

Cells in affine Weyl groups, III

Dedicated to Professor Nagayoshi Iwahori on his sixtieth birthday

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Let H_{r_0} be the Hecke algebra (over C) attached by Iwahori and Matsumoto [2] to an affine Weyl group W and to a parameter $r_0 = \sqrt{q} \in C^*$.

The simple H_{r_0} -modules have been recently classified (see [4]), when r_0 is not a root of 1, using methods from equivariant K -theory. Another (conjectural) approach to the same question, using cells in W , was given in [6]. In this paper, we shall use the results of [4] to answer some of the questions raised in [6, 9.10] concerning the relation between simple W -modules and simple modules for H_{r_0} . Our main tool is the *asymptotic Hecke algebra* \underline{J} defined in [7]; this is a C -algebra J whose structure constants are integers. (See 1.3.)

It turns out that \underline{J} contains all the algebras H_{r_0} as subalgebras (see 1.7), in such a way that the simple \underline{J} -modules restricted to any H_{r_0} form a basis for the Grothendieck group of H_{r_0} -modules of finite length, at least when r_0 is not a root of 1, or when $r_0=1$. (Theorem 3.4.)

One of the applications of our results is that for a large class of modules over the Hecke algebra there is a canonical direct sum decomposition (indexed by left cells) such that the action of the generators of the Hecke algebra is given in terms of this decomposition by a particularly simple formula (Theorem 3.8).

The results of this paper together with [4] imply the validity of several of the conjectures made in [6, 9.10]; more precisely conjecture A follows (without uniqueness of irreducible quotients), and also conjectures C , F and a variant of conjecture E . (Conjecture B in [*loc. cit.*] has been verified in [7]; conjecture D remains open.)

We shall often give references to [6], [7] to results which are proved there for ordinary affine Weyl groups and which we need for extended affine Weyl groups; the results we need for extended affine Weyl groups can be reduced trivially to those for ordinary affine Weyl groups.

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We make the convention that, unless otherwise specified, “module” means “left module”.

§ 1. The Hecke algebra and the asymptotic Hecke algebra

1.1. Let (W_1, S_1) be an irreducible affine Weyl group regarded as a Coxeter group; S_1 is the set of simple reflections. There is a unique subgroup Q of W_1 which is abelian of finite index and is maximal with these properties.

Let Ω be the group of all automorphisms of W_1 which leave S_1 stable and whose restriction to Q coincides with the restriction of some inner automorphism of W_1 . We form the semidirect product $W = \Omega \cdot W_1$ with W_1 normal. Let $l: W \rightarrow \mathbb{N}$ be the function extending the usual length function of W_1 and such that $l(\omega w_1) = l(w_1)$, $\omega \in \Omega$, $w_1 \in W_1$.

Let X be the centralizer of Q in W . It is a free abelian normal subgroup of W and $X/Q \cong \Omega$. We can find a simple reflection $s_0 \in S_1$ such that the set $S' = S_1 - \{s_0\}$ generates a finite subgroup W' of W which is complementary to X . Thus, W is a semidirect product $W' \cdot X$ with X normal.

Let w_0 be the longest element of W' .

Let $X^{++} = \{x \in X \mid l(sx) > l(x) \text{ for all } s \in S'\}$. We have

- (a) $l(w_0x) = l(w_0) + l(x)$, $l(x^{-1}w_0) = l(x^{-1}) + l(w_0)$ for all $x \in X^{++}$.
- (b) Any $x \in X$ can be written as $x = x_1x_2^{-1}$, with $x_1, x_2 \in X^{++}$.
- (c) For any $y \in W$ we can find $s_1, s_2, \dots, s_p \in S_1$ such that $y' = ys_1 \cdots s_p$ satisfies $l(y') = l(y) + p$ and $l(y's') > l(y')$ for all $s' \in S'$.

1.2. Let r be an indeterminate, and let $\underline{A} = \mathbb{C}[r, r^{-1}]$. Let \underline{H} be the Hecke algebra of W over \underline{A} , that is the free \underline{A} -module with basis $\tilde{T}_w (w \in W)$ and multiplication defined by $(\tilde{T}_s + r^{-1})(\tilde{T}_s - r) = 0$ if $s \in S_1$ and $\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'}$ if $l(ww') = l(w) + l(w')$.

Define a polynomial $P_{y,w} (y = \omega y_1, w = \omega' w_1, \omega, \omega' \in \Omega, y_1, w_1 \in W)$ to be P_{y_1, w_1} of [3] if $\omega = \omega'$ and to be 0 if $\omega \neq \omega'$.

For each $w \in W$, the element

$$C_w = \sum_y (-r)^{l(w)-l(y)} P_{y,w} (r^{-2}) \tilde{T}_y \in \underline{H}$$

is well defined, see [3].

If \underline{B} is a commutative \underline{A} -algebra with 1, we shall write $\underline{H}_B = \underline{H} \otimes_{\underline{A}} \underline{B}$;

we shall denote the images of $\tilde{T}_w \otimes 1, C_w \otimes 1$ in \underline{H}_B , again by \tilde{T}_w, C_w .

We write $C_x C_y = \sum_z h_{x,y,z} C_z \in \underline{H}$, $h_{x,y,z} \in \underline{A}$.

For each $z \in W$, there is a well defined integer $a(z) \geq 0$ such that

$$\begin{aligned} \gamma^{a(z)} h_{x,y,z} &\in C[r] && \text{for all } x, y \in W \\ \gamma^{a(z)-1} h_{x,y,z} &\notin C[r] && \text{for some } x, y \in W \end{aligned}$$

see [6, 7.3]. We have $a(z) = a(z^{-1})$ and $a(z) \leq l(w_0)$.

1.3. Let $\gamma_{x,y,z}$ be the constant term of $(-r)^{a(z)} h_{x,y,z^{-1}} \in C[r]$; we have $\gamma_{x,y,z} \in N$. Moreover,

(a)
$$\gamma_{x,y,z} \neq 0 \implies a(x) = a(y) = a(z).$$

Let \underline{J} be the C vector space with basis $(t_w)_{w \in W}$. This is an associative C -algebra with multiplication $t_x t_y = \sum_{z \in W} \gamma_{x,y,z} t_z^{-1}$. (This is a finite sum.) It has a unit element $1 = \sum_{d \in \mathcal{D}} t_d$ where \mathcal{D} is a certain finite subset of W_1 consisting of involutions. (See [7, 2.3].) For any C -algebra B with 1 we denote $\underline{J}_B = \underline{J} \otimes_C B$.

The \underline{A} -linear map $\phi: \underline{H} \rightarrow \underline{J}_A$ defined by

(b)
$$\phi(C_w) = \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ a(d) = a(z)}} h_{w,d,z} t_z$$

is a homomorphism of \underline{A} -algebras with 1, [7, 2.4].

(c) If $a(w) = i$, we have

$$\phi((-r)^i C_w) = \sum_{\substack{z \in W \\ a(z) = i}} \pi_{w,z} t_z + \sum_{\substack{z \in W \\ a(z) > i}} \pi'_{w,z} t_z$$

where $\pi'_{w,z} \in \underline{A}$, $\pi_{w,z} \in C[r]$ and the constant term of $\pi_{w,z}$ is 1 for $z = w$ and 0 for $z \neq w$, [7].

Let \underline{J}^i be the C -subspace of \underline{J} spanned by the t_w , $(a(w) = i)$.

(d) \underline{J}^i is a two sided ideal of \underline{J} , see (a), and we have clearly $\underline{J} = \bigoplus_i \underline{J}^i$.

If B is a commutative \underline{A} -algebra with 1 we shall write $\phi_B: \underline{H}_B \rightarrow \underline{J} \otimes_C B$ for the B -algebra homomorphism defined by $\phi: \underline{H}_B \rightarrow \underline{J} \otimes_C \underline{A}$.

1.4. For each $r_0 \in C^*$ we denote $\underline{H}_{r_0} = \underline{H} \otimes_{\underline{A}} C$ where C is regarded as an \underline{A} -algebra with r acting as scalar multiplication by r_0 .

Let $\phi_{r_0}: \underline{H}_{r_0} \rightarrow \underline{J}$ be the C -algebra homomorphism induced by $\phi: \underline{H} \rightarrow \underline{J}_A$.

Let $\underline{H}_{r_0}^{\geq i}$ be the C -subspace of \underline{H}_{r_0} spanned by all C_w ($w \in W, a(w) \geq i$). This is a two-sided ideal of \underline{H}_{r_0} . Let $\underline{H}_{r_0}^i = \underline{H}_{r_0}^{\geq i} / \underline{H}_{r_0}^{\geq i+1}$; this is an \underline{H}_{r_0} -bimodule. It has as C -basis the images $[C_w]$ of $C_w \in \underline{H}_{r_0}^{\geq i}$, ($a(w) = i$). Hence for $f \in \underline{H}_{r_0}^i, h, h' \in \underline{H}_{r_0}$, we have $(hf)h' = h(fh')$.

We may regard $\underline{H}_{r_0}^i$ as a \underline{J} -bimodule with multiplication defined by the rule

$$(a) \quad \begin{cases} t_x \circ [C_w] = \sum_{\substack{w' \\ a(w')=i}} \gamma_{x,w,w'}^{-1} [C_{w'}] \\ [C_w] \circ t_x = \sum_{\substack{w' \\ a(w')=i}} \gamma_{w,x,w'}^{-1} [C_{w'}], \quad (w, x \in W, a(w) = i); \end{cases}$$

this simply expresses the fact that \underline{J}^i is a two-sided ideal of \underline{J} . We have, for all $f \in \underline{H}_{r_0}^i, h \in \underline{H}_{r_0}, j \in \underline{J}$:

$$(b) \quad hf = \phi_{r_0}(h) \circ f, \quad (j \circ f)h = j \circ (fh), \quad (hf) \circ j = h(f \circ j).$$

This follows from [7, 2.4 (d)].

1.5. Note that \underline{H}_1 is naturally the group algebra of W over C . (The basis (\tilde{T}_w) of \underline{H}_1 is the standard basis of the group algebra.) Let Z be the centre of \underline{H}_1 . It is easy to see that

(a) \underline{H}_1 is finitely generated as a Z -module and Z is a finitely generated C -algebra.

1.6. PROPOSITION.

- (i) \underline{J} is a finitely generated module over its centre.
- (ii) The centre of \underline{J} is a finitely generated C -algebra.
- (iii) Any simple \underline{J} -module is finite dimensional over C .

PROOF. (i) Let Z be as in 1.5 and let Z' be the centre of \underline{J} . We first show that $\phi_1(Z)$ is contained in Z' . It is enough to show that $\phi_1(z)t_x = t_x\phi_1(z)$ for all $z \in Z, x \in W$. Assume that $a(x) = i$.

Let $f_i = \sum_{\substack{d \in \mathfrak{D} \\ a(d)=i}} [C_d] \in \underline{H}_1^i$. We have

$$(a) \quad t_x \circ f_i = f_i \circ t_x = [C_x],$$

see 1.4 (a). Since $z \in Z$, we have

$$(b) \quad zf = fz, \quad \text{for all } f \in \underline{H}_1^i.$$

We have

$$\begin{aligned} (\phi_1(z)t_x) \circ f_i &= \phi_1(z) \circ t_x \circ f_i = \phi_1(z) \circ [C_x] = z[C_x] \\ (t_x \phi_1(z)) \circ f_i &= t_x \circ (\phi_1(z) \circ f_i) = t_x \circ (z f_i) = t_x \circ (f_i z) = (t_x \circ f_i) z = [C_x] z = z[C_x] \end{aligned}$$

see 1.4 (a), (b). Hence

$$(c) \quad (\phi_1(z)t_x) \circ f_i = (t_x \phi_1(z)) \circ f_i.$$

Writing $\phi_1(z)t_x = \sum_{a(x')=i} \alpha_{x'} t_{x'}$, $t_x \phi_1(z) = \sum_{a(x')=i} \beta_{x'} t_{x'}$ ($\alpha_{x'}, \beta_{x'} \in C$), we see from (a), (c) that $\sum_{a(x')=i} \alpha_{x'} [C_{x'}] = \sum_{a(x')=i} \beta_{x'} [C_{x'}]$ hence $\alpha_{x'} = \beta_{x'}$ for all $x', a(x')=i$. Thus $\phi_1(z)t_x = t_x \phi_1(z)$, as required.

It is now enough to show that \underline{J} is finitely generated as a $\phi_1(Z)$ -module. Clearly, the left \underline{J} -module \underline{J} (left regular representation) is isomorphic to $\bigoplus_i \underline{H}_1^i$, with \underline{J} acting by $j : f \rightarrow j \circ f$. Hence it is enough to show that for each i , \underline{H}_1^i is a finitely generated $\phi_1(Z)$ -module. (We have $\underline{H}_1^i = 0$ for all but finitely many i .)

Since \underline{H}_1^i is a subquotient of \underline{H}_1 , we see from 1.5 (a) that \underline{H}_1^i is finitely generated as a Z -module; let ϕ_1, \dots, ϕ_N be generators. For any $\psi \in \underline{H}_1^i$, there exist $z_1, \dots, z_N \in Z$ such that $\psi = \sum_{i=1}^N z_i \phi_i$. By 1.4 (b) we have also $\psi = \sum_{i=1}^N \phi_1(z_i) \circ \phi_i$, hence ϕ_i are also generators of \underline{H}_1^i as a $\phi_1(Z)$ -module. This proves (i).

(ii) Z' contains $\phi_1(Z)$, a finitely generated C -algebra. Moreover, Z' is a $\phi_1(Z)$ -submodule of the finitely generated $\phi_1(Z)$ -module \underline{J} , hence Z' is a finitely generated $\phi_1(Z)$ -module. Hence, by 1.5 (a), Z' is a finitely generated C -algebra.

(iii) Let E be a simple \underline{J} -module. Since \underline{J} has countable dimension over C , a known argument of Dixmier, see [8], shows that $\text{End}_{\underline{J}}(E) = C$. Hence Z' acts on E by scalar multiplications. Using (i), it follows that the image of \underline{J} in $\text{End}_C(E)$ is a finite dimensional C -vector space.

Since E is simple, it follows that $\dim_C E < \infty$.

(d) REMARK. The previous proposition is also true for \underline{H}_{r_0} instead of \underline{J} . (Bernstein).

1.7. PROPOSITION. For any $r_0 \in C^*$, the map $\phi_{r_0} : \underline{H}_{r_0} \rightarrow \underline{J}$ is injective.

PROOF. Assume that $h \in \underline{H}_{r_0}$ is a non-zero element in the kernel of ϕ_{r_0} . We express h as a C -linear combination of basis elements \tilde{T}_w and

let $y \in W$ be an element such that \tilde{T}_y appears in h with $\neq 0$ coefficient, with $l(y)$ maximum possible. Let $y' = y s_1 s_2 \cdots s_p$ be as in 1.1 (c). Let $h' = h \tilde{T}_{s_1} \tilde{T}_{s_2} \cdots \tilde{T}_{s_p}$. Then h' is an element in the kernel of ϕ_{r_0} in which $\tilde{T}_{y'}$ appears with $\neq 0$ coefficient and $l(y') \geq l(y'')$ whenever $\tilde{T}_{y''}$ appears with $\neq 0$ coefficient in h' . We have $C_{w_0} = \sum_{z \in W'} (-r_0)^{l(w_0) - l(z)} \tilde{T}_z$.

We have $a(w_0) = l(w_0)$ and $a(w) \leq l(w_0)$ for all $w \in W$. By 1.4 (b), we have $h'f = 0$ for all $f \in \underline{H}_{r_0}^{a(w_0)}$, hence $h' \underline{H}_{r_0}^{\geq a(w_0)} \subset \underline{H}_{r_0}^{\geq a(w_0)+1} = 0$; in particular, $h' C_{w_0} = 0$. But the coefficient of \tilde{T}_{y'/w_0} in $h' C_{w_0}$ is the same as the coefficient of $\tilde{T}_{y'}$ in h' , hence is non-zero. Thus, we have $h' C_{w_0} \neq 0$, a contradiction. The proposition is proved.

1.8. Any left \underline{J} -module E gives rise, via $\phi_{r_0} : \underline{H}_{r_0} \rightarrow \underline{J}$ to a left \underline{H}_{r_0} -module E_{r_0} . We denote by $K(\underline{J})$ (resp. $K(\underline{H}_{r_0})$) the Grothendieck group of left \underline{J} -modules (resp. \underline{H}_{r_0} -modules) of finite length, or equivalently, of finite dimension over C . The correspondence $E \rightarrow E_{r_0}$ defines a homomorphism $(\phi_{r_0})_* : K(\underline{J}) \rightarrow K(\underline{H}_{r_0})$.

1.9. LEMMA. For any $r_0 \in C^*$, $(\phi_{r_0})_* : K(\underline{J}) \rightarrow K(\underline{H}_{r_0})$ is surjective.

PROOF. Let M be a simple \underline{H}_{r_0} -module. We attach to M an integer $a = a_M$ by the following two requirements:

$$\begin{aligned} C_w M &= 0 && \text{for all } w \in W, a(w) > a \\ C_w M &\neq 0 && \text{for some } w \in W, a(w) = a. \end{aligned}$$

This is well defined since $a(w)$ is bounded on W .

Let $\tilde{M} = \underline{H}_{r_0}^a \otimes_{\underline{H}_{r_0}} M$ where $\underline{H}_{r_0}^a$ is regarded as a right \underline{H}_{r_0} -module ($h : f \rightarrow fh$) and M as a left \underline{H}_{r_0} -module. Then \tilde{M} is an \underline{H}_{r_0} -module ($h : (f \otimes m) \rightarrow (hf) \otimes m$). We have a natural homomorphism $p : \tilde{M} \rightarrow M$ defined by $p(f \otimes m) = f \cdot m$ where $f \in \underline{H}_{r_0}^{\geq a}$ is a representative for $f \in \underline{H}_{r_0}^a$. This map is correctly defined, by the definition of $a = a_M$. It is clearly a homomorphism of left \underline{H}_{r_0} -modules. It is non-zero since, otherwise, $\underline{H}_{r_0}^{\geq a} M = 0$, contradicting the definition of a . Since M is simple, it follows that p is surjective. We now show that

$$(a) \quad \underline{H}_{r_0} \cdot \text{Ker}(\tilde{M} \xrightarrow{p} M) = 0.$$

Let $\sum_i f_i \otimes m_i \in \text{ker } p$, ($f_i \in \underline{H}_{r_0}^a$, $m_i \in M$) and let $\hat{f}_i \in \underline{H}_{r_0}^{\geq a}$ be representatives for f_i . Then $\sum_i \hat{f}_i m_i = 0$ in M . Let $f' \in \underline{H}_{r_0}^a$ and let $\hat{f}' \in \underline{H}_{r_0}^{\geq a}$ be a representative for f' . We have

$$\begin{aligned} f'(\sum_{\dot{i}} f_i \otimes m_i) &= \sum_{\dot{i}} (f' f_i) \otimes m = \sum_{\dot{i}} (f' f_i) \otimes m_i = \sum_{\dot{i}} (f' \otimes (f_i m_i)) \\ &= f' \otimes (\sum_{\dot{i}} f_i m_i) = 0 \end{aligned}$$

and (a) follows.

Next we show that \tilde{M} is finite dimensional over C . From 1.6 (d) we see that \underline{H}_{r_0} is finitely generated as a module over the centre of \underline{H}_{r_0} hence it is generated by, say, N elements as a right \underline{H}_{r_0} -module. The definition of \tilde{M} shows then that $\dim_C \tilde{M} \leq N \cdot \dim_C M < \infty$.

Let \tilde{M}_j be the \underline{J} -module whose underlying C -vector space is \tilde{M} and \underline{J} acts by $j: (f \otimes m) \rightarrow (j \circ f) \otimes m$. (This is well defined by 1.4 (b).) The image of \tilde{M}_j under $(\phi_{r_0})_*$ is the class of the \underline{H}_{r_0} -module with underlying C -vector space \tilde{M} and \underline{H}_{r_0} -action $h: f \otimes m \rightarrow (\phi_{r_0}(h) \circ f) \otimes m = (hf) \otimes m$ (see 1.4 (b)), hence it is just the \underline{H}_{r_0} -module \tilde{M} defined earlier. Thus, \tilde{M} is in the image of $(\phi_{r_0})_*$. From (a) we see that in $K(\underline{H}_{r_0})$, \tilde{M} is equal to M plus a sum of simple \underline{H}_{r_0} -modules M' satisfying $a_{M'} < a_M$. We may assume by induction that any M' with $a_{M'} < a_M$ is in the image of $(\phi_{r_0})_*$. Since \tilde{M} is in the image of $(\phi_{r_0})_*$, it follows that M is in the image of $(\phi_{r_0})_*$. (To begin the induction we note that if $a_M = 0$, we must have $\tilde{M} = M$.) The lemma is proved.

§ 2. Simple \underline{J} -modules

2.1. We consider a simply connected reductive algebraic group G over C with a fixed maximal torus $T_0 \subset G$ and a fixed Borel subgroup B_0 containing T_0 , such that (W', S') is identified with the Weyl group of G with respect to T_0 with simple reflections determined by B_0 , X is identified with the group of characters of T_0 ; the elements of X^{++} correspond to the inverses of the characters by which T_0 acts on the B_0 -stable lines of the various simple rational G -modules.

The complex varieties G, T_0, B_0, \dots will be generally identified with their sets of C -points.

Let K be an algebraic closure of $C(r)$. Any complex variety Z gives rise to an algebraic variety over K with set of K -points Z_K . We shall identify algebraic varieties over K with their sets of K -points. In particular $G_K, (T_0)_K, (B_0)_K, \dots$ are well defined.

2.2. We now consider a unipotent element $u \in G$; let $f: SL_2(C) \rightarrow G$ be a homomorphism of C -algebraic groups such that $f\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u$. Let

$$\begin{aligned}
M(u) &= \{(\gamma, \alpha) \in G \times C^* \mid \gamma u \gamma^{-1} = u^\alpha\} \\
M_f &= \left\{ (\gamma, \alpha) \in G \times C^* \mid \gamma f(A) \gamma^{-1} = f \left(\begin{pmatrix} \alpha^{1/2} & 0 \\ 0 & \alpha^{-1/2} \end{pmatrix} A \begin{pmatrix} \alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2} \end{pmatrix} \right) \right. \\
&\quad \left. \text{for all } A \in SL_2(C) \right\}.
\end{aligned}$$

Let (s, β) be a semisimple element in $M(u)$. Define

$$\begin{aligned}
M(u, s) &= \{(\gamma, \alpha) \in M(u) \mid \gamma s = s \gamma\} \\
M_f(s) &= \{(\gamma, \alpha) \in M_f \mid \gamma s = s \gamma\}.
\end{aligned}$$

It is known [4] that

- (a) M_f is a maximal reductive subgroup of $M(u)$.

This implies that

- (b) $M_f(s)$ is a maximal reductive subgroup of $M(u, s)$.

In particular:

- (c) $M_f(s)/M_f^0(s) \cong M(u, s)/M^0(u, s) \stackrel{\text{def}}{=} \bar{M}(u, s)$.

Let \mathcal{B} be the variety of all Borel subgroup of G with the natural $G \times C^*$ action $(\gamma, \alpha): B \rightarrow \gamma B \gamma^{-1}$. Let \mathcal{B}_u^s be the variety of all Borel subgroups of G which contain u and s . Then $M(u, s)$ (a subgroup of $G \times C^*$) leaves \mathcal{B}_u^s stable and induces an action of $\bar{M}(u, s)$ on the étale cohomology of \mathcal{B}_u^s . Let $\rho_0 \bar{M}(u, s)$ be the set of isomorphism classes of irreducible representations of the finite group $\bar{M}(u, s)$ which appear in the representation of $\bar{M}(u, s)$ on the total étale cohomology of \mathcal{B}_u^s .

2.3. Now let u be a unipotent element of G_K and let s be a semisimple element of G_K such that $sus^{-1} = u^\beta$ for some $\beta \in K^*$. We define a K -algebraic group

$$M^{(K)}(u, s) = \{(\gamma, \alpha) \in G_K \times K^* \mid \gamma u \gamma^{-1} = u^\alpha, \gamma s = s \gamma\};$$

let $\bar{M}^{(K)}(u, s)$ be the group of components of $M^{(K)}(u, s)$.

Let $(\mathcal{B}_K)_u^s$ be the variety of Borel subgroups of G_K which contain u and s . As in 2.2, $\bar{M}^{(K)}(u, s)$ acts naturally on the étale cohomology of $(\mathcal{B}_K)_u^s$ and in terms of this action we define $\rho_0 \bar{M}^{(K)}(u, s)$ to be the set of

isomorphism classes of irreducible representations of the finite group $\bar{M}^{(K)}(u, s)$ which appear in the representation of $\bar{M}^{(K)}(u, s)$ on the total étale cohomology of $(\mathcal{B}_K)_u$.

2.4. Given a semisimple conjugacy class \mathcal{C} in G_K , we define a new conjugacy class $\hat{\mathcal{C}}$ in G_K as follows. Let $s \in \mathcal{C}$; then $Z_{G_K}(s)$ has a unique open orbit on the variety of unipotent elements $u' \in G_K$ such that $su's^{-1} = u'^2$. Choose an element u in this open orbit. We can find a homomorphism of K -algebraic groups $f: SL_2(K) \rightarrow G_K$ such that $f\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u$ and $(s, r^2) \in M_f^{(K)}$. Then s commutes with $f\left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix}\right)$. We set $s_1 = sf\left(\begin{smallmatrix} r^{-1} & 0 \\ 0 & r \end{smallmatrix}\right)$. Then s and u are in $Z_{G_K}(s_1)$ and we define $\hat{\mathcal{C}}$ to be the G_K -conjugacy class of $s_1u = us_1$. One can check that $\hat{\mathcal{C}}$ is independent of the choices made; it depends only on \mathcal{C} .

2.5. Let M be a simple H_K -module. It is necessarily of finite dimension over K . We attach to M a semisimple class \mathcal{C}_M in G_K as follows. We can find a non-zero vector $\xi \in M$ and a homomorphism $\chi: X \rightarrow K^*$ such that $\tilde{T}_x\xi = \chi(x)\xi$ for all $x \in X^{++}$. (Note that the operators $\tilde{T}_x: M \rightarrow M$, ($x \in X^{++}$), commute with each other.) Then there is a unique element $s \in (T_0)_K$ such that $\chi(x) = x(s)$ for all $x \in X = \text{Hom}(T_0, K^*) = \text{Hom}((T_0)_K, K^*)$. Let \mathcal{C}_M be the conjugacy class of s in G_K . It is known that \mathcal{C}_M is an invariant of M (it is independent of the choice of ξ).

(a) Let Y be the set of isomorphism classes of simple H_K -modules M such that the conjugacy class $\hat{\mathcal{C}}_M$ in G_K contains some \mathcal{C} -point of G .

2.6. Let $V: K^* \rightarrow \mathbb{R}$ be a homomorphism such that $V(r) > 0$. An element g of G_K (or its conjugacy class in G_K) is said to be V -tempered if all eigenvalues of $\text{Ad}(g)$ on $\text{Lie}(G_K)$ (or, equivalently, all eigenvalues of g in all finite dimensional rational G_K -modules) are in the kernel of V .

An H_K -module M of finite dimension over K is said to be V -tempered (resp. V -antitempered) if all the eigenvalues λ of $\tilde{T}_x: M \rightarrow M$ satisfy $V(\lambda) \leq 0$ (resp. $V(\lambda) \geq 0$) for all elements $x \in X^{++}$.

One of the main results of [4] is:

(a) A simple H_K -module is V -tempered if and only if $\hat{\mathcal{C}}_M$ is a V -tempered conjugacy class in G_K .

(See [4, 7.12, 8.2]; these results are applicable since the field K is (non

canonically) isomorphic to C .)

For any commutative A -algebra B with unit, let $h \rightarrow h^*$ be the unique B -algebra automorphism of H_B such that

$$*\tilde{T}_s = -\tilde{T}_s^{-1}, (s \in S'), \quad *\tilde{T}_x = \tilde{T}_x^{-1}, (x \in X^{++}).$$

If M is an H_B -module, then composition with $*$ gives a new H_B -module $*M$. It is clear that

(b) $*$ interchanges V -tempered and V -antitempered simple H_K -modules.

2.7. Let $\bar{\cdot} : K \rightarrow K$ be any field automorphism inducing identity on C and satisfying $\bar{r} = r^{-1}$. Following [3], we extend $\bar{\cdot} : K \rightarrow K$ to a ring homomorphism $\bar{\cdot} : H_K \rightarrow H_K$ by $\bar{\tilde{T}}_w = \tilde{T}_w^{-1}$, ($w \in W$). For any C -variety Z , the map $\bar{\cdot} : K \rightarrow K$ induces a natural bijection $\bar{\cdot} : Z_K \rightarrow Z_K$. In particular, it induces a bijection $\bar{\cdot} : G_K \rightarrow G_K$ which is the identity on the C -points of G .

2.8. LEMMA. Let $V_0 : C(r)^* \rightarrow \mathbb{Z}$ be the homomorphism defined by attaching to a rational function its order of vanishing at $r=0$. Let $\lambda \in K^*$ be an integral element over $C[r, r^{-1}]$ such that for any valuation $V : K^* \rightarrow \mathbb{R}$ extending V_0 , we have $V(\lambda) = V(\bar{\lambda}) = 0$, ($\bar{\lambda}$ as in 2.7). Then $\lambda \in C^*$.

PROOF. We define V (as above) at $0 \in K$ by $V(0) = \infty$. Let $\lambda_1, \dots, \lambda_n$ be the conjugates of λ under the Galois group of K over $C(r)$, and let ε_i be the i -th elementary symmetric function in $\lambda_1, \dots, \lambda_n$. If V is as in the lemma we have $V(\lambda_1) = \dots = V(\lambda_n) = 0$ hence $V(\varepsilon_i) \geq 0$, so that $\varepsilon_i \in C(r)$ has no pole at 0. Similarly, we have $V(\bar{\lambda}_1) = \dots = V(\bar{\lambda}_n) = 0$ hence $V(\bar{\varepsilon}_i) \geq 0$ hence $\bar{\varepsilon}_i \in C(r)$ has no pole at ∞ . Since λ is integral over $C[r, r^{-1}]$, we see that ε_i is integral over $C[r, r^{-1}]$; this is integrally closed in $C(r)$, hence $\varepsilon_i \in C[r, r^{-1}]$. As ε_i has no poles at 0 and ∞ , it is in C . Since $\bar{\varepsilon}_i \in C$ for all i , we must have $\lambda_j \in C$ for all j and the lemma is proved.

2.9. PROPOSITION. Let M be a simple H_K -module. Assume that (a), (b), (c) below hold.

- (a) $\text{Tr}_K(\tilde{T}_w, M) \in C[r, r^{-1}]$ for all $w \in W$.
- (b) M is V -tempered for any V as in Lemma 2.8.
- (c) There exists a C -isomorphism $\bar{\cdot} : M \rightarrow M$ such that $\overline{hm} = \bar{h}\bar{m}$ for all $h \in H_K, m \in M$.

Then $M \in Y$. (See 2.5 (a).)

PROOF. From (a), it follows that $\text{Tr}_K(\tilde{T}_w^n, M) \in C[r, r^{-1}]$ for any $n \in \mathbb{Z}$.

Hence all eigenvalues of \tilde{T}_w and $\tilde{T}_w^{-1}: M \rightarrow M$ are integral over $C[r, r^{-1}]$ for any $w \in W$. Let $\xi \in M$ be a non-zero vector such that for some $s_0 \in (T_0)_K$, we have $\tilde{T}_x \xi = x(s_0) \xi$ for all $x \in X^{++}$. We then have also $\tilde{T}_x^{-1} \xi = x^{-1}(s_0) \xi$ for all $x \in X^{++}$. By the argument above, $x(s_0)$ and $x^{-1}(s_0)$ are integral over $C[r, r^{-1}]$ for all $x \in X^{++}$. Using 1.1 (b), we deduce that $x(s_0)$ is integral over $C[r, r^{-1}]$ for any $x \in X$.

By definition, we have $s_0 \in \mathcal{C}_M$. Starting with an element $s \in \mathcal{C}_M$, we define $u, f: SL_2(K) \rightarrow G_K$, s_1 as in 2.4; then $s_1 u \in \hat{\mathcal{C}}_M$. We can find an element $g \in G_K$ and a homomorphism of C -algebraic groups $f': SL_2(C) \rightarrow G$ such that $f'(A) = gf(A)g^{-1}$ for all $A \in SL_2(C)$. Hence, replacing s, u, f, s_1 by their conjugate under g , we can assume that f and u are defined over C .

We want to show that the G_K -conjugacy class of $s_1 u$ contains some C -point of G . For this, it is enough to show that for any rational G_K -module $\mathcal{C}\mathcal{V}$ of finite dimension over K , any eigenvalue $\lambda \in K^*$ of $s_1: \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$ is actually in C^* . We shall verify that λ satisfies the hypothesis of Lemma 2.8.

Since $x(s_0)$ is integral over $C[r, r^{-1}]$ for all $x \in X$, it follows that the eigenvalues of $s_0: \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$ are integral over $C[r, r^{-1}]$. Since s is conjugate to s_0 in G_K , the eigenvalues of $s: \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$ are integral over $C[r, r^{-1}]$. We have $s = s_1 f \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = f \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} s_1$ and the eigenvalues of $f \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ on $\mathcal{C}\mathcal{V}$ are clearly of form r^j ($j \in \mathbb{Z}$). Then λ is equal to r^{-j} times an eigenvalue of s on $\mathcal{C}\mathcal{V}$; hence λ is integral over $C[r, r^{-1}]$.

Now fix $V: K^* \rightarrow R$ as in Lemma 2.8. Since M is V -tempered we see from 2.6 (a) that $s_1 u$ is V -tempered, hence s_1 is V -tempered, hence $V(\lambda) = 0$.

To check that $V(\bar{\lambda}) = 0$ it will be enough to check that $\bar{\lambda}$ is an eigenvalue of $s_1: \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$.

For any $x \in X^{++}$, we have $\tilde{T}_x = \tilde{T}_{w_0} \tilde{T}_{w_0 x^{-1} w_0^{-1}} \tilde{T}_{w_0}^{-1}$. (This is equivalent to $\tilde{T}_{w_0} \tilde{T}_x = \tilde{T}_{w_0 x w_0^{-1}} \tilde{T}_{w_0}$; both expressions are equal to $\tilde{T}_{w_0 x}$, see 1.1 (a).)

If ξ, s_0 are as in the beginning of the proof, we set $\xi' = \tilde{T}_{w_0}^{-1} \xi$; using (c), we have

$$\begin{aligned} \tilde{T}_x \xi' &= \tilde{T}_x \tilde{T}_{w_0}^{-1} \xi = \overline{\tilde{T}_x \tilde{T}_{w_0} \xi} = \overline{\tilde{T}_{w_0} \tilde{T}_{w_0 x^{-1} w_0^{-1}} \xi} \\ &= \overline{(w_0 x^{-1} w_0^{-1})(s_0)^{-1} \tilde{T}_{w_0} \xi} = \overline{x(w_0(s_0)) \tilde{T}_{w_0}^{-1} \xi} = x(w_0(\bar{s}_0)) \xi' \end{aligned}$$

for all $x \in X^{++}$. It follows that $w_0(\bar{s}_0) \in (T_0)_K$ is in \mathcal{C}_M . (See 2.5.) Let $\sigma: G \rightarrow G$ be the unique automorphism of C -algebraic groups such that

$\sigma B_0 = B_0$, $\sigma T_0 = T_0$ and $\sigma(t) = w_0(t^{-1})$ for all $t \in T_0$. Then σ extends to an automorphism $\sigma : G_K \rightarrow G_K$ which commutes with $\bar{\cdot} : G_K \rightarrow G_K$. (See 2.7.) We have $\sigma(\bar{s}_0^{-1}) \in \mathcal{C}_M$. Since s_0, s are G_K -conjugate, we have also $\sigma(\bar{s}^{-1}) \in \mathcal{C}_M$.

We now apply the definition 2.4 of $\hat{\mathcal{C}}_M$ starting from $\sigma(\bar{s}^{-1})$ instead of s . Since u is in the open $Z_{G_K}(s)$ -orbit on the set of unipotents $u' \in G_K$ satisfying $su's^{-1} = u'^{r^2}$, we see that the element $\sigma(\bar{u}) = \sigma(u)$ is in the open $Z_{G_K}(\sigma(\bar{s}^{-1}))$ -orbit on the set of unipotents $u'' \in G_K$ satisfying $\sigma(\bar{s}^{-1})u''\sigma(\bar{s}^{-1})^{-1} = u''^{r^2}$.

Let ${}^{\circ}f = \sigma \circ f$. Since f is defined over \mathcal{C} , $f\left(\begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix}\right) = u$, and $(s, r^2) \in (M_f)_K$, we see that ${}^{\circ}f$ is defined over \mathcal{C} , ${}^{\circ}f\left(\begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix}\right) = \sigma(u)$ and $\sigma(\bar{s}^{-1}) \in (M_{{}^{\circ}f})_K$. We have $\sigma(\bar{s}^{-1}){}^{\circ}f\left(\begin{smallmatrix} r^{-1} & 0 \\ 0 & r \end{smallmatrix}\right) = \sigma\left(\left(\overline{{}^{\circ}f\left(\begin{smallmatrix} r^{-1} & 0 \\ 0 & r \end{smallmatrix}\right)}\right)^{-1}\right) = \sigma(\bar{s}_1^{-1})$. Hence the definition 2.4 of $\hat{\mathcal{C}}_M$ starting from $\sigma(\bar{s}^{-1})$ instead of s leads to $\sigma(\bar{s}_1^{-1})\sigma(u)$ instead of s_1u .

Since $s_1u, \sigma(\bar{s}_1^{-1})\sigma(u)$ are in the same G_K -conjugacy class $\hat{\mathcal{C}}_M$, it follows that $s_1, \sigma(\bar{s}_1^{-1})$ are conjugate in G_K . Since any semisimple element $t \in G_K$ is G_K -conjugate to $\sigma(t^{-1})$, it follows that s_1, \bar{s}_1 are G_K -conjugate. We can find a \mathcal{C} -linear isomorphism $\bar{\cdot} : \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$ such that $\overline{g\bar{y}} = \bar{g}y$ for all $g \in G_K, y \in \mathcal{C}\mathcal{V}$, and $\overline{\xi y} = \bar{\xi}y$ for all $\xi \in K, y \in \mathcal{C}\mathcal{V}$. It follows that the eigenvalues of $\bar{s}_1 : \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$ are obtained by applying $\bar{\cdot} : K \rightarrow K$ to the eigenvalues of $s_1 : \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$. Since λ is an eigenvalue of $s_1 : \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$, it follows that $\bar{\lambda}$ is an eigenvalue of $\bar{s}_1 : \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$. Since s_1, \bar{s}_1 are G_K -conjugate, $\bar{\lambda}$ is also an eigenvalue of $s_1 : \mathcal{C}\mathcal{V} \rightarrow \mathcal{C}\mathcal{V}$. Since s_1 is V -tempered we have $V(\bar{\lambda}) = 0$.

Thus, we have verified that λ satisfies the hypothesis of Lemma 2.8; it follows that $\lambda \in \mathcal{C}^*$. Hence the G_K -conjugacy class of s_1u contains some \mathcal{C} -point of G . The proposition is proved.

2.10. Let E be a simple \underline{J} -module and let \underline{B} be an \underline{A} -algebra with 1. Then $E_B = E \otimes_{\mathcal{C}} B$ is naturally a \underline{J}_B -module and it can be also regarded as an \underline{H}_B -module, via $\phi_B : \underline{H}_B \rightarrow \underline{J}_B$. Hence, for $B = K$, $*E_K$ (see 2.6) is again an \underline{H}_K -module.

2.11. PROPOSITION.

(a) *If E is a simple \underline{J} -module, then $*E_K$ is a simple \underline{H}_K -module in Y . (See 2.5 (a).)*

(b) *If E' is another simple \underline{J} -module and E'_K is isomorphic to E_K as an \underline{H}_K -module then E' is isomorphic to E as a \underline{J} -module.*

PROOF. (a) From 1.3 (d) and the fact that E is simple it follows that

there exists $i \in N$ such that $t_w E \neq 0 \implies a(w) = i$. By Burnside's theorem, the endomorphisms $t_w : E \rightarrow E$, ($a(w) = i$), span the C -vector space $\text{End}_C(E)$. Hence we can find $w_1, \dots, w_r \in W$, $r = \dim_C(E)^2$, $a(w_1) = \dots = a(w_r) = i$, such that $t_{w_1}, \dots, t_{w_r} : E \rightarrow E$ form a C -basis of $\text{End}_C(E)$. From 1.3 (c), we see that $\phi_K((-r)^i C_{w_j}) = \sum_{\substack{z \in W \\ a(z) = i}} \pi_{w_j, z} t_z$ as endomorphisms of $E \otimes_C K$, ($1 \leq j \leq r$);

moreover $\pi_{w_j, z} \in C[r]$ has constant term 1 if $z = w_j$ and 0 if $z \neq w_j$. Expressing $t_z : E \rightarrow E$ as C -linear combination of $t_{w_k} : E \rightarrow E$, we see that $\phi_K((-r)^i C_{w_j}) = \sum_{1 \leq k \leq r} \phi_{j, k} t_{w_k}$, as endomorphisms of E_K ($1 \leq j \leq r$); moreover $\phi_{j, k} \in C[r]$ has constant term 1 if $j = k$ and 0 if $j \neq k$. In particular, $\det(\phi_{j, k}) \in C[r]$ has constant term 1, so it is $\neq 0$. Thus $\phi_K((-r)^i C_{w_j}) : E_K \rightarrow E_K$ span $\text{End}_K(E_K)$ as a K -vector space. Using again Burnside's theorem we see that the \underline{H}_K -module E_K is simple; hence the \underline{H}_K -module $*E_K$ is simple.

From 1.3 (c) we see that $\phi_K((-r)^i C_w) = \sum_{\substack{z \in W \\ a(z) = i}} \pi_{w, z} t_z$ as endomorphisms of $E \otimes_C K$, ($w \in W$), $\pi_{w, z} \in C[r]$. Hence $\phi_K((-r)^i C_w)$ maps the $C[r]$ -submodule $E \otimes_C C[r]$ of E_K into itself, ($w \in W$). The same holds for $\phi_K((-r)^i \tilde{T}_w) : E_K \rightarrow E_K$, ($w \in W$) since $\tilde{T}_w \in H$ is a finite $C[r]$ -linear combination of elements $C_{w'}$. It follows that all the eigenvalues in K of $\phi_K((-r)^i \tilde{T}_w) : E_K \rightarrow E_K$ are integral over $C[r]$. We apply this to w a power x^n ($n \geq 1$) of an element $x \in X^{++}$. It is known that $l(x^n) = nl(x)$ hence $\tilde{T}_{x^n} = \tilde{T}_x^n$. Thus, all the eigenvalues of $\phi((-r)^i \tilde{T}_x^n) : E_K \rightarrow E_K$ are integral over $C[r]$. Hence if λ is an eigenvalue of $\phi(\tilde{T}_x) : E_K \rightarrow E_K$ then $(-r)^i \lambda^n$ is integral over $C[r]$ for $n = 1, 2, 3, \dots$. This implies that λ is integral over $C[r]$. Hence if $V : K^* \rightarrow \mathcal{R}$ is as in Lemma 2.8, then $V(\lambda) \geq 0$. (Note that $\lambda \neq 0$ since \tilde{T}_x is invertible in \underline{H} .) It follows that the \underline{H}_K -module E_K is V -antitempered (see 2.6) and therefore the \underline{H}_K -module $*E_K$ is V -tempered (see 2.6 (a)).

The \underline{H}_K -modules $E_K, *E_K$ come by extension of scalars from \underline{A} to K from \underline{H} -modules on the underlying \underline{A} -modules E_A . Hence $\text{Tr}_K(\tilde{T}_w, E_K)$ and $\text{Tr}_K(\tilde{T}_w, *E_K)$ are in \underline{A} for any $w \in W$.

Let $\bar{\cdot} : E \otimes_C K \rightarrow E \otimes_C K$ be the map defined by $e \otimes \xi \rightarrow e \otimes \bar{\xi}$, ($e \in E, \xi \in K$), where $\bar{\cdot} : K \rightarrow K$ is as in 2.7. The elements C_w are known to be fixed by $\bar{\cdot} : \underline{H}_K \rightarrow \underline{H}_K$. Hence the elements $h_{x, y, z} \in C[r, r^{-1}] \subset K$ (see 1.2) are fixed by $\bar{\cdot} : K \rightarrow K$. From 1.3 (b) it now follows that

$$\overline{\phi(C_w)(e \otimes \xi)} = \overline{\sum_{\substack{d \in \mathcal{D} \\ z \in W \\ a(d) = a(z)}} (t_z e) \otimes h_{w, d, z} \xi}$$

$$= \sum_{\substack{d \in \mathfrak{J} \\ z \in W \\ a(d)=a(z)}} (t_z e) \otimes h_{w,d,z} \bar{\xi} = \phi(C_w) \overline{(e \otimes \xi)} = \phi(\bar{C}_w) \overline{(e \otimes \xi)},$$

($w \in W, e \in E, \xi \in K$), hence $\overline{hm} = \bar{h}\bar{m}$ for all $h \in \underline{H}_K, m \in E_K$. We then have also $\overline{(*\bar{h})m} = *(\bar{h})\bar{m}$, ($h \in \underline{H}_K, m \in E_K$), since $\overline{(*\bar{h})} = *(\bar{h})$ for all $h \in \underline{H}_K$.

Thus we have verified that the \underline{H}_K -module $*E_K$ satisfies the hypothesis of 2.9. It follows that $*E_K \in Y$ and (a) is proved.

Now let E' be another simple \underline{J} -module such that E'_K, E_K are isomorphic as \underline{H}_K -modules.

It follows that $E'_{C(r)}, E_{C(r)}$ are isomorphic as $\underline{H}_{C(r)}$ -modules, hence $E'_{C((r))}, E_{C((r))}$ are isomorphic as $\underline{H}_{C((r))}$ -modules.

We must show that $E' \cong E$ as \underline{J} -modules.

As in the beginning of the proof, we can find $i, i' \in N$ such that $t_w E \neq 0 \implies a(w) = i$ and $t_w E' \neq 0 \implies a(w) = i'$. We may assume that $i' \leq i$.

Let $\alpha : E_{C((r))} \rightarrow E'_{C((r))}$ be an isomorphism of $\underline{H}_{C((r))}$ -modules. Replacing if necessary α by $r^k \alpha$, ($k \in \mathbb{Z}$), we may assume that α maps $\mathcal{E} = E \otimes_{\mathcal{C}} C[[r]]$ into $\mathcal{E}' = E' \otimes_{\mathcal{C}} C[[r]]$ but not into $r\mathcal{E}'$.

Hence, if we identify in the natural way $\mathcal{E}/r\mathcal{E} \cong E, \mathcal{E}'/r\mathcal{E}' \cong E'$, we see that there exists a \mathcal{C} -linear map $\bar{\alpha} : E \rightarrow E'$ such that

(c)
$$\alpha(e \otimes \xi) = \bar{\alpha}e \otimes \xi \pmod{r\mathcal{E}'}$$

for any $e \in E, \xi \in C[[r]]$.

Let $w \in W$ be such that $a(w) = i$. Since $i' \leq i$, we see from 1.3 (c) and the definition of i, i' that:

$\phi_{C(r)}((-r)^i C_w)$ maps \mathcal{E} into \mathcal{E} and \mathcal{E}' into \mathcal{E}' . Moreover, it commutes with $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$. We shall express this commutation using 1.3 (c):

$$\alpha \left(\sum_{\substack{z \in W \\ a(z)=i}} t_z(e) \otimes \pi_{w,z} \xi \right) = \left(\sum_{\substack{z \in W \\ a(z)=i}} \pi_{w,z} t_z \right) \alpha(e \otimes \xi)$$

for all $e \in E, \xi \in C[[r]]$. Using (c), this can be written as follows

$$\sum_{\substack{z \in W \\ a(z)=i}} \bar{\alpha}(t_z(e)) \otimes \pi_{w,z} \xi = \sum_{\substack{z \in W \\ a(z)=i}} t_z(\bar{\alpha}(e)) \otimes \pi_{w,z} \xi + \text{element in } r\mathcal{E}'.$$

Here $\pi_{w,z} \in C[[r]]$ has constant term 1 if $w = z$ and 0 if $w \neq z$; hence the previous equality implies

$$\bar{\alpha}(t_w(e)) \otimes \xi = t_w(\bar{\alpha}(e)) \otimes \xi + \text{element in } r\mathcal{E}'$$

hence

$$\tilde{\alpha}(t_w(e)) = t_w(\tilde{\alpha}(e)) \quad \text{for all } e \in E, w \in W, \alpha(w) = i.$$

Since $t_w : E \rightarrow E$, $(\alpha(w) = i)$ span $\text{End}_C(E)$, and $\tilde{\alpha} \neq 0$, we can find $e \in E, w \in W$ ($\alpha(w) = i$) such that $\tilde{\alpha}(t_w(e)) \neq 0$. For that e, w we have $t_w(\tilde{\alpha}(e)) \neq 0$, hence $t_w E' \neq 0$ hence $i = i'$ by the definition of i' .

From the definition of i, i' and the equality $i = i'$, we see that the equality $\tilde{\alpha}(t_w(e)) = t_w(\tilde{\alpha}(e))$, ($e \in E$), holds also if $\alpha(w) \neq i$. Hence it holds for all $w \in W$. Hence $\tilde{\alpha}$ is a J -module homomorphism. Since E, E' are simple and $\tilde{\alpha} \neq 0$, we see that $\tilde{\alpha}$ must be an isomorphism. The proposition is proved.

§ 3. The main results

3.1. Fix $r_0 \in C^*$. To any simple H_{r_0} -module M , we can attach a semi-simple element $s \in G$ (a C -point of G) well defined up to conjugacy, by the procedure of 2.5: we can find a non-zero vector $\xi \in M$ and a homomorphism $\chi : X \rightarrow C^*$ such that $\tilde{T}_x \xi = \chi(x)\xi$ for all $x \in X^{++}$; then there is a unique element $s \in T_0$ such that $\chi(x) = x(s)$ for all $x \in X = \text{Hom}(T_0, C^*)$. This is the required semisimple element. We then have a canonical decomposition

$$K(H_{r_0}) = \bigoplus_s K(H_{r_0})_s$$

indexed by a set of representatives for the semisimple classes in G , where $K(H_{r_0})_s$ is spanned by the simple H_{r_0} -modules M such that the semisimple class associated above to M is that of s .

Then $K(H_{r_0})_s$ is a free abelian group of finite rank.

If r_0 is not a root of 1, then

$$(a) \quad \text{rank } K(H_{r_0})_s = \sum_u \#(\rho_0 \bar{M}(u, s))$$

where u runs over a set of representatives for the $Z(s)$ -orbits on the set of unipotent elements $u' \in G$ such that $su's^{-1} = u'^{r_0^2}$ and $\rho_0 \bar{M}(u, s)$ is as in 2.2. This follows from [4].

The equality (a) remains true for $r_0 = 1$, by the Springer correspondence [9] for the irreducible representations of the Weyl group of $Z(s)$.

3.2. We now attach to each simple H_K -module $M \in Y$ (see 2.5 (a)) a semisimple element $s \in G$ (defined up to conjugacy). The definition of s is given in terms of a fixed $r_0 \in C^*$. Let s_1 be a semisimple element of

G , u a unipotent element of $Z(s_1)$ (both defined over C) such that $s_1u \in \hat{C}_M$. Choose a homomorphism of C -algebraic groups $f: SL_2(C) \rightarrow Z(s_1)$ such that $f\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u$ and define $s = s_1 f\left(\begin{smallmatrix} r_0 & 0 \\ 0 & r_0^{-1} \end{smallmatrix}\right)$. The conjugacy class of s in G depends only on M . The fibres of the map $M \rightarrow s$ will be denoted Y_s . Thus, we have a partition $Y = \coprod Y_s$, indexed by a set of representatives for the semisimple conjugacy classes in G .

From the results of [4] applied to H_K we see that

$$(a) \quad \#(Y_s) = \sum_f \rho_0 \bar{M}^{(K)} \left(f \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), s f_K \left(\begin{smallmatrix} r r_0^{-1} & 0 \\ 0 & r^{-1} r_0 \end{smallmatrix} \right) \right)$$

where f runs over a set of representatives for the $Z(s)$ -conjugacy classes of homomorphisms of C -algebraic groups $f: SL_2(C) \rightarrow G$ such that $(s, r_0) \in M_f$ and $f_K: SL_2(K) \rightarrow G_K$ is the natural extension of f . (See 2.2, 2.3.)

For any such f , we denote

$$\begin{aligned} u &= f \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), s_1 = s f \left(\begin{smallmatrix} r_0^{-1} & 0 \\ 0 & r_0 \end{smallmatrix} \right); \text{ then } M^{(K)} \left(u, s_1 f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) \right) \\ &= \left\{ (\gamma, \alpha) \in G_K \times K^* \mid \gamma u \gamma^{-1} = u^\alpha, \gamma s_1 = s_1 \gamma, \gamma f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) = f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) \gamma \right\} \\ &= \left\{ (\gamma, \alpha) \in G_K \times K^* \mid \gamma u \gamma^{-1} = u^\alpha, \gamma s = s \gamma, \gamma f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) = f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) \gamma \right\} = M_f(s)_K, \end{aligned}$$

(see 2.2). Hence $\bar{M}^{(K)} \left(u, s_1 f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) \right)$ may be identified with the group of components of $M_f(s)$, hence with $\bar{M}(u, s)$, (see 2.2 (c)). Moreover the variety $(\mathcal{B}_K)_u^s$ is obtained by extension of scalars from \mathcal{B}_u^s ; hence these two varieties (one over K , one over C) have the same étale cohomology; it follows that $\rho_0 \bar{M}^{(K)} \left(u, s_1 f_K \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix} \right) \right) \cong \rho_0 \bar{M}(u, s)$. Hence

$$\#(Y_s) = \sum_f \rho_0 \bar{M} \left(f \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), s \right)$$

where f is as in (a). It follows that

$$(b) \quad \#(Y_s) = \text{right hand side of 3.1 (a)}.$$

3.3. Let Y' be the set of isomorphism classes of $H_{C(r)}$ -modules M' with the properties (a), (b) below.

$$(a) \quad M' \otimes_{C(r)} K \text{ is a simple } H_K\text{-module in } Y \text{ (see 2.5 (a)).}$$

- (b) There exists an \underline{H} -submodule M'_1 of M such that M'_1 is of finite A -rank equal to $\dim_{C(r)} M'$.

Let Y'' be the set of isomorphism classes of simple \underline{J} -modules. We have natural maps

$$Y'' \xrightarrow{\alpha} Y' \xrightarrow{\beta} Y$$

defined as follows. If E is a simple \underline{J} -module, then $E_{C(r)} = E \otimes_C C(r)$ may be regarded as an $\underline{H}_{C(r)}$ -module, via $\phi_{C(r)} : \underline{H}_{C(r)} \rightarrow \underline{J}_{C(r)}$; by 2.11, we have $*E_{C(r)} \in Y'$ and we define $\alpha E = *E_{C(r)}$. If M' is an $\underline{H}_{C(r)}$ -module in Y' , then $M' \otimes_{C(r)} K$ is in Y and we define $\beta(M') = M' \otimes_{C(r)} K$.

Let $Z[Y'']$, $Z[Y']$, $Z[Y]$ be the free abelian groups with bases Y'' , Y' , Y respectively. For fixed $r_0 \in C^*$, we define homomorphisms

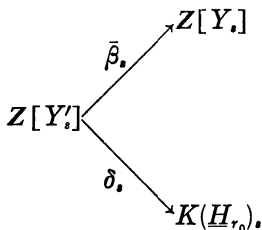
$$\begin{array}{ccccc} Z[Y''] & \xrightarrow{\bar{\alpha}} & Z[Y'] & \xrightarrow{\bar{\beta}} & Z[Y] \\ & \searrow \varepsilon & \downarrow \delta & & \\ & & K(\underline{H}_{r_0}) & & \end{array}$$

as follows: $\bar{\alpha}, \bar{\beta}$ are induced by α, β . If E is a simple \underline{J} -module we can regard E as an \underline{H}_{r_0} -module E_{r_0} via $\phi_{r_0} : \underline{H}_{r_0} \rightarrow \underline{J}$ and we define $\varepsilon(E) = *E_{r_0}$. If $M' \in Y'$, we choose M'_1 as in (b), and we let $\delta(M')$ be the class of the \underline{H}_{r_0} -module $M'_1 \otimes_A C$ where C is regarded as an A -module with r acting as multiplication by r_0 . (A standard argument shows that $\delta(M')$ is well defined, i.e., independent of the choice of M'_1 .) It is clear that $\varepsilon = \delta \circ \bar{\alpha}$.

3.4. THEOREM. *Assume that r_0 is not a root of 1, or that $r_0=1$. Then the maps $\alpha, \beta, \bar{\alpha}, \bar{\beta}, \varepsilon, \delta$ (in 3.3) are isomorphisms.*

PROOF. From 1.9 we see that ε is surjective. Since $\varepsilon = \delta \circ \bar{\alpha}$, it follows that δ is surjective.

The map β is clearly injective. Let $Y'_s = \beta^{-1}(Y_s) \cap Y'$ (s semisimple class in G), where Y_s is as in 3.2. Then we have a direct sum decomposition $Z[Y'] = \bigoplus_s Z[Y'_s]$. From the definitions it follows easily that δ (resp. $\bar{\beta}$) defines by restriction a homomorphism $\delta_s : Z[Y'_s] \rightarrow K(\underline{H}_{r_0})_s$, (see 3.1) resp. $\bar{\beta}_s : Z[Y'_s] \rightarrow Z[Y_s]$. Moreover, each δ_s is surjective, since δ is surjective. Also $\bar{\beta}_s$ is clearly injective. In the diagram



with $\bar{\beta}_s$ injective, δ_s surjective, the free abelian groups $Z[Y_s]$, $K(\underline{H}_{r_0})_s$ have the same rank. (See 3.1 (a), 3.2 (b).) It follows that δ_s is an isomorphism and $\bar{\beta}_s$ has image of finite index. Since $\bar{\beta}_s$ is induced by an imbedding $Y'_s \rightarrow Y_s$, it follows that $Y'_s \rightarrow Y_s$ is bijective. Hence $\beta: Y' \rightarrow Y$ is bijective. Since δ_s is an isomorphism for all s , we see that δ is an isomorphism.

From 2.11 (b) it follows that $\beta \circ \alpha$ is injective, hence α is injective, hence $\bar{\alpha}$ is injective. Since $\varepsilon = \delta \circ \bar{\alpha}$ with ε surjective, $\bar{\alpha}$ injective and δ bijective, it follows that ε and $\bar{\alpha}$ are both bijective. It also follows that α is bijective. The theorem is proved.

3.5. Let r_0 be as in 3.4. For any simple \underline{J} -module E we define $a = a_E \in N$ by the requirement that $\underline{J}^a E \neq 0$ (see 1.3 (d)). For any simple \underline{H}_{r_0} -module M we define $a = a_M$ as in the proof of 1.9; we define a \underline{J} -module \tilde{M}_J and an \underline{H}_{r_0} -module \tilde{M} , both with underlying C -vector space $\underline{H}_{r_0}^a \otimes_{\underline{H}_{r_0}} M$ as in the proof of 1.9.

3.6. COROLLARY. *In the setup of 3.5, the \underline{J} -module \tilde{M}_J is simple, for any simple \underline{H}_{r_0} -module M ; we have $a_M = a_{\tilde{M}_J}$. The \underline{H}_{r_0} -module \tilde{M} has M as a quotient and all its other simple constituents M' satisfy $a_{M'} > a_M$. The rule $M \rightarrow \tilde{M}_J$ defines a 1-1 correspondence between the set Z of simple \underline{H}_{r_0} -modules (up to isomorphism) and the set Y'' of simple \underline{J} -modules (up to isomorphism).*

PROOF. Let Y''_a (resp. Z_a) be the set of simple modules $E \in Y''$ (resp. $M \in Z$) such that $a_E = a$ (resp. $a_M = a$). For any \underline{J} -module E , we denote by $(\phi_{r_0})_* E$ the \underline{H}_{r_0} -module obtained from E , via $\phi_{r_0}: \underline{H}_{r_0} \rightarrow \underline{J}$.

(a) If $E \in Y''_a$, then all simple constituents of $(\phi_{r_0})_* E$ are in $\coprod_{a' \leq a} Z_{a'}$.

Indeed, we have $(\phi_{r_0})_* E = E$ as C -vector spaces. Assume that $C_w E \neq 0$ ($C_w \in \underline{H}_{r_0}$, $a(w) > a$). Then $\sum_{\substack{z \in W \\ d \in \mathcal{D} \\ a(z) = a(d) = a}} h_{w,d,z}|_{r=r_0} t_z E \neq 0$. It follows that there

exist $y, z \in W$, $a(z) = a$ such that $h_{w,y,z} \neq 0$. It is known that $h_{w,y,z} \neq 0$ implies $a(z) \geq a(w)$. We get $a > a$, contradiction. This proves (a).

(b) If $M \in Z_a$, then all simple constituents of \tilde{M}_J are in Y''_a .

Indeed, from the definition of \tilde{M}_J , it is clear that $t_w \tilde{M}_J \neq 0 \implies a(w) = a$, and (b) follows.

Let $\varepsilon_1: Z[Y''] \rightarrow Z[Z]$ be the homomorphism of group rings defined by associating to $E \in Y''$ the sum of simple constituents of $(\phi_{\tau_0})_* E$, with multiplicities. From 3.4, it follows that

(c) ε_1 is an isomorphism.

Note that ε_1 is compatible with the filtrations $\bigoplus_{a' \leq a} Z[Y''_{a'}]$, $\bigoplus_{a' \leq a} Z[Z_{a'}]$ of $Z[Y'']$, $Z[Z]$ (see (a)) hence it induces an isomorphism on the associated graded groups: $Z[Y''_a] \xrightarrow{\cong} Z[Z_a]$ for all a . In particular, it follows that

(d) If $E \in Y''_a$, then some simple constituent of $(\phi_{\tau_0})_* E$ is in Z_a .

Assume that for some $M \in Z_a$, \tilde{M}_J is not simple. From (b) and (d) it then follows that $(\phi_{\tau_0})_* \tilde{M}_J$ has at least two simple constituents in Z_a . But, as shown in the proof of 1.9, $(\phi_{\tau_0})_* \tilde{M}_J = \tilde{M}$ has exactly one simple constituent in Z_a . Thus, \tilde{M}_J is simple for all $M \in Z$. The second assertion of the corollary is proved in 1.9. It implies that the map $Z \rightarrow Y''$ defined by $M \rightarrow \tilde{M}_J$ is injective. It also implies that the restriction of ε_1 to $Z[Y''_0]$, (where Y''_0 is the image of $Z \rightarrow Y''$) is an isomorphism $Z[Y''_0] \xrightarrow{\cong} Z[Z]$. From (c) it now follows that $Y''_0 = Y''$ hence $Z \rightarrow Y''$ is bijective. This completes the proof.

3.7. Let M be a simple H_K -module such that $*M \in Y$ (see 2.5 (a)). We define a canonical direct sum decomposition $M = \bigoplus_{d \in \mathfrak{D}} M_d$ into K -linear subspaces, such that the action of the elements $C_s \in H_K$, ($s \in S_1$), on M is given in terms of this decomposition by a particularly simple formula.

By 3.4 we can find a simple J -module E such that $M \cong E \otimes_{\mathbb{C}} K$ where $E \otimes_{\mathbb{C}} K$ is regarded as a H_K -module via $\phi_K: H_K \rightarrow J_K$. We have a direct sum decomposition $E = \bigoplus_{d \in \mathfrak{D}} E_d$, $E_d = t_d E$. We define $M_d = E_d \otimes_{\mathbb{C}} K$; then $M = \bigoplus_{d \in \mathfrak{D}} M_d$.

For any $y, w \in W$, let $\mu(y, w) \in N$ be defined by $\mu(y, w) =$ coefficient of $\gamma^{l(w)-l(y)-1}$ in $P_{y, w}(\gamma^2)$, if $y < w$, $\mu(y, w) = \mu(w, y)$, if $w < y$, $\mu(y, w) = 0$,

otherwise. (See [3].) For $y, w \in W$, we shall write $y \sim_L w$ whenever y, w are in the same left cell of W .

Let $\delta', \delta \in \mathcal{D}$ be such that, for some $s \in S_1$, we have $s\delta > \delta$, $s\delta' < \delta'$. We define a K -linear map $f_{\delta', \delta} : M_\delta \rightarrow M_{\delta'}$ by

$$f_{\delta', \delta}(m) = \sum_{\substack{z \in W \\ z \sim \delta \\ L \\ z^{-1} \sim \delta' \\ L}} \mu(z, \delta) t_z m.$$

(Note that for z in the sum we have $t_z m = t_\delta t_z m$, since $z^{-1} \sim_L \delta'$, hence $t_z m \in M_{\delta'}$; by our assumption on δ', δ the sum is finite. It is independent of s .)

3.8. THEOREM. *In the setup of 3.7, for any $s \in S_1$, the action of $C_s \in \underline{H}_K$ on M is given by*

$$C_s m = \begin{cases} -(r+r^{-1})m, & \text{if } m \in M_\delta, s\delta < \delta \\ \sum_{\substack{\delta' \in \mathcal{D} \\ s\delta' < \delta'}} f_{\delta', \delta}(m), & \text{if } m \in M_\delta, s\delta > \delta. \end{cases}$$

PROOF. We have for any $m \in M_\delta$,

$$C_s m = \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ z \sim d \\ L}} h_{s, d, z} t_z m.$$

For any z in the sum we have $t_z m = t_\delta t_z m$; this is zero unless $z \sim_L \delta$, hence in our sum we can restrict ourselves to those z such that $d \sim_L z \sim_L \delta$; but it is known that two elements of \mathcal{D} are in the same left cell only if they are equal, hence:

$$C_s m = \sum_{\substack{z \in W \\ z \sim \delta \\ L}} h_{s, \delta, z} t_z m.$$

If $s\delta < \delta$, we have $C_s C_\delta = -(r+r^{-1})C_\delta$ hence $h_{s, \delta, z} = \begin{cases} -(r+r^{-1}) & \text{if } z = \delta \\ 0 & \text{if } z \neq \delta \end{cases}$ and $C_s m = -(r+r^{-1})t_\delta m = -(r+r^{-1})m$. Assume now that $s\delta > \delta$. Then $h_{s, \delta, z} = \begin{cases} \mu(z, \delta) & \text{if } sz < z \\ 0 & \text{if } sz > z \end{cases}$ (See [3].) Hence

$$C_s m = \sum_{\substack{z \in W \\ sz < z \\ z \sim \delta \\ L}} \mu(z, \delta) t_z m.$$

For any z in the sum there is a unique $\delta' \in \mathcal{D}$ such that $z^{-1} \sim_L \delta'$; we have

necessarily $s\delta' < \delta'$. Conversely, if for some z, δ' , we have $z^{-1} \underset{L}{\sim} \delta', s\delta' < \delta'$, then $sz < z$. Hence

$$C_s m = \sum_{\substack{\delta' \in \mathfrak{g} \\ s\delta' < \delta'}} \sum_{\substack{z \in W \\ z \underset{L}{\sim} \delta' \\ z^{-1} \underset{L}{\sim} \delta'}} \mu(z, \delta) t_z m = \sum_{\substack{\delta' \in \mathfrak{g} \\ s\delta' < \delta'}} f_{\delta', \delta}(m)$$

and the theorem is proved.

3.9. REMARKS. 1. We can interpret 3.8, as saying that the H_K -module M admits a W -graph in the sense of [3]. An analogous result for (finite) Weyl groups is proved in [1], using [5].

2. There is a two sided cell \mathfrak{c} of W (depending on M) such that $M_\delta \neq 0 \implies \delta \in \mathfrak{c}$.

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