}\alpha_wC_wC_t\\ &=\sum_{w'}\beta_{w'}C_{w'},\quad(\beta_w\in\mathscr{A})\end{aligned}$$

where

$$\begin{aligned}\beta_{z^{-1}}&=-(q^{1/2}+q^{-1/2})\alpha_{z^{-1}}+\delta,\\ \delta&=\sum_{\substack{w'\neq z^{-1}\\ w't>w'}}\tilde\mu(z^{-1},w')\alpha_{w'}.\end{aligned}$$

Let m_i, n_i, p_i be the coefficient of $q^{-i/2}$ in $\beta_{z^{-1}}$, $\alpha_{z^{-1}}$, δ, respectively. Then

$m_i = -n_{i-1} - n_{i+1} + p_i$. We now take $i = \boldsymbol{a}(z)+1$. Then $n_{i+1}=0$, by the definition of $\boldsymbol{a}(z)$ and $n_{i-1} \neq 0$ since $c_{x,y,z} \neq 0$. Moreover, by (3.1.1), we have $(-1)^{i-1} n_{i-1} \geq 0$, $(-1)^i p_i \geq 0$. It follows that $(-1)^{i-1} n_{i-1} > 0$ and $(-1)^i m_i = (-1)^i p_i + (-1)^{i-1} n_{i-1} \geq (-1)^{i-1} n_{i-1} > 0$. In particular, we have $m_i \neq 0$. Thus

$$q^{\boldsymbol{a}(z)/2} \sum_{\substack{y'-y \\ y't<y'}} \tilde{\mu}(y', y) \tau(C_x C_{y'} D_{z^{-1}}) \notin \mathscr{A}^+.$$

Hence for some y' in the last sum we have

$$q^{\boldsymbol{a}(z)/2} \tau(C_x C_{y'} D_{z^{-1}}) \notin \mathscr{A}^+.$$

This contradicts the definition of $\boldsymbol{a}(z)$. Thus we have proved that $\mathscr{L}(z) - \mathscr{R}(z)$ is empty, so that $\mathscr{L}(z) = \mathscr{R}(y)$. From the proof of 2.2, we see that $c_{x,y,z} = c_{y^{-1}, x^{-1}, z^{-1}}$. Hence we must also have $\mathscr{L}(z^{-1}) = \mathscr{R}(x^{-1})$, and therefore $\mathscr{R}(z) = \mathscr{L}(x)$. The lemma is proved.

We can now prove:

Theorem 5.4. *Assume that (W, S) is crystallographic. Let $z, z' \in W$ be such that $z' \underset{LR}{\leq} z$. Then $\boldsymbol{a}(z') \geq \boldsymbol{a}(z)$. In particular, the function $\boldsymbol{a}$ is constant on the two-sided cells of W.*

Proof. To show that $\boldsymbol{a}(z') \geq \boldsymbol{a}(z)$ we may assume that either

$$z' - z \quad \text{and} \quad \mathscr{R}(z') \not\subset \mathscr{R}(z)$$
or
$$z' - z \quad \text{and} \quad \mathscr{L}(z') \not\subset \mathscr{L}(z).$$

In the first case,we have $\boldsymbol{a}(z') \geq \boldsymbol{a}(z)$ by Lemma 5.3(a). In the second case, Lemma 5.3(a) is applicable to z'^{-1}, z^{-1}. (We have $z'^{-1} - z^{-1}$ and $\mathscr{R}(z'^{-1}) \not\subset \mathscr{R}(z^{-1})$.) It follows that $\boldsymbol{a}(z'^{-1}) \geq \boldsymbol{a}(z^{-1})$, hence, by Proposition 2.2, we have $\boldsymbol{a}(z') \geq \boldsymbol{a}(z)$. The theorem follows.

Corollary 5.5. *Assume that (W, S) is crystallographic. Let $z, z' \in W$ be such that $z' - z$, $\mathscr{R}(z') \not\subset \mathscr{R}(z)$ and $\mathscr{L}(z') \not\subset \mathscr{L}(z)$. Assume that $\boldsymbol{a}(z) < \infty$. Then $\boldsymbol{a}(z') > \boldsymbol{a}(z)$. In particular, z and z' are not in the same two-sided cell.*

Proof. From 5.4 it follows that $\boldsymbol{a}(z') \geq \boldsymbol{a}(z)$. Assume now that $\boldsymbol{a}(z') = \boldsymbol{a}(z)$. Let $x, y \in W$ be such that $c_{x,y,z} \neq 0$. Then the coefficient of $q^{-\boldsymbol{a}(z)/2}$ in $\tau(C_x C_y D_z)$ is non-zero. By 5.3(a), we can find $x' \in W$ such that the coefficient of $q^{-\boldsymbol{a}(z)/2}$ in $\tau(C_{x'} C_y D_{z'})$ is non-zero. Since $\boldsymbol{a}(z) = \boldsymbol{a}(z')$, we have $c_{x',y,z'} \neq 0$. Using now 5.3(b) it follows that $\mathscr{R}(y) = \mathscr{L}(z')$ and also that $\mathscr{R}(y) = \mathscr{L}(z)$. Thus, we have $\mathscr{L}(z') = \mathscr{L}(z)$, a contradiction. The corollary is proved.

This result was proved in [L_2, 4] in the special case where W is a finite Weyl group, using the known connection between $\underset{L}{\leq}$ and the order relation on the primitive ideals in an enveloping algebra. The present proof is quite different and applies in more general circumstances.

§6. The case of finite Weyl groups

Theorem 6.1. *Assume that (W, S) is a finite Weyl group. For any $x, y, z \in W$ we have*

$$c_{x,y,z} = c_{y,z,x} = c_{z,x,y}.$$

Proof. We set $c = c_{x,y,z}$. Assume first that $c \neq 0$. Then $q^{a(z)/2}\tau(\tilde{T}_x \tilde{T}_y D_z) \in \mathscr{A}^+$ has constant term c. For any $x' \in W$, $x' > x$, we have $q^{a(z)/2}\tau(\tilde{T}_{x'}\tilde{T}_y D_z) \in \mathscr{A}^+$. Since

$$D_x = \tilde{T}_x + \sum_{\substack{x' \\ x'>x}} \alpha_{x'} \tilde{T}_{x'}, \quad (\alpha_{x'} \in q^{1/2}\mathscr{A}^+)$$

it follows that $q^{a(z)/2}\tau(D_x \tilde{T}_y D_z) \in \mathscr{A}^+$ has constant term c, hence that $q^{a(z)/2}(\tilde{T}_y D_z D_x) \in \mathscr{A}^+$ has constant term c. (Since W is finite, we have $H = \hat{H}$, hence the products $D_x \tilde{T}_y D_z$, $\tilde{T}_y D_z D_x$ are defined.) We now substitute

$$D_z = \tilde{T}_z + \sum_{\substack{z' \\ z'>z}} \beta_{z'} \tilde{T}_{z'}, \quad (\beta_{z'} \in q^{1/2}\mathscr{A}^+).$$

Note that for $z' > z$, we have

$$q^{a(z)/2}\tau(\tilde{T}_y \tilde{T}_{z'} D_x) \in \mathscr{A}^+$$

since $\boldsymbol{a}(z) \geq \boldsymbol{a}(x)$ (by 5.2(c) and 5.4). It follows that

$$q^{a(z)/2}\tau(\tilde{T}_y(\sum_{z'>z} \beta_{z'}\tilde{T}_{z'})D_x) \in q^{1/2}\mathscr{A}^+$$

so that $q^{a(z)/2}\tau(\tilde{T}_y D_z D_x) \in \mathscr{A}^+$ has the same constant term as

$$q^{a(z)/2}\tau(\tilde{T}_y \tilde{T}_z D_x) \in \mathscr{A}^+.$$

Hence the constant term of $q^{a(z)/2}\tau(\tilde{T}_y \tilde{T}_z D_x) \in \mathscr{A}^+$ is c. By the definition of $\boldsymbol{a}(x)$, this implies $\boldsymbol{a}(z) \leq \boldsymbol{a}(x)$. Combining with $\boldsymbol{a}(z) \geq \boldsymbol{a}(x)$, we get $\boldsymbol{a}(z) = \boldsymbol{a}(x)$, hence $q^{a(x)/2}\tau(\tilde{T}_y \tilde{T}_z D_x) \in \mathscr{A}^+$ has constant term c. Thus, $c_{y,z,x} = c$, as required. The same argument applied to y, z, x instead of x, y, z shows that $c_{z,x,y} = c$.

Thus, if one of the three numbers $c_{x,y,z}$, $c_{y,z,x}$, $c_{z,x,y}$ is non-zero, then these three numbers are equal. If all three numbers are zero, they are again equal. The theorem is proved.

6.2. Remark. Although the conclusion of the theorem may be true also for infinite W, I don't see how to carry out the proof. If W is an affine Weyl group of type $\tilde{A}_1$ or $\tilde{A}_2$, then the elements $D_x \in \hat{H}$ are "square integrable" in the following sense: the coefficient of $\tilde{T}_u$ in D_x tends to zero for $l(u) \to \infty$ in the topology of formal power series in $q^{1/2}$. It follows that the products $D_x \tilde{T}_y D_z$, $\tilde{T}_y D_z D_x$ are defined (they are infinite sums of $\tilde{T}_u$ with coefficients formal power series in $q^{1/2}$) and the proof can still be carried out. However, if W is an affine Weyl group of type $\tilde{B}_2$, there exist elements $D_x \in \hat{H}$ which are not "square integrable".

Corollary 6.3. *Assume that (W, S) is a finite Weyl group.*

(a) *Let $x, y, z \in W$ be such that $c_{x,y,z} \neq 0$. Then $x \underset{L}{\sim} y^{-1}$, $y \underset{L}{\sim} z^{-1}$, $z \underset{L}{\sim} x^{-1}$.*

(b) *If $z' \underset{L}{\leq} z$ and $\boldsymbol{a}(z') = \boldsymbol{a}(z)$, then $z' \underset{L}{\sim} z$.*

(c) *If $z' \underset{L}{\leq} z$ and $z' \underset{LR}{\sim} z$, then $z' \underset{L}{\sim} z$.*

(d) *For any $y \in W$, there exists $x \in W$ such that $c_{x,y,y^{-1}} \neq 0$.*

(e) *If y belongs to a standard parabolic subgroup W' of W then $\boldsymbol{a}(y)$ computed with respect to W is equal to $\boldsymbol{a}(y)$ computed with respect to W'.*

(f) *For any $y \in W$, we have $\boldsymbol{a}(y) \leq l(y)$.*

Proof. (a) We have seen in 5.2(c) that $c_{x,y,z} \neq 0$ implies $z \underset{L}{\leq} x^{-1}$, $z^{-1} \underset{L}{\leq} y$. By 6.1, it also implies $c_{y,z,x} \neq 0$ (hence $x \underset{L}{\leq} y^{-1}$, $x^{-1} \underset{L}{\leq} z$) and $c_{z,x,y} \neq 0$ (hence $y \underset{L}{\leq} z^{-1}$, $y^{-1} \underset{L}{\leq} x$). Thus, we have $z \underset{L}{\sim} x^{-1}$, $x \underset{L}{\sim} y^{-1}$, $y \underset{L}{\sim} z^{-1}$.

(b) We must only show that from $z' - z$, $\mathscr{L}(z') \not\subset \mathscr{L}(z)$, $\boldsymbol{a}(z') = \boldsymbol{a}(z)$ it follows that $z' \underset{L}{\sim} z$. Let $x, y \in W$ be such that $c_{x,y,z^{-1}} \neq 0$. Applying 5.3(a) to z^{-1}, z'^{-1}, x, y and $i = \boldsymbol{a}(z)$ we see that there exists $x' \in W$ such that $q^{a(z)/2} \tau(C_{x'} C_y D_{z'^{-1}}) \in \mathscr{A}$ has non-zero constant term. Since $\boldsymbol{a}(z) = \boldsymbol{a}(z')$, by assumption, it follows that $c_{x',y,z'^{-1}} \neq 0$. From $c_{x,y,z^{-1}} \neq 0$, $c_{x',y,z'^{-1}} \neq 0$ and (a) it follows that $y \underset{L}{\sim} z$, $y \underset{L}{\sim} z'$, hence $z \underset{L}{\sim} z'$, as required.

(c) follows immediately from (b) and 5.4.

(d) Given $y \in W$, we can find $x, z \in W$ such that $c_{z,x,y} \neq 0$, see 5.2(b). By 6.1, we then have $c_{x,y,z} \neq 0$. We have $y \underset{L}{\sim} z^{-1}$, hence there exists a sequence $z^{-1} = y_0, y_1, \cdots, y_n = y$ such that for each j, $1 \leq j \leq n$, we have $y_j - y_{j-1}$, $\mathscr{L}(y_j) \not\subset \mathscr{L}(y_{j-1})$. We show that there exists a sequence

$x_0, x_1, \cdots, x_n$ in W such that $c_{x_j, y, y_j^{-1}} \neq 0$ for $0 \leq j \leq n$. We can take $x_0 = x$. Assume that for some $j \geq 1$, we have found x_{j-1} such that $c_{x_{j-1}, y, y^{-1}} \neq 0$. We apply 5.3(a) with z, z' replaced by y_{j-1}^{-1}, y_j^{-1} and with $i = \boldsymbol{a}(y) = \boldsymbol{a}(y_{j-1}^{-1})$. It follows that there exists $x' \in W$ such that $q^{\boldsymbol{a}(y)/2}\tau(C_{x'}C_y D_{y_j^{-1}}) \in \mathscr{A}$ has non-zero constant term. Since $\boldsymbol{a}(y) = \boldsymbol{a}(y_j^{-1})$, it follows that $c_{x', y, y_j^{-1}} \neq 0$, hence we may take $x_j = x'$. Thus, the required sequence $x_0, x_1, \cdots, x_n$ is constructed. We have $c_{x_n, y, y^{-1}} \neq 0$ and (d) is proved.

(e) By (d), we can find $x \in W$ such that $c_{x, y, y^{-1}} \neq 0$ (with $c_{x, y, y^{-1}}$ defined in terms of W). Then we have also $c_{y, y^{-1}, x} \neq 0$. Hence, if α is the coefficient of $C_{x^{-1}}$ in the product $C_y C_{y^{-1}}$ (expressed in the C-basis of H) then $q^{\boldsymbol{a}(x)/2}\alpha \in \mathscr{A}$ has non-zero constant term. (We shall write $\boldsymbol{a}(\ )$ (respectively $\boldsymbol{a}'(\ )$) for the $\boldsymbol{a}$-function computed in terms of W (respectively W').)

Note that $C_y C_{y^{-1}}$ belongs to the subalgebra H' of H spanned by the $\tilde{T}_u$ ($u \in W'$) or, equivalently, by the C_u, ($u \in W'$). (For $u \in W'$, C_u defined in terms of W' is the same as that defined in terms of W.) It follows that $x \in W'$ and $\boldsymbol{a}'(x)$ is defined. Since $q^{\boldsymbol{a}(x)/2}\alpha$ has non-zero constant term, we have $\boldsymbol{a}(x) \leq \boldsymbol{a}'(x)$. The reverse inequality $\boldsymbol{a}'(x) \leq \boldsymbol{a}(x)$ is obvious, hence $\boldsymbol{a}(x) = \boldsymbol{a}'(x)$. From this it follows that $q^{\boldsymbol{a}'(x)/2}\alpha$ has non-zero constant term, hence $c_{y, y^{-1}, x}$ (computed in terms of W') is non-zero. By (a) applied to W' it follows that y^{-1}, x^{-1} are in the same left cell of W' (hence also in the same left cell of W). From 5.4, we see then that $\boldsymbol{a}'(y) = \boldsymbol{a}'(x)$, $\boldsymbol{a}(y) = \boldsymbol{a}(x)$, so that $\boldsymbol{a}(y) = \boldsymbol{a}'(y)$.

(f) Let again $x \in W$ be such that $c_{x, y, y^{-1}} \neq 0$. If β is the coefficient of C_y in the product $C_x C_y$ (expressed in the C-basis of H), then $q^{\boldsymbol{a}(y)/2}\beta$ has non-zero constant term. From (4.3.2), we see by induction on $l(y_1)$ that for any $x_1, y_1 \in W$, we have $C_{x_1} \cdot C_{y_1} = \sum_z \gamma_z C_z$ where $\gamma_z \in \mathscr{A}$ satisfy $q^{l(y_1)/2}\gamma_z \in \mathscr{A}^+$. In particular, we have $q^{l(y)/2}\beta \in \mathscr{A}^+$, hence $l(y) \geq \boldsymbol{a}(y)$, as required.

The following result relates (for finite Weyl groups) the function $\boldsymbol{a}(w)$ to the function a_E defined in [L₅, 4.1] for any irreducible representation E of W over Q. For such E, we shall denote $E(q)$ the corresponding representation of $H \otimes Q(q^{1/2})$.

Proposition 6.4. *Let $\mathscr{C}$ be a two-sided cell in a finite Weyl group W. Let a be the constant value of the function $w \to \boldsymbol{a}(w)$ on $\mathscr{C}$. Let E be an irreducible representation of W appearing in the left W-module carried by $\mathscr{C}$. Then $a_E \leq a$ and, for at least one such E, we have $a_E = a$.*

Proof. For any $x \in W$, the trace $\mathrm{Tr}(\tilde{T}_x, E(q))$ can be expressed as a Q-linear combination of elements $\tau(\tilde{T}_x C_z D_{z'^{-1}})$ with $z, z' \in \mathscr{C}$, (see [L₃, 1.3]). From the definition of a, it follows that $q^{a/2}\tau(\tilde{T}_x C_z D_{z'}) \in \mathscr{A}^+$

hence $q^{a/2}\mathrm{Tr}(\tilde{T}_x, E(q)) \in Q[q^{1/2}]$. It follows that

$$q^a \sum_x \mathrm{Tr}(\tilde{T}_x, E(q))^2 \in Q[q^{1/2}],$$

hence, by the definition of a_E, we have $a_E \le a$.

To prove the second assertion, it is enough to show that for some E appearing in $\mathscr{C}$ and some $x \in W$, $q^{a/2}\mathrm{Tr}(\tilde{T}_x, E(q)) \in Q[q^{1/2}]$ has non-zero constant term. This would follow from the following statement: there exists $x \in W$ such that $q^{a/2}\mathrm{Tr}(\tilde{T}_x, [\mathscr{C}]) \in Q[q^{1/2}]$ has non-zero constant term, where $[\mathscr{C}]$ is the left H-module carried by $\mathscr{C}$. Since

$$C_x = \tilde{T}_x + \sum_{x'<x} \alpha_{x'}\tilde{T}_{x'} \quad (\alpha_{x'} \in q^{1/2}\mathscr{A}^+),$$

and $q^{a/2}\mathrm{Tr}(\tilde{T}_{x'}, [\mathscr{C}]) \in Q[q^{1/2}]$ for $x'<x$, we are reduced to proving that $q^{a/2}\mathrm{Tr}(C_x, [\mathscr{C}]) \in Q[q^{1/2}]$ has non-zero constant term for some $x \in W$. This is proved as follows.

Fix $y \in \mathscr{C}$ and choose $x \in W$ such that $c_{x,y,y^{-1}} \neq 0$ (see 6.3(d)). We have $C_x C_z = \sum_{z' \in \mathscr{C}} \alpha_{z,z'} C_{z'}$ $(z \in \mathscr{C})$, where $\alpha_{z,z'} \in \mathscr{A}$. Let n_z be the constant term of $q^{a/2}\alpha_{z,z}$ $(z \in \mathscr{C})$. Then $\mathrm{Tr}(C_x, [\mathscr{C}]) = \sum_{z \in \Omega} \alpha_{z,z}$ and it is enough to show that $\sum_{z \in \mathscr{C}} n_z \neq 0$. By (3.3.1), we have $(-1)^a n_z \ge 0$ for all $z \in \mathscr{C}$. Since $c_{x,y,y^{-1}} \neq 0$, we have $n_y \neq 0$, hence, $(-1)^a n_y > 0$. It follows that

$$(-1)^a \sum_{z \in \mathscr{C}} n_z > 0.$$

The proposition is proved.

6.5. Remark. It is known (see [L_5, 5.27]) that a_E is in fact constant when E runs through the irreducible W-modules appearing in $\mathscr{C}$. However, this can be proved at present only through case by case checking.

§ 7. An upper bound for $a(w)$ for w in an affine Weyl group

7.1. In this section, (W, S) denotes an irreducible affine Weyl group. Let ν be the number of positive roots in the corresponding root system. We shall prove:

Theorem 7.2. *For any $x, y, z \in W$, we have $\tilde{T}_x\tilde{T}_y = \sum_{z \in W} m_{x,y,z}\tilde{T}_{z^{-1}}$ where $m_{x,y,z}$ is a polynomial in $\xi = (q^{1/2} - q^{-1/2})$ with integral, ≥ 0 coefficients, of degree $\le \nu$.*

Before giving the proof, we note:

Corollary 7.3. *For any $z \in W$, we have $\boldsymbol{a}(z) \leq \nu$.*

Proof. From 7.2, we see that $q^{\nu/2} m_{x,y,z} \in \mathscr{A}^+$ for all x, y, z. On the other hand, $\tilde{T}_{z^{-1}} \in \sum_u \mathscr{A}^+ C_u$, (see (1.2.2)). It follows that

$$q^{\nu/2} \tilde{T}_x \tilde{T}_y \in \sum_{u \in W} \mathscr{A}^+ \cdot C_u$$

and the corollary follows.

For the proof of the theorem we shall need the following.

Lemma 7.4. *For any $x, y, z \in W$, $m_{x,y,z}$ is a polynomial in $\xi = (q^{1/2} - q^{-1/2})$ with integral ≥ 0 coefficients, of degree $\leq \min(l(x), l(y), l(z))$ in ξ.*

Proof. From (1.1.1) we see immediately, by induction on $l(x)$ that $m_{x,y,z}$ is a polynomial with integral ≥ 0 coefficients of degree $\leq l(x)$ in ξ. Similarly, by induction on $l(y)$ we see that $m_{x,y,z}$ has degree $\leq l(y)$ in ξ. We have $m_{x,y,z} = \tau(\tilde{T}_x \tilde{T}_y \tilde{T}_z) = \tau(\tilde{T}_y \tilde{T}_z \tilde{T}_x) = m_{y,z,x}$. By what we have proved so far, we have then $\deg(m_{y,z,x}) \leq l(z)$ hence $\deg(m_{x,y,z}) \leq l(z)$. The lemma is proved.

7.5. The affine Weyl group (W, S) can be obtained as follows (cf. [L₁, 1.1]). Let E be an affine euclidean space with a given set of hyperplanes $\mathscr{F}$. Let Ω be the group of affine motions in E generated by the orthogonal reflections in the various hyperplanes P in $\mathscr{F}$, regarded as acting on the right on E. We assume that Ω is an infinite discrete subgroup of the group of all affine motions of E acting irreducibly on the space of translations of E and leaving stable the set $\mathscr{F}$. Let X be the set of alcoves (=connected components of the set $E - \bigcup_{P \in \mathscr{F}} P$). Then Ω acts simply transitively on X. Let S_1 be the set of Ω-orbits in the set of codimension 1 facets of alcoves. Each $s \in S_1$ defines an involution $A \to sA$ of X, where, for an alcove A, sA is the alcove $\neq A$ which has with A a common face of type s. The maps $A \to sA$ generate a group of permutations of X. This group, together with its subset S_1 is a Coxeter group (an affine Weyl group). We shall assume that (W, S) is this particular Coxeter group, (thus $S = S_1$).

We regard W as acting on the left on X. (It acts simply transitively and commutes with the action of Ω on X.) A special point in E is a 0-dimensional facet v of an alcove such that the number of hyperplanes $P \in \mathscr{F}$ passing through v is maximum possible (it is equal to ν). For such v, we denote by W_v the subgroup of W which is the stabilizer of the set of alcoves containing v in their closure. Then W_v is a standard parabolic subgroup of W generated by $|S|-1$ elements of S. We denote by w_v the longest element of W_v; we have $l(w_v) = \nu$. We choose for each special point, a connected component C_v^+ of the set $E-\bigcup_{\substack{P\in\mathscr{F}\\ P\ni v}} P$ in such a way that for any two special points v, v' in E, $C_{v'}^+$ is a translate of C_v^+. Let A_v^+ be the unique alcove contained in C_v^+ and having v in its closure, and let $A_v^-=w_vA_v^+$.

To any alcove A we associate a subset $\mathscr{L}(A)\subset S$, as follows. Let $s\in S$ and let P be the hyperplane in $\mathscr{F}$ supporting the common face of type s of A and sA. We say that $s\in\mathscr{L}(A)$ if A is in that half space determined by P which meets C_v^+ for any special point v.

7.6. Following [L₁, 1.6] we consider the free $\mathscr{A}$-module $\mathscr{M}$ with basis corresponding to the alcoves in X. It can be regarded as a left H-module:

$$T_sA=\begin{cases} sA, & \text{if } s\in S-\mathscr{L}(A)\\ q(sA)+(q-1)A, & \text{if } s\in\mathscr{L}(A).\end{cases}$$

Let δ: $X\to\mathbf{Z}$ be a length function on X in the sense of [L₁, 2.11]; we have $\delta(A)=\delta(sA)+1$, if $s\in\mathscr{L}(A)$ and $\delta(A)=\delta(sA)-1$, if $s\in S-\mathscr{L}(A)$. It follows that if we set $\tilde{A}=q^{-\delta(A)/2}A$, then

$$\tilde{T}_s\tilde{A}=\begin{cases} \widetilde{sA}, & \text{if } s\in S-\mathscr{L}(A)\\ \widetilde{sA}+(q^{1/2}-q^{-1/2})\tilde{A}, & \text{if } s\in\mathscr{L}(A).\end{cases} \tag{7.6.1}$$

From this it follows by induction on $l(w)$ that

$$\tilde{T}_w\tilde{A}=\sum_B M_{w,A,B}\tilde{B}, \quad \text{(finite sum)}$$

where $M_{w,A,B}$ are polynomials in $\xi=(q^{1/2}-q^{-1/2})$ with integral, ≥ 0 coefficients.

Lemma 7.7. $\deg_\xi M_{w,A,B}\leq\nu$.

Proof. Given w, A, we choose a special point v in the closure of A. We can uniquely write $w=w'\cdot w_1$ where $w_1\in W_v\in w'$ has minimal length in $w'W_v$ and $l(w)=l(w')+l(w_1)$. We have $A=w_2(A_v^-)$ for some $w_2\in W_v$ and $\tilde{A}=\tilde{T}_{w_2}\tilde{A}_v^-$. We have

$$\tilde{T}_{w_1}\tilde{A}=\tilde{T}_{w_1}\tilde{T}_{w_2}\tilde{A}_v^-=\sum_{w_3\in W_v} m_{w_1,w_2,w_3^{-1}}\tilde{T}_{w_3}\tilde{A}_v^-=\sum_{w_3\in W_v} m_{w_1,w_2,w_3^{-1}}\widetilde{w_3(A_v^-)}$$

and $m_{w_1,w_2,w_3^{-1}}$ has degree at most $l(w_3)$ in ξ, (see 7.4). For a fixed w_3, let $C=w_3(A_v^-)$, and let $s_k\cdots s_2s_1$ be a reduced expression for w', ($s_i\in S$). It is clear from (7.6.1) that

$$\tilde{T}_{w'}\tilde{C}=\sum_{I}(q^{1/2}-q^{-1/2})^{p_I}\tilde{C}_I$$

where I ranges over all subsets $i_1<i_2<\cdots<i_{p_I}$ of $\{1, 2, \cdots, k\}$ such that

$$s_{i_t}\cdots\hat{s}_{i_{t-1}}\cdots\hat{s}_{i_2}\cdots\hat{s}_{i_1}\cdots s_1(C)<\hat{s}_{i_t}\cdots\hat{s}_{i_{t-1}}\cdots\hat{s}_{i_1}\cdots s_1(C)$$

for $t=1, \cdots, p_I$, and $C_I=s_k\cdots\hat{s}_{i_p}\cdots\hat{s}_{i_1}\cdots s_1(C)$, $p=p_I$.

According to [L_1, 4.3] we have $p_I=|I|\leq\nu-l(w_3)$. Hence

$$\begin{aligned}\tilde{T}_w\tilde{A}=\tilde{T}_{w'}\tilde{T}_{w_1}\tilde{A}&=\sum_{w_3\in W_v} m_{w_1,w_2,w_3^{-1}}\tilde{T}_{w'}(\widetilde{w_3A_v^-})\\&=\sum_{\substack{w_3\\I}} m_{w_1,w_2,w_3^{-1}}\xi^{p_I}\tilde{C}_I\end{aligned}$$

with $\deg_\xi(m_{w_1,w_2,w_3^{-1}}\xi^{p_I})\leq\deg(m_{w_1,w_2,w_3^{-1}})+p_I\leq l(w_3)+\nu-l(w_3)=\nu$. The lemma is proved.

Lemma 7.8. *Given $y\in W$, there exists an alcove A such that $\tilde{T}_y\tilde{A}=\widetilde{yA}$.*

Proof. Let v be a special point in E. Write $y=y'\cdot y_1$ with $y_1\in W_v$ and y' of minimal length in $y'W_v$. Let $A=(y_1^{-1}w_v)A_v^-$. Then $y_1A=A_v^+$, $\delta(y_1A)=\delta(A)+l(y_1)$ and $\delta(y'A_v^+)=\delta(A_v^+)+l(y')$ ([L_1, 3.6]) hence A has the required property.

7.9 Proof of Theorem 7.2. Given $x, y\in W$, we select A as in Lemma 7.8. Then

$$\begin{aligned}\tilde{T}_x\tilde{T}_y\tilde{A}=\tilde{T}_x\widetilde{yA}&=\sum_{B\in X}M_{x,yA,B}\tilde{B}=\sum_{z\in W}m_{x,y,z^{-1}}\tilde{T}_z\tilde{A}\\&=\sum_{z,B}m_{x,y,z^{-1}}M_{z,A,B}\tilde{B}\end{aligned}$$

hence

$$\sum_{z\in W}m_{x,y,z^{-1}}M_{z,A,B}=M_{x,yA,B}$$

for any $B\in X$. By Lemma 7.7, $M_{x,yA,B}\in\mathbf{Z}[\xi]$ has degree $\leq\nu$. Since $m_{x,y,z^{-1}}\cdot M_{z,A,B}\in\mathbf{Z}[\xi]$ have ≥ 0 coefficients it follows that $m_{x,y,z^{-1}}\cdot M_{z,A,B}$ has degree $\leq\nu$ in the variable ξ, for any z, B. We take $B=zA$; then $M_{z,A,B}$ is $\neq 0$ (its value for $\xi=0$ is equal to 1). It follows that $m_{x,y,z^{-1}}$ has degree $\leq\nu$ in ξ for any $z\in W$. This completes the proof.

For future reference we state

Corollary 7.10. *For any $x, y, z\in W$, the elements $q^{\nu/2}\tau(\tilde{T}_x\tilde{T}_y\tilde{T}_z)$, $q^{\nu/2}\tau(\tilde{T}_x\tilde{T}_yD_z)$ are in $\mathscr{A}^+$ and have the same constant term.*

Proof. The fact that they are in $\mathscr{A}^+$ is just a reformulation of Theorem 7.2 and Corollary 7.3. Let us write

$$D_z = \tilde{T}_z + \sum_{\substack{z' \\ z'>z}} \alpha_{z'} \tilde{T}_{z'} \in \hat{H}, \quad (\alpha_{z'} \in q^{1/2}\mathscr{A}^+).$$

It remains to prove that

$$q^{\nu/2} \sum_{\substack{z' \\ z'>z}} \alpha_{z'} \tau(\tilde{T}_x \tilde{T}_y \tilde{T}_{z'}) \in q^{1/2}\mathscr{A}^+.$$

(Note that all but finite terms in the sum are zero.) But this follows from $\alpha_{z'} \in q^{1/2}\mathscr{A}^+$ and from $q^{\nu/2}\tau(\tilde{T}_x \tilde{T}_y \tilde{T}_{z'}) \in \mathscr{A}^+$. The corollary is proved.

§ 8. The subset $W_{(\nu)}$ of an affine Weyl group

8.1. In this section, we preserve the notations from the previous section. Let $W_{(\nu)} = \{w \in W \mid \boldsymbol{a}(w) = \nu\}$. Consider an element w in our affine Weyl group with the following property: there exists a special point $v \in E$ and a decomposition $w = w' w_v w''$ of w such that $l(w) = l(w') + l(w_v) + l(w'')$. (Recall that w_v is the longest element in W_v.) By 2.4, we have $\boldsymbol{a}(w) \geq \nu$, and by 7.3, we have $\boldsymbol{a}(w) \leq \nu$. Hence $w \in W_{(\nu)}$.

8.2. This argument shows that almost all elements of W are in $W_{(\nu)}$. (More precisely, let B be a large ball in E with center v (a fixed special point) and let $B_{(\nu)}$ be the set of points of B which belong to an alcove wA_v^- ($w \in W_{(\nu)}$). Then $\mathrm{vol}(B_{(\nu)})/\mathrm{vol}(B)$ tends to 1 when the radius of B tends to ∞.)

Proposition 8.3. (a) *If $x, y, z \in W_{(\nu)}$ then $c_{x,y,z} = c_{y,z,x} = c_{z,x,y}$.*
(b) *If $x, y \in W$, $z \in W_{(\nu)}$ and $c_{x,y,z} \neq 0$, then $x \in W_{(\nu)}$ amd $y \in W_{(\nu)}$.*

Proof. (a) By 7.10, $c_{x,y,z}$ is equal to the constant term of $q^{\nu/2}\tau(\tilde{T}_x \tilde{T}_y \tilde{T}_z)$. It is therefore sufficient to check that $q^{\nu/2}\tau(\tilde{T}_x \tilde{T}_y \tilde{T}_z)$ is invariant under cyclic permutations of x, y, z. This follows from (1.4.2).

(b) Our assumptions and 7.10 imply that $q^{\nu/2}\tau(\tilde{T}_x \tilde{T}_y \tilde{T}_z)$ has non-zero constant term. If follows that $q^{\nu/2}\tau(\tilde{T}_y \tilde{T}_z \tilde{T}_x)$ has non-zero constant term. Using again 7.10, it follows that $q^{\nu/2}\tau(\tilde{T}_y \tilde{T}_z D_x)$ has non-zero constant term, hence $\boldsymbol{a}(x) \geq \nu$. On the other hand, $\boldsymbol{a}(x) \leq \nu$ by 7.3. Thus $\boldsymbol{a}(x) = \nu$. The proof of the equality $\boldsymbol{a}(y) = \nu$ is similar.

Corollary 8.4. (a) *Let $x, y, z \in W_{(\nu)}$ be such that $c_{x,y,z} \neq 0$. Then $x \underset{L}{\sim} y^{-1}$, $y \underset{L}{\sim} z^{-1}$, $z \underset{L}{\sim} x^{-1}$.*

(b) *If* $z, z' \in W_{(\nu)}$ *and* $z' \underset{L}{\leq} z$, *then* $z' \underset{L}{\sim} z$.

(c) *For any* $y \in W_{(\nu)}$ *there exists* $x \in W_{(\nu)}$ *such that* $c_{x,y,y^{-1}} \neq 0$.

(d) *For any* $y \in W_{(\nu)}$, *we have* $l(y) \geq \nu$.

Proof. The proof is the same as that of 6.3, once 8.3 is known.

Corollary 8.5. *Let* v *be a special point in* E. *The set*

$$\Gamma_v = \{w \in W \mid l(w) = l(ww_v) + l(w_v)\}$$

is a left cell in W.

Proof. The function $w \mapsto \mathscr{R}(w)$ from W to the set of subsets of S is constant on left cells, cf. (4.4.1). The set Γ_v is one particular fibre of this function, hence it is a union of left cells. Note also that by the discussion in 8.1, we have $\Gamma_v \subset W_{(\nu)}$.

We shall prove by induction on $l(w)$ that $w \underset{L}{\sim} w_v$ for any $w \in \Gamma_v$. The induction starts with the case $l(w) = \nu$; in this case the result is clear since $w = w_v$. Assume now that $l(w) \geq \nu + 1$. We can find $s \in S$ such that $w = sw'$, $w' \in \Gamma_v$, $l(w) = l(w') + 1$. We have clearly $w \underset{L}{\leq} w'$. As $w, w' \in \Gamma_v \subset W_{(\nu)}$, we may apply 8.4(b) and conclude that $w \underset{L}{\sim} w'$. By the induction hypothesis, we have $w' \underset{L}{\sim} w_v$. It follows that $w \underset{L}{\sim} w_v$. Thus, we have proved that $w \underset{L}{\sim} w_v$ for all $w \in \Gamma_v$, hence that Γ_v is exactly one left cell.

§ 9. Construction of n-tempered representations

9.1. In this section, (W, S) denotes again an irreducible affine Weyl group. Given a commutative ring R, and an integer $i \geq 0$ we define E_R^i to be the free R-module with basis (e_w), $w \in W_{(i)} = \{w \in W \mid \boldsymbol{a}(w) = i\}$. Similarly, we define $E_R^{\geqq i}$ to be the free R-module with basis (e_w), $w \in W_{(i)} \cup W_{(i+1)} \cup \cdots$.

If $\phi\colon \mathscr{A} \to R$ is a ring homomorphism, we denote by H_ϕ the R-algebra obtained from H by extension of scalars, via ϕ. The elements $\tilde{T}_w$ give rise to elements of H_ϕ denoted in the same way: $\tilde{T}_w$. The rule

$$\tilde{T}_s e_w = \begin{cases} -e_w, & \text{if } sw < w \\ \phi(q^{1/2}) e_w + \sum\limits_{\substack{y - w \\ sy < y}} \tilde{\mu}(y, w) e_y, & \text{if } sw > w \end{cases} \qquad (s \in S,\ w \in W,\ a(w) \geq i)$$

makes $E_R^{\geqq i}$ into a left H_ϕ-module. (For each y in the sum, we have automatically $\boldsymbol{a}(y) \geq i$, (see 5.4).)

Similarly, if $\psi\colon \mathscr{A} \to R$ is a ring homomorphism, the rule

$$e_w \tilde{T}_s = \begin{cases} -e_w, & \text{if } ws < w \\ \psi(q^{1/2})e_w + \sum\limits_{\substack{y-w \\ ys<y}} \tilde{\mu}(y, w)e_y, & \text{if } ws > w \end{cases} \quad (s \in S,\ w \in W,\ \boldsymbol{a}(w) \geq i)$$

makes $E_R^{\geqq i}$ into a right H_ψ-module.

The left H_ϕ-module structure on $E_R^{\geqq i}$ doesn't commute, in general, with the right H_ψ-module structure. Since $E_R^{\geqq i+1}$ is a left H_ϕ-submodule of $E_R^{\geqq i}$ and a right H_ψ-module, we may regard $E_R^i = E_R^{\geqq i}/E_R^{\geqq i+1}$ in a natural way both as a left H_ϕ-module and a right H_ψ-module.

Theorem 9.2. *The left H_ϕ-module structure on E_R^i commutes with the right H_ψ-module structure on E_R^i.*

Proof. It is enough to prove the following statement.

Let $w \in W_{(i)}$, $s, s' \in S$, and consider the basis element e_w of $E_R^{\geqq i}$. Then

$$(\tilde{T}_s e_w)\tilde{T}_{s'} - \tilde{T}_s(e_w \tilde{T}_{s'}) \tag{9.2.1}$$

is in $E_R^{\geqq i+1}$. (Here $\tilde{T}_s$ is in H_ϕ and $\tilde{T}_{s'}$ is in H_ψ.) A simple computation (compare [L_2, 2]) shows that (9.2.1) is zero unless $s \notin \mathscr{L}(w)$ and $s' \notin \mathscr{R}(w)$ in which case it is an R-linear combination of elements $e_{w'}$ ($w' \in W$) such that $w' - w$, $s \in \mathscr{L}(w')$, $s' \in \mathscr{R}(w')$. By 5.5, (which is applicable since $\boldsymbol{a}(w) \leq \nu < \infty$) all these elements w' satisfy $\boldsymbol{a}(w') \geq i+1$. Hence (9.2.1) is in $E_R^{\geqq i+1}$ and the theorem is proved.

9.3. From now on, we assume that $\psi\colon \mathscr{A} \to R$ is the ring homomorphism such that $\psi(q^{1/2}) = 1$; in this case, H_ψ is the group algebra $R[W]$ of W over R; its basis $\tilde{T}_w$ becomes the standard basis of the group algebra. If $i \geq 0$, then E_R^i is a right $R[W]$-module. Let V be a right $R[W]$-module. We associate to V and $i \geq 0$ the R-module $\hat{V}_R^i = (E_R^i \otimes_R V)_W$ (=space of W-coinvariants on $E_R^i \otimes V$); here W acts on $E_R^i \otimes_R V$ by $(\varepsilon \otimes v)w = (\varepsilon w) \otimes (vw)$. With these definitions, we have

Lemma 9.4. *Assume that R is noetherian and that V is finitely generated as an R-module. Then $\hat{V}_R^i$ is finitely generated as an R-module.*

Proof. Let T be the group of translations in W. Then $\hat{V}_R^i$ is a quotient of the space $(E_R^i \otimes_R V)_T$ of T-coinvariants, which is a quotient of $(E_R^{\geqq i} \otimes_R V)_T$. Thus it is enough to show that $E_R^{\geqq i} \otimes_R V$ is finitely generated as an $R[T]$-module. This is an $R[T]$-submodule of $E_R^{\geq 0}\otimes_R V$. Since $R[T]$ is a noetherian ring it is enough to show that $E_R^{\geq 0}\otimes_R V$ is a finitely generated $R[T]$-module. This is clear, since $E_R^{\geq 0}$ is a free $R[T]$-module of finite rank (equal to the index of T in W).

9.5. Assume now that $\phi\colon \mathscr{A}\to R$ is a ring homomorphism. Then, as we have seen in 9.1, E_R^i is a left H_ϕ-module. It follows that, if V is a right $R[W]$-module, then $E_R^i\otimes_R V$ is a left H_ϕ-module: $h(\varepsilon\otimes v)=(h\varepsilon)\otimes v$, ($h\in H_\phi$, $\varepsilon\in E_R^i$, $v\in V$). By 9.2 this left H_ϕ-module structure commutes with the right $R[W]$-module structure, hence the space $\hat{V}_R^i$ of W-coinvariant s inherits a left H_ϕ-module structure.

9.6. In 9.6–9.7 we shall assume that R is the quotient field of a discrete valuation ring $\mathcal{O}$ and that $\phi\colon \mathscr{A}\to R$ is a ring homomorphism such that $\phi(q^{1/2})\in\mathcal{O}$.

(9.6.1) *A left H_ϕ-module M is said to be n-tempered, ($n\in\mathbf{N}$), if it is finite dimensional as an R-vector space and if there exists an $\mathcal{O}$-lattice $\mathscr{L}$ in M such that $\phi(q^{n/2})\tilde{T}_w$ maps $\mathscr{L}$ into itself for any $w\in W$.*

(Here, an $\mathcal{O}$-lattice means a finitely generated $\mathcal{O}$-submodule of M which generates M as an R-vector space.)

Let ω be a translation in $T\subset W$ such that $l(x\omega)=l(x)+l(\omega)$ for all $x\in W_v$ (for a fixed special point $v\in E$). It is known that for any integer $j\geq 0$, we have $l(\omega^j)=jl(\omega)$. It follows that $\tilde{T}_{\omega^j}=(\tilde{T}_\omega)^j$ for all $j\geq 0$. If M is an n-tempered H_ϕ-module and λ is an eigenvalue of $\tilde{T}_\omega\colon M\to M$ (in an extension of R) then λ is *integral over* $\mathcal{O}$. (This justifies the name "tempered"; in the usual definition of tempered representations (over $\mathbf{C}$) one assumes that λ has always absolute value ≤ 1.) To prove this, we observe that $\phi(q^{n/2})\tilde{T}_\omega^j$ preserves the lattice $\mathscr{L}$ for all $j\geq 0$. Hence $\phi(q^{n/2})\lambda^j$ is integral over $\mathcal{O}$ for all $j\geq 0$; hence λ is integral over $\mathcal{O}$.

Theorem 9.7. *Let V_0 be a right $\mathcal{O}[W]$-module which is free of finite rank as an $\mathcal{O}$-module and let $V=V_0\otimes_{\mathcal{O}} R$ be the corresponding right $R[W]$-module. Then for any $n\geq 0$, the left H_ϕ-module $\hat{V}_R^n$ (see 9.5) is n-tempered.*

Proof. By 9.4, the $\mathcal{O}$-module $(\hat{V}_0)_{\mathcal{O}}^i=(E_{\mathcal{O}}^i\otimes_{\mathcal{O}} V_0)_W$ is finitely generated and the R-module $\hat{V}_R^i=(E_R^i\otimes_R V)_W$ is finitely generated. It is clear that $(\hat{V}_0)_{\mathcal{O}}^i\otimes_{\mathcal{O}} R=\hat{V}_R^i$ as an R-module. Hence the image of $(\hat{V}_0)_{\mathcal{O}}^i$ in $\hat{V}_R^i$ is an $\mathcal{O}$-lattice $\mathscr{L}$, which is generated (as an $\mathcal{O}$-module) by the images of the elements $e_w \otimes v \in E^i_{\mathcal{O}} \otimes_{\mathcal{O}} V_0$. For any $y \in W$, and any $w \in W_{(i)}$, we have

$$q^{n/2}\tilde{T}_y C_w \in \sum_{\substack{w' \\ a(w')=n}} \mathscr{A}^+ \cdot C_w + \sum_{\substack{w' \\ a(w')>n}} \mathscr{A} \cdot C_{w'}$$

(identity in H), by the definition of the $\boldsymbol{a}$-function. Hence, we have

$$\phi(q^{n/2})\tilde{T}_y e_w \in \sum_{\substack{w' \\ a(w')=n}} \mathcal{O} \cdot e_{w'} + \sum_{\substack{w' \\ a(w')>n}} Re_{w'}$$

(identity in $E_R^{\geqq n}$) and, therefore,

$$\phi(q^{n/2})\tilde{T}_y e_w \in \sum_{\substack{w' \\ a(w')=n}} \mathcal{O} \cdot e_{w'}$$

(identity in E_R^n). This shows that $\mathscr{L}$ is stable under $\phi(q^{n/2})\tilde{T}_y$ for all $y \in W$. The theorem is proved.

9.8. Remark. The same proof shows that in the case where the set $W_{(n)}$ is finite, the H_ϕ-module E_R^n itself is n-tempered.

9.9. In 9.9–9.11 we shall assume that R is an algebraically closed field of characteristic zero and $\phi: \mathscr{A} \to R$ is a ring homomorphism such that $\phi(q^{1/2})$ is not a root of 1 in R.

Let V be an irreducible right $R[W]$-module. The sequence of canonical surjective maps

$$E_R^{\geqq 0} \longrightarrow E_R^{\geqq 0}/E_R^{\geqq \nu} \longrightarrow E_R^{\geqq 0}/E_R^{\geqq \nu-1} \to \cdots \to E_R^{\geqq 0}/E_R^{\geqq 1} \longrightarrow E_R^{\geqq 0}/E_R^{\geqq 0} = 0$$

gives rise to a sequence of surjective maps

$$\begin{aligned}(9.9.1) \qquad & (E_R^{\geqq 0} \otimes_R V)_W \longrightarrow ((E_R^{\geqq 0}/E_R^{\geqq \nu}) \otimes_R V)_W \\ & \to \cdots \to ((E_R^{\geqq 0}/E_R^{\geqq 1}) \otimes_R V)_W \longrightarrow ((E_R^{\geqq 0}/E_R^{\geqq 0}) \otimes_R V)_W = 0\end{aligned}$$

(W-coinvariants are taken with respect to the right $R[W]$-module structure.) Each of the R-modules in this sequence is a left $R[W]$-module since $E_R^{\geqq i}$ is a left $R[W]$-module (replacing temporarily ϕ by the homomorphism ψ as in 9.3. The first $R[W]$-module in the sequence (9.9.1) is irreducible, since V is irreducible and $E_R^{\geqq 0}$ is the two-sided regular representation of W. The last $R[W]$-module in (9.9.1) is zero. Since all maps in (9.9.1) are surjective maps of left $R[W]$-modules, it follows that there is a unique integer n, $0 \leq n \leq \nu$ such that the map α_i in the natural exact sequence

$$((E_R^{\geq i}/E_R^{\geq i+1})\otimes V)_W \longrightarrow ((E_R^{\geq 0}/E_R^{\geq i+1})\underset{R}{\otimes} V)_W \xrightarrow{\alpha_i} (E_R^{\geq 0}/E_R^{\geq i})\underset{R}{\otimes} V)_W \longrightarrow 0$$

is an isomorphism ($\neq 0$) for $i \geq n+1$ and is zero for $i \leq n$. It follows for this n, the natural map

$$\hat{V}_R^n = ((E_R^{\geq n}/E_R^{\geq n+1})\underset{R}{\otimes} V)_W \longrightarrow ((E_R^{\geq 0}/E_R^{\geq n+1})\underset{R}{\otimes} V)_W$$

is surjective and $((E_R^{\geq 0}/E_R^{\geq n+1})\underset{R}{\otimes} V)_W \approx (E_R^{\geq 0}\underset{R}{\otimes} V)_W$. Thus, we have associated to V an integer $n \geq 0$ such that $\hat{V}_R^n$ is non-zero (indeed $\dim_R \hat{V}_R^n \geq \dim_R V$). We shall denote this $\hat{V}_R^n$ simply as $\hat{V}$. It is a left H_ϕ-module (see 9.5). The integer n just defined will be denoted a_V.

9.10. We shall now state a number of conjectures.

Conjecture A. *If V is an irreducible right $R[W]$-module, then the left H_ϕ-module $\hat{V}$ (see 9.9) has a unique irreducible quotient $\tilde{V}$. All other composition factors of $\hat{V}$ are of form $\tilde{V}'$ where V' are irreducible $R[W]$-modules such that $a_{V'} < a_V$. The correspondence $V \to \tilde{V}$ is a bijection between the set of isomorphism classes of irreducible right $R[W]$-modules and the set of isomorphism classes of irreducible left H_ϕ-modules.*

Conjecture B. *For any x, y, z in the affine Weyl group W, we have $c_{x,y,z} = c_{y,z,x} = c_{z,x,y}$ (see 6.1).*

This would imply that all statements 6.3(a) to (f) hold for affine Weyl groups. It would also imply the following statement.

(9.10.1) *W is a union of finitely many left cells (hence of finitely many two-sided cells).*

We now show how (9.10.1) can be deduced from the statement 6.3(b) for the affine Weyl group W. For each right cell Γ contained in $W_{(i)}$, the R-subspace E_R^Γ of E_R^i spanned by the $e_w (w \in \Gamma)$ is a right $R[W]$-submodule of E_R^i; this follows from the assumption 6.3(b). Hence E_R^i is direct sum of its right $R[W]$-submodules E_R^Γ for the various right cells Γ in $W_{(i)}$. If T is the group of translations in W, then $R[T]$ is a noetherian ring, hence E_R^i is a finitely generated right $R[T]$-module (as a subquotient of $E_R^{\geq 0}$), hence also finitely generated right $R[W]$-module. It follows that there can be only finitely many summands E_R^Γ in E_R^i. Hence $W_{(i)}$ is a union of finitely many right cells. Hence $W = \bigcup_{0 \leq i \leq \nu} W_{(i)}$ is a union of finitely many right cells and (9.10.1) follows. This proof shows also that (assuming Conjecture B), for any irreducible right $R[W]$-module V, the corresponding H_ϕ-module $\hat{V}$ has a canonical direct sum decomposition (as an R-vector space): $\hat{V}=\bigoplus_{\Gamma}\hat{V}^{\Gamma}$ where Γ runs through the right cells in $W_{(n)}$, $(n=a_V)$, and

(9.10.2) $$\hat{V}^{\Gamma}=(E_R^{\Gamma}\otimes V)_W.$$

Conjecture C. *If V is as above, then the union of all right cells $\Gamma\subset W_{(n)}$ such that $\hat{V}^{\Gamma}\neq 0$ (see (9.10.2)) is contained in a single two-sided cell $C=\mathscr{C}_V$.*

Now let G be a simple (adjoint) algebraic group over R which has Hom (T, R^*) as a maximal torus and whose Weyl group is isomorphic to W/T (the action of the Weyl group of G on the maximal torus being that induced by the action of W/T on T by conjugation.)

Using Springer's correspondence between Weyl group representations and unipotent classes, S. Kato [Kt] has attached to each irreducible $R[W]$-module V a conjugacy class in G. (All classes in G arise from some V.) Let g_V be an element in this class and let u_V be the unipotent part of g_V. (For example, when V is a generic representation or the sign representation of W than $u_V=1$; if V is the unit representation of W, then u_V is a regular unipotent element in G.)

Conjecture D. *Given two irreducible right $R[W]$-modules V, V', the following two conditions are equivalent:* (a) *the two-sided $\mathscr{C}_V$, $\mathscr{C}_{V'}$ of W (see Conjecture* C) *satisfy $\mathscr{C}_V\underset{LR}{\leq}\mathscr{C}_{V'}$,* (b) *$u_{V'}$ is contained in the closure of the conjugacy class of u_V in G. Hence there is a canonical one-to-one correspondence $\mathscr{C}_V\leftrightarrow u_V$ between the set of two-sided cells in $W_{(i)}$ and the set of unipotent classes in G. If $\mathscr{C}$ is a two-sided cell in $W_{(i)}$ and u is a corresponding unipotent element in G, then i is equal to the dimension of the variety $\mathscr{B}_u$ of Borel subgroups of G containing u.*

Conjecture E. *Let g be an element of G and let ρ be an irreducible representation of the finite group $A(g)=Z_G(g)/Z_G^0(g)$ which appears in the permutation representation of $A(g)$ on the top homology of the variety $\mathscr{B}_g$ of Borel subgroups in G containing g. Let $V_{g,\rho}$ be the irreducible $R[W]$-module associated in* [Kt, 4.1] *to (g, ρ). Then* dim $\hat{V}_{g,\rho}$ *is equal to the sum*

$$\sum_i \dim (H_{2i}(\mathscr{B}_g)\otimes\rho)^{A(g)}$$

(space of $A(g)$-invariants). More precisely, the $R[W]$-module obtained from $\hat{V}_{g,\rho}$ by letting $q^{1/2}\to 1$ is equal (in the Grothendieck group of $R[W]$-modules to the $R[W]$-module $\bigoplus_i (H_{2i}(\mathscr{B}_g)\otimes\rho)^{A(g)}$, (*see* [Kt, 3.2]).

Let T^{++} be the semigroup of translations ω in $T\subset W$ satisfying the equality $l(x\omega)=l(x)+l(\omega)$ for all $x\in W_v$ (for a fixed special point v).

With the notations in Conjecture E, we consider a complete flag of subspaces in $\hat{V}_{g,\rho}$ stable under the commutative semigroup of transformations $\tilde{T}_\omega(\omega \in T^{++})$. Given any (one dimensional) subquotient of this flag, there is a unique homomorphism $\alpha: T \to R^*$, such that $\tilde{T}_\omega$ acts on the subquotient as scalar multiplication by $\alpha(\omega)$ for all $\omega \in T^{++}$. Let s_α be the corresponding element of the maximal torus of G, (see Conjecture D). The following conjecture relates s_α to the Jordan decomposition $g=g_s \cdot g_u$ of g in G. (Here g_s is the semisimple part of g and g_u is the unipotent part.) Let χ: $\mathrm{SL}(2, \mathbf{R}) \to Z_G^0(g_s)$ be a homomorphism such that $g_u=\chi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.

Conjecture F. *The elements s_α and $g_s \cdot \chi\begin{pmatrix}\phi(q^{1/2}) & 0\\ 0 & \phi(q^{-1/2})\end{pmatrix}$ are conjugate in G.*

§ 10. Left cells and dihedral subgroups

10.1. In this section we shall give some methods which allow in certain cases to show that two elements in a Coxeter group are in the same left cell, or to construct new left cells from a given one. This method, which was inspired by Vogan's use of the "generalized τ-invariant" in [V], has been used in [KL_1, § 5] to describe the left cells of the symmetric groups. We shall generally omit proofs since they are similar to those in [KL_1, § 4].

10.2. Given the Coxeter group (W, S), we fix a subset $S' \subset S$ consisting of two elements s, t such that st has order $m<\infty$ and we denote by W' the subgroup generated by s, t. Each coset $W'w$ can be decomposed into four parts: one consists of the unique element x of minimal length, one consists of the unique element y of maximal length, one consists of the $(m-1)$ elements $sx, tsx, stsx, \cdots$ and one consists of the $(m-1)$ elements $tx, stx, tstx, \cdots$. The last two subsets are called *strings*. We shall regard them as sequences (as above) rather than subsets.

10.3. We shall extend the definition of the function $\tilde{\mu}(y, w)$ (see § 4): in the case where two elements $y, w \in W$ do not satisfy $y - w$, we set $\tilde{\mu}(y, w)=0$.

10.4. We assume that we are given two strings $x_1, x_2, \cdots, x_{m-1}$ and $y_1, y_2, \cdots, y_{m-1}$ (with respect to S'). We set

$$a_{ij}=\begin{cases}\tilde{\mu}(x_i, y_j), & \text{if } S' \cap \mathscr{L}(x_i)=S' \cap \mathscr{L}(y_j)\\ 0, & \text{otherwise.}\end{cases}$$

The integers a_{ij} satisfy a number of identities.

(10.4.1) *Assume first that* $m=3$.
Then: $a_{11}=a_{22}$ *and* $a_{12}=a_{21}$.

(10.4.2) *Assume next that* $m=4$.
Then: $a_{11}=a_{33}$, $a_{13}=a_{31}$, $a_{22}=a_{11}+a_{13}$, $a_{12}=a_{21}=a_{23}=a_{32}$.

(10.4.3) *Finally, assume that* $m=6$.
Then: $a_{11}=a_{55}$, $a_{13}=a_{31}=a_{35}=a_{53}$, $a_{15}=a_{51}$, $a_{22}=a_{44}=a_{11}+a_{13}$, $a_{33}=a_{11}+a_{13}+a_{15}$, $a_{24}=a_{42}=a_{13}+a_{15}$, $a_{12}=a_{21}=a_{45}=a_{54}$, $a_{14}=a_{41}=a_{25}=a_{52}$, $a_{23}=a_{32}=a_{34}=a_{43}=a_{12}+a_{14}$.

(In the case $m=3$, this is proved in [KL_1, § 4]. In the other cases, the proof is similar. An analogous result holds for arbitrary m.)

10.5. Note that $\{x_1, \cdots, x_{m-1}\}$ is contained in a left cell Γ. (Indeed, $x_{i-1}-x_i$ and $\mathscr{L}(x_{i-1})\not\subset\mathscr{L}(x_i)\not\subset\mathscr{L}(x_{i-1})$, hence $x_{i-1}\underset{L}{\sim}x_i$ for $i=2, 3, \cdots, m-1$.) Similarly, $\{y_1, \cdots, y_{m-1}\}$ is contained in a left cell Γ'. In certain cases it is possible to show using (10.4.1), (10.4.2) or (10.4.3) that $\Gamma=\Gamma'$. Assume for example that we know that for some i_0, j_0 we have $x_{i_0}=s_1y_{j_0}$, $s_1\in\mathscr{L}(x_{i_0})-\mathscr{L}(y_{j_0})$. Then $a_{i_0,j_0}=1$ and $x_{i_0}\underset{L}{\leq}y_{j_0}$. Using then (10.4.1), (10.4.2) or (10.4.3) we can deduce that several other $a_{i,j}$ are $\neq 0$. (In the cases $m=4$ or 6, one gets stronger conclusions if one assumes that (W, S) is crystallographic since then $a_{ij}\geq 0$ and therefore $a_{ij}\neq 0$, $a_{i'j'}\neq 0$ imply $a_{ij}+a_{i'j'}\neq 0$.) It may happen that for one of these i, j for which $a_{ij}\neq 0$ was have $\mathscr{L}(y_j)\not\subset\mathscr{L}(x_i)$; we then have $y_j\underset{L}{\leq}x_i$ and it follows that $\Gamma=\Gamma'$.

10.6. The identities (10.4.1), (10.4.2), (10.4.3) can also be used in a different way. Let Γ be a subset in W such that for any $w\in\Gamma$, $\mathscr{R}(w)\cap S'$ consists of a single element; an equivalent assumption is that for any $w\in\Gamma$, the element w^{-1} is contained in a string $\sigma_{w^{-1}}$ (with respect to S'). We then define $\Gamma^*=(\bigcup_{w\in\Gamma}(\sigma_{w^{-1}})^{-1})-\Gamma$.

In the case $m=4$ or 6, we define $\tilde{\Gamma}$ as follows. For each $w\in\Gamma$, there is a well defined number i, $1\leq i\leq m-1$ such that w^{-1} is the i^{th} element of the string $\sigma_{w^{-1}}$; we define $\tilde{w}$ to be the element such that $\tilde{w}^{-1}$ is the $(m-i)^{\text{th}}$ element of the string $\sigma_{w^{-1}}$. Then $\tilde{\Gamma}$ is the set of all $\tilde{w}$, where w runs through Γ.

Proposition 10.7. *Assume that* $m=3$ *or that* (W, S) *is crystallographic, and let* Γ, Γ^* *be as above. If* Γ *is a union of left cells, then so is* Γ^*. *More precisely, if* Γ *is left cell, then* Γ^* *is a union of at most* $(m-2)$ *left cells and, if* $m=4$ *or 6, then* $\tilde{\Gamma}$ *is a left cell.*

In the case $m=3$, this is proved in [KL_1, 4.3]. The proof in the other cases is similar. It is based on (10.4.2), (10.4.3). The hypothesis that (W, S) is crystallographic is used in the same way as in 10.5.

§ 11. Left cells in the affine Weyl groups $\tilde{A}_2$, $\tilde{B}_2$, $\tilde{G}_2$

11.1. In this section, (W, S) is an affine Weyl group of type $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$. We denote the elements of S by s_1, s_2, s_3. In the case $\tilde{B}_2$, we assume $(s_1s_3)^4=(s_2s_3)^4=(s_1s_2)^2=1$. In the case $\tilde{G}_2$, we assume that $(s_1s_3)^3=(s_2s_3)^6=(s_1s_2)^2=1$. For any subset J of $\{1, 2, 3\}$ we denote by W^J the set of all $w \in W$ such that $R(w)$ consists of the s_j, $(j \in J)$.

11.2. We shall define a partition of W into finitely many subsets, as follows.

Type $\tilde{A}_2$: $A_{13}=W^{13}$, $A_{12}=W^{12}$, $A_{23}=W^{23}$, $A_2=A_{13}s_2$, $A_3=A_{12}s_3$, $A_1=A_{23}s_1$, $B_1=W^1-A_1$, $B_2=W^2-A_2$, $B_3=W^3-A_3$, $C_\phi=W^\phi$.

Type $\tilde{B}_2$: $A_{13}=W^{13}$, $A_{12}=A_{13}s_2$, $A_1=A_3s_1$, $A_{23}=W^{23}$, $A'_{12}=A_{23}s_1$, $A'_3=A'_{12}s_3$, $A_2=A'_3s_2$, $B_{12}=W^{12}-(A_{12}\cup A'_{12})$, $B_3=B_{12}s_3$, $B_1=B_3s_1$, $B_2=B_3s_2$, $C_1=W^1-(A_1\cup B_1)$, $C_2=W^2-(A_2\cup B_2)$, $C_3=W^3-(A_3\cup A'_3\cup B_3)$, $D_\phi=W^\phi$.

Type $\tilde{G}_2$: $A_{23}=W^{23}$, $A_{12}=A_{23}s_1$, $A_{13}=A_{12}s_3$, $A'_{12}=A_{13}s_2$, $A_3=A'_{12}s_3$, $A_2=A_3s_2$, $A'_{13}=A_3s_1$, $A''_{12}=A'_{13}s_2$, $A'_3=A''_{12}s_3$, $A'_2=A'_3s_2$, $A''_3=A'_2s_3$, $A_1=A''_3s_1$.
$B_{13}=W^{13}-(A_{13}\cup A'_{13})$, $B_{12}=B_{13}s_2$, $B_3=B_{12}s_3$, $B_2=B_3s_2$, $B'_3=B_2s_3$, $B_1=B_3s_1$.
$C_{12}=W^{12}-(A_{12}\cup A'_{12}\cup A''_{12}\cup B_{12})$, $C_3=C_{12}s_3$, $C_2=C_3s_2$, $C'_3=C_2s_3$, $C_1=C'_3s_1$, $C'_2=C'_3s_2$.
$D_1=W^1-(A_1\cup B_1\cup C_1)$, $D_2=W^2-(A_2\cup A'_2\cup B_2\cup C_2\cup C'_2)$, $D_3=W^3-(A_3\cup A'_3\cup A''_3\cup B_3\cup B'_3\cup C_3\cup C'_3)$, $E_\phi=W^\phi$.

Each of the subsets in the partition is contained in some W^J, (with J indicated as a subscript.)

Theorem 11.3. *The partition of W just described coincides with the partition of W into left cells.*

(I have announced this result in a lecture at the Santa Cruz conference on finite groups in 1979. The proof was based on the techniques of strings in Section 10. However, in the case of $\tilde{G}_2$, there was a gap in the proof which I can now overcome, using results on the $\boldsymbol{a}$-function.)

We shall sketch a proof. We start with the case of $\tilde{G}_2$. By 8.5, A_{23} is a left cell. Using 10.7 with $\Gamma=A_{23}$, $S'=\{s_1, s_3\}$ we see that A_{12} is a left cell. Using 10.7 with $\Gamma=A_{12}$, $S'=\{s_2, s_3\}$, we see that $A_{13}\cup A'_{12}\cup A_3\cup A_2$ is a union of at most 4 left cells. Since $w\to\mathscr{R}(w)$ is constant on left cells, each of $A_{12}\cup A'_{12}$, A_{13}, A_3, A_2 is a union of left cells. Moreover, by 10.7 with $\Gamma=A_3$, $S'=\{s_1, s_3\}$, A'_{12} is a union of left cells. This forces each of A_{12}, A'_{12}, A_{13}, A_3, A_2 to be a left cell.

The set A'_{13} is contained in a left cell. (The proof is the same as that of 8.5 using the fact that $A'_{13}\subset W_{(\nu)}$, $\nu=6$). The set B_{13} is also contained in a left cell. (This is proved easily by the technique of strings in 10.5). However, the sets A'_{13}, B_{13} cannot be contained in the same left cell. Indeed, by 8.1 the $\boldsymbol{a}$-function is equal to 6 on A'_{13}. On the other hand we cannot have $\boldsymbol{a}(s_1s_3)=6$ since this would imply $l(s_1s_3)\geq 6$ (see 8.4(d)), a contradiction. Since $s_1s_3\in B_{13}$ and the $\boldsymbol{a}$-function is constant on left cells, it follows that A'_{13}, B_{13} are contained in distinct left cells. Since W^{13} is a union of left cells and A_{13} is a left cell, the difference $W^{13}-A_{13}=A'_{13}\cup B_{13}$ is a union of left cells. It follows that each of A'_{13}, B_{13} is a left cell.

Using 10.7 with $\Gamma=A'_{13}$, $S'=\{s_2, s_3\}$ we see that $A''_{12}\cup A'_3\cup A'_2\cup A''_3$ is a union of at most 4 left cells. It follows that each of A''_{12}, $A'_3\cup A''_3$, A'_2 is a union of left cells. Moreover, by 10.7 with $\Gamma=A''_{12}$, $S'=\{s_1, s_3\}$ the set A'_3 is a union of left cells. It follows that each of A''_{12}, A'_3, A''_3, A'_2 is a left cell.

Using 10.7 with $\Gamma=A''_3$, $S'=\{s_1, s_3\}$, we see that A_1 is a left cell.

Using 10.7 with $\Gamma=B_{13}$, $S'=\{s_2, s_3\}$, we see that $B_{12}\cup B_3\cup B_2\cup B'_3$ is a union of at most 4 left cells. It follows that each of B_{12}, $B_3\cup B'_3$, B_2 is a union of left cells. Using 10.7 with $\Gamma=B_{12}$, $S'=\{s_1, s_3\}$, we see that $B_{12}\cup B_3$ is a union of left cells. It follows that each of B_{12}, B_3, B'_3, B_2 is a left cell.

Using 10.7 with $\Gamma=B'_3$, $S'=\{s_1, s_3\}$ we see that B_1 is a left cell.

Using the technique of strings in 10.5 one can show easily that C_{12} is contained in a left cell. Since W^{12} is a union of left cells and A_{12}, A'_{12}, A''_{12}, B_{12} are left cells it follows that C_{12} is a union of left cells, hence it is a left cell.

Using 10.7 with $\Gamma=C_{12}$, $S'=\{s_1, s_3\}$ we see that C_3 is a left cell. Using 10.7 with $\Gamma=C_{12}$, $S'=\{s_2, s_3\}$ we see that $C_3\cup C_2\cup C'_3\cup C'_2$ is a union of at most 4 left cells, hence $C_2\cup C'_3\cup C'_2$ is a union of at most 3 left cells. 10.7 shows also that $C'_2=\tilde{C}_{12}$ is a left cell. Hence C'_3, C_2 and C'_2 are left cells.

Using now 10.7 with $\Gamma=C'_3$, $S'=\{s_1, s_3\}$, we see that C_1 is a left cell. Since W^1, W^2, W^3 are unions of left cells, so must be D_1, D_2, D_3. Using strings, we see easily that each of D_1, D_2, D_3 is contained in a left cell hence each of them is a left cell. The set D_ϕ is clearly a left cell. This completes the proof of the Theorem in case $\tilde{G}_2$.

We now consider the case of $\tilde{B}_2$. By 8.5, A_{13} is a left cell. Using 10.7 with $\Gamma = A_{13}$, $S' = \{s_2, s_3\}$, we see that $A_{12} \cup A_3$ is a union of at most 2 left cells. Since $w \to \mathscr{R}(w)$ is constant on left cells, it follows that both A_{12}, A_3 are left cells. Using 10.7 with $\Gamma = A_{12}$, $S' = \{s_1, s_3\}$ we see that $A_3 \cup A_1$ is a left cell. Hence A_1 is a left cell.

Since s_1, s_2 play a symmetric role, it follows automatically that A_{23}, A'_{12}, A'_3, A_2 are left cells.

Since W^{12} is a union of left cells, we see that B_{12} is a union of left cells. Using the technique of strings 10.5, we see easily that B_{12} is contained in a left cell. Hence B_{12} is a left cell. Using 10.7 with $\Gamma = B_{12}$, $S' = \{s_1, s_3\}$, we see that $B_3 \cup B_1$ is a union of at most 2 left cells. It follows that both B_3, B_1 must be left cells. Since s_1, s_2 play a symmetric role, the fact that B_1 is a left cell implies that B_2 is a left cell.

The sets C_1, C_2, C_3 are left cells by the argument used for D_1, D_2, D_3 in case $\tilde{G}_2$. The set D_ϕ is clearly a left cell.

Finally, we consider the case $\tilde{A}_2$. By 8.5, A_{13} is a left cell. Using 10.7 with $\Gamma = A_{13}$, $S' = \{s_1, s_2\}$, we see that A_2 is a left cell. By symmetry, A_{23}, A_{12}, A_3, A_1 are also left cells. The sets B_1, B_2, B_3 are left cells by the argument used for D_1, D_2, D_3 in case $\tilde{G}_2$. The set C_ϕ is clearly a left cell. This completes the proof.

11.4. Remark. I understand that recently J. Y. Shi (a student of R. W. Carter at Warwick University) has described explicitly the left cells of the affine Weyl group of type $\tilde{A}_n$.

11.5. We now consider the union of all left cells in W whose name contains a fixed capital letter; we denote this union by that capital letter. (For example, for type $\tilde{G}_2$, we have $C = C_{12} \cup C_3 \cup C_2 \cup C'_3 \cup C'_2 \cup C_1$.) Thus we have a partition into pieces:

$$W = A \cup B \cup C \text{ (for type } \tilde{A}_2), \quad W = A \cup B \cup C \cup D \text{ (for type } \tilde{B}_2),$$
$$W = A \cup B \cup C \cup D \cup E \text{ (for type } \tilde{G}_2).$$

Proposition 11.6. *The pieces in this partition of W are just the two-sided cells of W.*

We can check directly that each piece in our partition is stable under $w \to w^{-1}$ and that any left cell in a piece meets the image under $w \to w^{-1}$ of any left cell in the same piece. It follows that each piece is contained in a two-sided cell of W. The piece denoted A has the property that the $\boldsymbol{a}$-function on it has the constant value ν ($\nu = 3, 4, 6$ for $\tilde{A}_2$, $\tilde{B}_2$, $\tilde{G}_2$), (see 8.1); all other pieces contain elements of length $< \nu$ hence the value of the

$\boldsymbol{a}$-function on them must be $<\nu$, (see 8.4(d)). It follows that A is a two-sided cell. The piece denoted C (respectively, D, E) for $\tilde{A}_2$ (respectively, $\tilde{B}_2, \tilde{G}_2$) is clearly a two-sided cell. The piece denoted B (respectively, C, D) for $\tilde{A}_2$ (respectively, $\tilde{B}_2, \tilde{G}_2$) is a two-sided cell by [L_6, 3.8]. It follows that the piece B (for type $\tilde{B}_2$) is a two-sided cell and that the union $B \cup C$ (for type $\tilde{G}_2$) is a union of two-sided cells. It remains to show that $B \cup C$ (for type $\tilde{G}_2$) cannot be a single two-sided cell. This is shown as follows. Let $x = s_1 s_3 \in B$, $y = s_1 s_2 s_3 s_2 s_1 \in C$ (for type $\tilde{G}_2$). It is easy to compute $P_{x,y} = 1+q$. Ir follows that $x - y$. We have $\mathscr{L}(x) \not\subset \mathscr{L}(y)$, $\mathscr{R}(x) \not\subset \mathscr{R}(y)$. Using now 5.5, it follows that x, y belong to distinct two-sided cells. This completes the proof.

11.7. We shall describe the left cells and two-sided cells for W of type $\tilde{A}_2, \tilde{B}_2, \tilde{C}_2$ in three figures. We represent the elements of W by alcoves in E, (see 7.5): we choose a special point $v \in E$ and we attach to $w \in W$, the alcove $w \cdot A_v^- \subset E$. Then a left cell (or a two-sided cell) will be represented by a subset of E: the union of all closed alcoves corresponding to the elements in that cell. A two-sided cell is represented by the union of all closed alcoves of the same colour. If we remove from this union all facets of codimension ≥ 2, the remaining set will have finitely many connected components; the closures of these components will be the subsets of E corresponding to the various left cells.

References

[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque, vol. **100**.

[IM] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rnig of p-adic Chevalley groups, Inst. Haute Études Sci. Publ. Math., **25** (1965), 237–280.

[KP] V. G. Kac and D. H. Peterson, Infinite flag varieties and conjugacy theorems, Proc. Natl. Acad. Sci., **80** (1983), 1778–1782.

[Kt] S. Kato, A realization of irreducible representations of affine Weyl groups, Proc. Kon. Nederl. Akad., **A 86** (2) (1983), 193–201.

[KL_1] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math., **53** (1979), 165–184.

[KL_2] D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, Proc. Symp. Pure Math., vol. **36** 185–203, Amer. Math. Soc. 1980.

[L_1] G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, Adv. in Math., **37** (1980), 121–164.

[L_2] ——, On a theorem of Benson and Curtis, J. Algebra, **71** (1981), 490–498.

[L_3] ——, Unipotent characters of the symplectic and odd orthogonal groups over a finite field, Invent. Math., **64** (1981), 490–498.

[L_4] ——, Singularities, character formulas and a q-analog of weight multiplicities, Astérisque, vol. **101–102** (1983), 208–229.

[L_5] ——, Characters of reductive groups over a finite field, Ann. of Math. Studies, **107** Princeton University Press, 1984.

[L₆] ——, Some examples of square integrable representations of semisimple p-adic groups, Trans. Amer. Math. Soc., **277** (1983), 623–653.

[S] T. A. Springer, Quelques applications de la cohomologie d'intersection, Sém. Bourbaki, **589**, Fév. 1982.

[T] J. Tits, Résumé du cours, Annuaire du Collège de France 1981–1982.

[V] D. Vogan, A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra, Math. Ann., **242** (1979), 209–224.

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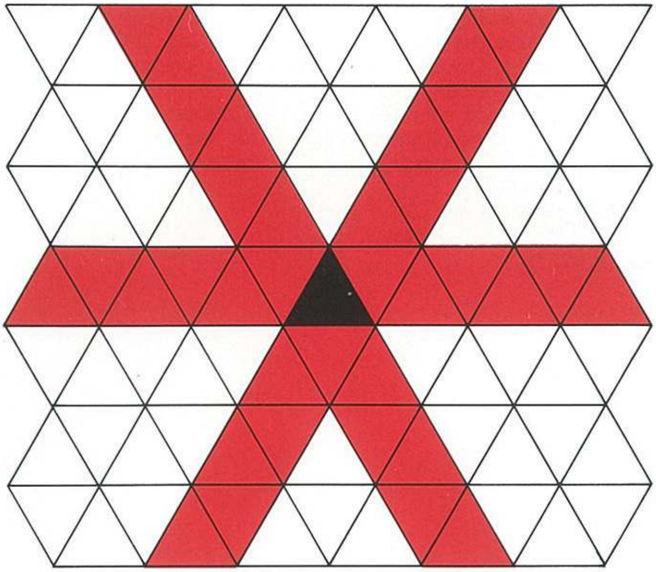

Fig. 1. $\tilde{A}_2$

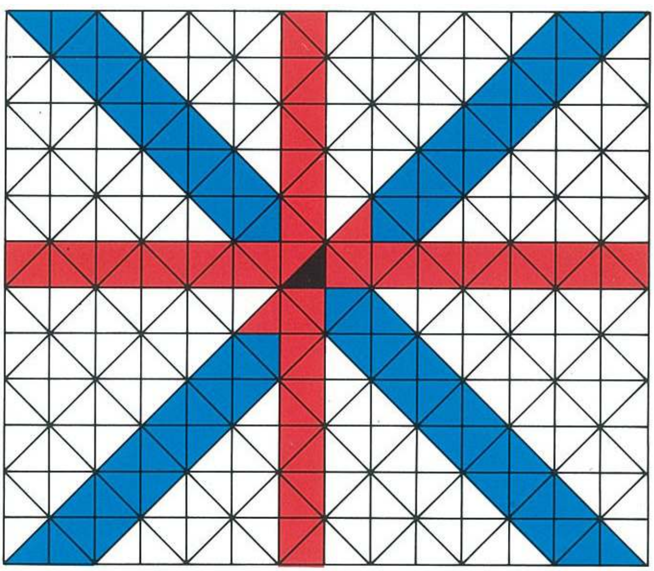

Fig. 2. $\tilde{B}_2$

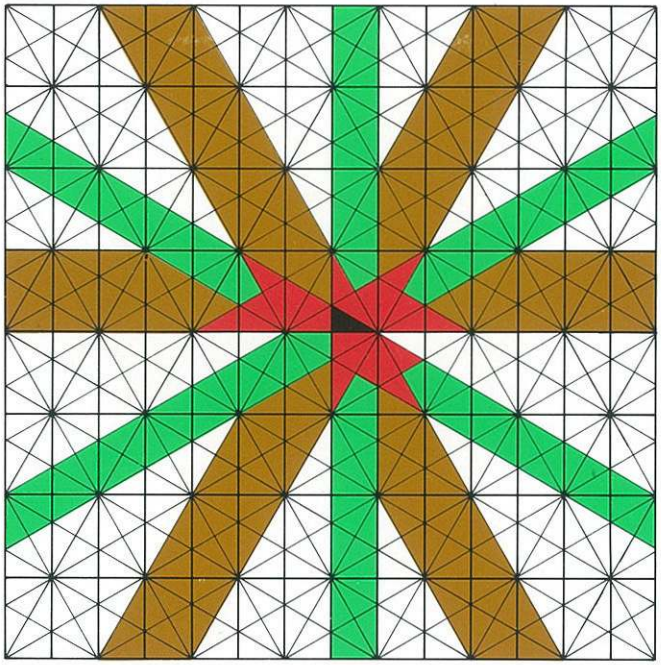

Fig. 3. $\tilde{G}_2$