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Cellular Automata in Cryptographic Random Generators

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A thesis submitted in partial fulfillment of
the requirements for the degree of

Master of Science in Computer Science

May 1, 2013

Abstract

Cryptographic schemes using one-dimensional, three-neighbor cellular automata as a primitive have been put forth since at least 1985. Early results showed good statistical pseudorandomness, and the simplicity of their construction made them a natural candidate for use in cryptographic applications. Since those early days of cellular automata, research in the field of cryptography has developed a set of tools which allow designers to prove a particular scheme to be as hard as solving an instance of a well-studied problem, suggesting a level of security for the scheme. However, little or no literature is available on whether these cellular automata can be proved secure under even generous assumptions. In fact, much of the literature falls short of providing complete, testable schemes to allow such an analysis.

In this thesis, we first examine the suitability of cellular automata as a primitive for building cryptographic primitives. In this effort, we focus on pseudorandom bit generation and noninvertibility, the behavioral heart of cryptography. In particular, we focus on cyclic linear and non-linear automata in some of the common configurations to be found in the literature. We examine known attacks against these constructions and, in some cases, improve the results.

Finding little evidence of provable security, we then examine whether the desirable properties of cellular automata (i.e. highly parallel, simple construction) can be maintained as the automata are enhanced to provide a foundation for such proofs. This investigation leads us to a new construction of a finite state cellular automaton (FSCA) which is NP-Hard to invert. Finally, we introduce the Chasm pseudorandom generator family built on this construction and provide some initial experimental results using the NIST test suite.

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This work is dedicated to my girls, without whose sacrifice it would not have been possible.

1 Introduction

"Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin."–John von Neumann.

The modern world increasingly hinges on communication. Business is ruled more and more by e-Commerce. Our computers run largely on downloaded open-source software. Students turn in their distance-learning homework through e-mail and websites. Nations attack each other's domestic infrastructure over the internet. Notice that at least one party in each of these scenarios has an interest in ensuring the secrecy and/or the authenticity of the communication. In an age where information itself becomes a prime mover, protecting that information becomes more important. So it is seems fair to say that *secure* communication is increasingly critical in our daily life. And though security requires an array of solutions to many challenging problems, certainly good cryptography is one cornerstone.

The foundation of much of modern security and cryptography is good random number generation. Pseudorandom generators (PRGs) are used for exchanging session keys, creating public/private key pairs, creating symmetric keys from user input, and generating nonces and initialization vectors for various modes of encryption. Good PRGs are also at the heart of many cryptographic primitives such as stream ciphers, block ciphers, and hash functions. PRGs used in this setting must be carefully designed lest they compromise the entire cryptosystem. Even the best crypto primitives become useless when operated with poorly generated pseudorandom data. We mention the cases of Netscape [15], Kerberos [30], the GSM wireless network [6], and the Sony PlayStation 3 [13] as evidence. These were all failures in seeding PRGs or generating random values. While block ciphers and the like get much of the attention, these deterministic components are almost boring when not fed sufficiently random data.

Randomness is crucial to many activities besides cryptography. Monte Carlo simulations allow mathematical modeling of systems and functions that are too complicated or expensive to solve directly. Such a simulation gives us a statistically qualified numerical value by evaluating the system or function at a number of randomly chosen inputs. This random selection is often done with a pseudorandom generator for reasons of cost and speed. Monte Carlo methods are a mainstay in fields as varied as computational physics, financial modeling, numerical optimization, and a number of engineering disciplines.

In fact, it was the application of Monte Carlo methods to integrated circuit

(IC) testing in the 1980's that drove a body of research to find better and cheaper pseudorandom bits [4, 10, 14, 16, 18, 36, 41]. Because of the combinatoric nature of the possible failures in an IC, most problems in this kind of testing are NP-Complete. Monte Carlo methods offered a path to high-confidence test results without the need to wrestle with these combinatoric problems. The common solution was to add some circuitry to accept static test vectors from a test fixture and produce some sort of pass/fail check inside the IC. In addition, a separate mode was supported where a short seed sent to the IC would generate a much longer yet deterministic sequence of pseudorandom test vectors. These would be run and the same pass/fail check performed. This cut down transmitted data to the IC and greatly increased the speed of the test.

Since this circuitry was on the IC itself but was not directly valuable to the customer, there was pressure to reduce the resources it consumed–transistors, connection lengths, CMOS area, etc. The PRG used had traditionally been a linear feedback shift register (LFSR), but in the late 1980's researchers began to compare LFSRs with one-dimensional cellular automata (CA). After understanding some behavioral basics such as how to achieve a maximal period, experiments showed that CA gave better test coverage with lower cost in ICs than LFSRs [14, 18]. Their abilities as PRGs were established.

It was natural then to investigate CA as primitives for cryptographic schemes, and through the 1990's and 2000's, many such schemes were put forward. As is often the case with cryptographic schemes, many have been broken and none have really garnered widespread attention. This may be due in part to the lack of specifics in many of these schemes about key scheduling and initialization that would facilitate implementation and focused cryptanalysis.

If so, it would be somewhat ironic since this same period has seen the opening and continued growth of standards-based algorithm selection (for AES, SHA-1, 2, & 3, and eStream), governing body standardization, and the increasing popularity of provable security in cryptographic theory. This last area began in the early 1980's as the application of the techniques from complexity theory to problems in cryptography, even in concrete situations of constant size. Proofs concerning cryptographic primitives and protocols are given in relation to mathematically precise definitions and (hopefully) minimal assumptions about hardness. Proofs of protocol security are often provided as reductions from breaking the primitive they employ. This allows assumptions to be minimal and explicit so that effort spent on cryptanalysis can be focused and re-used for these primitives.

In this thesis, we will attempt to bring a provable security approach to cryptography based on CA. Specifically, we will consider the ability of CA constructions presented in the literature to act as cryptographic PRGs since PRGs are fundamental to all of cryptography. As we will see, this ability depends almost entirely on the noninvertibility of CA. The remainder of this section gives the necessary definitions and criteria for this evaluation. In Section 2 we present an overview of CA and related terminology then review some important schemes and seminal breaks of CA constructions from the literature. Section 3 presents new analysis of CA using non-linear rules, including a new algorithm to invert certain non-linear rules in two-state, three-neighbor cyclic CA as well as a proposed algorithm to derandomize cryptanalysis of non-linear CA when an output sequence over time from a single cell is known. Unconvinced of the suitability of simple CA for use as primitives in secure cryptographic systems, we then examine how they may be enhanced to allow for proofs of hardness and, eventually, security. A new construction called finite state cellular automaton which adds a minimal amount of complexity to cells is shown to be provably hard to invert in Section 5. We then use this theoretical construction to build the Chasm family of concrete PRGs and give some experimental results in Section 6.

1.1 Definitions of Security

Let us first make clear the context in which we consider PRGs and exactly what we mean by that term. Randomness in modern cryptography begins with Shannon's information theory [42], where randomness is something of a measure of the lack of information. Kolmogorov (and Chaitin) added the notion of descriptive complexity [21], which classifies a string to be no more random than the program required to generate it. Both of these notions allow us to quantify randomness, but when we do so we find "perfect" randomness only at extreme, theoretical limits. These concepts do not help us create or assess practical randomness.

A third notion defines randomness relative to an observer. A string is "random" to an observer if it cannot be distinguished from a truly random string within the bounds of that observer's computational resources. This in a sense defines randomness as the extent to which the observer is unable to compute any meaningful information from a string. Since this is a different concept from truly random, we instead use the term *pseudorandom*. The observers we are concerned with are algorithms whose running time are bounded in some way.

Pseudorandom generators were the first primitives to be defined using this notion

of pseudorandomness. The original contemporaneous definitions are due to Yao [52] and also Blum and Micali [8]. We give a more common, modern definition.

Definition 1.1. A deterministic polynomial-time algorithm $G : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ is a *pseudorandom generator* with stretching function $\ell : \mathbb{N} \to \mathbb{N}$ such that $\ell(n) > n$ if for any probabilistic polynomial-time algorithm A, for any positive polynomial p:

$$\left|\Pr\left[A(G(U_n)) = 1\right] - \Pr\left[A(U_{\ell(n)}) = 1\right]\right| < \frac{1}{p(n)}$$

for all sufficiently large n where U_k is a k-bit string drawn uniformly at random from $\{0,1\}^k$ and the probabilities are taken over the respective U_k and over the coins of A.

Use of the term pseudorandom in this thesis should be assumed to imply this notion of computational indistinguishability. Pseudorandomness subject to a fixed set of statistical tests will be referred to as *statistical pseudorandomness*.

Using this definition, we can see that a statistical test against the output of a generator G is simply a special kind of distinguishing algorithm A. While we can run a battery of statistical tests, we still cannot truly satisfy this definition without some assumption on the hardness of inverting G. If G were easy for A to invert, A could:

- Assume its input I_A is the output of G, and invert G for that output, to arrive at an assumed input I_G.
- Run $G(I_G)$ and compare the result to I_A. If they are the same, A outputs a 1.

This algorithm would let A distinguish the outputs of G quite easily and so G would not be a pseudorandom generator as defined. Thus, some notion of noninvertibility or one-wayness of G is essential. We define this notion as follows:

Definition 1.2. A function $f : \{0,1\}^* \to \{0,1\}^*$ is said to be *one-way* if f is polynomial-time computable and for every probabilistic polynomial-time algorithm A, for any positive polynomial p:

$$\Pr_{x \in \{0,1\}^n}\left[f(A(f(x))) = f(x)\right] < \frac{1}{p(n)}$$

for all sufficiently large n where the probability is taken uniformly over the choices of x and the coins of A.

It turns out that this notion is the key to PRGs. Impagliazzo, Levin, and Luby prove in [19] that the existence of one-way functions is a necessary and sufficient

condition for the existence of PRGs. Note that f must be generally hard to invert as the probabilities are taken over all x. This is juxtaposed with NP-Complete problems, where having occasional strings which cannot be decided in polynomial-time is sufficient to consider the whole language "hard." Note also that by this definition, A need not compute x exactly, just any pre-image of $f(x)$. If this pre-image is not unique the function is noninvertible by some definitions of that term. To be clear, we will use noninvertible to mean one-way and inversion to mean finding any pre-image of $f(x)$.

Other properties of PRGs are very useful in practice, especially in scenarios where the internal state of the generator (especially software-based generators) may become known to an attacker. In this scenario, the attacker may be able to predict future pseudorandom outputs and/or recreate past outputs. If the generator is shared among users (as is /dev/random on Linux systems), and if those outputs are used in another user's cryptosystem (as a key or initialization vector), that system can easily be compromised. Security against this scenario is captured by the following properties:

Backward-Secure: Future outputs are secure against a compromise of the internal state of the generator which occurred in the past. Equivalently, given the current internal state of the generator, an attacker is unable to predict future outputs with non-negligible success. As PRGs are deterministic algorithms, this property is difficult for a generator to display intrinsically, and is usually achieved only be external re-seeding of the generator.

Forward-Secure: Previous outputs are secure against a compromise of the internal state of the generator which may occur in the future. Given the current internal state, an attacker is unable to guess past outputs with non-negligible success. This implies the PRG's function is effectively one-way.

The notion of forward security was first formalized by Bellare and Yee [5], and is also applied to symmetric encryption schemes regarding key compromise. As seen in our example of shared /dev/random, this property is very powerful, and is to be expected in modern PRGs.

We can see that we need noninvertibility to guarantee pseudorandom behavior going forward, and we also need it to protect previous outputs. Thus noninvertibility is a must-have property of any primitive used as a cryptographic PRG.

2 Cellular Automata

A Cellular Automaton is a discrete time and space dynamical system consisting of an array of cells, each of which implements a (usually simple) automaton. These cells can be arranged over one or two dimensions and connected to their neighbors in various configurations. The cells use the state of their neighbors and their own state to decide a next state to transition to. In some cases, very simple configurations of cells using very simple rules display surprisingly complex behavior.

Cellular Automata were first proposed by John von Neumann while working at Los Alamos on the problem of building self-replicating systems[1]. His 1953 construction had 200,000 cells over two-dimensions where each cell used 29 states to model various operations of the robot. Stanislaw Ulam picked up this concept again in the 1960's with work on recursively-defined geometrical objects. He noted that in two-state two-dimensional CA with simple rules, a single non-conforming cell generated complex patterns which may model biologic interactions. John Conway experimented further with connected cells using simple rules and developed "The Game of Life," which popularized two-state two-dimensional CA. Many other constructions of one and two dimensions are useful for a host of applications too numerous to mention.

Pseudorandom generators and cryptography using CA flow from work by Stephen Wolfram in the 1980s and the response to it. This is the trail we will follow.

2.1 Definitions

Formally, a CA is a vector $F = \langle f_1, f_2, \ldots, f_n \rangle$ of functions $f_i : \{0,1\}^N \mapsto \{0,1\}$ for some $N \in \mathbb{N}$. Each f_i is evaluated at some discrete time t to produce a vector $S^{(t)} = \langle s_1^{(t)}, s_2^{(t)}, \ldots, s_n^{(t)} \rangle$ of values $s_i^{(t)}$. The values s_i are also known as the *state* (in the sense of stored value) of cell i, and so S is sometimes called the *state vector*. When each value $s_i^{(t)} \in \{0,1\}$, the CA is referred to as a *two-state* CA. Each f_i is traditionally called the *transition function* of the cell (as in Finite State Automata). The N inputs to cell i normally come from its own output at the previous time step as well as $(N-1)/2$ of its immediate neighbors on either side. Thus, in an N-*neighbor* CA, the value of cell i at time $t+1$ is defined by

$$s_i^{(t+1)} = f_i(s_{i-(N-1)/2}^{(t)}, \ldots, s_i^{(t)}, \ldots, s_{i+(N-1)/2}^{(t)})$$

[1]See [51] and [38] for a more complete history.

For convenience, we will denote a neighborhood of values at time t from cell i to cell j inclusive with $S_{i:j}^{(t)}$, and likewise for neighborhoods of functions.

To determine what happens at the outer-most cells, a CA must specify an input for the missing neighbor. A *null-boundary* CA provides a constant 0 for these neighbors, while a *cyclic-boundary* CA provides the value from the outer-most cell from the opposite end of the array. We will be concerned only with cyclic-boundary CA.

Two-state, 3-neighbor, cyclic-boundary CA are of particular interest in the literature. This is the simplest configuration shown to have complex behavior, depending on F. Most authors refer to the transition function of the CA as the "rule," following the numbering convention of Wolfram[50]. Rules are numbered by considering the output bits for each of the 2^N possible inputs as a binary number, then interpreting that number in decimal. For example, a function that, on input strings of $111, 110, \ldots, 000$, produces output bits $0, 0, 0, 1, 1, 1, 1, 0$ respectively is called rule 30. Using this scheme, the vector F will be referred to as the *rule set* or the *rule vector*.

We will often work in standard Boolean algebra to describe these rules, using $\cdot$ or concatenation to indicate AND, $+$ for OR, and $\oplus$ for XOR. Rules using only XOR giving f the form $f(x_1, x_2, x_3) = a_0 \oplus a_1 x_1 \oplus a_2 x_2 \oplus a_3 x_3$ we define as *affine* in GF(2). Affine rules having $a_0 = 0$ are called linear. For a linear rule, complementing one of the inputs is equivalent to setting $a_0 = 1$, and so affine rules are linear rules with an additive offset (as in other domains). This leads to the occasional use of the term *additive* as a synonym for affine. Rules that are not linear are called *nonlinear*. Though all rules have three arguments by definition in a 3-neighbor CA, not all rules make use of all inputs. We will describe those rules that make use of only two inputs as *binary* and those that use all three as *ternary*. The rules that have attracted the most attention are:

$$
\begin{aligned}
\text{rule 30:} \quad s_i^{(t+1)} &= s_{i-1}^{(t)} \oplus (s_i^{(t)} + s_{i+1}^{(t)}) \\
\text{rule 150:} \quad s_i^{(t+1)} &= s_{i-1}^{(t)} \oplus s_i^{(t)} \oplus s_{i+1}^{(t)} \\
\text{rule 90:} \quad s_i^{(t+1)} &= s_{i-1}^{(t)} \oplus s_{i+1}^{(t)} \\
\text{rule 105:} \quad s_i^{(t+1)} &= 1 \oplus s_{i-1}^{(t)} \oplus s_i^{(t)} \oplus s_{i+1}^{(t)} \\
\text{rule 165:} \quad s_i^{(t+1)} &= 1 \oplus s_{i-1}^{(t)} \oplus s_{i+1}^{(t)}
\end{aligned}
$$

These and some slight variations will be the focus of our analysis.

Even such a small set of rules still allows a variety of options in constructing CAs. A *uniform* CA is one where each cell applies the same rule at each time step. In such

cases, the rule vector may be denoted F_r to indicate that Wolfram rule r is used for each cell. Alternatively, a *hybrid* CA may assign different rules to different cells. A hybrid CA where the rules are symmetric about a center cell will be referred to as *symmetric*; those hybrid CA without this property are called *asymmetric*. Uniform and hybrid CAs keep the assigned rules constant across time steps. CAs which vary a given cell's rule over time according to some scheme are called *programmable*. A CA (uniform or hybrid) which uses only linear rules is known as a *linear CA*, and likewise for nonlinear rules.

The use of XOR on just a single input can be of critical importance. A rule f is said to be *left-toggle* if the following property holds for all $S \in \{0,1\}^3$:

$$1 \oplus s_i^{(t+1)} = f(1 \oplus s_{i-1}^{(t)}, s_i^{(t)}, s_{i+1}^{(t)})$$

A rule is *right-toggle* if instead

$$1 \oplus s_i^{(t+1)} = f(s_{i-1}^{(t)}, s_i^{(t)}, 1 \oplus s_{i+1}^{(t)})$$

holds. Rules combining the left or right bits with XOR will be left- or right-toggle, respectively. It will be important in the analysis of toggle rules to have the following lemma.

Proposition 2.1. *Let $S \in \{0,1\}^n$ be the state vector of an n-cell CA, $f : \{0,1\}^3 \to \{0,1\}$ be a left-toggle rule, and $g : \{0,1\}^3 \to \{0,1\}$ be a right-toggle rule. After applying f to cell i at time t, $s_{i-1}^{(t)} = f(s_i^{(t+1)}, s_i^{(t)}, s_{i+1}^{(t)})$. Similarly, after applying g to cell i at time t, $s_{i+1}^{(t)} = f(s_{i-1}^{(t)}, s_i^{(t)}, s_i^{(t+1)})$.*

Proof. We first address the case of a left-toggle rule, and consider

$$(s_i^{(t+1)} \oplus s_{i-1}^{(t)}) \oplus s_i^{(t+1)} = f((s_i^{(t+1)} \oplus s_{i-1}^{(t)}) \oplus s_{i-1}^{(t)}, s_i^{(t)}, s_{i+1}^{(t)}). \tag{1}$$

Either $s_i^{(t+1)} = s_{i-1}^{(t)}$ or $s_i^{(t+1)} \neq s_{i-1}^{(t)}$. In the first case, we have

$$0 \oplus s_i^{(t+1)} = f(0 \oplus s_{i-1}^{(t)}, s_i^{(t)}, s_{i+1}^{(t)})$$

which is just the definition of f. But since $s_i^{(t+1)} = s_{i-1}^{(t)}$, we can exchange these values and write (1) as

$$s_{i-1}^{(t)} = f(s_i^{(t+1)}, s_i^{(t)}, s_{i+1}^{(t)})$$

which is the identity we seek. In the second case, we have

$$1 \oplus s_i^{(t+1)} = f(1 \oplus s_{i-1}^{(t)}, s_i^{(t)}, s_{i+1}^{(t)}).$$

Since $s_i^{(t+1)} \neq s_{i-1}^{(t)}$, $1 \oplus s_i^{(t+1)} = s_{i-1}^{(t)}$ and $1 \oplus s_{i-1}^{(t)} = s_i^{(t+1)}$. Substituting these identities gives us

$$s_{i-1}^{(t)} = f(s_i^{(t+1)}, s_i^{(t)}, s_{i+1}^{(t)})$$

This proves the Lemma for all f. The proof for right-toggle rules is similar. ■

Thus, a left-toggle rule allows us to substitute the rule output for the left neighbor in order to solve for that left neighbor. The reflected observation holds for right-toggle rules as well.

A *temporal sequence* is a sequence of the output values of a single cell taken over multiple time steps. We say a CA *produces* a temporal sequence (or just sequence) σ_i at cell i if the values s_i over time match the elements of σ_i. Of particular interest in the literature is the central temporal sequence, used as a pseudorandom stream in many CA-based constructs. Unless otherwise noted, we assume n is odd to make clear which is the central temporal sequence. The *right-adjacent* sequence of a temporal sequence is the temporal sequence one position to its right, and similarly for the *left-adjacent* sequence.

Finally, it bears mentioning that cellular automata can do some funny things. We refer to the state vector at time step $t = 0$ as the *initial state* and also the *seed*, a term frequently used for the initial input to random generators. Some CA are not capable of generating their initial state at a later time step. Such a state vector is known as a *garden of Eden* state as it can only exist in the beginning. Some CA evolve into a single, fixed state (usually all 0s), which is known as a *dead-end state* for that CA.

2.2 A Brief Overview of Research on Pseudorandom Generation with CA

CA entered the cryptography and random number generation domains with Stephen Wolfram's claims about rule 30. In [50], he proposed a uniform n-cell cyclic arrangement using rule 30 with the temporal sequence of the center cell used as a random stream. The periods of sequences produced depend on the number of cells, but also on the initial seed. The maximal period was estimated to be $2^{0.61(n+1)}$. This generator was shown to pass a suite of 7 statistical tests, performing best when the sequence was

much shorter than the period of the CA. The CA performed better than an LFSR of the same size, but not as well as a linear congruential generator or the bytes of $\sqrt{2}$, e, or π.

Hortensius, et. al. [18] compared this rule 30-based generator with a hybrid configuration using rules 90 and 150, based on work in [36], and also with traditional Linear Feedback Shift Registers (LFSRs). The focus of these experiments was generation of test vectors in VLSI manufacturing where layout concerns are important. They showed that both kinds of CA performed better than LFSRs in statistical tests. The null boundary hybrid CA had on average longer periods than cyclic boundary uniform rule 30 CAs. They also catalog the configuration of rule sets which produce maximal periods in the hybrid configurations, along with periods for other configurations tested. These results were arrived at through exhaustive search.

Serra, et. al. took up the question of how to synthesize a null-boundary linear CA given a primitive polynomial in [41]. Given a linear CA, each new state vector can be described as a linear system, and so can be represented as a matrix operation over GF(2). This matrix is called the transition matrix of the automaton. The authors first establish an isomorphism between the transition matrix of an LFSR with that of a CA by showing they are similar (i.e. have the same characteristic polynomial) and so describe the same linear transform under different bases. They then give an algorithm to produce the transition matrix for a null-boundary linear CA using only rules 90 and 150 (from which the rule set is clear) given a characteristic polynomial. While this algorithm is based on searching a space of $2^{\lfloor n/2 \rfloor}$ vectors, later algorithms in [10] and [11] improve these results. The first solves a quadratic congruence on subpolynomials using Euclid's algorithm in a finite field, the second uses a revised Lanczos tridiagonalization method in GF(2) to find one of two possible CA for the given polynomial.

These results are only for null-boundary linear CA. Bardell in [4] shows that the outputs of linear CA and LFSRs are identical when a phase shift between output bit sequences is accounted for. Bardell also conjectures that no *cyclic*-boundary linear CA has maximal period. This conjecture is proved by Nandi in [32], who reported that the characteristic polynomial of any transition matrix using cyclic boundary conditions is factorizable, and so cannot be primitive. Nandi also claims that periodic boundary linear CA provide better statistical randomness than null boundary, due to the fixed 0s at the ends. This claim is supported by [43].

Nandi, et. al. examine the group behavior of hybrid CAs over rules 51, 153, and 195 with null boundary conditions in [33] and show these rules lead to CA which

are even permutations. By combining several rule sets in a programmable CA, they create transformations which generate an alternating group of even permutations. These transformations then become primitives on which they base block and stream ciphers.These cryptosystems were broken in [7], which showed that the groups formed are actually a subset of the affine group, not the alternating group, of degree n and therefore are easily recreated with sufficient plaintext/ciphertext pairs. This was improved in [29] using ciphertext only.

In the same year, Sipper and Tomassini gave a genetic algorithm approach to evolving a single "good" CA by using an entropy metric and introducing mutations of rule changes randomly. They looked at 3-neighbor, two-state, cyclic hybrid CA with $n = 50$ over 300 random initial states run for 4096 steps. The entropy of each cell is computed over time, and with probability 0.001, cells would mix rules with a neighbor by swapping the neighbor rule's output value assigned to certain input combinations. Two resulting CA (a mixture of rules 165, 90, and 150 in one case and rules 165 and 225 in the other) were compared to a uniform rule 30 CA and a hybrid 150/90 CA over 4 statistical tests, showing favorable results.

The approach of evolving CA gave rise to a series of papers. See [17, 40, 45, 46]. Most of these end up focusing on the main 4 linear rules, 150, 105, 90, and 165.

In [49], Wolfram proposed a cryptosystem based on his rule 30 temporal sequence random generator from [50] as a key stream with which the plaintext could be XORed. This scheme was broken by Meier and Staffelbach in [28] . The authors first showed that the temporal sequence is not hard to recover in the case of known plaintext attacks. Then, given this temporal sequence, they showed that the left half of the CA's computational history was uniquely determined if the right-adjacent sequence could be guessed due to the left-toggle property of rule 30. Further, the right-adjacent sequence could be determined by guessing the right half of the seed and running the CA forward for $n/2$ time steps. They also showed that not all right halves of the seed are equiprobable, so far fewer than $n/2$ guesses are required. For $n = 300$, for instance, a probabilistic algorithm requires 18.1 bits of entropy to recover the seed with a probability of 0.5. This algorithm remains a seminal one in cryptanalysis of CA cryptosystems, and one which we will seek to improve.

In [20], Koç and Apohan investigate a claim made by Wolfram that recovering a seed value given a sequence of states over an n-cell automaton using rule 30 was NP-Complete. Koç and Apohan present an inversion algorithm which finds the best affine approximation of the transition function and then solves an n-variable Boolean linear equation to get a good approximation $S^*$of $S^{(t-1)}$. It then checks the affine

approximation by using S^* to re-compute $S^{(t)}$. If there are errors, the algorithm resorts to a search for combinations that correct them. Using this algorithm, Koç and Apohan shows that rule 30 can be inverted in time $O(n)$ for some seeds and $O(2^{n/2})$ worst case. We will improve this result.

Sen et. al. [39] present a (rare) fairly complete description of an entire cryptosystem based on CAs. The algorithm is multistage, with some key management and both linear and non-linear rules applied to the plaintext. The system was broken by Bao in [2] using only hundreds of chosen plaintexts and very little computation. Bao presents an equivalent transform and searches a small space of one of the parameters of the cryptosystem to decrypt any ciphertext with probability 0.5.

A series of papers propose using programmable CAs, wherein the rule vector F applied at each time step can be controlled by external circuitry. These schemes usually focus on 2 or 4 of the linear rules (150, 105, 90, 165). This seems to do very well in statistical testing, but makes little or no claims about use in cryptography. See [16].

Shin et. al. [43] analyzed the conditional probability distributions of different combinations of binary operations and showed that only XOR produces "cryptographic" PRGs, since all other combinations are skewed. They show that 64-cell hybrid CA based on rules having uniformly distributed outputs pass almost all tests from the Diehard [23] statistical test battery.

3 Analysis of Non-Linear CA

When viewed from a computational complexity perspective, it's not clear that a 2 state, 3 neighbor CA is capable of hardness at all. Any resolution to this question would certainly depend on the rule set of the CA. Many rules simply do not generate any complex output patterns, and some degenerate to very small cycles or constant patterns very quickly. The authors of [22] conclude that no CA using only uniform rules is suitable for use in cryptographic applications by demonstrating the following: Of all uniform CA subjected to frequency, serial, poker, gap, and auto-correlation statistical randomness tests using the evolution of the center cell as a random stream, only 22 rules passed. These rules were then subjected to a linear complexity test. The linear complexity of a sequence is defined as the length of the shortest LFSR that produces the sequence. We know via the Berlekamp/Massey algorithm [24] that an LFSR of length no more than $\ell/2$ can be synthesized for any sequence of length

ℓ. The linear complexity test showed that only the following rules generate ℓ-bit sequences whose linear complexity approaches $\ell/2$ (the ideal): 30, 45, 75, 86, 89, 101, 106, 120, 135, 149, 169, 225. The authors then observe that all of these rules are left- or right-toggle rules, and adapt the Meier-Staffelbach algorithm to work on either side and recover the initial state of the CA with no more than $2^{\lfloor n/2 \rfloor}$ trials.

These results and the algorithms of Koç/Apohan [20] and Meier/Staffelbach [28] for attacking rule 30 all seem to imply that $2^{n/2}$ bits are the most one would have to guess to know everything about the CA's history one wanted. While this is still an exponential bound, we'd like to know for certain that the security of the PRG is related to the full seed length. Further, both algorithms often do much better than worst case. This apparent weakness invites further investigation into whether $2^{n/2}$ is the tightest upper bound that can be achieved. We will focus our efforts on rule 30 due to the large body of literature for this rule. The other rules listed above are in most cases simple variations using negation of a term or reflection of the inputs, and we would expect results against rule 30 to also apply there as well. Unless otherwise noted, we are concerned with cyclic CA as they are most commonly used in the literature.

We therefore examine the common structures using rule 30 in search of techniques to improve the worst case bounds. Specifically, we examine the case of a known state vector over n cells for a given time step (the problem addressed by Koç/Apohan) and the case of a known temporal sequence from one of n cell over for $n/2$ time steps (addressed by Meier/Staffelbach). In both cases we investigate techniques to recover the initial state.

3.1 Improvements to Koç and Apohan

The algorithm presented in [20] selects the best affine approximation of the rule used in a CA and applies the inverse affine transform to estimate the previous state. The success of this technique depends primarily on the rule(s) used in the CA. Those CAs using affine rules can be represented by a linear system in GF(2). In this case, the system can be solved for the state vector at time $t-1$ if the entire state vector is known at time t. This solution requires only $O(N^2 n)$ operations for an N-neighbor CA with n cells using Gaussian elimination. To recover the state t time steps ago, this process must be repeated t times. However, for affine rules, the inverse transform can also be represented as a linear recurrence $S^{(t-1)} = MS^{(t)} + b$ in GF(2), where M is an $n \times n$ Boolean matrix and the $1 \times n$ offset vector b models any negation operations in the rule of each cell. Given some $S^{(t)}$, this recurrence can be composed t times and

3-neighbor state	Possible 5-neighbor predecessors	# of prior states with 1 in each position
000	(00000) (11101) (11110) (11111)	[3, 3, 3, 2, 2]
001	(00001) (11010) (11011) (11100)	[3, 3, 1, 2, 2]
010	(10101) (10110) (10111) (11000)	[4, 1, 3, 2, 2]
011	(00010) (00011) (10100) (11001)	[2, 1, 1, 2, 2]
100	(01101) (01110) (01111) (10000)	[1, 3, 3, 2, 2]
101	(01010) (01011) (01100) (10001)	[1, 3, 1, 2, 2]
110	(00101) (00110) (00111) (01000)	[0, 1, 3, 2, 2]
111	(00100) (01001) (10010) (10011)	[2, 1, 1, 2, 2]

Table 1: Prior states of 3 neighbors in uniform rule 30 CA

solved to recover $s^{(0)}$ directly. The run time of inverting uniform CA using known affine rules then is clearly polynomially bounded.

The affine approximation does, however, have difficulty with non-linear rules. Applying the inverse of an affine approximation to $S^{(t)}$ gives an estimate S^* which can differ from the true $S^{(t-1)}$. If so, S^* may not be a valid predecessor of $S^{(t)}$ under the non-linear rule. The algorithm of Koç/Apohan resolves this by searching through templates of these mismatches in the context of the neighborhood in which they occur to find possible modifications to the estimate vector S^* which resolves the differences. It is this search that pushes the bound on the running time up to $O(2^{n/2})$.

We observe that not all prior state vectors are equiprobable under non-linear rules. Prior probabilities for 3-neighbor cells are given in Table 1. Note that any sequence of length 3 has only 4 possible prior states of length 5 under rule 30. Experimentation shows this holds for other sequence lengths as well. Since rule 30 is left-toggle, choosing $s_i^{(t)}$ and $s_{i+1}^{(t)}$ uniquely determine $s_{i-1}^{(t)}$ when $s_i^{(t+1)}$ is fixed by Proposition 2.1. But then $s_{i-1}^{(t)}$ and $s_i^{(t)}$ are known, so $s_{i-2}^{(t)}$ is determined if $s_{i-1}^{(t+1)}$ is known, and so on. Therefore, the two right bits are sufficient to determine the $k+2$ predecessor bits of any sequence of length k. This information can be used to optimize the search for erroneous predecessor bits in the templates. For instance, if the center value $s_i^{(t)}$ of a 3-cell neighborhood is 1, there is a $3/4$ chance that $s_{i-1}^{(t-1)}$ is 0. If the affine approximation does not immediately yield a unique prior state, using such probabilities may inform the template search and reduce the average-case search space of the algorithm.

More noteworthy is that rule 30 has certain patterns that always have fixed bits in the prior state. Notice the patterns 010 and 110 centered on cell $s_i^{(t)}$ have a fixed value in cell $s_{i-2}^{(t-1)}$ for all possible prior 5-neighbor states. The following proposition

shows why this must always be true.

Proposition 3.1. *Let* $(abcde) \in \{0,1\}^5$ *be the values in a 5-cell neighborhood at time* t *in a uniform rule 30 CA of* $n \geq 5$ *cells. Let* $(xyz) \in \{0,1\}^3$ *be the vales of the 3-cell neighborhood resulting from evaluating the rule 30 function* f *as* $f(a,b,c)$, $f(b,c,d)$ $f(c,d,e)$ *respectively. Then* $(xyz = 010)$ *implies* $a = 1$ *and* $(xyz = 110)$ *implies* $a = 0$.

Proof. First we show $(xyz = 010) \implies (a = 1)$. In rule 30, note the following identities:

$$x = 0 = a \oplus (b + c) \implies a = b + c \tag{2}$$
$$y = 1 = b \oplus (c + d) \implies b = \overline{c + d} \tag{3}$$
$$z = 0 = c \oplus (d + e) \implies c = d + e \tag{4}$$

Then, by substituting (4) into (2), we have

$$a = b + d + e \tag{5}$$

Thus a can only take the value 0 when $b = d = e = 0$. But substituting (4) into (3), we get

$$\begin{aligned} b &= \overline{d + e + d} \\ &= \overline{d + e} \end{aligned}$$

So when $d = e = 0$, b cannot be 0. Therefore, a can never be 0 and must always be 1.

To see that $(xyz = 110) \implies (a = 0)$,

$$x = 1 = a \oplus (b + c) \implies a = \overline{b + c} \tag{6}$$
$$y = 1 = b \oplus (c + d) \implies b = \overline{c + d} \tag{7}$$
$$z = 0 = c \oplus (d + e) \implies c = d + e \tag{8}$$

In particular, we note that (6) is just the complement of (2). For completeness, we can see

$$a = \overline{b + d + e}$$

by substituting for c in (6). But by (7) and (8), $b \neq d + e$, so a can never be 1. ∎

This information can be used to invert the entire state vector in rule 30 anytime the pattern 010 occurs by using the left-toggle property of the rule. We capture this in the following proposition.

Proposition 3.2. *Let $S^{(t)} \in \{0,1\}^n$ be the known state vector in a cyclic boundary n-cell uniform rule 30 CA at time t. Suppose $S^{(t)}_{i-1:i+1} = 010$ for some $0 \leq i < n$. Then $S^{(t-1)}$ is uniquely determined.*

Proof. First, note that

$$s^{(t-1)}_{i-3} = s^{(t)}_{i-2} \oplus (s^{(t-1)}_{i-2} + s^{(t-1)}_{i-1})$$

and that $s^{(t-1)}_{i-2} = 1$ by Proposition 3.1. Then $s^{(t-1)}_{i-2} + s^{(t-1)}_{i-1} = 1$ and so

$$s^{(t-1)}_{i-3} = s^{(t)}_{i-2} \oplus 1 = \overline{s^{(t)}_{i-2}}$$

Since $s^{(t-1)}_{i-2}$ and $s^{(t-1)}_{i-3}$ are known, we can then evaluate

$$s^{(t-1)}_{i-4} = s^{(t)}_{i-3} \oplus (s^{(t-1)}_{i-3} + s^{(t-1)}_{i-2})$$

and likewise for $i - k$ where $k = 1, \ldots, n - i$ where $i - i - 1$ evaluates to n because of the cyclic boundary. This allows the calculation of all $s^{(t-1)}_i$ for $0 \leq i < n$. ∎

Note that this technique does not work for the 110 pattern, since knowing the middle bit $s^{(t-1)}_i$ in a 3-neighborhood when its value is 0 does not determine $s^{(t-1)}_i + s^{(t-1)}_{i+1}$. However, if we examine longer predecessor patterns we may be able to guess two adjacent bits in the predecessor with high probability. Then we can evaluate

$$s^{(t-1)}_{i-1} = s^{(t)}_i \oplus (s^{(t-1)}_i + s^{(t-1)}_{i+1})$$

and so on until we complete the previous state vector. Looking at the 32 possible 5-neighbor blocks under rule 30, 11 have a fixed 1 position in their 7-neighbor predecessors, and another 16 reveal two adjacent positions with probability 3/4. Only 5 of the possible values leave the probability of guessing two adjacent predecessor bits at 1/2.

We observe that the analog of Proposition 3.1 also holds for all left- or right-toggle rules, and so each such rule can be inverted using the techniques discussed above.

Table 2 shows the patterns of interest for each of the rules identified. These results suggest some general weakness in toggle rules that can be used to find predecessors for any given state vector.

Rule 30 is known to have some garden of Eden state vectors and is not strictly injective [1, 26, 47]. In practice, those $S \in \{0,1\}^n$ for which $F_{30}^{-1}(S)$ is not defined appear only as initial conditions, so a partial inversion is sufficient to recover a seed. Various authors [9, 35] have addressed a means to realize a partial inversion once a rule 30 CA is run past time step 1, but none give a specific algorithm to find any possible predecessor in a uniform cyclic rule 30 CA. The observation that toggle rules require only two bits to make the full preceding state vector known leads to such a partial inversion algorithm that runs in time $\Theta(n)$. While the general mechanism may be known, there seemed to still be some question as recently as 2011 (see chapter 10 by Wolfram in [53]). We give a formal statement and proof of the algorithm here for completeness.

Proposition 3.3. *Let $S \in \{0,1\}^n$ and let f_i be a function either of the form $f_i(S_{i-1:i+1}) = S_{i-1} \oplus g(S_i, S_{i+1})$ or $f_i(S_{i-1:i+1}) = g(S_{i-1}, S_i) \oplus S_{i+1}$ for some $g : \{0,1\}^2 \to \{0,1\}$. Let r be the rule number of f_i and let F_r be the rule vector of a cyclic boundary n-cell CA having state vector $S^{(t)} = S$ at time t. Then all valid $S^{(t-1)}$ if any exist are computable in time $\Theta(n)$.*

Proof. We first consider the case of the left-toggle function $f_i(S_{i-1:i+1}) = S_{i-1} \oplus g(S_i, S_{i+1})$. For $S^{(t-1)}$ to exist, it must satisfy $S_i^{(t)} = S_{i-1}^{(t-1)} \oplus g(S_i^{(t-1)}, S_{i+1}^{(t-1)})$ for $2 \leq i < n$ and $S_1^{(t)} = S_n^{(t-1)} \oplus g(S_1^{(t-1)}, S_2^{(t-1)})$ as well as $S_n^{(t)} = S_{n-1}^{(t-1)} \oplus g(S_n^{(t-1)}, S_1^{(t-1)})$ due to boundary conditions. Consider Algorithm 3.1, InvertToggleRule.

Algorithm 3.1 InvertToggleRule

Input: S

Output: $\mathcal{P} = \{P \in \{0,1\}^n \mid F_r(P) = S\}$

1: $\mathcal{P} \leftarrow \emptyset$
2: $R \leftarrow 0^{n+2}$
3: **for** $\eta \in \{(0,0),(0,1),(1,0),(1,1)\}$ **do**
4: $\quad R_{n+1,n+2} \leftarrow \eta$
5: $\quad$ **for** $i = n$ downto 1 **do**
6: $\quad\quad R_i \leftarrow S_i \oplus g(R_{i+1}, R_{i+2})$
7: $\quad$ **if** $R_1 = R_{n+1} \wedge g(R_1, R_2) = g(R_{n+1}, R_{n+2})$ **then**
8: $\quad\quad \mathcal{P} \leftarrow \mathcal{P} \cup R_{2:n+1}$
9: **return** $\mathcal{P}$

InvertToggleRule runs in time $\Theta(n)$, as controlled by line 6 which executes n

Rule	3-neighbor state	Possible 5-neighbor predecessors	# of prior states with 1 in each position
30	010	10101 10110 10111 11000	[4, 1, 3, 2, 2]
	110	00101 00110 00111 01000	[0, 1, 3, 2, 2]
45	000	11001 11100 11110 11111	[4, 4, 3, 2, 2]
	001	10010 10011 11000 11101	[4, 2, 1, 2, 2]
	100	01001 01100 01110 01111	[0, 4, 3, 2, 2]
	101	00010 00011 01000 01101	[0, 2, 1, 2, 2]
75	001	01000 01001 01011 11110	[1, 4, 1, 2, 2]
	010	10010 10111 11100 11101	[4, 2, 3, 2, 2]
	011	10000 10001 10011 10110	[4, 0, 1, 2, 2]
	101	01110 11000 11001 11011	[3, 4, 1, 2, 2]
	110	00010 00111 01100 01101	[0, 2, 3, 2, 2]
	111	00000 00001 00011 00110	[0, 0, 1, 2, 2]
86	010	00011 01101 10101 11101	[2, 2, 3, 1, 4]
	011	00010 01100 10100 11100	[2, 2, 3, 1, 0]
89	010	00111 01001 10111 11101	[2, 2, 3, 2, 4]
	011	00110 01000 10110 11100	[2, 2, 3, 2, 0]
	110	00001 01101 10001 11001	[2, 2, 1, 0, 4]
	111	00000 01100 10000 11000	[2, 2, 1, 0, 0]
101	000	00111 01111 10011 11111	[2, 2, 3, 4, 4]
	001	00110 01110 10010 11110	[2, 2, 3, 4, 0]
	010	00100 01100 10001 11100	[2, 2, 3, 0, 1]
	011	00101 01101 10000 11101	[2, 2, 3, 0, 3]
	100	00011 01001 10111 11001	[2, 2, 1, 2, 4]
	101	00010 01000 10110 11000	[2, 2, 1, 2, 0]
106	010	00010 01010 10010 11100	[2, 2, 1, 3, 0]
	011	00011 01011 10011 11101	[2, 2, 1, 3, 4]
120	010	00111 01000 01001 01010	[0, 3, 1, 2, 2]
	110	10111 11000 11001 11010	[4, 3, 1, 2, 2]
135	001	10111 11000 11001 11010	[4, 3, 1, 2, 2]
	101	00111 01000 01001 01010	[0, 3, 1, 2, 2]
149	100	00011 01011 10011 11101	[2, 2, 1, 3, 4]
	101	00010 01010 10010 11100	[2, 2, 1, 3, 0]
169	100	00010 01100 10100 11100	[2, 2, 3, 1, 0]
	101	00011 01101 10101 11101	[2, 2, 3, 1, 4]
225	001	00101 00110 00111 01000	[0, 1, 3, 2, 2]
	101	10101 10110 10111 11000	[4, 1, 3, 2, 2]

Table 2: Fixed position patterns in toggle rules

times for each of 4 possible values of η. We must show that $\mathcal{P}$ is exactly the set of possible prior state vectors leading to S under F_r.

Suppose $\mathcal{P} \neq \emptyset$ and let $P \in \mathcal{P}$. Notice that P is the inner n bits of the full R computed by line 6, and so $P_1 = R_2$ and $P_n = R_{n+1}$. We show that for all P, $F_r(P) = S$. It is clear that for any $P \in \mathcal{P}$, $S_i = P_{i-1} \oplus g(P_i, P_{i+1})$ holds for $2 \leq i < n$ since $R_i = P_{i-1}$ is assigned to be $S_i \oplus g(P_i, P_{i+1})$ based on the left-toggle property of f. That leaves us to verify just the boundaries. Since $P \in \mathcal{P}$, the condition on line 7 must hold, and so we know that $R_{n+1} = R_1$. From line 6, we have $R_1 = S_1 \oplus g(R_2, R_3)$, so $R_{n+1} = S_1 \oplus g(R_2, R_3)$, and therefore $P_n = S_1 \oplus g(R_2, R_3)$. XORing $g(R_2, R_3)$ to both sides gives $S_1 = P_n \oplus g(P_1, P_2)$, which is the definition of f_1 with cyclic boundaries. Also from line 6, we have $R_n = S_n \oplus g(R_{n+1}, R_{n+2})$. By line 6 we know $R_n = S_n \oplus g(R_1, R_2)$ and by line 7, this becomes $P_{n-1} = S_n \oplus g(P_n, P_1)$ (i.e. the left toggle of f_n) since $R_1 = R_{n+1} = P_n$. Therefore, $S_n = P_{n-1} \oplus g(P_n, P_1)$, the definition of f_n. This proves $S_i = P_{i-1} \oplus g(P_i, P_{i+1})$ for $1 \leq i \leq n$, so $S = F_r(P)$ for any $P \in \mathcal{P}$.

Suppose that there exists $Q \in \{\mathtt{0}, \mathtt{1}\}^n$ such that $F_r(Q) = S$. We can construct $Q' = Q_n Q_1 Q_2 \ldots Q_n Q_1$ in which $Q'_1 = Q'_{n+1}$ and $Q'_2 = Q'_{n+2}$ so $g(Q'_1, Q'_2) = g(Q'_{n+1}, Q'_{n+2})$. Notice that, by the definition of f_i, $Q_{i-1} = S_i \oplus g(Q_i, Q_{i+1})$ for $2 \leq i < n$ and also that $Q_n = S_1 \oplus g(Q_1, Q_2)$ and $Q_{n-1} = S_n \oplus g(Q_n, Q_1)$. Then by our construction, $Q'_i = S_i \oplus (Q'_{i+1} + Q'_{i+2})$ must hold for all $1 \leq i \leq n$. Thus Q' has exactly the form of some R generated on line 6 since any such Q' must end in one of the possible η considered on line 3. Further, Q' meets the criteria on line 7 of the algorithm and therefore Q must be in $\mathcal{P}$.

It is easy to see that reversing the direction of the algorithm proves the case of a right-toggle $f_i(S_{i-1:i+1}) = g(S_{i-1}, S_i) \oplus S_{i+1}$. ∎

Algorithm 3.1 inverts all rules in Table 2 except rule 101 with only rules 135 and 149 requiring a slight modification to handle negation of the toggle input. The algorithm is successful whenever a given state vector has at least one predecessor. This would seem to be the majority of state vectors: under rule 30 for $n = 6$, there are 12 state vectors with no predecessor, 41 states with exactly 1, 10 with 2, and 1 with 3. For $n = 9$, these numbers are 57 with no predecessor, 393 with 1, 61 with 2, and 1 with 3.

Another implication of this algorithm is that a state vector with no predecessor discovered mid-stream in a pseudorandom generator must be the result of re-seeding the generator or updating its entropy. Leaking this information may be more damaging than knowledge of the internal state itself: if entropy timing and values can be

t	c-3	c-2	c-1	c	c+1	c+2	c+3
0	1	1	0	0	1	1	0
1		0	1	1	1	0	
2			1	0	0		
3				1			

Table 3: Example of applying the Meier-Staffelbach algorithm in rule 30. Step 1: The seed values on the right half (in green) are guessed. Step 2: The right triangle (in blue) is computed. Step 3: The left triangle (in magenta) is solved to complete the left half of the seed (in red).

discerned, they may become controllable.

Finally, this algorithm improves the running time to invert a rule 30 CA given by Koç/Apohan from $O(2^{n/2})$ to $\Theta(n)$. We are not aware of any previous bound on the running time to invert other toggle rules as presented above.

3.2 Improvements to Meier and Staffelbach

The Meier/Staffelbach algorithm can be used to recover the CA state $S^{(0)}$ given a temporal sequence of at least $\lceil n/2 \rceil$ time steps starting at time $t = 0$ for the central cell s_c where $c = \lceil n/2 \rceil$. The algorithm can be summarized as:

- Guess $\lfloor n/2 \rfloor$ bits for cells $S^{(0)}_{c+1:n}$ for time $t = 0$.
- Evaluate the cells $S^{(t)}_{c+1:n-t}$ for $1 \leq t < c$ time steps. Each time step, one fewer cell on the right end can be computed since the boundary neighbor value is not known. This leaves only the central cell known at time $\lceil n/2 \rceil$. Plotted as a two-dimensional chart over time, the computed cells now form a triangle between the central cell's values and the right half of the initial state.
- Using the computed right-adjacent sequence $S^{(0)}_{c+1}, \ldots, S^{(\lfloor n/2 \rfloor)}_{c+1}$, solve the left triangle from $t = \lfloor n/2 \rfloor$ backward up to $t = 0$ to complete the seed.

This algorithm is illustrated in Table 3. Meier and Staffelbach show for $n = 300$, the center temporal sequence of a uniform rule 30 CA requires about 18 bits of entropy to guess a compatible seed. We would like to see if taking advantage of the observations made above allows us to improve this result.

Meier and Staffelbach note in [28] that where the temporal sequence is a sequence of 0s, the right-adjacent sequence must match 0^*1^*. We make a related observation on the left-adjacent sequence.

t	i-4	i-3	i-2	i-1	i
0					0
1				1	1
2	0				0
3		1	1	0	1
4			0	0	1
5				1	1
6					0
7					0
8					1

Table 4: Example of solved neighbors in rule 30

Proposition 3.4. *Any occurrence of* `10` *in the temporal sequence starting at time t in a uniform rule 30 CA must have a left-adjacent value of* `1` *at t; any occurrence of* `11` *must have a left-adjacent value of* `0` *at t.*

Proof. Let i be the index of the cell for which the temporal sequence is known. By the definition of rule 30,

$$s_{i-1}^{(t)} = s_i^{(t+1)} \oplus (s_i^{(t)} + s_{i+1}^{(t)})$$

and we know that $s_i^{(t)} = 1$. Then $s_i^{(t)} + s_{i+1}^{(t)} = 1$ and so

$$s_{i-1}^{(t)} = s_i^{(t+1)} \oplus 1 = \overline{s_i^{(t+1)}}.$$

When `10` occurs in the temporal sequence at cell i, $s_i^{(t+1)} = \mathtt{0}$ and so the left-adjacent $s_{i-1}^{(t)}$ must be `1`. When `11` occurs, $s_{i-1}^{(t)}$ must be `0`. ∎

Further, if two such occurrences are temporally adjacent (as must always be the case when `11` appears since the third bit is either `1` or `0`) starting at time t for a sequence in cell i, then two left-adjacent cells are known and can be solved backwards to produce the value of $s_{i-2}^{(t)}$. Table 4 gives an example temporal sequence at cell i and the solved values in red. In addition, the known predecessor patterns can also be used to solve even more of the CA history. By using the prior states from Table 1, cells in green have been filled in as well.

There are a couple of applications of these observations to improving the Meier/Staffelbach algorithm. First, let σ be the temporal sequence of length n from cell i in a rule 30 CA

beginning at $t = 0$ and let j be the number of 1s in σ. Then no more than $n - j - 1$ random bits, or coins, would be necessary to chose those unknown left neighbors that would allow us to solve the entire left triangle back to $t = 0$ and thus recover the entire seed. If σ is evenly distributed, then we would expect to need $n/2$ coins. This is comparable to the Meier-Staffelbach algorithm. If, however, we spend those coins to fill in missing values of s_{i-1} from the bottom up, we may reach a point were examining predecessors of varying widths in different positions fills in prior time steps deterministically. This leads us to the following algorithm:

1. Rotate the CA cells to the right, placing the known temporal sequence on the right edge, leaving only a left triangle of size n to solve.

2. Repeat until a full state vector is known or no changes are possible:

   (a) Recursively apply Proposition 3.4 to fill in left-adjacent cells where $\sigma_i = 1$.

   (b) Recursively apply Proposition 3.2 to fill in fixed predecessor values.

   (c) Run the CA forward to fill in any cell for which all three inputs are known.

   (d) For any $s_{i-1}^{(t)} = 1$ and $s_i^{(t+1)} = 1$, set $s_i^{(t)} = s_{i+1}^{(t)} = 0$, accounting for boundaries.

3. If the full state vector is known:

   (a) Run the CA forward to fill in unknown cells. Check all known cells for correctness.

   (b) If discrepancies arise:

      i. Backtrack to revisit random bits chosen in step 4 and flip them.

   (c) Otherwise, use Algorithm 3.1 to invert the CA back to $t = 0$.

4. Otherwise, if no changes are possible, choose a random bit for the bottom-most empty cell adjacent to the left edge of known cell values.

5. Return to step 2.

Pencil and paper experimentation show this algorithm to hold promise, but a more complete effort is required to determine limits on the number of random bit selections needed.

Table 5 illustrates the use of coins (in blue) and predecessor information to complete the previous example. Once the state contains a 010 string, all predecessors to the left

t	i-8	i-7	i-6	i-5	i-4	i-3	i-2	i-1	i	i-8
0	0	0	1	1	1	1	0	1	0	0
1		1	1	0	0	0	0	1	1	
2			0	1	0	0	1	1	0	
3				1	1	1	1	0	1	
4					0	0	0	0	1	
5						0	1	1	1	
6							1	1	0	
7								0	0	
8									1	

Table 5: Example of solved neighbors in rule 30. In this example, the sequence 111101 in $s^{(3)}$ with 0 at $s_i^{(2)}$ doesn't quite determine that $s^{(2)}$ so spending a coin on $s_{i-1}^{(2)}$ is still required. In other cases, a 010 may appear in $s^{(2)}$, saving the coin. All of $s^{(1)}$ is determined at the right edge and by $s^{(2)}$.) Since $s^{(1)}$ begins with 110, $s_{i-8}^{(0)} = 0$. Suppose $n = 9$; then $s_i^{(0)}$ is determined and so on to the left.

of that string are known for as long and the successor is known to the left. This means that coins are only required where the temporal sequence cannot fill in those positions to the right until the boundary conditions are known. Past that point, these holes can also be filled in by solving forward with knowledge of the right-adjacent values. Cells colored in magenta show cells which can be determined once the boundary conditions are known.

4 Analysis of Linear and Affine CAs

Over and again, the literature shows interesting results around rules 90, 105, 150, and 165 [16–18, 33, 40, 44–46]. A cryptanalyst with a firm command of constructions based on these rules stands to gain good advantage over the majority of random number generators and cryptosystems based on these CAs. Therefore, we would like to understand the extent to which we can apply the same techniques to the case of linear and affine CAs based on these four rules.

There are some obvious challenges to using linear CA for cryptography. As previously mentioned, knowledge of the complete state vector in a linear CA makes its entire history solvable—simply solve the linear equations backwards for the number of time steps desired. If the rule vector is known, using the full state vector seems unwise. Using only a temporal sequence would keep knowledge of the full state vector

secret, but in linear CA, Meier/Staffelbach approach seems only to get easier as we now have full linearity in both directions.

We will focus on the approach taken by Tomassini and Perrenoud [45] to deal with these problems. Their scheme is as follows: Select via the key one of these four rules (details of how to do this are not given) as well as the initial state for each of n cells in a cyclic boundary CA. Use the central temporal sequence as a random stream and XOR it with the plaintext. The key space would be $4^n \times 2^n = 2^{3n}$ making the key $3n$ bits with suggested values of n around 100 (for 2001 compute power). It is claimed that $2^{(5n-9)/2}$ guesses over rules and values would be necessary to solve the sequence backwards to find a single, unique rule set and seed that produces that sequence. This is one bound we seek to improve.

4.1 Analysis of CAs with Symmetric Rule Sets

Much of the key space proposed in [45] will result in CA whose rule sets are symmetric about a center cell. Recall that a CA is symmetric if rule $f_i = f_{n-i+1}$ for all $1 \leq i \leq n$. Experiments performed on these configurations show very low periods, making them very inefficient. We capture this observation in the following conjecture.

Conjecture 4.1. *Let n be an odd number and Σ be an n-cell symmetric hybrid linear CA over rules $R = 90, 105, 150, 165$ with cyclic boundaries. There are at most $2^{\lfloor n/2 \rfloor}$ initial states of Σ that give the same temporal sequence of length $\ell \geq \lceil n/2 \rceil$ at cell $\lceil n/2 \rceil$.*

Evidence for this conjecture begins with the following proposition.

Proposition 4.1. *There are $2^{\lfloor n/2 \rfloor}$ initial states of Σ that give the same temporal sequence of length $\lceil n/2 \rceil$ at cell $\lceil n/2 \rceil$.*

Proof. Let $k = \lceil n/2 \rceil$ and let $\sigma = s_k^{(0)}, \ldots, s_k^{(k)}$ be the first k bits of the center temporal sequence at cell k. Choose any $\rho \in \{0,1\}^{k-1}$ as the right-adjacent sequence of cell k beginning at $t = 0$. Since each transition function in Σ is affine, both the right and left triangles of Σ are easily solved for, resulting in a full initial state. This state necessarily produces σ, regardless of the choice of ρ. ∎

This shows that the bound on the number of initial states holds at $\ell = \lceil n/2 \rceil$, giving a sort of base case for the argument.

Further evidence for Conjecture 4.1 can be seen in the experimental results in Table 6, and suggest that the period of any linear CA (symmetric or asymmetric) is

n	Max Period	$2^{\lceil n/2 \rceil}$
5	8	8
7	14	16
9	30	32
11	62	64
51	67108860	67108864

Table 6: Maximum periods of linear cyclic CA. The value for $n = 51$ is based on a single observation of $F = \langle \{150\}^{12}, 90, \{150\}^{25}, 90, \{150\}^{12} \rangle$ with $s^{(0)} = 562964991182857$. A few other rule sets have been tried with only a few other seeds, all having far lower periods. This is a symmetric CA, and larger periods may be possible for asymmetric CA. Lower values for n are the results from exhaustive computation.

no more than $2^{\lceil n/2 \rceil}$. If this is true and some period p can be derived from the rule set, then a proof need only deal with values of $k < \ell < p$. For $t \geq p$, the sequence must be fixed.

A likely approach to proving Conjecture 4.1 uses a linear system in GF(2) to model the evolution of the CA. Since all rules in R are affine, we can represent Σ's transition function across all cells as

$$\begin{bmatrix} w_{1,2} & w_{1,3} & 0 & 0 & & w_{1,1} \\ w_{2,1} & w_{2,2} & w_{2,3} & 0 & & 0 \\ 0 & w_{3,1} & w_{3,2} & w_{3,3} & & 0 \\ 0 & 0 & w_{4,1} & w_{4,2} & \cdots & 0 \\ & & & \vdots & \ddots & \\ 0 & 0 & 0 & w_{n-1,1} & w_{n-1,2} & w_{n-1,3} \\ w_{n,3} & 0 & 0 & 0 & w_{n,1} & w_{n,2} \end{bmatrix} \begin{bmatrix} s_1^{(t)} \\ s_2^{(t)} \\ \\ \vdots \\ \\ \\ s_n^{(t)} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \\ \vdots \\ \\ \\ b_n \end{bmatrix} = \begin{bmatrix} s_1^{(t+1)} \\ s_2^{(t+1)} \\ \\ \vdots \\ \\ \\ s_n^{(t+1)} \end{bmatrix} \tag{9}$$

where $w_{i,j}$ is the weight for cell i on input j in the cells transition function $f_i(x_1, x_2, x_3)$ and b_i is an offset for each cell to affect a complement rule. If M is the matrix of $w_{i,j}$, then the recurrence $S^{(t+1)} = MS^{(t)} + b$ models the evolution of Σ. If this recurrence is periodic, so is Σ.

The matrix M has a few interesting properties in the cases of interest. First, for all rules in R, $w_{i,1} = w_{i,3} = 1$. Therefore, M is symmetric. The values on the diagonal are determined by whether $f_i \in \{90, 165\}$(where $w_{i,2} = 0$) or $f_i \in \{150, 105\}$ ($w_{i,2} = 1$). Second, since Σ is symmetric, M is symmetric relative to both diagonals, i.e. $w_{i,2} = w_{n-i+1,2}$ for $i < n/2$. Lastly, M is a band matrix[2].

[2]The strict definition of band matrix precludes the non-zero values at $(1, n)$ and $(n, 1)$, but some

If the recurrence is periodic, then applying the recurrence some number of times must yield the starting value. That is

$$M(M(\ldots(M(MS^{(t)}+b)+b)\ldots)+b)+b = MS^{(t)}+b.$$

Showing that repeating the recurrence leads to an earlier result of the recurrence rather than just the original $S^{(t)}$ allows for proper handling of garden of Eden state vectors. Collecting terms over p applications of the recurrence, we see that the period is p when

$$M^{p+1}S^{(t)} + M^p b + \ldots + Mb + b = MS^{(t)} + b$$

This may be further broken down to showing that raising a matrix with these properties to the power p is idempotent, i.e. $M^{p+1} = M$, followed by showing that $M^p b + \ldots + Mb = \mathbf{0}$. This last part can be re-written as

$$\left(\sum_{i=1}^{p} M^i\right) b = \mathbf{0}. \tag{10}$$

In fact, the condition in (10) also implies $M^{p+1} = M$. Suppose $Q(x)$ is the characteristic polynomial of M so that $Q(M) = 0$. So also $M \cdot Q(M) = 0$ and $M^2 \cdot Q(M) + M \cdot Q(M) = \mathbf{0}$. Since Q is over GF(2), the coefficients of Q can be written as a binary vector c of length n. Multiplying $M \cdot Q(M)$, as each iteration of the recurrence does, gives a polynomial with coefficients $c \ll 1$ which also evaluates to $\mathbf{0}$ at M. Here, $\ll$ denotes the left-shift operator. We can likewise sum any number of shifts of c and arrive at a polynomial which evaluates to $\mathbf{0}$.

Now suppose that through this process, we create a polynomial with a coefficient vector $c' = 1^p$. We conjecture it is always possible to create such a polynomial if $Q(x)$ has terms $1x^1 + 0x^0$. If $Q(x)$ contains the term $1x^0$, we simply start from $c \ll 1$. We know the polynomial represented by c' evaluates to $\mathbf{0}$ at M. Therefore, this polynomial satisfies 10. To check whether $M^{p+1} = M$, we can see

authors refer to matrices of this form as band matrices in the literature. To avoid this technicality, we can simply duplicate s_1 and s_n on either side, append 0's to b, and increase the matrix dimension to match.

$$
\begin{aligned}
\sum_{i=1}^{p} M^i &= \mathbf{0} \\
(M+I)\sum_{i=1}^{p} M^i &= \mathbf{0} \\
\sum_{i=2}^{p+1} M^i + \sum_{i=1}^{p} M^i &= \mathbf{0} \\
M^{p+1} &= M
\end{aligned}
$$

The reverse implication also holds as long as $M + I$ is invertible. An algorithm to build such a c' exists (by repeatedly shifting the lowest order 1-bit up to the lowest order 0-bit in the running sum over GF(2)), but there is no proof of a bound on the resulting p. Recall that Nandi, et. al. prove the characteristic polynomial for a cyclic linear CA must have x or $(x + 1)$ as a factor, but it is not clear how this leads to a bound of $2^{n/2}$.

As an example, we illustrate the case for a 5-cell uniform rule 90 CA, which has the following transition matrix:

$$
M_{90} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

with successive powers

$$
M_{90}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \; M_{90}^3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \; M_{90}^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} = M_{90}.
$$

Notice in this case that

$$
\sum_{i=1}^{3} M_{90}^i = \mathbf{0}
$$

so even if b were not $\mathbf{0}$, the period would still be 3. Thus any 5-cell hybrid CA over

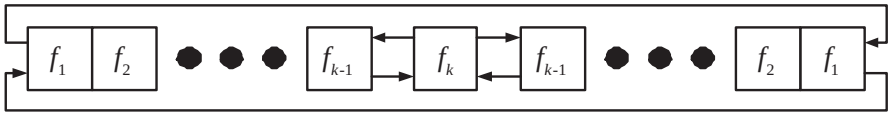


Figure 1: Symmetric CA viewed as having right- and left-side blocks

rules in $\{90, 165\}$ has period 3 for any starting seed.

Remark 4.1. There are 2^n symmetric CAs. To see this, we choose $n/2$ bits for the diagonal of M to select between binary ternary rules at each cell, then reflect them to the other half. We then choose $n/2$ bits for the vector b to select between normal and complementary rules and reflect those bits as well. These n bits cover the range of all possible values of M and b for symmetric CAs.

Another possible approach to proving Conjecture 4.1 is to think of a symmetric CA as a left block of $k-1$ cells, a center cell, and a right block of $k-1$ cells. We notice the left and right blocks are equivalent in construction: they have inner and outer neighbors that apply the same function, and they deliver their outputs to inner and outer neighbors after applying the same function as inputs move from one side to the other. (See Figure 1.) Their only difference is their initial state and the order in which the initial state vector $s^{(0)}$ is acted upon. This model may explain the reflection seen in cell values over several steps. Further, notice that at $t = k$, each cell is a function of all cells' initial state. In particular, each block is a function of its initial state, the first $k-1$ bits of σ, and the output of the opposite block. With the functions so similar, the intuition is that the period must be small (certainly no more than $(k-1)^2$), but a proof is not known.

Other considerations may further reduce the period. Spatially symmetric seeds can generate periods of no greater than $2^{\lfloor n/2 \rfloor}$ since both blocks will be exactly the same. It may also be of interest to know what the spacial period is in the case of symmetric seeds; some combinations dead-end at 0, some have very low periods. McIntosh covers the effects of spatially periodic seeds in depth in [25].

Remark 4.2. Assuming Conjecture 4.1, some sequences of length $\ell > \lceil n/2 \rceil$ are not possible in a given symmetric CA. This is supported by a pigeon-hole argument: If each sequence is produced by $2^{\lfloor n/2 \rfloor}$seeds then all seeds lead to one of $2^{\lceil n/2 \rceil}$ sequences, leaving $2^{\lfloor n/2 \rfloor}$ sequences with no possible seeds. This is born out by experimentation. See Appendix B.1 and Appendix B.3.

Remark 4.3. Assuming Conjecture 4.1, sequences which are not eventually periodic

with eventual period $\leq 2^{\lceil n/2 \rceil}$ cannot be produced by a symmetric CA. If such a CA were used as a key stream, a single known plaintext of more than $2^{\lceil n/2 \rceil}$ bits would quickly be distinguishable from random.

In summary, using a symmetric CA does not seem advisable. Suppose n is large enough to provide for an adequate period. There are still many seeds that would allow an attacker to compute past and future bits, defeating the whole system. Excluding symmetric CAs (both in arity and complementarity) reduces the rule vector space by 2^n. To guess the rule vector and initial state for only asymmetric CAs given a temporal sequence of length $n/2$, we would first need to choose from $2^{2n} - 2^n = 2^n(2^n - 1)$ rule vectors. Then we would need to choose one bit per time step for $(n-1)/2$ time steps to allow solving the state vectors backwards once the rule set is fixed. Multiplying these gives us a bound of $2^{(3n-1)/2}(2^n - 1)$ guesses for the whole key space, reducing the bound in [45] by $2^{(3n-1)/2}$ trials.

4.2 Analysis of CAs with Asymmetric Rule Sets

Asymmetric CAs prove more difficult to recover a seed from. Many seem to have unique seeds for each temporal sequence and periods generally seem longer in asymmetric CA, though these statements are not true in all cases. Deriving any identities concerning periods, number of distinct sequences, relation to other rule sets, etc. has proved very difficult.

To explore these constructs further, we instead examine data on particular qualities. As documented in Table 6, exhaustive computation gives us the maximum eventual period for all n-cell CAs for $n \leq 11$. Taking a 128-ruleset sample from the data from $n = 9$, we can chart the number of CA having a certain maximum period which generate a certain number of distinct sequences. The rule sets were chosen by looking at all possible rule sets over rules 90 and 150. This seems justified since the complementing of terms in rules 105 and 165 has been shown to add little if any value at all in most experiments[3]. Fixing the two left rules at 150 gives 128 possible rule sets to examine. For each of these, its maximum period and the total number of distinct sequences it generates over all possible seeds were computed. The results are shown in the bubble chart in Figure 2 and raw data for the table is in Appendix B.6.

Some interesting observations are possible looking at this data. First, 41% of the rule sets sampled generate a distinct central temporal sequence for every possible seed.

[3]But not none. There are cases where simply complementing certain rules can increase a rule set's period

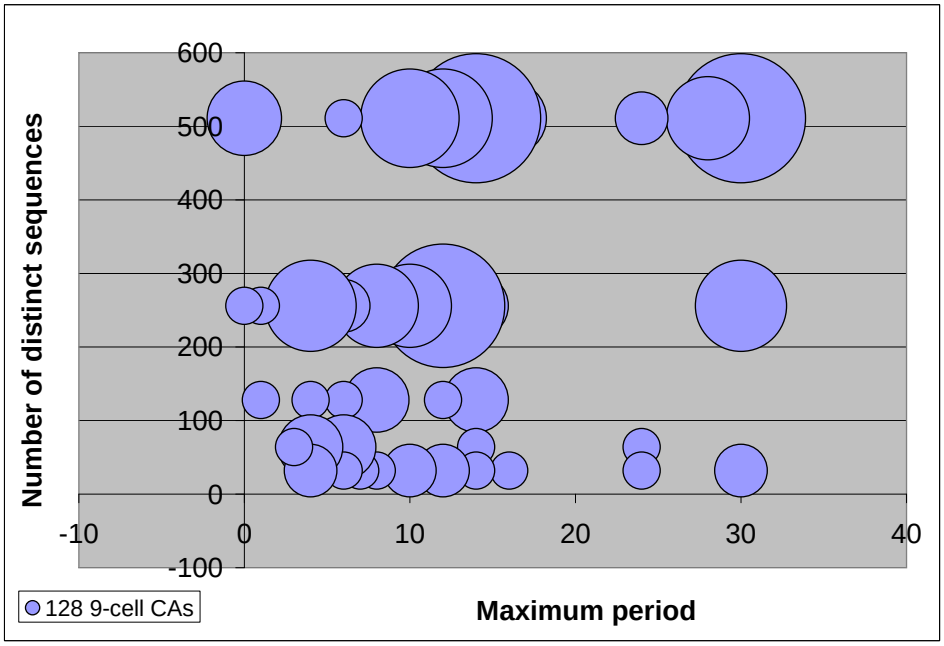


Figure 2: Number (indicated by bubble area, maximum of 12) of Asymmetric CA rule sets having a given period and generating a number of distinct sequences

Next, 29/128 have a period greater than or equal to 24. And finally, 19/128 have both of these properties. These numbers represent the maximums for the rule sets; many seeds under a given rule set may do worse than the maximum.

This suggests that at least roughly 85% of the key space over $R \times \{0,1\}^n$ has some considerable weakness.

4.3 Open Problems

The following sections record open investigations that may hold some promise for improving cryptanalysis in various CA. The ideas are rather unpolished, and a time-constrained reader may certainly skip this section.

4.3.1 Solving an Arbitrary Temporal Sequence in an Arbitrary Symmetric CA

Conjecture 4.1 suggests the workings for an algorithm analogous to that of Meier-Staffelbach to recover a seed given a central temporal sequence generated in a symmetric CA. What's missing is a method to select rules under which to solve the triangles of the CA backward (or run forward, for that matter.) Such a method is not obvious

since different symmetric CA have different periods. It may be possible to solve a part of a sequence under a chosen rule set, then map to another rule set with a longer period. This new rule set can then be run forward to check for a match. Mapping between rule sets seems to be possible, as suggested by experiments in Appendix B.2.

4.3.2 Mapping Seeds from Symmetric to Asymmetric CA

Let A be an asymmetric CA that generates a temporal sequence α from seed $s_A^{(0)}$. Suppose α is also generated by a symmetric CA B (which can be tested using the algorithm stated above) using seed $s_B^{(0)}$. We would like to know if there a mapping from the seed $s_B^{(0)}$ to $s_A^{(0)}$ or, short of a mapping, if there is an efficient algorithm for finding $s_A^{(0)}$ given $s_B^{(0)}$. Such an algorithm would reduce the problem of recovering a seed in an asymmetric CA to recovering a seed in a symmetric CA, which we've already solved.

Experiments in Appendix B.4 show that each of the $2^{\lfloor n/2 \rfloor}$possible $s_B^{(0)}$ has a unique difference from $s_A^{(0)}$. Further, their difference is the same for $\overline{\alpha}$, which may reduce the search space by half. Of course, α may not be a possible temporal sequence of B, which may also be useful information.

4.3.3 Mapping Sequences from Symmetric to Asymmetric CA

We approach the problem orthogonally to Section 4.3.2: for all seeds $s_B^{(0)}$ that generate each sequence σ in symmetric CA B, examine the sequence α generated by A using the same seed $s_B^{(0)}$. The difference between σ and α may provide information about the rule set F_A. Experiments in Appendix B.5 show example difference patterns. In the specific case shown, these differences only occur after the first $\lceil n/2 \rceil$ time steps, which indicates that the asymmetry only affects the temporal sequence in positions where contributing seed bits have been used as input to cells symmetric about the sequence, i.e. after wrapping around the boundary. This suggests that the asymmetry is one sided. If we think of the information from the initial state as "flowing" through the cells on any of their connection paths towards the central temporal sequence, we notice conditions that might change the information that arrives in the central cell compared to a uniform or symmetric CA. For all temporal sequence positions before $t = \lceil n/2 \rceil$, an asymmetric seed could correct for rule differences on just one side. But after passing through both symmetric cells, the effect of asymmetry changes the information contribution to the central cell when it arrives there. Other pairs of rule sets in this same experiment show differences earlier than $t = \lceil n/2 \rceil$, and indeed, have

rule changes on both sides and closer to the central cell. This supports the general idea, though the actual mechanics are not fully understood.

If an efficient algorithm could be found to identify the actual values of the asymmetry in the rule set, then recovering the original seed is reduced to solving a known sequence under known hybrid affine rules, which can be done with the Meier/Staffelbach algorithm. Recovering the rule values will likely require comparing the sequences from an unknown, asymmetric CA to those of uniform rule 90, 105, 150, and 165 CAs.

5 A New Cellular Construction

Given the observations made previously and the body of literature showing weakness in 3-neighbor, fixed-rule cellular automata (CA), we may begin to wonder if *any* CA construction is capable of exhibiting provably NP-hard behavior. The 3-neighbor uniform CA using rule 110 has been proven to be universal (i.e. capable of simulating an arbitrary Turing machine) [12]. Unfortunately, this rule does not seem capable of passing basic statistical tests by itself since it is highly non-linear and therefore biased in its output. It seems all 3-neighbor fixed-rule set CAs, uniform or hybrid, are unsuitable for cryptographic applications.

Yet the highly parallel nature and simple operation of CA are still appealing. Whereas block ciphers and other cryptographic primitives must be specifically designed for a predetermined block width, a strong CA construction can allow scalability simply by adding more cells. Since cells are identical, they can be packaged in ASICs or programmable logic blocks and configured at a width just adequate for the job at hand. For software implementations, the ubiquity of vector register operations on most CPUs and the advent of GPUs give CA constructions an easy path to performance improvements. Most block ciphers, in contrast, do not naturally decompose into parallel tasks in an obvious way. Lastly, many cryptographic primitives have no proofs of security properties or only derive provable security properties through reduction to other primitives which have no such proofs (see e.g. [3]). The simple, regular operations of CA, on the other hand, seem more likely to lend themselves to proving certain properties than ad-hoc combinations of shifts and XORs. If so, and if the implementation is efficient and scalable, such a construct would have natural advantages for cryptographic application designers.

Therefore it seems worth understanding what might be required to produce hard-to-invert CA. We might first ascribe the observed failings to the 3-neighbor construct after noting that any two neighbors share 2 of their 3 predecessors in common. This construct leaks information about a cell's value to its three descendants. With knowledge of the function applied at each cell, an attacker can build a system of equations, even if its non-linear in some cells. One way to plug this leak is to protect the knowledge of the function applied at each cell. Suppose we fix the neighborhood of cells at 3 but allow the rule applied at each step to be selected uniformly at random from all possible 3-neighbor rules and we seed the CA with n bits, also sampled uniformly at random. It is clear that knowing the full state of the CA at time $t + 1$ would provide no information about the state at time t. Each iteration of such a CA would realize

Shannon's notion of *perfect secrecy* [42].

Unfortunately, requiring truly random rule selection leaves us no better off than where we started–in need of good random bits. We might consider using pseudorandom functions to preserve some notion of semantic security, that is, perfect secrecy under computational bounds. Even so, that would leave us requiring one pseudorandom primitive to produce another. We may then wonder: Can a 1-D CA with enhanced cells which modulate their rule by a simple process (e.g. a weak random number generator or a fixed permutation) be provably NP-hard to invert and still produce cryptographic-strength pseudorandom bits?

5.1 Finite State Transducers

To understand the capabilities of cellular constructs, we must first formalize a computational model to evaluate. There are a few factors that guide the selection of our model. First, we'd like to find the simplest model possible which is still capable of the required computation. This is a general principle but also a practical concern since simple models are easiest to reason about. Second, we'd like physical implementations to be able to match the computational model closely. This allows any provable properties which exist in theory to also be claimed by the implementations (up to differences required of the mapping to the physical world.)

The simplest computational model in theory is the finite state machine (FSM), which is ostensibly the model for each cell. We consider FSMs with greater than two states and the capabilities of the overall automaton when we place various restrictions on the construction of the FSMs in the cells.

Historically, the output of a two-state cell has been referred to as its *state* since the output directly reflects the current state of the automaton. This terminology becomes confusing when we consider many-state machines which still produce only two outputs. We will therefore adopt the notion of a cell producing an *output* as distinct from its current state. This notion is captured nicely in the model known as finite state transducers (FSTs), which originated with Mealy [27] and Moore [31] after whom the popular variants are named. Using these models provides the advantage of having well known ways to map such transducers into combinatorial logic, making practical applications more straightforward.

Conceptually, an FST comprises two tapes, an input tape and an output tape, and computes a function that maps strings on the input tape to strings on the output tape. For a cellular FST operating repeatedly in discrete time steps with instantaneous

communication of outputs to neighbors, the concept of tapes does not seem a natural fit. Some awkward constructions would be needed to copy outputs from neighbors' tapes to each cell's input tape or for them to be shared some how.

To address this difficulty, we will provide our cellular FSTs with N direct, discrete inputs and a single output. These inputs and outputs can be routed and connected together to allow various constructions just as if they were wires, similar to traditional CA. More formally, we define a finite state cell (FSC) as a quadruple (Q, Σ, q_s, δ), where:

- Q is a finite set of states,
- Σ is the alphabet (input and output) of the cell,
- $q_s \in Q$ is the start state,
- $\delta : Q \times \Sigma^N \to Q \times \Sigma$ is the transition function.

Again, we limit our discussion to $N = 3$. For convenience, we name the 3 input values read each time step λ, ω, and ρ, where ω is the output of the cell routed back as an input. An FSC requires an initial value $\omega_s \in \Sigma$ to begin operation. Once received, the cell outputs ω_s and enters q_s. At each time step, the cell computes a function from Σ^3 to Σ depending on the current state and then changes to the next state, all according to δ. The ordered string $\lambda\omega\rho$ defines the input to this function, and its result becomes the next value of ω. FSCs have no final states, and simply operate continuously after initialization.

It is easy to see the analogy to traditional CA. FSCs, however, are not fixed in the function they compute at each time step. Instead, these cells may have an arbitrary but finite number of states, the transitions between each of which compute different functions. The path through the states may be dependent on the inputs received or may be fixed.

We will be concerned only with the case of $\Sigma = \{0, 1\}$, though it is easy to imagine FSCs with a larger Σ. We could also consider larger values of N. These variations will not be necessary for our present purposes and so will not be considered.

5.2 Cellular Automata based on FSCs

We can now consider a cellular automaton which aggregates n such cells. We define a finite state cellular automaton, or FSCA, $A = \langle a_1, a_2, \ldots, a_n \rangle$ as an array of n finite state cells as defined above. The left neighbor input $\lambda_i = \omega_{i-1}$ and the right neighbor

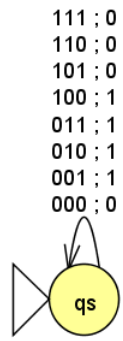


Figure 3: FSC diagram for an elementary rule 30 cell.

input $\rho_i = \omega_{i+1}$ for $i = 2, \ldots, n-1$. A cyclic-boundary FSCA connects the end inputs to the opposite end's output, so that $\lambda_1 = \omega_n$ and $\rho_n = \omega_1$. We say that A has a value $\mathbf{s} \in \{0,1\}^n$ when $\mathbf{s} = \omega_1|\omega_2|\ldots|\omega_n$ and more specifically we denote the value of A at time step t by $\mathbf{s}^{(t)}$.

It is useful to have a notion of the current configuration of the entire FSCA which describes the current output values and state for each cell. Let Q_i be the set of states for a_i, and $\mathcal{Q} = Q_1 \times Q_2 \times \ldots \times Q_n$. We say an FSCA A has *configuration* $C^{(t)} = (\mathbf{q}^{(t)}, \mathbf{s}^{(t)})$ at time step t for $\mathbf{q}^{(t)} \in \mathcal{Q}$ if $\mathbf{q}_i^{(t)}$ is the current state of a_i at time t for $i \leq n$, and $\mathbf{s}^{(t)}$ is the value of A at time t. We say $C^{(t)}$ *yields* $C^{(t+1)}$, written $C^{(t)} \vdash_A C^{(t+1)}$ if operating A with current configuration $C^{(t)}$ for one time step produces configuration $C^{(t+1)}$. For short hand, we may also write $C^{(t)} \vdash_A^k C^{(t+k)}$ to show the operation of A for k time steps.

We can represent elementary CA as a special kind of FSCA where each cell has only a single state. We simply define 8 self-transitions that map all possible $\lambda\omega\rho$ inputs to a new output ω'. This collection of transitions then defines a function from 3 bits to 1 bit, which is the rule of the cell over all time steps. An example FSC for an elementary rule 30 cell is shown in Figure 3.

5.3 Variations on FSCA

We have seen that single-state cells create FSCA which are equivalent to traditional elementary CA and it seems clear that two states can mimic alternating between two such rules (as proposed in [16]). We now explore the computational complexity of an n-cell FSCA with cells having for example 256 or n or n^2 states. Specifically, we would like to know at what number of cells does an FSCA become computationally non-invertible. We may also wonder about the effects of other limiting properties,

such as the effect of a limited branching factor on the transition paths through the states.

To capture these limiting notions, we use a function $B : \mathbb{N} \mapsto \mathbb{N}$ to provide an upper bound on the number of states any cell in an FSCA can have as a function of the number of cells in the FSCA, so that $|Q_i| \leq B(n)$ for $i \leq n$.

We define an *elementary* cell as one which obeys $B(n) = 1$ having only one state, q_s. With only one state, an elementary cell can have only one transition type: $\delta(q_s, \lambda\omega\rho) = (q_s, f(\lambda\omega\rho))$ where $f : \Sigma^3 \to \Sigma$ is the function defined by the set of self transitions of q_s. Therefore, the output of the cell at each time step is the result of a single, fixed function. Thus an FSCA with elementary cells is equivalent to an elementary CA.

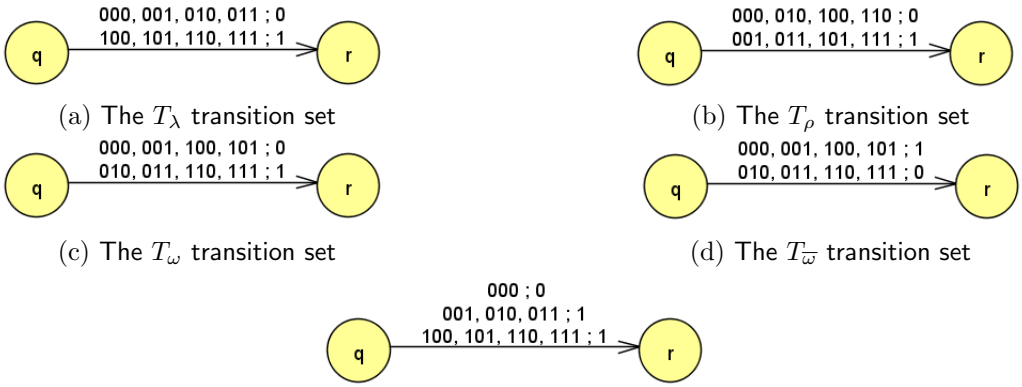
We say δ is *simple* if there is a $\delta' : Q \to Q$ and an $f : \Sigma^3 \to \Sigma$ such that for all $q \in Q$, $\delta(q, \lambda\omega\rho) = (\delta'(q), f(\lambda\omega\rho))$. Intuitively, this means that if there is any transition from q to another state r, then all input combinations take an FSC a from q to r. The inputs have no effect on selecting the next state and only the output bit on these transitions may vary. Under a simple δ, every state has exactly one successor state with f as the *time step function* computed at that time step. We say a cell is simple if its δ is simple, and an FSCA is simple if all cells are simple.

This notion of simple cells turns out to be quite an interesting one. If we construct special purpose cells to affect a specific function at each time step, we can have two neighboring cells swap their values at a certain time step, or compute the sum (XOR) and a carry (AND) of their two values for example. If an FSCA has only simple cells, it will perform the same computation without regard to the value of the cells at any time step. This begins to have the feel of a machine capable of universal computation.

It will be convenient to have a shorthand notation defining transitions which compute a given function for all combinations of $Q \times \Sigma^3$. We define a *transition set* $T_f(q, r)$ from a state q to a state r with respect to a time step function f as the set $\{((q, \lambda\omega\rho), (r, f(\lambda\omega\rho)))\}$. Where the implicit definition of f is simple, we will use its expression in the notation, e.g. $T_{\overline{\omega}}(q, r)$ denotes the use of the complement as the time step function. Such transition sets can be combined to define δ as a function. Defining some commonly used transition sets illustrates the concept and will also be useful in our discussion of FSC capabilities to follow.

$T_\lambda(q, r)$: $\{((q, \lambda\omega\rho), (r, \lambda))\}$.

$T_\rho(q, r)$: $\{((q, \lambda\omega\rho), (r, \rho))\}$.



(a) The T_λ transition set

(b) The T_ρ transition set

(c) The T_ω transition set

(d) The $T_{\overline{\omega}}$ transition set

(e) The $T_{\lambda+\omega+\rho}$ transition set

Figure 4: Transition sets for useful functions in FSCs. Input combinations separated by a comma in the diagrams is short-hand for distinct inputs which share the same resulting output which appears after a semi-colon.

$T_\omega(q,r)$: $\{((q,\lambda\omega\rho),(r,\omega))\}$.

$T_{\overline{\omega}}(q,r)$: $\{((q,\lambda\omega\rho),(r,\overline{\omega}))\}$.

$T_{\lambda+\omega+\rho}(q,r)$: $\{((q,\lambda\omega\rho),(r,\lambda+\omega+\rho))\}$ where $+$ denotes Boolean OR.

$T_\alpha(q,r)$: $\{((q,\lambda\omega\rho),(r,\alpha)) \mid \alpha \in \Sigma\}$.

These transition sets are shown in Figure 4 in diagrammatic form.

5.4 Evaluating 3-CNF formulas with FSCA

Given these functions, it's not hard to imagine building a simple FSCA capable of performing basic computational tasks, such as evaluating a Boolean formula for instance. Variable assignments are input as initial values in certain cells. Cells route these assignments to positions in each clause of the formula using the transition sets T_λ, T_ρ, and T_ω, all to arrive at some fixed time step. Literals that are complements of their variables would require the transition set $T_{\overline{\omega}}$. When all variables are positioned correctly and complemented according to the formula, the evaluation of the clauses begins. If the clauses have only three literals (i.e. the formula is in 3-CNF), this can be accomplished in one time step with the transition set $T_{\lambda+\omega+\rho}$ defined above. Those cells which do not compute the disjunction of a clause can simply use a transition set T_α with $\alpha = 0$. The conjunction of clauses then follows, with each clause value being routed again down to a single cell which holds the final formula value.

	cell											
t	1	2	3	4	5	6	7	8	9	10	11	12
0	α_1	0	α_2	0	0	α_3	0	α_4	0	0	α_5	0
1	α_1	α_1	α_2	α_2	α_3	α_3	α_3	α_4	α_4	α_5	α_5	α_5
2	α_1	α_2	α_1	α_3	α_2	α_3	α_3	α_4	α_4	α_5	α_5	α_5
3	α_1	α_2	α_3	α_1	α_3	α_2	α_4	α_3	α_5	α_4	α_5	α_5
4	α_1	α_2	α_3	α_1	α_3	α_2	α_4	α_5	α_3	α_5	α_4	α_5
5	α_1	α_2	α_3	α_1	α_3	α_2	α_5	α_4	α_5	α_3	α_4	α_5
6	α_1	α_2	α_3	α_1	α_3	α_5	α_2	α_4	α_5	α_3	α_4	α_5
7	α_1	α_2	$\overline{\alpha_3}$	α_1	α_3	α_5	$\overline{\alpha_2}$	$\overline{\alpha_4}$	α_5	α_3	α_4	$\overline{\alpha_5}$
8	0	c_1	0	0	c_2	0	0	c_3	0	0	c_4	0

Table 7: Computational history of A for formula ϕ with assignments $\alpha_1, \ldots, \alpha_5$. $c_1, \ldots, c_4$ give the values of each clause in ϕ for these assignments.

We illustrate this concept with an example. Suppose we have a 3-CNF formula over 5 variables in 4 clauses given by:

$$\phi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee x_3 \vee x_5) \wedge (\overline{x_2} \vee \overline{x_4} \vee x_5) \wedge (x_3 \vee x_4 \vee \overline{x_5}).$$

An FSCA A might use twelve cells to evaluate ϕ, one for each literal. At time step 0, five specially selected cells would be provided the assignments to $x_1, \ldots, x_5$ while other cells get 0. We will label the actual assignment values as $\alpha_1, \ldots, \alpha_5$. A computational history of A, showing the cell values at each time step, might look like Table 7. Here, evaluation is carried out only up to clause evaluation, at which point satisfiability is clear. Notice that at time step 1, all the dummy 0 values are gone, and each assignment value appears as many times as its variable appears in ϕ. From this point, routing is just a matter of permuting the contents of the cells. At time step 6, all cells contain the correct assignments ignoring negation. This is accounted for in step 7. Finally, step 8 performs the ORing of clauses. Cells not evaluating a clause get a fixed 0 value.

To formalize this example, we give the following theorem and constructive proof in which we consider 3-CNF formulas where no clause uses the same variable twice and no clause appears more than once (as these are easily reducible.) The theorem will only cover evaluation of the clauses of the formula. We adopt the following conventions in all pseudocode:

- [ ] denotes array/table indexing

- for a 3-CNF formula ϕ, $\langle\phi\rangle$ denotes an encoding of ϕ as an array of literals such that $\langle\phi\rangle[i] \vee \langle\phi\rangle[i+1] \vee \langle\phi\rangle[i+2]$ is a clause in ϕ if $i \equiv 1 \bmod 3$. Each literal ℓ in ϕ is encoded as a pair $\langle j, c\rangle$ where x_j is a variable in ϕ and $c = 1$ if $\ell = \overline{x_j}$ or $c = 0$ if $\ell = x_j$. $\langle\phi\rangle[8][1]$ is then the encoded variable of the second literal in the third clause of ϕ.

- Variables will appear as single symbols or as "$Variable$."

- Sub-algorithms and procedures will appear as "SubAlgorithm."

Theorem 5.1. *Let ϕ be a 3-CNF formula with c unique clauses and v variables $x_1, x_2, \ldots, x_v$ such that no variable appears twice in the same clause. There exists a simple FSCA A that evaluates the clauses of ϕ for any encoded assignments $\alpha_1, \alpha_2, \ldots, \alpha_v$.*

Proof. The proof is by construction. We wish to show we can create a machine whose computation will correctly evaluate ϕ. To do so, we will perform the computation in the abstract and construct the machine to mirror each step. The construction will follow the same steps as the example above:

1. Create an array of cells over which to perform the computation. Every group of 3 cells will correspond to a clause.

2. Determine the cells which will accept the assignments as initial values.

3. Duplicate the assignment values so there is one copy for each literal that needs it.

4. Distribute those values to their positions in the clauses.

5. Account for negation in the literals where necessary

6. OR the literals in the same clause together.

Consider Algorithm 5.1, 3-SatToFsca, on page 42 and its sub-algorithms GrowSpans (Algorithm 5.2 on page 43), FindDestinations (Algorithm 5.3 on page 44), and OrderAssignments (Algorithm 5.4 on page 45) which we now sketch. 3-SatToFsca takes as input a formula ϕ and constructs a machine which takes as input $3c$ Boolean values, v of which are the assignments to variables. As we cannot know what assignments will be provided, the algorithm simply identifies the locations which will contain an assignment α_j by storing the value j. The algorithm first constructs

the FSCA cells and computes the indices of cells whose initial values will be the assignments to evaluate. These indices are determined by the number of times each assignment is needed, and that number is stored in the array $Count$. If, for example, some α_j is used five times in the formula, it will be assigned a starting position $p[j]$ that is in the middle of a span of five cells, and the span for α_{j+1} will immediately follow. The array V will store the contents of each cell—either an index j of the indeterminate assignment value, or 0. The values in V at each stage of the construction indicate the value of the FSCA at the corresponding stage of its computational history.

GrowSpans replicates the initial assignment values for each cell in that assignment's span so that there are as many cells with an assignment α_j as there are appearances of its variable x_j in ϕ. As it does so, it adds states and transitions to the cells of the constructed FSCA A which perform the same copy operation. When complete, all cells will contain the assignment for the span to which they are assigned.

FindDestinations examines where in ϕ each assignment copy in V is needed and assigns that position as a destination. These destinations are stored in the array D which is one-to-one with V. If each assignment in V were moved to its corresponding destination in D, it would be in the same position as its variable in ϕ.

Creating the machinery to actually move these elements is the responsibility of OrderAssignments. This algorithm uses an Odd-Even sort on the destinations in D to re-order the contents in V. While doing so, it creates states and transitions in the cells of A to perform the same reordering. The result is the list of assignments which exactly mirrors the variables in ϕ.

Finally, 3-SatToFsca adds states and transitions to account for complementing literals that require it, and then for evaluating clauses.

These algorithms are in turn aided by SplitLeft (Procedure 5.1 on page 50), SplitRight (Procedure 5.2 on page 50), SplitLeftRight (Procedure 5.3 on page 50), Swap (Procedure 5.4 on page 50), and Remain (Procedure 5.5 on page 50). These procedures perform the moving of elements in V paired with creating states and transitions to perform the same moving of values in the corresponding cells of A. This helps to ensure the computational history of A will mirror what has happened in V.

We will develop the proof through a series of lemmata which establish some necessary properties.

Lemma 5.1. 3-SatToFsca *computes an evolution of V from an initial condition, having a single element for each encoded assignment and* 0*s otherwise, to a final condition at the return of* OrderAssignments *in which V contains the variables of*

Algorithm 5.1 3-SatToFsca

Input: $\langle\phi\rangle$, an encoding of a 3-CNF Boolean formula over the v variables $x_1, x_2, \ldots, x_v$.
Output: FSCA A, and $k \in \mathbb{N}$

$n = 3|\langle\phi\rangle|$
for $j = 1, \ldots, v$ **do** $Count[i] \leftarrow 0$
for $i = 1, \ldots, n$ **do** {initialize data structures used to create A}
  $Count[\langle\phi\rangle[i][1]] \leftarrow Count[\langle\phi\rangle[i][1]] + 1$
  $V[i] \leftarrow 0$
  Create an initial state $q_{i,0}$
  $Q_i \leftarrow \{q_{i,0}\}$
  Create a finite state cell a_i with output $\omega_i, Q = Q_i, \Sigma = \{0, 1\}, q_s = q_{i,0}, \delta = \emptyset$

  $Last[i] \leftarrow q_{i,0}$
$i \leftarrow 1$
for $j = 1, \ldots, |Count|$ **do** {determine initial positions for variables}
  $p[j] \leftarrow i + \lfloor (Count[j] - 1)/2 \rfloor$
  $V[p[j]] \leftarrow j$
  $i \leftarrow i + Count[j]$
$\lambda_1 \leftarrow \omega_n$; $\rho_1 \leftarrow \omega_2$ {connect all cells using cyclic boundaries}
$\rho_n \leftarrow \omega_1$; $\lambda_n \leftarrow \omega_{n-1}$
for $i = 2, \ldots, n-1$ **do**
  $\lambda_i \leftarrow \omega_{i-1}$
  $\rho_i \leftarrow \omega_{i+1}$
$A \leftarrow \langle a_1, a_2, \ldots, a_n \rangle$
$t \leftarrow$ GrowSpans$(A, V, p, Count, Last)$
$D \leftarrow$ FindDestinations$(\langle\phi\rangle, v)$
$t \leftarrow$ OrderAssignments$(A, V, D, t, Last)$
for $i = 1, \ldots, n$ **do** {account for complementation, evaluate all clauses}
  create states $q_{i,t+1}, q_{i,t+2}$
  $Q_i \leftarrow Q_i \cup \{q_{i,t+1}, q_{i,t+2}\}$
  if $\langle\phi\rangle[i][2] = 1$ **then**
    $\delta_i \leftarrow \delta_i \cup T_{\overline{\omega}}(Last[i], q_{i,t+1})$
  else
    $\delta_i \leftarrow \delta_i \cup T_{\omega}(Last[i], q_{i,t+1})$
  if $i \equiv 2 \mod 3$ **then**
    $\delta_i \leftarrow \delta_i \cup T_{\lambda+\omega+\rho}(q_{i,t+1}, q_{i,t+2})$
  else
    $\delta_i \leftarrow \delta_i \cup T_0(q_{i,t+1}, q_{i,t+2})$
  $\delta_i \leftarrow \delta_i \cup T_{\omega}(q_{i,t+2}, q_{i,t+2})$ {self-transition forever with same value}
return $(A, t+2)$

Algorithm 5.2 GROWSPANS

Input: $A, V, p, Count, Last$ where:
A is the array of cells,
V is the array of variables held in each cell,
p is the starting cell index of each variable,
$Count$ is the number of occurrences of each variable,
$Last$ is the last state created in each $a_i \in A$.

Output: t, the number of time steps used

```
for j = 1, ..., |Count| do
    Need[j] ← Count[j] − 1
    Span[j] ← ⟨p[j], p[j]⟩
t ← 0
while ∑_{N∈Need} N > 0 do
    t ← t + 1
    i ← 1
    for j = 1, ..., |Need| do
        if Need[j] = 0 then {skip to next span}
            next j
        else if Need[j] ≥ 2 then
            for k = i, ..., Span[j][1] − 2 do {visit all elements since last split up to next split}
                REMAIN(A, k, t, Last)
            if Span[j][1] = Span[j][2] then {for first split, grow in both directions}
                SPLITLEFTRIGHT(A, V, Span[j][1], t, Last)
                Span[j] ← ⟨Span[j][1] − 1, Span[j][2] + 1⟩ {adjust both ends of span}
                Need[j] ← Need[j] − 2
                i ← Span[j][2] + 1 {next element to consider is just right of the new span end}
                next j {begin work on the next span}
            else
                SPLITLEFT(A, V, Span[j][1], t, Last) {Grow the left side}
                Span[j] ← ⟨Span[j][1] − 1, Span[j][2]⟩ {adjust the left span end}
                Need[j] ← Need[j] − 1
                i ← Span[j][1] + 2
        {Need[j] > 0 implies the right end needs to grow, whether or not left end grows}
        for k = i, ..., Span[v][2] − 1 do
            REMAIN(A, k, t, Last) {keep current values up to end of span}
        SPLITRIGHT(A, V, Span[j][2], t, Last) {Grow right}
        Span[j] ← ⟨Span[j][1], Span[j][2]+1⟩ {adjust the right span end}
        Need[j] ← Need[j] − 1
        i ← Span[j][2] + 1
    for k = i, ..., |V| do
        REMAIN(A, k, t, Last) {keep current values to end of array}
return t
```

Algorithm 5.3 FindDestinations

Input: $\langle\phi\rangle, v$ where:

$\langle\phi\rangle$ is the 3-CNF formula, encoded as an array of literals, whose variables are targets for each cell,

v is the number of variables in ϕ.

Output: D, an array of destinations for each cell.

```
1: for j = 1, ..., v do Dest[j] ← []
2: for i = 1, ..., |⟨φ⟩| do
3:     Dest[⟨φ⟩[i][1]] ← Dest[⟨φ⟩[i][1]] || [i]  {append destinations for each variable to
       its own list}
4: i ← 0
5: for all d ∈ Dest do {concatenate the destinations}
6:     for j = i, ..., |d| do
7:         D[i + j] ← d[j]
8:     i ← i + |d|
9: return D
```

ϕ matching in order of appearance without regard to negation. That is, 3-SatToFsca *evolves V to the value $\langle\langle\phi\rangle[1][1], \langle\phi\rangle[2][1], \ldots, \langle\phi\rangle[n][1]\rangle$.*

Proof. Let the function $\#(j)$ be defined as the number of occurrences of x_j in ϕ and so must have value greater than 0 for all j. This is also the number of occurrences of α_j which will be needed to evaluate ϕ. We first show that V's initialization consists only of 0s and a single encoded element for each α_j. It is clear that $Count[j] \equiv \#(j)$ for $1 \leq j \leq v$ since $Count[j]$ is initialized to 0 in line 2 and is increased on line 4 of 3-SatToFsca by 1 for every occurrence of j (the encoding of x_j) in $\langle\phi\rangle$. We can also see that lines 10-14 create a span in V of length $\#(j)$ having the contents

$$0^{\lfloor\frac{1}{2}(\#(j)-1)\rfloor} j 0^{\lceil\frac{1}{2}(\#(j)-1)\rceil}$$

since: V is initialized to all 0s on line 5; j ranges over every variable which was counted in $\langle\phi\rangle$; $p[j]$ is initialized to the position $i + \lfloor(\#(j) - 1)/2\rfloor$ on line 12, and $V[p[j]]$ is assigned j on line 13. These spans begin at index 1 in V by line 10, and are all adjacent since each time through the loop, i is assigned $i + Count[j]$. Therefore, V on completing initialization has the form

$$\left\langle 0^{\lfloor\frac{1}{2}(\#(j)-1)\rfloor} j 0^{\lceil\frac{1}{2}(\#(j)-1)\rceil} \right\rangle_{j=1}^{v} \tag{11}$$

so that each assignment appears exactly once. This satisfies the initial condition of the theorem.

Algorithm 5.4 ORDERASSIGNMENTS

Input: $A, V, D, t, Last$ where:
A is the array of cells,
V is the array of variables held in each cell,
D is the array of destinations for each variable in V,
t is the time step at which distributing variables begins.
$Last$ is the array of last states created in each $a_i \in A$.

Output: t, the time step at which all variables are at their destinations.

```
Sorted ← false
while not Sorted do
    Sorted ← true
    t ← t + 1
    for i = 1 to |V| − 1 in steps of 2 do
        if D[i] > D[i + 1] then
            Swap(A, V, D, i, t, Last)
            Sorted ← false
        else
            Remain(A, i, t, Last)
            Remain(A, i + 1, t, Last)
    if |V| ≡ 0 mod 2 then
        Remain(A, |V|, t, Last)
    if Sorted then
        break
    Sorted ← true
    t ← t + 1
    Remain(A, 1, t, Last)
    for i = 2 to |V| − 1 in steps of 2 do
        if D[i] > D[i + 1] then
            Swap(A, V, D, i, t, Last)
            Sorted ← false
        else
            Remain(A, i, t, Last)
            Remain(A, i + 1, t, Last)
    if |V| ≡ 1 mod 2 then
        Remain(A, |V|, t, Last)
```

We next show that on return from GrowSpans, V has the value $\langle 1^{\#(1)}, 2^{\#(2)}, \ldots, v^{\#(v)} \rangle$ where $1, \ldots, v$ encode $\alpha_1, \ldots, \alpha_v$ and b^k denotes k sequential occurrences of b for $k > 0$ and the empty string otherwise. In GrowSpans, clearly $Need[j]$ is initialized to $\#(j) - 1$ for $1 \leq j \leq v$ by line 2. Since each span around the assignments in V has exactly one non-zero element, $Need[j]$ counts the number of 0s in each assignment's span. Notice that if $\#(j)$ is odd, $\#(j) - 1$ is even and so evenly divisible by 2, meaning there are an equal number of 0s on either side of an assignment j. Conversely, If $\#(j)$ is even, there is one more 0 on the right side of the assignment j. It is also clear that $Span[j]$ is initialized with a pair of indices of the first and last occurrence of j in V.

Let $\Delta(\langle x, y \rangle) = y - x + 1$ be a function which computes the number of elements in a span. We can see by induction that $\#(j) = \Delta(Span[j]) + Need[j]$ for all j at each pass through the while loop. For the base case of $\#(j) = 1$, we have $Span[j] = \langle p[j], p[j] \rangle$ so $\Delta(Span[j]) = 1$ and $Need[j] = 0$ for all j by line 2. Assume the relation holds at an arbitrary pass through the loop. On the next pass, there are three possibilities:

- $Need[j] \geq 2$, for which their are two possibilities: $\Delta(Span[j]) = 1$ and so SplitLeftRight is called, or $\Delta(Span[j]) > 1$ and SplitLeft then SplitRight is called. For the first of these, a 0 is replaced with j on both ends of the span, $\Delta(Span[j])$ is increased by 2, and $Need[j]$ is decreased by 2. No further action is taken for this case since the loop is abbreviated on line 19. For the second possibility, the loop first replaces a single 0 on the left end of the span by calling SplitLeft and then increases $\Delta(Span[j])$ by one and decreases $Need[j]$ by one. The loop then continues until it calls SplitRight, replacing a single 0 on the right end of the span, and then increases $\Delta(Span[j])$ by one and decreases $Need[j]$ by one. In either possibility, $\Delta(Span[j])$ is increased by the same amount as $Need[j]$ is decreased, so their sum is unchanged and the relation holds. Note also that if there were one 0 more on the right side of the span, that will still be the case after the pass through the loop since growth in this case is symmetric.

- $Need[j] = 1$, in which case there must be a single 0 on the right end of the span. The algorithm calls SplitRight, $\Delta(Span[j])$ is increased by one on the right end, and $Need[j]$ is decreased by one. Since $Span[j]$ is increased by the same amount $Need[j]$ is decreased, their sum is unchanged and the relation holds.

- $Need[j] = 0$, in which case no changes are made to $Span[j]$ or $Need[j]$ and so the relation still holds.

The for loop on line 8 repeats this process for all $j \leq v$. Since the inductive step holds,

the relation is true for every iteration of the while loop.

It is clear that each $Span[j]$ grows by replacing 0s with j each pass through the loop until $\Delta(Span[j]) = \#(j)$ and $Need[j] = 0$. Recall that $Need[j]$ began as the number of 0s in span j. Therefore when $Need[j] = 0$, the span must contain $j^{\#(j)}$, and so V must contain $\langle 1^{\#(1)}, 2^{\#(2)}, \ldots, v^{\#(v)} \rangle$ when $\Sigma_j Need[j] = 0$.

Next, we look at D returned by FindDestinations and show that $V[i] = j$ if and only if $\langle \phi \rangle [D[i]][1] = j$. First, Assume $V[i] = j$ and let $i - k$ be the left-most occurrence of j in V for some k. Then there must be $j - 1$ other assignment spans that appear to the left of $V[i - k]$. By the construction of D, $D[i]$ must be in $Dest[j]$ created on line 3 since $D[1, \ldots, i - k - 1]$ is the concatenation of the first $j - 1$ arrays in $Dest$ with the jth array to follow. But $D[i] \in Dest[j]$ implies there was discovered a literal in $\langle \phi \rangle$ having variable j at position i by the construction of $Dest[j]$ on line 3. Now suppose instead that $\langle \phi \rangle [D[i]][1] = j$. Then $Dest[j]$ contains $D[i]$, and so there is exactly $j - 1$ assignment spans in V before index i. Being in the jth span of V, $V[i]$ must have the value j.

Lastly, we examine the result of OrderAssignments. Ignoring the calls to Remain, which do not affect D or V, it is clear that OrderAssignments is exactly Odd-Even Sort on the elements of D with every swap carried out by Swap. Considering for the moment only the effect on D and V in Swap, it is clear that each change to one is duplicated in the other so that if $D[i]$ and $D[i + 1]$ are swapped on any given pass through the array, $V[i]$ and $V[i + 1]$ are also swapped and not otherwise. Since each $D[i]$ contains the index of a literal in $\langle \phi \rangle$ which requires the assignment $V[i]$, sorting the elements of $D[i]$ so that the corresponding elements in $V[i]$ are moved in exactly the same way must result in $D = \langle 1, 2, \ldots, n \rangle$ and correspondingly $V = \langle \langle \phi \rangle [1][1], \langle \phi \rangle [2][1], \ldots, \langle \phi \rangle [n][1] \rangle$. This proves the lemma. ∎

Lemma 5.2. *Let (A, k) be the result of* 3-SatToFsca *on input $\langle \phi \rangle$.* 3-SatToFsca *makes $k - 2$ passes through the array V and, for each pass, adds exactly one state to each $a_i \in A$.*

Proof. To see this, it is first useful to note that states are only added in procedures Remain, SplitLeftRight, SplitLeft, SplitRight, and Swap. Further, each of these adds exactly one new state to the cells they affect: Remain affects only the cell a_i for the given parameter i, SplitLeftRight affects cells a_{i-1}, a_i, and a_{i+1}, SplitLeft affects cells a_{i-1} and a_i, and SplitRight and Swap both affect cells a_i and a_{i+1}. In all cases, a single new state $q_{i,t}$ is added to Q_i, a transition set from $Last[i]$ to $q_{i,t}$ is defined for all possible inputs to the cell, and $Last[i]$ is updated to

refer to $q_{i,t}$ for all affected a_i. Therefore it is sufficient to show that one and only one of these procedures is called to affect, or *cover*, each cell on every pass through V. For this, we need only to consider only GrowSpans and OrderAssignments as V is not accessed elsewhere after initialization.

In GrowSpans, if $\Sigma_j Need[j] = 0$, then the number of passes equals the number of time steps added to A since both are 0. Otherwise, for each iteration of the while loop, there must be a next and, independently, a last changed span j in V. Assume i is set to the index of the first position not yet covered by one of the state-adding procedures. Suppose the next changed span j has $Need[j] \geq 2$. Then all positions from i to $Span[j][1] - 2$ remain unchanged in V, and Remain is called for each of those positions in the loop on line 12. The element of V at position $Span[j][1] - 1$ is then changed in one of two ways: either $\Delta(Span[j]) = 1$ and so SplitLeftRight is called, covering positions $Span[j][1] - 1, \ldots, Span[j][1] + 1$ and i is set one passed the new end of the span; or $\Delta(Span[j])) > 1$, in which case SplitLeft is called to cover positions $Span[j][1] - 1$ and $Span[j][1]$ and i gets assigned $Span[j][1] + 1$. In this second case, the algorithm will call Remain for every position up to $Span[j][2] - 1$, then cover positions $Span[j][2]$ and $Span[j][2] + 1$ with a call to SplitRight, and finally set i to one passed the new end of the span. In either case, all positions from the starting value of i up to the new end of the span are covered and i is set to the first position not yet covered. This is the condition in which we began.

Suppose instead that the next changed span j has $Need[j] = 1$. Then all positions from 1 to $Span[j][2] - 1$ remain unchanged in V, and Remain is called for each of those positions in the loop on line 26. Positions $Span[j][2]$ and $Span[j][2] + 1$ will be covered with a call to SplitRight, and i will be set to one passed the new end of the span. Again, we are in the starting condition.

Since it is clear for a base case where $i = 1$ and the next span to change has any $Need[j] > 0$ that all positions are covered from the starting i up to the new right end of the span, then by induction we see that all spans are so covered.

Now suppose the last changed span has been covered. Then i is set to the first position not yet covered and no other spans with $Need[j] > 0$ remain. Then all positions from i to $|V|$ are covered by calls to Remain in the loop on line 32. Notice that no position was covered more than once. Therefore, each pass through the while loop adds exactly 1 state to each a_i.

Since t is incremented each time through the while loop, and each pass through the loop adds exactly one state to each cell, t counts the number of states added in GrowSpans.

In OrderAssignments, coverage is easier to see. Either positions i and $i+1$ are swapped or they are both covered by a call to Remain in both the odd and even phase of the sort. In the even phase, position 1 is covered explicitly on line 18, and in both phases, any unpaired element at the end is covered conditionally in lines 13 and 27 with calls to Remain. Therefore, each pass of OrderAssignments adds exactly 1 state to each a_i.

Since t is passed to OrderAssignments holding the number of states added in GrowSpans, and t is incremented for each pass of the odd-even sort, and each pass adds exactly one state to each cell, t counts the number of states added in GrowSpans and OrderAssignments. Since $k = t + 2$ for the states added to each cell by the loop on line 24, GrowSpans and OrderAssignments make exactly $k - 2$ passes through V. ∎

Corollary 5.1. *A is simple.*

Proof. Since only one state is added to each cell for each time step; each newly added state becomes the last state; and only simple transitions are added from the last state to any new state, it is easy to see by induction that A is simple for all $t \leq k-2$. Finally, it is clear that the last three transition sets added by lines 27-35 only take a_i from $Last[i]$ to $q_{i,t+1}$, from $q_{i,t+1}$ to $q_{i,t+2}$, and from $q_{i,t+2}$ to $q_{i,t+2}$ respectively. Therefore, A is simple for all t. ∎

Lemma 5.3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_v$ be assignments to $x_1, x_2, \ldots, x_v$ in ϕ and let (A, k) be the result of 3-SatToFsca on input $\langle\phi\rangle$. If A is provided the initial values in $\mathbf{s}$ defined by*

$$\mathbf{s} = \left\langle 0^{\lfloor \frac{1}{2}(\#(j)-1) \rfloor} \alpha_j 0^{\lceil \frac{1}{2}(\#(j)-1) \rceil} \right\rangle_{j=1}^{v} \tag{12}$$

and operated for $k-2$ time steps, the resulting value of A is $\left\langle \alpha_{\langle\phi\rangle[1][1]}, \alpha_{\langle\phi\rangle[2][1]}, \ldots, \alpha_{\langle\phi\rangle[n][1]}, \right\rangle$.

Proof. We will denote the value in $V[i]$ at pass t by $V_i^{(t)}$ to make clear differences in value of the same location in different passes. Similarly, we will use $\omega_i^{(t)}$ to note the output of cell a_i at time step t . We show by induction that $V_i^{(t)} = j$ implies that $\omega_i^{(t)} = \alpha_j$ for $t \leq k-2$. For the base case of $t = 0$, the implication holds by comparison of (11) and (12).

Assume the implication holds for arbitrary $t < k - 2$. To show the implication holds for $t + 1$, we must consider two cases. Let t_E be the value of t returned from GrowSpans. First, suppose $t < t_E$, in which case pass $t + 1$ will be made inside GrowSpans. We know from Lemma 5.2 that every element in V is either explicitly

Procedure 5.1 SplitLeft

Parameters: $A, V, i, t, Last$

$V[i-1] \leftarrow V[i]$
Create a state $q_{i,t}$ in a_i
$Q_i \leftarrow Q_i \cup \{q_{i,t}\}$
$\delta_i \leftarrow \delta_i \cup T_\omega(Last[i], q_{i,t})$
$Last[i] \leftarrow q_{i,t}$
Create a state $q_{i-1,t}$ in a_{i-1}
$Q_{i-1} \leftarrow Q_{i-1} \cup \{q_{i-1,t}\}$
$\delta_{i-1} \leftarrow \delta_{i-1} \cup T_\rho(Last[i-1], q_{i-1,t})$
$Last[i-1] \leftarrow q_{i-1,t}$

Procedure 5.2 SplitRight

Parameters: $A, V, i, t, Last$

$V[i+1] \leftarrow V[i]$
Create a state $q_{i,t}$ in a_i
$Q_i \leftarrow Q_i \cup \{q_{i,t}\}$
$\delta_i \leftarrow \delta_i \cup T_\omega(Last[i], q_{i,t})$
$Last[i] \leftarrow q_{i,t}$
Create a state $q_{i+1,t}$ in a_{i+1}
$Q_{i+1} \leftarrow Q_{i+1} \cup \{q_{i+1,t}\}$
$\delta_{i+1} \leftarrow \delta_{i+1} \cup T_\lambda(Last[i+1], q_{i+1,t})$
$Last[i+1] \leftarrow q_{i+1,t}$

Procedure 5.3 SplitLeftRight

Parameters: $A, V, i, t, Last$

$V[i-1] \leftarrow V[i]; V[i+1] \leftarrow V[i]$
Create a state $q_{i,t}$ in a_i
$Q_i \leftarrow Q_i \cup \{q_{i,t}\}$
$\delta_i \leftarrow \delta_i \cup T_\omega(Last[i], q_{i,t})$
$Last[i] \leftarrow q_{i,t}$
Create a state $q_{i-1,t}$ in a_{i-1}
$Q_{i-1} \leftarrow Q_{i-1} \cup \{q_{i-1,t}\}$
$\delta_{i-1} \leftarrow \delta_{i-1} \cup T_\rho(Last[i-1], q_{i-1,t})$
$Last[i-1] \leftarrow q_{i-1,t}$
Create a state $q_{i+1,t}$ in a_{i+1}
$Q_{i+1} \leftarrow Q_{i+1} \cup \{q_{i+1,t}\}$
$\delta_{i+1} \leftarrow \delta_{i+1} \cup T_\lambda(Last[i+1], q_{i+1,t})$
$Last[i+1] \leftarrow q_{i+1,t}$

Procedure 5.4 Swap

Parameters: $A, V, D, i, t, Last$

$tmp \leftarrow V[i]$
$V[i] \leftarrow V[i+1]$
$V[i+1] \leftarrow tmp$
$tmp \leftarrow D[i]$
$D[i] \leftarrow D[i+1]$
$D[i+1] \leftarrow tmp$
Create a state $q_{i,t}$ in a_i
$Q_i \leftarrow Q_i \cup \{q_{i,t}\}$
$\delta_i \leftarrow \delta_i \cup T_\rho(Last[i], q_{i,t})$
$Last[i] \leftarrow q_{i,t}$
Create a state $q_{i+1,t}$ in a_{i+1}
$Q_{i+1} \leftarrow Q_{i+1} \cup \{q_{i+1,t}\}$
$\delta_{i+1} \leftarrow \delta_{i+1} \cup T_\lambda(Last[i+1], q_{i+1,t})$
$Last[i+1] \leftarrow q_{i+1,t}$

Procedure 5.5 Remain

Parameters: $A, i, t, Last$

Create a state $q_{i,t}$ in a_i
$Q_i \leftarrow Q_i \cup \{q_{i,t}\}$
$\delta_i \leftarrow \delta_i \cup T_\omega(Last[i], q_{i,t})$
$Last[i] \leftarrow q_{i,t}$

changed or it is explicitly not changed as the index i ranges over the positions of V. If an element is changed, the change must happen in a call to one of SplitLeftRight,

SplitLeft, or SplitRight. If it is *not* changed, Remain is called for position i. We examine each in turn.

In SplitLeftRight, notice that $V_{i-1}^{(t+1)}$ is assigned the value of the position to its right, $V_{i+1}^{(t+1)}$ is assigned the value of the position to its left, and $V_i^{(t+1)}$ keeps its previous value. So $V_{i-1}^{(t+1)} = V_{i+1}^{(t+1)} = V_i^{(t+1)} = V_i^{(t)}$. Correspondingly, the new transition set in a_{i-1}, on any input, takes the value of its right neighbor so that $\omega_{i-1}^{(t+1)} = \omega_i^{(t)}$. We know by the inductive hypothesis that a time step t, $\omega_i^{(t)} = \alpha_{V_i^{(t)}}$, so then $\omega_{i-1}^{(t+1)} = \alpha_{V_i^{(t)}}$. We also know that $V_{i-1}^{(t+1)} = V_i^{(t)}$, so it must be that $\omega_{i-1}^{(t+1)} = \alpha_{V_{i-1}^{(t+1)}}$. Likewise, a_{i+1} gets a new state and transition which takes its left neighbor's value, so $\omega_{i+1}^{(t+1)} = \omega_i^{(t)} = \alpha_{V_i^{(t)}}$ by hypothesis. But we also have $V_{i+1}^{(t+1)} = V_i^{(t)}$, so $\omega_{i+1}^{(t+1)} = \alpha_{V_{i+1}^{(t+1)}}$. Finally, a_i gets a new state and transition which keeps ω_i constant, and so $\omega_i^{(t+1)} = \omega_i^{(t)}$ which is by hypothesis equal to $\alpha_{V_i^{(t)}}$. And since $V_i^{(t)} = V_i^{(t+1)}$, we have $\omega_i^{(t+1)} = \alpha_{V_i^{(t+1)}}$. The implication holds for all cells affected by SplitLeftRight.

Similar arguments show that the implication also holds for SplitLeft, SplitRight, and Remain. Thus the hypothesis is true for $t < t_E$.

Now suppose $t \geq t_E$. Then pass $t+1$ will be made in OrderAssignments. As shown in Lemma 5.2, every element in V is either changed in Swap or it is left unchanged, in which case Remain is called. We have already shown the implication holds in Remain, so we have only Swap to contend with. The argument is very similar to the one above. Clearly $V_i^{(t+1)} = V_{i+1}^{(t)}$ and $V_{i+1}^{(t+1)} = V_i^{(t)}$. Since a_i adds the the transition set T_λ from $Last[i]$ to $q_{i,t+1}$, we know $\omega_i^{(t+1)} = \omega_{i+1}^{(t)}$. Likewise, for a_{i+1}, we know $\omega_{i+1}^{(t+1)} = \omega_i^{(t)}$. By the induction hypothesis, we have $\omega_i^{(t)} = \alpha_{V_i^{(t)}}$ and $\omega_{i+1}^{(t)} = \alpha_{V_{i+1}^{(t)}}$. Therefore, $\omega_{i+1}^{(t+1)} = \omega_i^{(t)} = \alpha_{V_i^{(t)}} = \alpha_{V_{i+1}^{(t+1)}}$ and $\omega_i^{(t+1)} = \omega_{i+1}^{(t)} = \alpha_{V_{i+1}^{(t)}} = \alpha_{V_i^{(t+1)}}$, so the implication holds for Swap as well, and by extension, for all $t_E \leq t \leq k-2$.

Since $(V_i^{(t)} = j) \implies (\omega_i^{(t)} = \alpha_j)$, then A at time $k-2$ has the value $\langle \omega_1^{(k-2)}, \omega_2^{(k-2)}, \ldots, \omega_n^{(k-2)} \rangle = \langle \alpha_{V_1^{(k-2)}}, \alpha_{V_2^{(k-2)}}, \ldots, \alpha_{V_n^{(k-2)}} \rangle$. By Lemma 5.1, $V = \langle \langle \phi \rangle [1][1], \langle \phi \rangle [2][1], \ldots, \langle \phi \rangle [n][1] \rangle$ after its last pass and by Lemma 5.2 there are $k-2$ passes through V. Therefore, A at time $k-2$ has the value $\left\langle \alpha_{\langle \phi \rangle [1][1]}, \alpha_{\langle \phi \rangle [2][1]}, \ldots, \alpha_{\langle \phi \rangle [n][1]} \right\rangle$.

■

We can now complete the proof of Theorem 5.1. Let (A, k) be the result of 3-SatToFsca on input $\langle \phi \rangle$ and let $\alpha_j = \mathtt{1}$ encode a `true` assignment and $\alpha_j = \mathtt{0}$ encode a `false` assignment to x_j. Define the initial values for A as

$$\mathbf{s} = \left\langle \mathtt{0}^{\lfloor \frac{1}{2}(\#(j)-1) \rfloor} \alpha_j \mathtt{0}^{\lceil \frac{1}{2}(\#(j)-1) \rceil} \right\rangle_{j=1}^{v}.$$

We claim $(\langle q_{0,0}, q_{1,0}, \ldots, q_{n,0}\rangle, \mathtt{s}) \vdash_A^k (\langle q_{0,k}, q_{1,k}, \ldots, q_{n,k}\rangle, \langle(\mathtt{010})^c\rangle)$ if and only iff ϕ is satisfied by the assignments $\alpha_1, \ldots, \alpha_v$.

First, assume the assignments encoded as $\alpha_1, \ldots, \alpha_v$ satisfy ϕ. By Lemma 5.3, we know the $k-2^{\text{nd}}$ value of A consists of the values $\alpha_1, \ldots, \alpha_v$ ordered as they appear in ϕ. Lines 24 through 30 in 3-SatToFsca make it clear that each a_i will complement its value if and only if the corresponding literal in ϕ is complemented, and so will have the opposite value at time $k-1$. Therefore, the value of A at time $k-1$ is exactly the encoded literals of ϕ evaluated for the assignments $\alpha_1, \ldots, \alpha_v$. Lines 31 through 34 show that every third cell starting with the second cell will compute the OR function, and all other cells will compute the 0 function at time k. So then ω_i will take the value $\omega_{i-1} \vee \omega_i \vee \omega_{i+1}$ for $i \equiv 2 \bmod 3$ and 0 otherwise. But these ORs exactly evaluate the clauses of ϕ when considering it's literals as an array of length $n = 3c$. Since ϕ is satisfied by $\alpha_1, \ldots, \alpha_v$, each such ORing of literals in this grouping must result in `true`, and so the OR of their encoding must be `1`. This implies the value of a_i will be `1` for $i \equiv 2 \bmod 3$ and `0` otherwise. Therefore, after k time steps, A has the configuration $(\langle q_{0,k}, q_{1,k}, \ldots, q_{n,k}\rangle, \langle(\mathtt{010})^c\rangle)$.

Conversely, suppose A with initial value $\mathtt{s}$ has the configuration $(\langle q_{0,k}, q_{1,k}, \ldots, q_{n,k}\rangle, \langle(\mathtt{010})^c\rangle)$ after k time steps. We know by the construction of each cell on lines 31 through 34 that those cells a_i for $i \equiv 2 \bmod 3$ with output value $\omega_i = \mathtt{1}$ are the result of the OR of their three inputs from the previous time step, and so $\omega_i^{(k)} = \omega_{i-1}^{(k-1)} \vee \omega_i^{(k-1)} \vee \omega_{i+1}^{(k-1)}$. We also know that $\omega_i^{(k-2)} = \alpha_{\langle\phi\rangle[i][1]}$ and that $\omega_i^{(k-1)} = \overline{\omega_i^{(k-2)}}$ if and only if the ith literal in ϕ is complemented. Then the values of a_i at time step $k-1$ are exactly the corresponding literals of ϕ when evaluated for the assignments $\alpha_1, \ldots, \alpha_v$. Since $\omega_{i-1}^{(k-1)} \vee \omega_i^{(k-1)} \vee \omega_{i+1}^{(k-1)} = \mathtt{1}$, the clauses over those literals must also evaluate to `true`, and so ϕ is satisfied.

Since a_i has value `1` for $i \equiv 2 \bmod 3$ only when ϕ is satisfied, A correctly evaluates the clauses of ϕ. ∎

5.5 Comparing Computational Ability

It can be difficult to get a sense of the computational efficiency of simple FSCA since it is a parallel construction on one hand, but a construction of machines much less powerful than Turing machines on the other. To facilitate a comparison, we will consider the number of time steps required to perform certain computations relative to other computational models. A Turing machine with alphabet $\Sigma = \{0, 1\}$, for instance, would require roughly $5n/3$ operations to evaluate the clauses of a 3-CNF formula

having n literals: checking and inverting each literal (n operations), then performing two OR operations for every three literals ($2n/3$ operations). This number would grow at least by a factor of n if we considered individual head movements. Our FSCA as constructed above, however, does somewhat better (without using the cyclic boundary property.) We examine this formally in the following theorem.

Theorem 5.2. *Let (A, k) be as returned from* 3-SatToFsca *on input $\langle\phi\rangle$ such that A has n cells. $k \leq 3n/2 + 2$.*

Proof. Lemma 5.2 shows that there is one state in each a_i for every pass through V, and that there are exactly $k - 2$ such passes. We simply bound $k - 2$ as a function of n. This is complicated somewhat by having no fixed relationship between the number of clauses and the number of variables. However, a coarse bound is still possible.

In GrowSpans, we note that each span can grow by two at every time step except possibly its last (if its ultimate size is even.) At worst, there is only one variable that needs to grow, and that variable is centered in its span by (11). Since the variable occupies one element of V at initialization (by Lemma 5.1), GrowSpans can require no more than $\lceil (n-1)/2 \rceil \leq n/2$ passes to fill the entire array.

The remaining passes are made by OrderAssignments. Recall that this algorithm functions exactly as Odd-Even sort. It is well known than Odd-Even sort can sort n variables in n passes.

Finally, k is assigned the number of passes made in GrowSpans and OrderAssignments plus 2. Therefore, $k \leq n/2 + n + 2 = 3n/2 + 2$, and so the bound holds. ∎

It appears in the case of evaluating 3-CNF formulas, even simple FSCA are capable of reasonably efficient operation.

Comparing simple FSCA to elementary FSCA, it is not clear an elementary FSCA could be built to evaluate a 3-CNF efficiently. If a cell is constructed to perform the OR of a clause, then it cannot also invert a literal or exchange a value with a neighbor as each cell is allowed only one function. Any such solution would require a more clever mixture of functions over neighborhoods of cells, and may end up relying on the ability to simulate a Turing machine, as shown for rule 110. [12]

5.6 Invertibility of FSCA

Having formally defined FSCA and examined some of their computational capability, we return to the issue of invertibility. We first need a formal definition of the problem,

which we provide for the general case of FSCA. We define the k-INVERT decision problem as follows:

$$k\text{-INVERT} = \{(A, C^{(t)}, k) | A \text{ is an FSCA of } n \text{ cells}, \exists\, C^{(t-k)}(C^{(t-k)} \vdash_A^k C^{(t)}) \text{ for } k \in \mathbb{N}\}$$

We would like to know whether deciding k-INVERT is NP-Hard. If so, then an FSCA could serve as a primitive on which to build a provably secure PRG provided its construction and operation are efficient. The efficiency condition motivates us to seek the simplest FSCA for which inversion is provably NP-Hard.

Elementary FSCAs do not inspire great confidence in this regard. Apart from the attacks already shown, the difficulty in evaluating a simple Boolean formula suggests a fundamental lack of ability to withstand analysis from an opponent armed with a Turing machine.

Simple FSCA, however, seem to hold more promise. If running such a machine forward from an assignment produces the evaluation of a formula, then running it backwards (i.e. inverting its operation) from an evaluation must produce an assignment. If the evaluation were a satisfying one, such an ability could be used to decide 3-SAT. This might be a little surprising. Notice that the simple restriction reduces the k-INVERT problem to just finding an appropriate $\mathbf{s}^{(t-k)}$, since $\mathbf{q}^{(t-k)}$ is easily deduced from $\mathbf{q}^{(t)}$: simply follow the transitions backwards from $\mathbf{q}^{(t)}$ for k states. However, it may be that the mixing behavior of the 3-neighbor construction over sufficiently many time steps provides the hardness we need. Therefore, we will examine simple FSCAs to determine if there is a $B(n)$ for which they are NP-Hard to k-invert.

We now formalize this intuition in a reduction from 3-SAT to a k-INVERT variant for simple FSCA.

Theorem 5.3. *Let k-SIMPLE-INVERT, abbreviated kSI, be defined as*

$$\{(A, C^{(t)}, k) \mid A \textit{ is a simple FSCA of } n \textit{ cells}, \exists\, C^{(t-k)}(C^{(t-k)} \vdash_A^k C^{(t)}), k \in \mathbb{N}\}.$$

kSI is NP-Complete for $k \geq 3n/2 + 2$.

Proof. We show that kSI $\in$ NP and that 3-SAT is polynomial-time reducible to kSI. Since 3-SAT is NP-Complete, this will prove kSI is NP-Complete.

The first condition is easy to see. Recall that $\mathbf{q}^{(t-k)}$ is easy to deduce from $\mathbf{q}^{(t)}$ and A which are encoded in the input string. We can create a decider for kSI which, given a certificate $\mathbf{s}^{(t-k)}$, performs the following: run A with configuration $(\mathbf{q}^{(t-k)}, \mathbf{s}^{(t-k)})$ for

k time steps. If the result is $C^{(t)}$, accept. Otherwise, reject. Since this decider requires only $O(kn)$ steps, we can verify kSI in polynomial time, and so kSI is in NP.

Now we show 3-SAT $\leq_{\mathrm{P}}$ kSI. Let ϕ be a 3-CNF formula having c unique clauses over v variables such that no clause repeats. Let $n = 3c$. Consider Algorithm 5.5, 3-SatToKSi.

Algorithm 5.5 3-SatToKSi

Input: $\langle\phi\rangle$, an encoding of ϕ.
Output: FSCA A, a configuration C, and $\kappa \in \mathbb{N}$

1: $(A, \kappa) \leftarrow$3-SatToFsca$(\langle\phi\rangle)$
2: $C \leftarrow (\mathbf{q}^{(\kappa)}, \langle(\mathtt{010})^c\rangle)$
3: **return** (A, C, κ)

First, note that 3-SatToKSi runs in time $O(n^2)$ since, by Lemma 5.2, 3-SatToFsca makes κ (which is $O(n)$) passes over an array of length n and C can be created in time $O(n)$.

Next, we show that the returned $(A, C, \kappa) \in k\mathrm{SI} \iff \phi \in$ 3-SAT. Suppose $(A, C, \kappa) \in k\mathrm{SI}$ and that $k = \kappa$. Then there exists a $C^{(0)}$ such that $C^{(0)} \vdash_A^\kappa C$. As C has a value where each $\mathbf{s}_i = \mathtt{1}$ for $i \equiv 2 \bmod 3$, and since by Theorem 5.1 A evaluates ϕ, ϕ must be satisfiable and so $\phi \in$ 3-SAT. Now consider any $k > \kappa$. By the construction of A, the value after time step κ never changes since all $q_{i,\kappa}$ transition only to themselves, keeping the same value. If A has value $\langle(\mathtt{010})^c\rangle$ at time step k, it must have had the same value at time step κ, and so ϕ is satisfiable and in 3-SAT.

Suppose $\phi \in$ 3-SAT and $k = \kappa$. Then there is a satisfying assignment, $\alpha_1, \ldots, \alpha_v$ for ϕ. Let

$$\mathbf{s} = \left\langle \mathtt{0}^{\lfloor \frac{1}{2}(\#(j)-1)\rfloor} \alpha_j \mathtt{0}^{\lceil \frac{1}{2}(\#(j)-1)\rceil} \right\rangle_{j=1}^{v}$$

Since by Theorem 5.1 A evaluates ϕ, then it must be that $(\mathbf{q}^{(0)}, \mathbf{s})) \vdash_A^\kappa (\mathbf{q}^{(\kappa)}, \langle(\mathtt{010})^c\rangle) = C$, and so $(A, C, \kappa) \in k\mathrm{SI}$. Since $C^{(\kappa)} \vdash_A^* C^{(\kappa)}$, $(A, C, \kappa) \in k\mathrm{SI}$ for all $k > \kappa$.

Finally, by Theorem 5.2, we know that $\kappa \leq 3n/2 + 2$. ∎

We can generalize this a bit further using the technique of padding as often done for other NP-Complete problems. Notice that 3-SatToFsca constructs a cyclic boundary FSCA, but never makes use of the boundary connections. We can therefore break the boundary connections, insert dummy cells, and connect those cyclically without affecting the operation of the FSCA in the original n cells. This changes the number of cells while leaving the number of time steps constant, allowing the ratio between the two to be an arbitrary one. This gives us the following theorem.

Theorem 5.4. *kSI is NP-Complete for $k \geq n/\sigma$ for any arbitrary $\sigma \in \mathbb{N}$.*

Proof. We again reduce 3-SAT to kSI. Let ϕ be a 3-CNF formula having c clauses. Our goal is to construct an FSCA of n cells which evaluates ϕ and a $k \in \mathbb{N}$ such that $n/\sigma \leq k$. Consider the following algorithm:

Algorithm 5.6 3-SatToσkSi

Input: $\langle\phi\rangle$, an encoding of ϕ,
$\sigma \in \mathbb{N}$.
Output: FSCA A, a configuration C, and $k \in \mathbb{N}$

```
(A', κ) ← 3-SatToFsca(⟨φ⟩)
n ← (9cσ/2) + 2σ
for i = 1, ..., n − 3c do {create n − 3c dummy cells}
    Create a cell a_{3c+i}
    Add states q_{3c+i,0}, ..., q_{3c+i,κ} with transition sets T_0(q_{3c+i,t}, q_{3c+i,t+1}), 1 ≤ t < κ
    Add transition set T_0(q_{3c+i,κ}, q_{3c+i,κ})
    ρ_{3c+i−1} ← ω_{3c+i}
    λ_{3c+i} ← ω_{3c+i−1}
ρ_1 ← ω_{3c+n}
λ_{3c+n} ← ω_1
A ← ⟨a_1, ..., a_{3c}, ..., a_{3c+n}⟩
C ← (q^(κ), ⟨(010)^c 0^{n−3c}⟩)
return (A, C, κ)
```

3-SatToσkSi performs $O(9\sigma c) + O(3c)$ operations, and so runs in time polynomial in c. Further, A clearly evaluates ϕ since A' evaluates ϕ without any communication from λ_1 or ρ_{3c} by Theorem 5.1. Thus A reaches the κth time step with a value of $(\texttt{010})^c\texttt{0}^{n-3c}$ iff there is an assignment $\alpha_1, \ldots, \alpha_v$ to the v variables of ϕ which satisfy it. Lastly, by Theorem 5.2, we know that $\kappa \leq 3(3c)/2 + 2$. Since $n = 9c\sigma/2 + 2\sigma = \sigma\kappa$, and A has a total of $3c$ cells (from A') plus $n - 3c$ dummy cells, A has $n = \sigma\kappa$ cells. Therefore, (A, C, κ) obeys $\kappa \geq n/\sigma$. As A's value never changes after time step κ, the theorem holds for all $k \geq \kappa \geq n/\sigma$. ∎

6 A PRG Based on FSCA

"One of the most singular characteristics of the art of deciphering is the strong conviction possessed by every person, even moderately acquainted with it, that he is able to construct a cipher which nobody else can decipher."–Charles Babbage.

6.1 Design of an FSCA-based PRG

Recall our thought experiment from Section 5, where the rule of each cell is selected uniformly at random. Under this scheme, we cannot know anything about the previous output values even one time step back, nor can we guess what functions will be used next and so have no ability to guess the next output. We would like to approach this ideal behavior to the extent possible in a PRG based on FSCA.

While our approximation of the ideal must be lacking in some aspects, uniform distribution of each cell's outputs is a required property. This suggests that the functions applied by a cell over time must on the whole have a uniform distribution as well. Suppose each cell generates an 8-bit value to specify the function at each time step. Any single such function may be clearly biased, but if each cell cycles through all possible 8-bit values, its output distribution is uniform in the aggregate. This is because the number of '1' bits in any fixed bit position across all possible 8-bit strings is 128, meaning there are 128 functions that yield a '1' on any fixed 3-bit input and 128 that yield a '0' for that same input. We can thus guarantee uniform distribution by having each cell cycle through all possible 256 time step functions.

Another desired property from our thought experiment is that of independent, randomly selected functions. While we do not aspire to truly random selection and we are limited to cycling through the 256 possible functions, we can at a minimum apply the functions in the order of some permutation which resists cryptanalysis. The S-Boxes of cryptographic primitives provide such permutations, and are commonly studied for resistance to linear and differential attacks. We choose the S-Box from AES as a basis for selecting the time step functions [34].

Following such a permutation does not achieve independence, though. At some point, the permutation will choose the 0 function. If all cells do so at the same time, the value would be fixed at **0**. We prevent this by having each cell cycle through the permutation in a different order and from a different starting position. We assign to

each cell an 8-bit offset value which is taken from the seed. This offset specifies the starting index in the S-Box when expressed as a lookup table. The function for the cell is then taken as the offset value XORed with the S-Box value at the cell's current index. XORing in a constant causes the time step function to use values from the S-Box in an order unique to that constant.

One difficulty in requiring the time step function to cycle through the 256 possible functions is the issue of biased functions at the end of the cycle. Suppose the last function for a cell in the cycle is the 0 function. On restarting the cycle, that cell will have its value stuck at 0. Even moderately biased functions at the end of the cycle lead to stuck bits in the value for the next cycle. Early experiments showed that most seeds became periodic after just 1 or 2 cycles. While the uniform distribution property requires the PRG to operate in cycles of 256, there is no requirement that they be the *same* cycle. We can change the cycle simply by changing the offsets. At the end of each cycle, we generate new offsets by:

1. Saving away the current value.
2. Stepping the FSCA for 8 time steps, saving the value away at each one to generate 8 bits for each cell.
3. XORing those 8 bits into the offsets for each cell.
4. Restoring the current value.

This process effectively changes the simple FSCA each 256 time steps, giving each cell a new time step function for each of its 256 states.

There are practical concerns in generating output bits from this construction. First, a cell's path through its cycle of functions may have periods of extreme bias, affecting the distribution of its output and that of its neighbors. If we simply return the FSCA's value during such a period, we may notice a bias in the resulting output. Further, we would also be revealing a considerable portion of the stored state (i.e. the value) of the PRG and so weaken its cryptographic strength.

Generating output by XORing FSCA values separated by a number of time steps seems to solve these problems. XORing two FSCA values leaves a slight bias in long sequences, but four seems to be sufficient in practice to remove bias. Different trade-offs between security and efficiency can be made here. Outputting the XOR of four FSCA values would also avoid directly revealing the internals of the PRG. Since n-cell simple FSCA are only hard to k-invert for some k as a function of n, we

would like to choose a number of time steps which is derived from n and also balances efficiency and security. We note that at $t = t_0 + n/2$, every cell is affected by every value at t_0, and so the value $\mathbf{s}^{(t_0)}$ is fully diffused over the FSCA. Paranoia inspires us to choose n rather than $n/2$ time steps as a minimum between generating outputs. Combining this with the necessary conditions to avoid bias, we generate output bits by XORing together four FSCA values, each separated by $n/4$ time steps.

Another practical consideration is allowing for regular or even constant (e.g. all 0) seeds. In this case, the offset is the same for all cells, resulting in a short period for the FSCA. In a fit of irony, we ensure entropy and chaos in the offsets by first XORing in the bytes of the most[4] harmonious algebraic number, φ, the golden ratio. This ensures each cell has some disordered offset from the S-Box values.

We name the resulting PRG "Chasm", owing to the etymology of the Greek word "chaos," originally meaning "void" or "chasm." We represent the current state of a Chasm PRG as a 5-tuple $(n, \mathbf{s}, \mathbf{o}, \mathbf{i}, c)$ where:

- $n \in \mathbb{N}$ is the number of cells in the PRG
- $\mathbf{s} \in \{0,1\}^n$ is the current value of the underlying FSCA
- $\mathbf{o} \in \{0,1\}^{8n}$ is the vector of 8-bit offsets for each cell, indexed as $\langle \mathbf{o}_1, \ldots, \mathbf{o}_{8n} \rangle$ where each $\mathbf{o}_i \in \{0,1\}^8$
- $\mathbf{i} \in \{0,1\}^{8n}$ is the vector of 8-bit indexes of each cell in the S-Box permutation, indexed as $\langle \mathbf{i}_1, \ldots, \mathbf{i}_n \rangle$ where each $\mathbf{i}_i \in \{0,1\}^8$
- $c \in \mathbb{N}$ is the cycle step counter

The algorithms ChasmInitialize (Algorithm 6.1), ChasmTimeStep (Procedure 6.1), and ChasmNext (Algorithm 6.2) fully specify the Chasm PRG operation. We use the notation $\mathbf{s} \lll k$ and $\mathbf{s} \ggg k$ to denote the cyclic shifting (rotating) of $\mathbf{s}$ by k bits to the left and right respectively. We use $\gg$ as right shift, $\oplus$ as XOR, $\cdot$ as AND, and $+$ as Boolean OR. These operations are done element-wise when the terms are vectors. We will also use φ to mean the golden ratio $(1+\sqrt{5})/2$ and φ_i to mean the ith bit of φ when represented in binary. $B[\mathbf{i}]$ denotes the vector of values B_j stored in the AES S-Box lookup table B for each index $j \in \mathbf{i}$. See Table 10 for the table values.

Note that ChasmNext requires modification when $n \nmid 256$ or when $n > 256$ to ensure the cycle is restarted at the correct time step. This presentation is simplified for clarity.

[4] Based on an incomplete survey.

Algorithm 6.1 ChasmInitialize

Input: $\sigma \in \{0,1\}^{9n}$, the seed.
Output: a Chasm PRG

1: $\mathbf{s} \leftarrow \langle \sigma_1, \ldots, \sigma_n \rangle \oplus \langle \varphi_1, \ldots, \varphi_n \rangle$
2: $\mathbf{o} \leftarrow \langle \sigma_{n+1}, \ldots, \sigma_{9n} \rangle \oplus \langle \varphi_{n+1}, \ldots, \varphi_{9n} \rangle$
3: $\mathbf{i} \leftarrow \mathbf{o}$
4: $c \leftarrow 0$
5: **return** $(n, \mathbf{s}, \mathbf{o}, \mathbf{i}, c)$

Procedure 6.1 ChasmTimeStep

Parameters: $g = (n, \mathbf{s}, \mathbf{o}, \mathbf{i}, c)$, a Chasm PRG

1: $\mathbf{x} \leftarrow ((\mathbf{s} \ggg 1) \ll \langle 2^n \rangle) + (\mathbf{s} \ll \langle 1^n \rangle) + \mathbf{s} \lll 1$ {{Collect the neighbors for each cell.}}
2: $\mathbf{f} \leftarrow B[\mathbf{i}] \oplus \mathbf{o}$
3: $\mathbf{s} \leftarrow (\mathbf{f} \gg \mathbf{x}) \cdot \langle 1^n \rangle$
4: **for** $j = 1, \ldots, n$ **do**
5: $\quad \mathbf{i}_j \leftarrow \mathbf{i}_j + 1$
6: $c \leftarrow c + 1$

6.2 Security of Chasm

Recall that two things are required to meet the definition of a forward-secure PRG:

1. The next output is hard to predict given previous outputs, and
2. The previous output is hard to compute given the current (stored) state of the generator.

Proof of either of these properties based on reasonable assumptions remains open. We conjecture, however, that computing previous outputs given the current state is closely related to the k-SIMPLE-INVERT problem.

6.3 Statistical Testing of Chasm

We have submitted Chasm to the Statistical Test Suite (STS) version 2.1.1 of the National Institute of Standards and Technology [37]. STS takes a number of sequences generated by the PRG in question and runs a battery of statistical tests. Each statistic has a distribution of possible values assuming the null hypothesis that the sequence is random is true. STS analyzes the observed distribution of statistics for each test and uses that to draw a conclusion about the null hypothesis.

A significance level α is chosen to help in drawing this conclusion. STS determines a p-value for each statistic run over a single test sequence. This p-value is the probability that a perfect random generator would produce a seemingly less random (or "worse") sequence than the one tested. The more extreme the p-value, the further out in the "tails" the test sequence is in the distribution of the possible statistic values when those values are computed using truly random sequences. If the p-value is less than our

Algorithm 6.2 ChasmNext

Input: $g = (n, \mathbf{s}, \mathbf{o}, \mathbf{i}, c)$, a Chasm PRG
Output: $\mathbf{v} \in \{0,1\}^n$

```
1:  v ← ⟨0^n⟩
2:  for j = 1, ..., 4 do
3:      for k = 1, ..., n/4 do
4:          ChasmTimeStep(g)  {update s}
5:      v ← v ⊕ s
6:  if c ≥ 256 then {check end of cycle}
7:      t ← s
8:      u ← ⟨⟩
9:      for j = 1, ..., 8 do
10:         ChasmTimeStep(g)  {update s}
11:         u ← u||s
12:     o ← o ⊕ u
13:     s ← t
14:     c ← 0
15: return v
```

significance level α, STS rejects the null hypothesis that the test sequence is random; it is simply too unlikely that it came from a random source.

While a single test sequence may result in a statistic outside the expected natural range, we must remember that randomness is a probabilistic property. A certain amount of failing sequences should be expected from any truly random generator; they should just happen according to the distribution of the statistic. For this reason, STS runs a group of sequences through each test and looks for a proportion of sequences passing a given statistic according to a confidence interval and also a uniform distribution of p-values in the group for that statistic. These two metrics provide for a high-level conclusion to be made on the PRG when tested with sufficient seeds, test sequences, and sequence length.

The 15 tests employed by the STS are as follows:

1. The Frequency (Monobit) Test,
2. Frequency Test within a Block,
3. The Runs Test,
4. Tests for the Longest-Run-of-Ones in a Block,
5. The Binary Matrix Rank Test,

6. The Discrete Fourier Transform (Spectral) Test,
7. The Non-overlapping Template Matching Test,
8. The Overlapping Template Matching Test,
9. Maurer's "Universal Statistical" Test,
10. The Linear Complexity Test,
11. The Serial Test,
12. The Approximate Entropy Test,
13. The Cumulative Sums (Cusums) Test,
14. The Random Excursions Test, and
15. The Random Excursions Variant Test.

Several of these tests are run a number of times with slightly different parameters, leading to a total of 188 individual tests. Each group comprises 100 sequences, each 1,000,000 bits in length, giving a total of 18,800 individual test results. STS reports the proportion of sequences which pass each individual statistic and compares that to the expected proportion computed at a chosen significance level. If the observed proportion is less than expected, we say the group has a *proportion* failure. STS also reports a χ^2 test on the distribution of the p-values to check that each statistic is uniformly distributed. If the resulting p-value of the distribution of p-values is less than 0.0001, we say the group has a *uniformity* failure. All testing uses a significance level $\alpha = 0.01$.

We wish to examine the behavior of Chasm using a number of different seeds and bit length configurations in order to build general confidence in the scheme. We chose 18 "structured" seeds having various regular patterns and 20 random seeds for Chasm PRGs having bit lengths of $n = 8$, 16, and 128. Structured seed patterns are listed in Table 8. Random seeds were obtained from the Hotbits radioactive decay generator [48]. We generated 12.5 MB of data from each seed for each n and used that data for the 100×1,000,000 bit test outlined above. Sequences were generated from a python implementation using numpy. For each group of 100 sequences, we recorded the STS uniformity p-value and the proportion of passed tests.

Seed	Pattern	Seed	Pattern	Seed	Pattern
1	0^*	7	$(1100)^*$	13	$(0^71)^*$
2	$(01)^*$	8	$0^{n/2}1^{n/2}$	14	$(1^70)^*$
3	$(010)^*$	9	$(1110)^*$	15	$(10^7)^*$
4	$(101)^*$	10	$(0001)^*$	16	$(01^7)^*$
5	1^*	11	$(1011)^*$	17	$(1^40^4)^*$
6	$(111000)^*$	12	$(0100)^*$	18	$(0^21^40^2)^*$

Table 8: Structured Seed patterns. These patterns are repeated as necessary to provide a $9n$-bit seed for each test.

6.4 Test Results

We will examine the test results from a number of viewpoints. Overall, there were 2,025,992 individual test statistics computed over 11,400 sequences of 1,000,000 bits each. Of these, 2,005,166 passed their individual test which is 141 less than the 2,005,307 we would expect at $\alpha = 0.01$. In straight percentages, this is a 98.972% passing rate. For comparison, we ran a 1,000-sequence test of 1,000,000 bits each on the Blum-Blum-Shub (BBS) generator, widely regarded as a strong PRG. This test passed 98.996% of individual tests.

Figure 5 charts how each tested Chasm generator configuration performed. The left bar in each category shows the number of observed failures of individual tests, while the right bar shows the maximum number of failures expected at $\alpha = 0.01$. Only the 128-bit configuration using random seeds goes beyond the expected value, failing 5 tests more than allowed. Notice there is a slightly different number of tests in each category. This is due to the variable nature of the Random Excursion tests, which use sequences of different lengths and so can produce a variable number of sequences to test. The lower numbers for structured configurations is simply due to testing only 18 structured seeds vs. 20 random seeds.

These sequences were tested in 21,432 groups of 100 sequences each. Of these, 20 groups had uniformity failures and 92 had proportion failures. Table 9 shows how these failures were distributed among the test configurations. For a closer look at the effect of various seed patterns, Figure 6 shows the number of each failure type for each seed. Seed numbers 19 and above are random, and are different for each n. Seed numbers 18 and below are of different lengths, but use the same pattern as described in Table 8 for all n. No clear pattern seems to emerge from this data.

We made a few other observations of a 4-bit configuration of Chasm. We generated 3,750,000 bytes of data from this generator seeded with 0s and subjected it to STS on

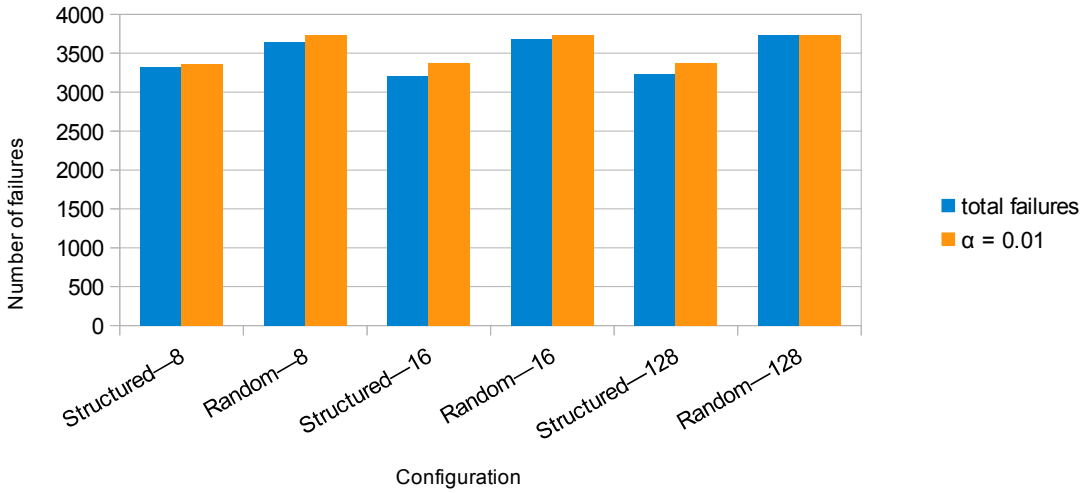


Figure 5: Individual test failures by Configuration

n	seed type	uniformity failures	proportion failures	% of groups
8	structured	11	27	0.13%
	random	4	18	0.08%
16	structured	1	11	0.05%
	random	1	8	0.04%
128	structured	3	12	0.06%
	random	0	16	0.07%

Table 9: Group test results by category

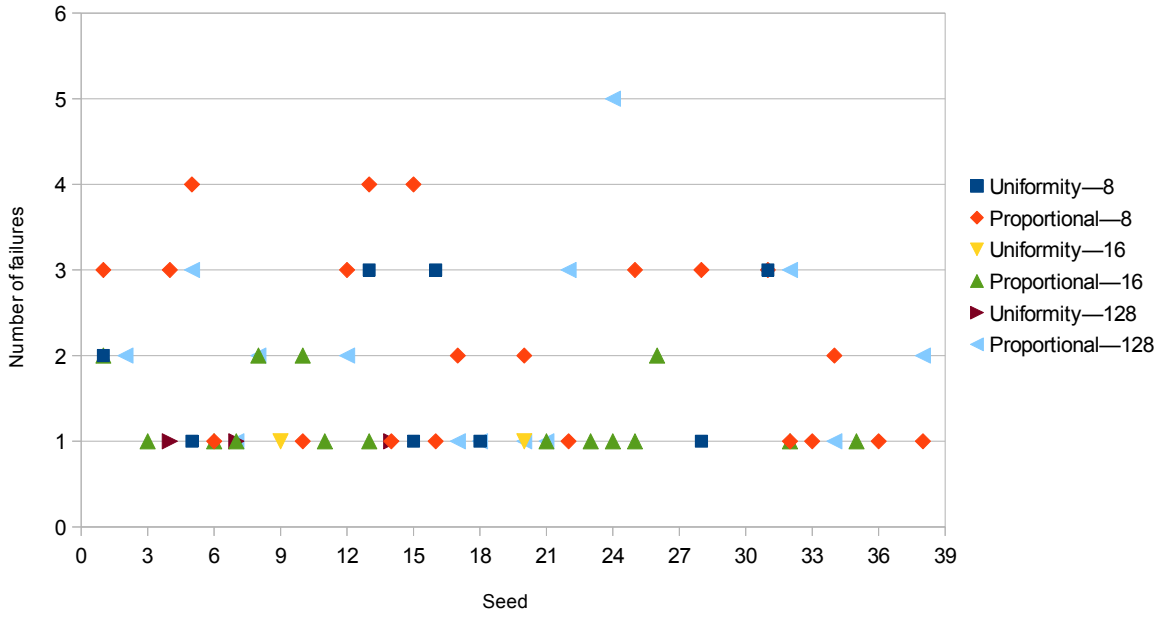


Figure 6: Group failures by seed

30 sequences of 1,000,000 bits each. The output had an average 0.49% bias towards '1' bits, and so failed 13 out of 30 groups in the frequency test, 10 and 14 groups in the two Cumulative Sums tests, and 8 groups in the Runs test. However, all other tests passed with no proportion or uniformity errors. These are impressive results for a 4-cell construction, especially considering the similar results of 18-cell constructions in [16]. Further, another run was observed to generate over 50 megabytes of data without exhibiting strictly periodic behavior. No statistical measurements were made on the result.

Finally, we note the python implementation was able to generate 475 megabytes in roughly 12 hours while the STS C implementation of the BBS generator took close to 72 hours to generate 125 megabytes.

7 Conclusions

The results of Section 3 and Section 4 make it hard to have confidence in the ability of one-dimensional two-state three-neighbor cyclic CA to provide secure cryptographic primitives according to modern definitions. When the rule set is known and linear and the entire state vector is known, inverting the CA is of course straight forward. Algorithm 3.1 extends this condition to uniform CA whose rule is a non-linear toggle

rule, the only kind shown to perform well in statistical testing. When a known temporal sequence is produced by a known rule set, linear or not, the techniques presented in Section 3.2 seem likely to provide a deterministic algorithm for recovering the seed of that sequence, improving on the previously known probabilistic algorithm. Only CA which choose their rules dynamically have no impending fatal cryptanalytic attack. Section 4 shows these CA are at best very inefficient when the dynamic rules are limited to linear rules. It may be that dynamically choosing from some set of non-linear rules provides better efficiency.

Returning to cellular automaton's historical roots by allowing multiple states per cell, on the other hand, provides a computational model whose power is determined by whether P=NP. Simple FSCA seem to offer a candidate one-way function for use in cryptographic primitives. We may imagine secure PRGs, hash functions, block ciphers, and stream ciphers based on carefully designed FSCA. These may also allow a security parameter which lets designers choose a security level appropriate for the application.

The Chasm family of PRGs presented in Section 6 approaches one such primitive. Chasm allows for a security parameter whereas block-based primitives, such as AES- or SHA-1-based PRGs require fixed sizes. An application can use Chasm at 3 bits and up. If the quality of data produced by low-n implementations can be shored up, this may be an attractive option for resource-constrained applications such as VLSI testing, Bluetooth/wireless applications, RFID readers, Key fobs, etc. The hardware requirements are a 256-byte lookup table, n-byte offset table (for the bytes of φ), n 8-to-1 MUXes, and an XOR accumulator. While demonstration that Chasm satisfies the definition of a forward-secure PRG assuming the one-wayness of FSCA remains open, it does not seem too far off. If shown, Chasm would offer a nicely parallel PRG suitable for hardware, GPU, and vector register implementations. Its performance even in single threaded interpreted languages seems far superior to the BBS generator, so a parallel implementation of Chasm with security proofs would be a very attractive primitive. Setting these proofs aside, our

test results suggest that Chasm is certainly a viable option for a statistical pseudorandom generator. While there are slightly more failures in the STS suite than one would like to see, it is quite conceivable that small adjustments to the Chasm algorithm can correct this. Simply mixing more values per output may be adequate.

Beyond cryptography, it may be worth considering other applications of more or less bounded FSCA. For instance, are their problems in NC0 which might be modeled as FSCA computations and examined from a different perspective? What would a poly(n) bounded FSCA be capable of? Are there applications to problems in PSPACE

or EXPTIME? It seems there are new questions for the adherents of cellular automata to tackle.

8 Future Work

There are quite a few open questions raised above. First, it may be interesting to consider a decision problem related to Proposition 3.2: Given the state vector for a uniform cyclic CA over any rule, is there a deterministic polynomial-time algorithm to compute its predecessor state if one exists? We've shown that, when certain patterns exist in the state vector for certain rules, such an algorithm exists. It may be interesting to consider the possibilities left when those patterns *do not* exist.

Second, the algorithm in Section 3.2 to improve the bounds on the Meier/Staffelbach algorithm remains to be fully developed and tested. Such an improvement would have an impact on much of the literature related to current CA-based cryptosystems. Related to this are all the open problems discussed in Section 4.3.

With respect to FSCA and Chasm, proofs of forward and backward security would be greatly beneficial to instill confidence in a new primitive. Also, further study of the linearity and differential properties of the generator is needed, as well as a more complete assessment of the potential for weak keys. Finally, applying FSCA to other cryptographic primitives seems likely to yield interesting results.

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A AES S-Box

The Values of the AES S-Box are provided for reference. See [34] for the full AES specification.

	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
0	63	7c	77	7b	f2	6b	6f	c5	30	01	67	2b	fe	d7	ab	76
10	ca	82	c9	7d	fa	59	47	f0	ad	d4	a2	af	9c	a4	72	c0
20	b7	fd	93	26	36	3f	f7	cc	34	a5	e5	f1	71	d8	31	15
30	04	c7	23	c3	18	96	5	9a	07	12	80	e2	eb	27	b2	75
40	09	83	2c	1a	1b	6e	5a	a0	52	3b	d6	b3	29	e3	2f	84
50	53	d1	00	ed	20	fc	b1	5b	6a	cb	be	39	4a	4c	58	cf
60	d0	ef	aa	fb	43	4d	33	85	45	f9	02	7f	50	3c	9f	a8
70	51	a3	40	8f	92	9d	38	f5	bc	b6	da	21	10	ff	f3	d2
80	cd	0c	13	ec	5f	97	44	17	c4	a7	7e	3d	64	5d	19	73
90	60	81	4f	dc	22	2a	90	88	46	ee	b8	14	de	5e	0b	db
a0	e0	32	3a	0a	49	06	24	5c	c2	d3	ac	62	91	95	e4	79
b0	e7	c8	37	6d	8d	d5	4e	a9	6c	56	f4	ea	65	7a	ae	08
c0	ba	78	25	2e	1c	a6	b4	c6	e8	dd	74	1f	4b	bd	8b	8a
d0	70	3e	b5	66	48	03	f6	0e	61	35	57	b9	86	c1	1d	9e
e0	e1	f8	98	11	69	d9	8e	94	9b	1e	87	e9	ce	55	28	df
f0	8c	a1	89	0d	bf	e6	42	68	41	99	2d	0f	b0	54	bb	16

Table 10: Hexadecimal values of the AES S-Box as a lookup table.

B Experiments

The sections in this appendix demonstrate representative experiments showing some property or other. These experiments were conducted with python 2.7.1 using Numpy 1.6.1. In some cases, actual python interpreter session are captured.

B.1 Effects of Rule Symmetry on Multiple Seeds for a Given Sequence

The following experiments show that symmetry in the rule set seems to affect the number of seeds that can generate the same temporal sequence. The function Other-

MatchingSeeds finds all seeds that produce the same temporal sequence over n steps as a given seed under a given rule set.

```
>>> OtherMatchingSeeds([150,150,150,150,90,150,150,150,150], SeedFromStr('010110110'))
[0 0 0 0 1 1 1 0 0]
[0 0 0 1 1 0 1 0 0]
[0 0 1 0 1 1 0 0 0]
[0 0 1 1 1 0 0 0 0]
[0 1 0 0 1 1 1 1 0]
[0 1 0 1 1 0 1 1 0]
[0 1 1 0 1 1 0 1 0]
[0 1 1 1 1 0 0 1 0]
[1 0 0 0 1 1 1 0 1]
[1 0 0 1 1 0 1 0 1]
[1 0 1 0 1 1 0 0 1]
[1 0 1 1 1 0 0 0 1]
[1 1 0 0 1 1 1 1 1]
[1 1 0 1 1 0 1 1 1]
[1 1 1 0 1 1 0 1 1]
[1 1 1 1 1 0 0 1 1]
>>> OtherMatchingSeeds([150,150,150,150,150,150,150,90,150], SeedFromStr('110010110'))
[0 1 1 1 1 1 0 1 1]
[1 1 0 0 1 0 1 1 0]
>>> OtherMatchingSeeds([150,150,150,150,150,150,150,150,90], SeedFromStr('110010110'))
[1 1 0 0 1 0 1 1 0]
```

Severely asymmetrical rules limit the matches severly–one rule 90 at the edge means there's a single seed that generates the temporal sequence.

```
>>> OtherMatchingSeeds([150,150,150,150,150,150,150,150,90], SeedFromStr('011001011'))
[0 1 1 0 0 1 0 1 1]
>>> OtherMatchingSeeds([150,150,150,150,150,150,150,150,90], SeedFromStr('101100101'))
[1 0 1 1 0 0 1 0 1]
>>> OtherMatchingSeeds([150,150,150,150,150,150,150,150,165], SeedFromStr('101100101'))
[1 0 1 1 0 0 1 0 1]
```

As long as that mismatched rule is a two-input rule, there's only one seed. If it's changed to a 3-input rule, even complementary, many matching seeds exist.

```
>>> OtherMatchingSeeds([150,150,150,150,150,150,150,150,105], SeedFromStr('101100101'))
[0 0 0 0 0 1 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 1 0 0]
[0 0 1 1 0 0 1 0 0]
[0 1 0 0 0 1 0 1 0]
[0 1 0 1 0 0 0 1 0]
[0 1 1 0 0 1 1 1 0]
[0 1 1 1 0 0 1 1 0]
[1 0 0 0 0 1 0 0 1]
[1 0 0 1 0 0 0 0 1]
[1 0 1 0 0 1 1 0 1]
```

```
[1 0 1 1 0 0 1 0 1]
[1 1 0 0 0 1 0 1 1]
[1 1 0 1 0 0 0 1 1]
[1 1 1 0 0 1 1 1 1]
[1 1 1 1 0 0 1 1 1]
>>> OtherMatchingSeeds([150,150,150,150,150,150,105,150,150], SeedFromStr('101100101'))
[0 0 0 0 0 1 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 1 0 0]
[0 0 1 1 0 0 1 0 0]
[0 1 0 0 0 1 0 1 0]
[0 1 0 1 0 0 0 1 0]
[0 1 1 0 0 1 1 1 0]
[0 1 1 1 0 0 1 1 0]
[1 0 0 0 0 1 0 0 1]
[1 0 0 1 0 0 0 0 1]
[1 0 1 0 0 1 1 0 1]
[1 0 1 1 0 0 1 0 1]
[1 1 0 0 0 1 0 1 1]
[1 1 0 1 0 0 0 1 1]
[1 1 1 0 0 1 1 1 1]
[1 1 1 1 0 0 1 1 1]
>>> OtherMatchingSeeds([150,150,150,150,105,150,105,150,150], SeedFromStr('101100101'))
[0 0 0 0 0 1 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 1 0 0]
[0 0 1 1 0 0 1 0 0]
[0 1 0 0 0 1 0 1 0]
[0 1 0 1 0 0 0 1 0]
[0 1 1 0 0 1 1 1 0]
[0 1 1 1 0 0 1 1 0]
[1 0 0 0 0 1 0 0 1]
[1 0 0 1 0 0 0 0 1]
[1 0 1 0 0 1 1 0 1]
[1 0 1 1 0 0 1 0 1]
[1 1 0 0 0 1 0 1 1]
[1 1 0 1 0 0 0 1 1]
[1 1 1 0 0 1 1 1 1]
[1 1 1 1 0 0 1 1 1]
>>> OtherMatchingSeeds([105,150,150,150,105,150,105,150,150], SeedFromStr('101100101'))
[0 0 0 0 0 1 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 1 0 0]
[0 0 1 1 0 0 1 0 0]
[0 1 0 0 0 1 0 1 0]
[0 1 0 1 0 0 0 1 0]
[0 1 1 0 0 1 1 1 0]
[0 1 1 1 0 0 1 1 0]
[1 0 0 0 0 1 0 0 1]
[1 0 0 1 0 0 0 0 1]
[1 0 1 0 0 1 1 0 1]
[1 0 1 1 0 0 1 0 1]
[1 1 0 0 0 1 0 1 1]
[1 1 0 1 0 0 0 1 1]
```

```
[1 1 1 0 0 1 1 1 1]
[1 1 1 1 0 0 1 1 1]
```

So placing complementary 3-input rules in various asymetric locations appears not to change the results. Here, different seeds are shown to produce the same temporal sequence.

```
>>> TempSeqFromSeed([105,150,150,150,105,150,105,150,150], SeedFromStr('100001001'), 18)
array([0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 0], dtype=uint8)
>>> TempSeqFromSeed([105,150,150,150,105,150,105,150,150], SeedFromStr('101100101'), 18)
array([0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 0], dtype=uint8)
```

The location of the asymmetry appears to affect the number of matching seeds. The following shows how the number of matches drops as we go from perfectly symmetrical through different asymmetry patterns. Notice that in the extreme cases, the only seed that matches is the original seed, stored as s1.

```
>>> s1 = SeedFromStr('101100101')
>>> rules = [150, 90, 150, 150, 150, 150, 150, 90, 150]
>>> OtherMatchingSeeds(rules, s1)
[0 0 0 0 0 1 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 1 0 0]
[0 0 1 1 0 0 1 0 0]
[0 1 0 0 0 1 0 1 0]
[0 1 0 1 0 0 0 1 0]
[0 1 1 0 0 1 1 1 0]
[0 1 1 1 0 0 1 1 0]
[1 0 0 0 0 1 0 0 1]
[1 0 0 1 0 0 0 0 1]
[1 0 1 0 0 1 1 0 1]
[1 0 1 1 0 0 1 0 1]
[1 1 0 0 0 1 0 1 1]
[1 1 0 1 0 0 0 1 1]
[1 1 1 0 0 1 1 1 1]
[1 1 1 1 0 0 1 1 1]
>>> rules = [90, 150, 150, 150, 150, 150, 150, 90, 150]
>>> OtherMatchingSeeds(rules, s1)
[1 0 1 1 0 0 1 0 1]
>>> rules = [90,  90, 150, 150, 150, 150, 150, 90, 150]
>>> OtherMatchingSeeds(rules, s1)
[1 0 1 1 0 0 1 0 1]
>>> rules = [150, 90,  90, 150, 150, 150, 150, 90, 150]
>>> OtherMatchingSeeds(rules, s1)
[0 1 0 1 0 0 0 0 1]
[0 1 1 0 0 1 1 1 0]
[1 0 0 0 0 1 0 1 0]
[1 0 1 1 0 0 1 0 1]
>>> rules = [150, 90,  90, 150, 150, 150,  90, 90, 150]
>>> OtherMatchingSeeds(rules, s1)
[0 0 0 0 0 1 0 0 0]
```

```
[0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 1 0 0]
[0 0 1 1 0 0 1 0 0]
[0 1 0 0 0 1 0 1 0]
[0 1 0 1 0 0 0 1 0]
[0 1 1 0 0 1 1 1 0]
[0 1 1 1 0 0 1 1 0]
[1 0 0 0 0 1 0 0 1]
[1 0 0 1 0 0 0 0 1]
[1 0 1 0 0 1 1 0 1]
[1 0 1 1 0 0 1 0 1]
[1 1 0 0 0 1 0 1 1]
[1 1 0 1 0 0 0 1 1]
[1 1 1 0 0 1 1 1 1]
[1 1 1 1 0 0 1 1 1]
>>> rules = [150, 90,  90, 150, 150, 150, 90, 150, 150]
>>> OtherMatchingSeeds(rules, s1)
[0 1 0 0 0 1 0 1 1]
[1 0 1 1 0 0 1 0 1]
>>> rules = [150,150,150,150,150,90,150,150,150]
>>> OtherMatchingSeeds(rules, s1)
[1 0 1 1 0 0 1 0 1]

>>> rules = [150,90,90,150,150,150,90,150,150,90,150,150,150,150,90,150,150]
>>> s3 = SeedFromStr('00101001110111101')
>>> OtherMatchingSeeds(rules, s3)
[0 0 1 0 1 0 0 1 1 1 0 1 1 1 1 0 1]
```

B.2 Solving Temporal Sequences Under Arbitrary Symmetrical Rulesets

This experiment examines how local rule differences affect a three-cell neighbor hood by looking at difference patterns in the output under different rule sets from a uniform rule 150 CA. The function prints ${}^{H}S^{t}_{i-1:i+1}$, ${}^{H}S^{t-1}_{i-2:i+2}$ for each predecessor, and ${}^{150}S^{t-1}_{i-2:i+2} \oplus {}^{H}S^{t-1}_{i-2:i+2}$ for all predecessors under a hybrid rule set H. The notation ${}^{R}S_{i:j}$ denotes the state vector values from cell i to j under rule set R.

This information shows how to get the same successor values under a different rule set by changing the predecessor state values. Notice that those rules that differ only in their complementarity have constant difference patterns for all successor values. There is no rule set with unique differences in all 8 successor values–4 seems to be the maximum.

```
Key:
s_t |-- S_i-2:i+2 at t-1 -| |--xor with preds(150)--|
>>> rs.hybridPredDiffs()
Hybrid Diff Rules: 90 90 90 :
000 00000 01010 10101 11111 00000 00111 00011 00100
```

```
001 00001 01011 10100 11110 00000 00111 00011 00100
010 00010 01000 10111 11101 00001 00110 00010 00101
011 00011 01001 10110 11100 00001 00110 00010 00101
100 00101 01111 10000 11010 00011 00100 00000 00111
101 00100 01110 10001 11011 00011 00100 00000 00111
110 00111 01101 10010 11000 00010 00101 00001 00110
111 00110 01100 10011 11001 00010 00101 00001 00110
Hybrid Diff Rules: 90 90 150 :
000 00000 01011 10101 11110 00000 00110 00011 00101
001 00001 01010 10100 11111 00000 00110 00011 00101
010 00011 01000 10110 11101 00000 00110 00011 00101
011 00010 01001 10111 11100 00000 00110 00011 00101
100 00101 01110 10000 11011 00011 00101 00000 00110
101 00100 01111 10001 11010 00011 00101 00000 00110
110 00110 01101 10011 11000 00011 00101 00000 00110
111 00111 01100 10010 11001 00011 00101 00000 00110
Hybrid Diff Rules: 90 90 105 :
000 00001 01010 10100 11111 00001 00111 00010 00100
001 00000 01011 10101 11110 00001 00111 00010 00100
010 00010 01001 10111 11100 00001 00111 00010 00100
011 00011 01000 10110 11101 00001 00111 00010 00100
100 00100 01111 10001 11010 00010 00100 00001 00111
101 00101 01110 10000 11011 00010 00100 00001 00111
110 00111 01100 10010 11001 00010 00100 00001 00111
111 00110 01101 10011 11000 00010 00100 00001 00111
Hybrid Diff Rules: 90 90 165 :
000 00001 01011 10100 11110 00001 00110 00010 00101
001 00000 01010 10101 11111 00001 00110 00010 00101
010 00011 01001 10110 11100 00000 00111 00011 00100
011 00010 01000 10111 11101 00000 00111 00011 00100
100 00100 01110 10001 11011 00010 00101 00001 00110
101 00101 01111 10000 11010 00010 00101 00001 00110
110 00110 01100 10011 11001 00011 00100 00000 00111
111 00111 01101 10010 11000 00011 00100 00000 00111
Hybrid Diff Rules: 90 150 90 :
000 00000 01010 10111 11101 00000 00111 00001 00110
001 00001 01011 10110 11100 00000 00111 00001 00110
010 00010 01000 10101 11111 00001 00110 00000 00111
011 00011 01001 10100 11110 00001 00110 00000 00111
100 00111 01101 10000 11010 00001 00110 00000 00111
101 00110 01100 10001 11011 00001 00110 00000 00111
110 00101 01111 10010 11000 00000 00111 00001 00110
111 00100 01110 10011 11001 00000 00111 00001 00110
Hybrid Diff Rules: 90 150 150 :
000 00000 01011 10110 11101 00000 00110 00000 00110
001 00001 01010 10111 11100 00000 00110 00000 00110
010 00011 01000 10101 11110 00000 00110 00000 00110
011 00010 01001 10100 11111 00000 00110 00000 00110
100 00110 01101 10000 11011 00000 00110 00000 00110
101 00111 01100 10001 11010 00000 00110 00000 00110
110 00101 01110 10011 11000 00000 00110 00000 00110
111 00100 01111 10010 11001 00000 00110 00000 00110
Hybrid Diff Rules: 90 150 105 :
000 00001 01010 10111 11100 00001 00111 00001 00111
```

```
001 00000 01011 10110 11101 00001 00111 00001 00111
010 00010 01001 10100 11111 00001 00111 00001 00111
011 00011 01000 10101 11110 00001 00111 00001 00111
100 00111 01100 10001 11010 00001 00111 00001 00111
101 00110 01101 10000 11011 00001 00111 00001 00111
110 00100 01111 10010 11001 00001 00111 00001 00111
111 00101 01110 10011 11000 00001 00111 00001 00111
Hybrid Diff Rules: 90 150 165 :
000 00001 01011 10110 11100 00001 00110 00000 00111
001 00000 01010 10111 11101 00001 00110 00000 00111
010 00011 01001 10100 11110 00000 00111 00001 00110
011 00010 01000 10101 11111 00000 00111 00001 00110
100 00110 01100 10001 11011 00000 00111 00001 00110
101 00111 01101 10000 11010 00000 00111 00001 00110
110 00100 01110 10011 11001 00001 00110 00000 00111
111 00101 01111 10010 11000 00001 00110 00000 00111
Hybrid Diff Rules: 90 105 90 :
000 00010 01000 10101 11111 00010 00101 00011 00100
001 00011 01001 10100 11110 00010 00101 00011 00100
010 00000 01010 10111 11101 00011 00100 00010 00101
011 00001 01011 10110 11100 00011 00100 00010 00101
100 00101 01111 10010 11000 00011 00100 00010 00101
101 00100 01110 10011 11001 00011 00100 00010 00101
110 00111 01101 10000 11010 00010 00101 00011 00100
111 00110 01100 10001 11011 00010 00101 00011 00100
Hybrid Diff Rules: 90 105 150 :
000 00011 01000 10101 11110 00011 00101 00011 00101
001 00010 01001 10100 11111 00011 00101 00011 00101
010 00000 01011 10110 11101 00011 00101 00011 00101
011 00001 01010 10111 11100 00011 00101 00011 00101
100 00101 01110 10011 11000 00011 00101 00011 00101
101 00100 01111 10010 11001 00011 00101 00011 00101
110 00110 01101 10000 11011 00011 00101 00011 00101
111 00111 01100 10001 11010 00011 00101 00011 00101
Hybrid Diff Rules: 90 105 105 :
000 00010 01001 10100 11111 00010 00100 00010 00100
001 00011 01000 10101 11110 00010 00100 00010 00100
010 00001 01010 10111 11100 00010 00100 00010 00100
011 00000 01011 10110 11101 00010 00100 00010 00100
100 00100 01111 10010 11001 00010 00100 00010 00100
101 00101 01110 10011 11000 00010 00100 00010 00100
110 00111 01100 10001 11010 00010 00100 00010 00100
111 00110 01101 10000 11011 00010 00100 00010 00100
Hybrid Diff Rules: 90 105 165 :
000 00011 01001 10100 11110 00011 00100 00010 00101
001 00010 01000 10101 11111 00011 00100 00010 00101
010 00001 01011 10110 11100 00010 00101 00011 00100
011 00000 01010 10111 11101 00010 00101 00011 00100
100 00100 01110 10011 11001 00010 00101 00011 00100
101 00101 01111 10010 11000 00010 00101 00011 00100
110 00110 01100 10001 11011 00011 00100 00010 00101
111 00111 01101 10000 11010 00011 00100 00010 00101
Hybrid Diff Rules: 90 165 90 :
000 00010 01000 10111 11101 00010 00101 00001 00110
```

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001 00011 01001 10110 11100 00010 00101 00001 00110
010 00000 01010 10101 11111 00011 00100 00000 00111
011 00001 01011 10100 11110 00011 00100 00000 00111
100 00111 01101 10010 11000 00001 00110 00010 00101
101 00110 01100 10011 11001 00001 00110 00010 00101
110 00101 01111 10000 11010 00000 00111 00011 00100
111 00100 01110 10001 11011 00000 00111 00011 00100
Hybrid Diff Rules: 90 165 150 :
000 00011 01000 10110 11101 00011 00101 00000 00110
001 00010 01001 10111 11100 00011 00101 00000 00110
010 00000 01011 10101 11110 00011 00101 00000 00110
011 00001 01010 10100 11111 00011 00101 00000 00110
100 00110 01101 10011 11000 00000 00110 00011 00101
101 00111 01100 10010 11001 00000 00110 00011 00101
110 00101 01110 10000 11011 00000 00110 00011 00101
111 00100 01111 10001 11010 00000 00110 00011 00101
Hybrid Diff Rules: 90 165 105 :
000 00010 01001 10111 11100 00010 00100 00001 00111
001 00011 01000 10110 11101 00010 00100 00001 00111
010 00001 01010 10100 11111 00010 00100 00001 00111
011 00000 01011 10101 11110 00010 00100 00001 00111
100 00111 01100 10010 11001 00001 00111 00010 00100
101 00110 01101 10011 11000 00001 00111 00010 00100
110 00100 01111 10001 11010 00001 00111 00010 00100
111 00101 01110 10000 11011 00001 00111 00010 00100
Hybrid Diff Rules: 90 165 165 :
000 00011 01001 10110 11100 00011 00100 00000 00111
001 00010 01000 10111 11101 00011 00100 00000 00111
010 00001 01011 10100 11110 00010 00101 00001 00110
011 00000 01010 10101 11111 00010 00101 00001 00110
100 00110 01100 10011 11001 00000 00111 00011 00100
101 00111 01101 10010 11000 00000 00111 00011 00100
110 00100 01110 10001 11011 00001 00110 00010 00101
111 00101 01111 10000 11010 00001 00110 00010 00101
Hybrid Diff Rules: 150 90 90 :
000 00000 01111 10101 11010 00000 00010 00011 00001
001 00001 01110 10100 11011 00000 00010 00011 00001
010 00010 01101 10111 11000 00001 00011 00010 00000
011 00011 01100 10110 11001 00001 00011 00010 00000
100 00101 01010 10000 11111 00011 00001 00000 00010
101 00100 01011 10001 11110 00011 00001 00000 00010
110 00111 01000 10010 11101 00010 00000 00001 00011
111 00110 01001 10011 11100 00010 00000 00001 00011
Hybrid Diff Rules: 150 90 150 :
000 00000 01110 10101 11011 00000 00011 00011 00000
001 00001 01111 10100 11010 00000 00011 00011 00000
010 00011 01101 10110 11000 00000 00011 00011 00000
011 00010 01100 10111 11001 00000 00011 00011 00000
100 00101 01011 10000 11110 00011 00000 00000 00011
101 00100 01010 10001 11111 00011 00000 00000 00011
110 00110 01000 10011 11101 00011 00000 00000 00011
111 00111 01001 10010 11100 00011 00000 00000 00011
Hybrid Diff Rules: 150 90 105 :
000 00001 01111 10100 11010 00001 00010 00010 00001
```

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001 00000 01110 10101 11011 00001 00010 00010 00001
010 00010 01100 10111 11001 00001 00010 00010 00001
011 00011 01101 10110 11000 00001 00010 00010 00001
100 00100 01010 10001 11111 00010 00001 00001 00010
101 00101 01011 10000 11110 00010 00001 00001 00010
110 00111 01001 10010 11100 00010 00001 00001 00010
111 00110 01000 10011 11101 00010 00001 00001 00010
Hybrid Diff Rules: 150 90 165 :
000 00001 01110 10100 11011 00001 00011 00010 00000
001 00000 01111 10101 11010 00001 00011 00010 00000
010 00011 01100 10110 11001 00000 00010 00011 00001
011 00010 01101 10111 11000 00000 00010 00011 00001
100 00100 01011 10001 11110 00010 00000 00001 00011
101 00101 01010 10000 11111 00010 00000 00001 00011
110 00110 01001 10011 11100 00011 00001 00000 00010
111 00111 01000 10010 11101 00011 00001 00000 00010
Hybrid Diff Rules: 150 150 90 :
000 00000 01101 10111 11010 00000 00000 00001 00001
001 00001 01100 10110 11011 00000 00000 00001 00001
010 00010 01111 10101 11000 00001 00001 00000 00000
011 00011 01110 10100 11001 00001 00001 00000 00000
100 00111 01010 10000 11101 00001 00001 00000 00000
101 00110 01011 10001 11100 00001 00001 00000 00000
110 00101 01000 10010 11111 00000 00000 00001 00001
111 00100 01001 10011 11110 00000 00000 00001 00001
Hybrid Diff Rules: 150 150 150 :
000 00000 01101 10110 11011 00000 00000 00000 00000
001 00001 01100 10111 11010 00000 00000 00000 00000
010 00011 01110 10101 11000 00000 00000 00000 00000
011 00010 01111 10100 11001 00000 00000 00000 00000
100 00110 01011 10000 11101 00000 00000 00000 00000
101 00111 01010 10001 11100 00000 00000 00000 00000
110 00101 01000 10011 11110 00000 00000 00000 00000
111 00100 01001 10010 11111 00000 00000 00000 00000
Hybrid Diff Rules: 150 150 105 :
000 00001 01100 10111 11010 00001 00001 00001 00001
001 00000 01101 10110 11011 00001 00001 00001 00001
010 00010 01111 10100 11001 00001 00001 00001 00001
011 00011 01110 10101 11000 00001 00001 00001 00001
100 00111 01010 10001 11100 00001 00001 00001 00001
101 00110 01011 10000 11101 00001 00001 00001 00001
110 00100 01001 10010 11111 00001 00001 00001 00001
111 00101 01000 10011 11110 00001 00001 00001 00001
Hybrid Diff Rules: 150 150 165 :
000 00001 01100 10110 11011 00001 00001 00000 00000
001 00000 01101 10111 11010 00001 00001 00000 00000
010 00011 01110 10100 11001 00000 00000 00001 00001
011 00010 01111 10101 11000 00000 00000 00001 00001
100 00110 01011 10001 11100 00000 00000 00001 00001
101 00111 01010 10000 11101 00000 00000 00001 00001
110 00100 01001 10011 11110 00001 00001 00000 00000
111 00101 01000 10010 11111 00001 00001 00000 00000
Hybrid Diff Rules: 150 105 90 :
000 00010 01111 10101 11000 00010 00010 00011 00011
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001 00011 01110 10100 11001 00010 00010 00011 00011
010 00000 01101 10111 11010 00011 00011 00010 00010
011 00001 01100 10110 11011 00011 00011 00010 00010
100 00101 01000 10010 11111 00011 00011 00010 00010
101 00100 01001 10011 11110 00011 00011 00010 00010
110 00111 01010 10000 11101 00010 00010 00011 00011
111 00110 01011 10001 11100 00010 00010 00011 00011
Hybrid Diff Rules: 150 105 150 :
000 00011 01110 10101 11000 00011 00011 00011 00011
001 00010 01111 10100 11001 00011 00011 00011 00011
010 00000 01101 10110 11011 00011 00011 00011 00011
011 00001 01100 10111 11010 00011 00011 00011 00011
100 00101 01000 10011 11110 00011 00011 00011 00011
101 00100 01001 10010 11111 00011 00011 00011 00011
110 00110 01011 10000 11101 00011 00011 00011 00011
111 00111 01010 10001 11100 00011 00011 00011 00011
Hybrid Diff Rules: 150 105 105 :
000 00010 01111 10100 11001 00010 00010 00010 00010
001 00011 01110 10101 11000 00010 00010 00010 00010
010 00001 01100 10111 11010 00010 00010 00010 00010
011 00000 01101 10110 11011 00010 00010 00010 00010
100 00100 01001 10010 11111 00010 00010 00010 00010
101 00101 01000 10011 11110 00010 00010 00010 00010
110 00111 01010 10001 11100 00010 00010 00010 00010
111 00110 01011 10000 11101 00010 00010 00010 00010
Hybrid Diff Rules: 150 105 165 :
000 00011 01110 10100 11001 00011 00011 00010 00010
001 00010 01111 10101 11000 00011 00011 00010 00010
010 00001 01100 10110 11011 00010 00010 00011 00011
011 00000 01101 10111 11010 00010 00010 00011 00011
100 00100 01001 10011 11110 00010 00010 00011 00011
101 00101 01000 10010 11111 00010 00010 00011 00011
110 00110 01011 10001 11100 00011 00011 00010 00010
111 00111 01010 10000 11101 00011 00011 00010 00010
Hybrid Diff Rules: 150 165 90 :
000 00010 01101 10111 11000 00010 00000 00001 00011
001 00011 01100 10110 11001 00010 00000 00001 00011
010 00000 01111 10101 11010 00011 00001 00000 00010
011 00001 01110 10100 11011 00011 00001 00000 00010
100 00111 01000 10010 11101 00001 00011 00010 00000
101 00110 01001 10011 11100 00001 00011 00010 00000
110 00101 01010 10000 11111 00000 00010 00011 00001
111 00100 01011 10001 11110 00000 00010 00011 00001
Hybrid Diff Rules: 150 165 150 :
000 00011 01101 10110 11000 00011 00000 00000 00011
001 00010 01100 10111 11001 00011 00000 00000 00011
010 00000 01110 10101 11011 00011 00000 00000 00011
011 00001 01111 10100 11010 00011 00000 00000 00011
100 00110 01000 10011 11101 00000 00011 00011 00000
101 00111 01001 10010 11100 00000 00011 00011 00000
110 00101 01011 10000 11110 00000 00011 00011 00000
111 00100 01010 10001 11111 00000 00011 00011 00000
Hybrid Diff Rules: 150 165 105 :
000 00010 01100 10111 11001 00010 00001 00001 00010
```

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001 00011 01101 10110 11000 00010 00001 00001 00010
010 00001 01111 10100 11010 00010 00001 00001 00010
011 00000 01110 10101 11011 00010 00001 00001 00010
100 00111 01001 10010 11100 00001 00010 00010 00001
101 00110 01000 10011 11101 00001 00010 00010 00001
110 00100 01010 10001 11111 00001 00010 00010 00001
111 00101 01011 10000 11110 00001 00010 00010 00001
Hybrid Diff Rules: 150 165 165 :
000 00011 01100 10110 11001 00011 00001 00000 00010
001 00010 01101 10111 11000 00011 00001 00000 00010
010 00001 01110 10100 11011 00010 00000 00001 00011
011 00000 01111 10101 11010 00010 00000 00001 00011
100 00110 01001 10011 11100 00000 00010 00011 00001
101 00111 01000 10010 11101 00000 00010 00011 00001
110 00100 01011 10001 11110 00001 00011 00010 00000
111 00101 01010 10000 11111 00001 00011 00010 00000
Hybrid Diff Rules: 105 90 90 :
000 00101 01010 10000 11111 00101 00111 00110 00100
001 00100 01011 10001 11110 00101 00111 00110 00100
010 00111 01000 10010 11101 00100 00110 00111 00101
011 00110 01001 10011 11100 00100 00110 00111 00101
100 00000 01111 10101 11010 00110 00100 00101 00111
101 00001 01110 10100 11011 00110 00100 00101 00111
110 00010 01101 10111 11000 00111 00101 00100 00110
111 00011 01100 10110 11001 00111 00101 00100 00110
Hybrid Diff Rules: 105 90 150 :
000 00101 01011 10000 11110 00101 00110 00110 00101
001 00100 01010 10001 11111 00101 00110 00110 00101
010 00110 01000 10011 11101 00101 00110 00110 00101
011 00111 01001 10010 11100 00101 00110 00110 00101
100 00000 01110 10101 11011 00110 00101 00101 00110
101 00001 01111 10100 11010 00110 00101 00101 00110
110 00011 01101 10110 11000 00110 00101 00101 00110
111 00010 01100 10111 11001 00110 00101 00101 00110
Hybrid Diff Rules: 105 90 105 :
000 00100 01010 10001 11111 00100 00111 00111 00100
001 00101 01011 10000 11110 00100 00111 00111 00100
010 00111 01001 10010 11100 00100 00111 00111 00100
011 00110 01000 10011 11101 00100 00111 00111 00100
100 00001 01111 10100 11010 00111 00100 00100 00111
101 00000 01110 10101 11011 00111 00100 00100 00111
110 00010 01100 10111 11001 00111 00100 00100 00111
111 00011 01101 10110 11000 00111 00100 00100 00111
Hybrid Diff Rules: 105 90 165 :
000 00100 01011 10001 11110 00100 00110 00111 00101
001 00101 01010 10000 11111 00100 00110 00111 00101
010 00110 01001 10011 11100 00101 00111 00110 00100
011 00111 01000 10010 11101 00101 00111 00110 00100
100 00001 01110 10100 11011 00111 00101 00100 00110
101 00000 01111 10101 11010 00111 00101 00100 00110
110 00011 01100 10110 11001 00110 00100 00101 00111
111 00010 01101 10111 11000 00110 00100 00101 00111
Hybrid Diff Rules: 105 150 90 :
000 00111 01010 10000 11101 00111 00111 00110 00110
```

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001 00110 01011 10001 11100 00111 00111 00110 00110
010 00101 01000 10010 11111 00110 00110 00111 00111
011 00100 01001 10011 11110 00110 00110 00111 00111
100 00000 01101 10111 11010 00110 00110 00111 00111
101 00001 01100 10110 11011 00110 00110 00111 00111
110 00010 01111 10101 11000 00111 00111 00110 00110
111 00011 01110 10100 11001 00111 00111 00110 00110
Hybrid Diff Rules: 105 150 150 :
000 00110 01011 10000 11101 00110 00110 00110 00110
001 00111 01010 10001 11100 00110 00110 00110 00110
010 00101 01000 10011 11110 00110 00110 00110 00110
011 00100 01001 10010 11111 00110 00110 00110 00110
100 00000 01101 10110 11011 00110 00110 00110 00110
101 00001 01100 10111 11010 00110 00110 00110 00110
110 00011 01110 10101 11000 00110 00110 00110 00110
111 00010 01111 10100 11001 00110 00110 00110 00110
Hybrid Diff Rules: 105 150 105 :
000 00111 01010 10001 11100 00111 00111 00111 00111
001 00110 01011 10000 11101 00111 00111 00111 00111
010 00100 01001 10010 11111 00111 00111 00111 00111
011 00101 01000 10011 11110 00111 00111 00111 00111
100 00001 01100 10111 11010 00111 00111 00111 00111
101 00000 01101 10110 11011 00111 00111 00111 00111
110 00010 01111 10100 11001 00111 00111 00111 00111
111 00011 01110 10101 11000 00111 00111 00111 00111
Hybrid Diff Rules: 105 150 165 :
000 00110 01011 10001 11100 00110 00110 00111 00111
001 00111 01010 10000 11101 00110 00110 00111 00111
010 00100 01001 10011 11110 00111 00111 00110 00110
011 00101 01000 10010 11111 00111 00111 00110 00110
100 00001 01100 10110 11011 00111 00111 00110 00110
101 00000 01101 10111 11010 00111 00111 00110 00110
110 00011 01110 10100 11001 00110 00110 00111 00111
111 00010 01111 10101 11000 00110 00110 00111 00111
Hybrid Diff Rules: 105 105 90 :
000 00101 01000 10010 11111 00101 00101 00100 00100
001 00100 01001 10011 11110 00101 00101 00100 00100
010 00111 01010 10000 11101 00100 00100 00101 00101
011 00110 01011 10001 11100 00100 00100 00101 00101
100 00010 01111 10101 11000 00100 00100 00101 00101
101 00011 01110 10100 11001 00100 00100 00101 00101
110 00000 01101 10111 11010 00101 00101 00100 00100
111 00001 01100 10110 11011 00101 00101 00100 00100
Hybrid Diff Rules: 105 105 150 :
000 00101 01000 10011 11110 00101 00101 00101 00101
001 00100 01001 10010 11111 00101 00101 00101 00101
010 00110 01011 10000 11101 00101 00101 00101 00101
011 00111 01010 10001 11100 00101 00101 00101 00101
100 00011 01110 10101 11000 00101 00101 00101 00101
101 00010 01111 10100 11001 00101 00101 00101 00101
110 00000 01101 10110 11011 00101 00101 00101 00101
111 00001 01100 10111 11010 00101 00101 00101 00101
Hybrid Diff Rules: 105 105 105 :
000 00100 01001 10010 11111 00100 00100 00100 00100
```

```
001 00101 01000 10011 11110 00100 00100 00100 00100
010 00111 01010 10001 11100 00100 00100 00100 00100
011 00110 01011 10000 11101 00100 00100 00100 00100
100 00010 01111 10100 11001 00100 00100 00100 00100
101 00011 01110 10101 11000 00100 00100 00100 00100
110 00001 01100 10111 11010 00100 00100 00100 00100
111 00000 01101 10110 11011 00100 00100 00100 00100
Hybrid Diff Rules: 105 105 165 :
000 00100 01001 10011 11110 00100 00100 00101 00101
001 00101 01000 10010 11111 00100 00100 00101 00101
010 00110 01011 10001 11100 00101 00101 00100 00100
011 00111 01010 10000 11101 00101 00101 00100 00100
100 00011 01110 10100 11001 00101 00101 00100 00100
101 00010 01111 10101 11000 00101 00101 00100 00100
110 00001 01100 10110 11011 00100 00100 00101 00101
111 00000 01101 10111 11010 00100 00100 00101 00101
Hybrid Diff Rules: 105 165 90 :
000 00111 01000 10010 11101 00111 00101 00100 00110
001 00110 01001 10011 11100 00111 00101 00100 00110
010 00101 01010 10000 11111 00110 00100 00101 00111
011 00100 01011 10001 11110 00110 00100 00101 00111
100 00010 01101 10111 11000 00100 00110 00111 00101
101 00011 01100 10110 11001 00100 00110 00111 00101
110 00000 01111 10101 11010 00101 00111 00110 00100
111 00001 01110 10100 11011 00101 00111 00110 00100
Hybrid Diff Rules: 105 165 150 :
000 00110 01000 10011 11101 00110 00101 00101 00110
001 00111 01001 10010 11100 00110 00101 00101 00110
010 00101 01011 10000 11110 00110 00101 00101 00110
011 00100 01010 10001 11111 00110 00101 00101 00110
100 00011 01101 10110 11000 00101 00110 00110 00101
101 00010 01100 10111 11001 00101 00110 00110 00101
110 00000 01110 10101 11011 00101 00110 00110 00101
111 00001 01111 10100 11010 00101 00110 00110 00101
Hybrid Diff Rules: 105 165 105 :
000 00111 01001 10010 11100 00111 00100 00100 00111
001 00110 01000 10011 11101 00111 00100 00100 00111
010 00100 01010 10001 11111 00111 00100 00100 00111
011 00101 01011 10000 11110 00111 00100 00100 00111
100 00010 01100 10111 11001 00100 00111 00111 00100
101 00011 01101 10110 11000 00100 00111 00111 00100
110 00001 01111 10100 11010 00100 00111 00111 00100
111 00000 01110 10101 11011 00100 00111 00111 00100
Hybrid Diff Rules: 105 165 165 :
000 00110 01001 10011 11100 00110 00100 00101 00111
001 00111 01000 10010 11101 00110 00100 00101 00111
010 00100 01011 10001 11110 00111 00101 00100 00110
011 00101 01010 10000 11111 00111 00101 00100 00110
100 00011 01100 10110 11001 00101 00111 00110 00100
101 00010 01101 10111 11000 00101 00111 00110 00100
110 00001 01110 10100 11011 00100 00110 00111 00101
111 00000 01111 10101 11010 00100 00110 00111 00101
Hybrid Diff Rules: 165 90 90 :
000 00101 01111 10000 11010 00101 00010 00110 00001
```

```
001 00100 01110 10001 11011 00101 00010 00110 00001
010 00111 01101 10010 11000 00100 00011 00111 00000
011 00110 01100 10011 11001 00100 00011 00111 00000
100 00000 01010 10101 11111 00110 00001 00101 00010
101 00001 01011 10100 11110 00110 00001 00101 00010
110 00010 01000 10111 11101 00111 00000 00100 00011
111 00011 01001 10110 11100 00111 00000 00100 00011
Hybrid Diff Rules: 165 90 150 :
000 00101 01110 10000 11011 00101 00011 00110 00000
001 00100 01111 10001 11010 00101 00011 00110 00000
010 00110 01101 10011 11000 00101 00011 00110 00000
011 00111 01100 10010 11001 00101 00011 00110 00000
100 00000 01011 10101 11110 00110 00000 00101 00011
101 00001 01010 10100 11111 00110 00000 00101 00011
110 00011 01000 10110 11101 00110 00000 00101 00011
111 00010 01001 10111 11100 00110 00000 00101 00011
Hybrid Diff Rules: 165 90 105 :
000 00100 01111 10001 11010 00100 00010 00111 00001
001 00101 01110 10000 11011 00100 00010 00111 00001
010 00111 01100 10010 11001 00100 00010 00111 00001
011 00110 01101 10011 11000 00100 00010 00111 00001
100 00001 01010 10100 11111 00111 00001 00100 00010
101 00000 01011 10101 11110 00111 00001 00100 00010
110 00010 01001 10111 11100 00111 00001 00100 00010
111 00011 01000 10110 11101 00111 00001 00100 00010
Hybrid Diff Rules: 165 90 165 :
000 00100 01110 10001 11011 00100 00011 00111 00000
001 00101 01111 10000 11010 00100 00011 00111 00000
010 00110 01100 10011 11001 00101 00010 00110 00001
011 00111 01101 10010 11000 00101 00010 00110 00001
100 00001 01011 10100 11110 00111 00000 00100 00011
101 00000 01010 10101 11111 00111 00000 00100 00011
110 00011 01001 10110 11100 00110 00001 00101 00010
111 00010 01000 10111 11101 00110 00001 00101 00010
Hybrid Diff Rules: 165 150 90 :
000 00111 01101 10000 11010 00111 00000 00110 00001
001 00110 01100 10001 11011 00111 00000 00110 00001
010 00101 01111 10010 11000 00110 00001 00111 00000
011 00100 01110 10011 11001 00110 00001 00111 00000
100 00000 01010 10111 11101 00110 00001 00111 00000
101 00001 01011 10110 11100 00110 00001 00111 00000
110 00010 01000 10101 11111 00111 00000 00110 00001
111 00011 01001 10100 11110 00111 00000 00110 00001
Hybrid Diff Rules: 165 150 150 :
000 00110 01101 10000 11011 00110 00000 00110 00000
001 00111 01100 10001 11010 00110 00000 00110 00000
010 00101 01110 10011 11000 00110 00000 00110 00000
011 00100 01111 10010 11001 00110 00000 00110 00000
100 00000 01011 10110 11101 00110 00000 00110 00000
101 00001 01010 10111 11100 00110 00000 00110 00000
110 00011 01000 10101 11110 00110 00000 00110 00000
111 00010 01001 10100 11111 00110 00000 00110 00000
Hybrid Diff Rules: 165 150 105 :
000 00111 01100 10001 11010 00111 00001 00111 00001
```

```
001 00110 01101 10000 11011 00111 00001 00111 00001
010 00100 01111 10010 11001 00111 00001 00111 00001
011 00101 01110 10011 11000 00111 00001 00111 00001
100 00001 01010 10111 11100 00111 00001 00111 00001
101 00000 01011 10110 11101 00111 00001 00111 00001
110 00010 01001 10100 11111 00111 00001 00111 00001
111 00011 01000 10101 11110 00111 00001 00111 00001
Hybrid Diff Rules: 165 150 165 :
000 00110 01100 10001 11011 00110 00001 00111 00000
001 00111 01101 10000 11010 00110 00001 00111 00000
010 00100 01110 10011 11001 00111 00000 00110 00001
011 00101 01111 10010 11000 00111 00000 00110 00001
100 00001 01011 10110 11100 00111 00000 00110 00001
101 00000 01010 10111 11101 00111 00000 00110 00001
110 00011 01001 10100 11110 00110 00001 00111 00000
111 00010 01000 10101 11111 00110 00001 00111 00000
Hybrid Diff Rules: 165 105 90 :
000 00101 01111 10010 11000 00101 00010 00100 00011
001 00100 01110 10011 11001 00101 00010 00100 00011
010 00111 01101 10000 11010 00100 00011 00101 00010
011 00110 01100 10001 11011 00100 00011 00101 00010
100 00010 01000 10101 11111 00100 00011 00101 00010
101 00011 01001 10100 11110 00100 00011 00101 00010
110 00000 01010 10111 11101 00101 00010 00100 00011
111 00001 01011 10110 11100 00101 00010 00100 00011
Hybrid Diff Rules: 165 105 150 :
000 00101 01110 10011 11000 00101 00011 00101 00011
001 00100 01111 10010 11001 00101 00011 00101 00011
010 00110 01101 10000 11011 00101 00011 00101 00011
011 00111 01100 10001 11010 00101 00011 00101 00011
100 00011 01000 10101 11110 00101 00011 00101 00011
101 00010 01001 10100 11111 00101 00011 00101 00011
110 00000 01011 10110 11101 00101 00011 00101 00011
111 00001 01010 10111 11100 00101 00011 00101 00011
Hybrid Diff Rules: 165 105 105 :
000 00100 01111 10010 11001 00100 00010 00100 00010
001 00101 01110 10011 11000 00100 00010 00100 00010
010 00111 01100 10001 11010 00100 00010 00100 00010
011 00110 01101 10000 11011 00100 00010 00100 00010
100 00010 01001 10100 11111 00100 00010 00100 00010
101 00011 01000 10101 11110 00100 00010 00100 00010
110 00001 01010 10111 11100 00100 00010 00100 00010
111 00000 01011 10110 11101 00100 00010 00100 00010
Hybrid Diff Rules: 165 105 165 :
000 00100 01110 10011 11001 00100 00011 00101 00010
001 00101 01111 10010 11000 00100 00011 00101 00010
010 00110 01100 10001 11011 00101 00010 00100 00011
011 00111 01101 10000 11010 00101 00010 00100 00011
100 00011 01001 10100 11110 00101 00010 00100 00011
101 00010 01000 10101 11111 00101 00010 00100 00011
110 00001 01011 10110 11100 00100 00011 00101 00010
111 00000 01010 10111 11101 00100 00011 00101 00010
Hybrid Diff Rules: 165 165 90 :
000 00111 01101 10010 11000 00111 00000 00100 00011
```

```
001 00110 01100 10011 11001 00111 00000 00100 00011
010 00101 01111 10000 11010 00110 00001 00101 00010
011 00100 01110 10001 11011 00110 00001 00101 00010
100 00010 01000 10111 11101 00100 00011 00111 00000
101 00011 01001 10110 11100 00100 00011 00111 00000
110 00000 01010 10101 11111 00101 00010 00110 00001
111 00001 01011 10100 11110 00101 00010 00110 00001
Hybrid Diff Rules: 165 165 150 :
000 00110 01101 10011 11000 00110 00000 00101 00011
001 00111 01100 10010 11001 00110 00000 00101 00011
010 00101 01110 10000 11011 00110 00000 00101 00011
011 00100 01111 10001 11010 00110 00000 00101 00011
100 00011 01000 10110 11101 00101 00011 00110 00000
101 00010 01001 10111 11100 00101 00011 00110 00000
110 00000 01011 10101 11110 00101 00011 00110 00000
111 00001 01010 10100 11111 00101 00011 00110 00000
Hybrid Diff Rules: 165 165 105 :
000 00111 01100 10010 11001 00111 00001 00100 00010
001 00110 01101 10011 11000 00111 00001 00100 00010
010 00100 01111 10001 11010 00111 00001 00100 00010
011 00101 01110 10000 11011 00111 00001 00100 00010
100 00010 01001 10111 11100 00100 00010 00111 00001
101 00011 01000 10110 11101 00100 00010 00111 00001
110 00001 01010 10100 11111 00100 00010 00111 00001
111 00000 01011 10101 11110 00100 00010 00111 00001
Hybrid Diff Rules: 165 165 165 :
000 00110 01100 10011 11001 00110 00001 00101 00010
001 00111 01101 10010 11000 00110 00001 00101 00010
010 00100 01110 10001 11011 00111 00000 00100 00011
011 00101 01111 10000 11010 00111 00000 00100 00011
100 00011 01001 10110 11100 00101 00010 00110 00001
101 00010 01000 10111 11101 00101 00010 00110 00001
110 00001 01011 10100 11110 00100 00011 00111 00000
111 00000 01010 10101 11111 00100 00011 00111 00000
```

B.3 Correlation of Sequences to Seeds

These experiments show the relationship between seeds and the sequences they generate under various rule configurations. The SeedDiffFromTemp function below creates a dictionary with all possible sequences as keys and a list of the seeds that generate them as values. Numbers here are often stored as native integers for performance reasons; results showing decimal numbers are meant to be interpreted as the binary equivalent of that number.

This first experiment shows that rule sets having rules of the same arity in each position generate the same temporal sequences with only a constant factor added into the seed to account for rules that differ only in their complimentarity. Rulesets having different arity in corresponding rules generate the same sequences only from seeds

with no constant relationship across the rule sets. Indeed, depending on the difference in rules, the same sequence may not even be possible in another rule set.

```
>>> hr1 = [90, 90, 90, 90, 90, 90, 90, 90, 150]
>>> hr2 = [90, 90, 90, 90, 90, 90, 90, 90, 105]
>>> hr3 = [90, 90, 90, 90, 90, 90, 150, 90, 150]
>>> d1 = hca.SeedDiffFromTemp(hr1, 9)
>>> d2 = hca.SeedDiffFromTemp(hr2, 9)
>>> d3 = hca.SeedDiffFromTemp(hr3, 9)
>>> len(d3)
511
>>> len(d2)
511
>>> len(d1)
511
>>> for i in range(1, 10):
print 'd1 ^ d2:',
pb(d1[i][0] ^ d2[i][0], 9)
print 'd2 ^ d3:',
pb(d2[i][0] ^ d3[i][0], 9)
print 'd1 ^ d3:',
pb(d1[i][0] ^ d3[i][0], 9)
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101
d1 ^ d2: 101101101

d2 ^ d3: 111101111
d2 ^ d3: 000000000
d2 ^ d3: 010000010
d2 ^ d3: 000101001
d2 ^ d3: 010101011
d2 ^ d3: 101000100
d2 ^ d3: 111000110
d2 ^ d3: 101101110
d2 ^ d3: 111101100

d1 ^ d3: 010000010
d1 ^ d3: 101101101
d1 ^ d3: 111101111
d1 ^ d3: 101000100
d1 ^ d3: 111000110
d1 ^ d3: 000101001
d1 ^ d3: 010101011
d1 ^ d3: 000000011
d1 ^ d3: 010000001
```

This holds regardless of mixing arity or complimentarity, as shown by the following rule sets.

```
>>> hr4 = [90] * 8 + [105]
>>> d4 = SeedDiffFromTemp(hr4, 9)
>>> hr5 = [165] * 8 + [150]
>>> d5 = SeedDiffFromTemp(hr5, 9)
>>> hr1 = [90] * 8 + [150]
>>> for j, d in enumerate([d1, d4, d5]):
for k, e in enumerate([d1, d4, d5]):
if d != e:
print 'd['+str(j)+'] ^ e['+str(k)+']'
for i in range(1, 10):
if i in d and i in e:
pb(d[i][0] ^ e[i][0], 9)
elif i in d:
print 'right does not have',
pb(i, 9)
else:
print 'left does not have',
pb(i,9)
d[0] ^ e[1]
101101101
101101101
101101101
101101101
101101101
101101101
101101101
101101101
101101101
d[0] ^ e[2]
001001001
001001001
001001001
001001001
001001001
001001001
001001001
001001001
001001001
d[1] ^ e[0]
<...>
d[1] ^ e[2]
100100100
100100100
100100100
100100100
100100100
100100100
100100100
100100100
100100100
d[2] ^ e[0]
<...>
001001001
d[2] ^ e[1]
```

```
<...>
100100100
```

It is possible for two rule sets to generate the same sequence up to $t = n$, then diverge. This can happen when one rule set "dead-ends" (i.e. zero's out) or has a different period than another. Continuing the environment from above, we see both of these conditions in two rule sets:

```
>>> hr2 = [90, 90, 90, 90, 90, 90, 90, 90, 105]
>>> hr3 = [90, 90, 90, 90, 90, 90, 150, 90, 150]
>>> d2[141]
[417]
>>> d3[141]
[137]
>>> t2 = hca.TempSeqFromSeed(hr2, hca.SeedFromInt(417,9),18)
>>> t3 = hca.TempSeqFromSeed(hr3, hca.SeedFromInt(137,9),18)
>>> t2
array([0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0], dtype=int8)
>>> t3
array([0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], dtype=int8)
```

The next experiment examines the hamming distance between seeds in two asymmetric CA which generate the same temporal sequence. Here, the sequence is printed in decimal next to the difference in seeds for that sequence between the two rule sets. Some editing of the results has been done to gather similar results. In the resulting table below, all decimal sequences in each row share the same low 6 bits. The top two sequences of each group of four in a column share the same upper 3 bits, and again for the bottom two of four. The middle two of each group of four have the same last 3 bits, and the outer two have the same last 4 bits.

The fact that these are groups of four seems to be related to the rule set at play. For rule set hr1 (same as below) and hr2 $= \{90\}^6 + \{150, 90, 150\}$, the groups have 8 members of the same hamming distance. hr2 has an eventual period of 0, hr1 of 28. Both have unique seeds for all sequences.

```
>>> hr1 = [90, 90, 90, 90, 90, 90, 90, 90, 150]
>>> (p,d) = MaxPeriod(hr1); print p
28
>>> len(d1)
511
>>> hr3 = [150] * 5 + [90] * 4
>>> (p,d) = MaxPeriod(hr3); print p
30
>>> d3 = SeedDiffFromTemp(hr3, 9)
>>> len(d3)
511
>>> for i in range(1,len(d1)):
```

```
if d1[i] == d3[i]:
print i,
59 195 248
>>> for i in range(1,len(d1)):
print i, pb(d1[i][0] ^ d3[i][0])
  0 000000000      64 011101010     256 001001100     320 010100110
 59 000000000     123 011101010     315 001001100     379 010100110
195 000000000     131 011101010     451 001001100     387 010100110
258 000000000     184 011101010     504 001001100     440 010100110

  1 011101110      65 000000100     257 010100010     321 001001000
 58 011101110     122 000000100     314 010100010     378 001001000
194 011101110     130 000000100     450 010100010     386 001001000
249 011101110     185 000000100     505 010100010     441 001001000

  2 110000101      66 101101111     258 111001001     322 100100011
 57 110000101     121 101101111     313 111001001     377 100100011
193 110000101     129 101101111     449 111001001     385 100100011
250 110000101     186 101101111     506 111001001     442 100100011

  3 101101011      67 110000001     259 100100111     323 111001101
 56 101101011     120 110000001     312 100100111     376 111001101
192 101101011     128 110000001     448 100100111     384 111001101
251 101101011     187 110000001     507 100100111     443 111001101

  4 111000111      68 100101101     260 110001011     324 101100001
 63 111000111     127 100101101     319 110001011     383 101100001
199 111000111     135 100101101     455 110001011     391 101100001
252 111000111     188 100101101     508 110001011     444 101100001

  5 100101001      69 111000011     261 101100101     325 110001111
 62 100101001     126 111000011     318 101100101     382 110001111
198 100101001     134 111000011     454 101100101     390 110001111
253 100101001     189 111000011     509 101100101     445 110001111

  6 001000010      70 010101000     262 000001110     326 011100100
 61 001000010     125 010101000     317 000001110     381 011100100
197 001000010     133 010101000     453 000001110     389 011100100
254 001000010     190 010101000     510 000001110     446 011100100

  7 010101100      71 001000110     263 011100000     327 000001010
 60 010101100     124 001000110     316 011100000     380 000001010
196 010101100     132 001000110     452 011100000     388 000001010
255 010101100     191 001000110   ?_511 011100000_?   447 000001010

  8 110000010      72 101101000     264 111001110     328 100100100
 51 110000010     115 101101000     307 111001110     371 100100100
203 110000010     139 101101000     459 111001110     395 100100100
240 110000010     176 101101000     496 111001110     432 100100100

  9 101101100      73 110000110     265 100100000     329 111001010
 50 101101100     114 110000110     306 100100000     370 111001010
202 101101100     138 110000110     458 100100000     394 111001010
241 101101100     177 110000110     497 100100000     433 111001010
```

10	000000111	74	011101101	266	001001011	330	010100001
49	000000111	113	011101101	305	001001011	369	010100001
201	000000111	137	011101101	457	001001011	393	010100001
242	000000111	178	011101101	498	001001011	434	010100001
11	011101001	75	000000011	267	010100101	331	001001111
48	011101001	112	000000011	304	010100101	368	001001111
200	011101001	136	000000011	456	010100101	392	001001111
243	011101001	179	000000011	499	010100101	435	001001111
12	001000101	76	010101111	268	000001001	332	011100011
55	001000101	119	010101111	311	000001001	375	011100011
207	001000101	143	010101111	463	000001001	399	011100011
244	001000101	180	010101111	500	000001001	436	011100011
13	010101011	77	001000001	269	011100111	333	000001101
54	010101011	118	001000001	310	011100111	374	000001101
206	010101011	142	001000001	462	011100111	398	000001101
245	010101011	181	001000001	501	011100111	437	000001101
14	111000000	78	100101010	270	110001100	334	101100110
53	111000000	117	100101010	309	110001100	373	101100110
205	111000000	141	100101010	461	110001100	397	101100110
246	111000000	182	100101010	502	110001100	438	101100110
15	100101110	79	111000100	271	101100010	335	110001000
52	100101110	116	111000100	308	101100010	372	110001000
204	100101110	140	111000100	460	101100010	396	110001000
247	100101110	183	111000100	503	101100010	439	110001000
16	110101101	80	101000111	272	111100001	336	100001011
43	110101101	107	101000111	299	111100001	363	100001011
211	110101101	147	101000111	467	111100001	403	100001011
232	110101101	168	101000111	488	111100001	424	100001011
17	101000011	81	110101001	273	100001111	337	111100101
42	101000011	106	110101001	298	100001111	362	111100101
210	101000011	146	110101001	466	100001111	402	111100101
233	101000011	169	110101001	489	100001111	425	111100101
18	000101000	82	011000010	274	001100100	338	010001110
41	000101000	105	011000010	297	001100100	361	010001110
209	000101000	145	011000010	465	001100100	401	010001110
234	000101000	170	011000010	490	001100100	426	010001110
19	011000110	83	000101100	275	010001010	339	001100000
40	011000110	104	000101100	296	010001010	360	001100000
208	011000110	144	000101100	464	010001010	400	001100000
235	011000110	171	000101100	491	010001010	427	001100000
20	001101010	84	010000000	276	000100110	340	011001100
47	001101010	111	010000000	303	000100110	367	011001100
215	001101010	151	010000000	471	000100110	407	011001100

236	001101010	172	010000000	492	000100110	428	011001100
21	010000100	85	001101110	277	011001000	341	000100010
46	010000100	110	001101110	302	011001000	366	000100010
214	010000100	150	001101110	470	011001000	406	000100010
237	010000100	173	001101110	493	011001000	429	000100010
22	111101111	86	100000101	278	110100011	342	101001001
45	111101111	109	100000101	301	110100011	365	101001001
213	111101111	149	100000101	469	110100011	405	101001001
238	111101111	174	100000101	494	110100011	430	101001001
23	100000001	87	111101011	279	101001101	343	110100111
44	100000001	108	111101011	300	101001101	364	110100111
212	100000001	148	111101011	468	101001101	404	110100111
239	100000001	175	111101011	495	101001101	431	110100111
24	000101111	88	011000101	280	001100011	344	010001001
35	000101111	99	011000101	291	001100011	355	010001001
219	000101111	155	011000101	475	001100011	411	010001001
224	000101111	160	011000101	480	001100011	416	010001001
25	011000001	89	000101011	281	010001101	345	001100111
34	011000001	98	000101011	290	010001101	354	001100111
218	011000001	154	000101011	474	010001101	410	001100111
225	011000001	161	000101011	481	010001101	417	001100111
26	110101010	90	101000000	282	111100110	346	100001100
33	110101010	97	101000000	289	111100110	353	100001100
217	110101010	153	101000000	473	111100110	409	100001100
226	110101010	162	101000000	482	111100110	418	100001100
27	101000100	91	110101110	283	100001000	347	111100010
32	101000100	96	110101110	288	100001000	352	111100010
216	101000100	152	110101110	472	100001000	408	111100010
227	101000100	163	110101110	483	100001000	419	111100010
28	111101000	92	100000010	284	110100100	348	101001110
39	111101000	103	100000010	295	110100100	359	101001110
223	111101000	159	100000010	479	110100100	415	101001110
228	111101000	164	100000010	484	110100100	420	101001110
29	100000110	93	111101100	285	101001010	349	110100000
38	100000110	102	111101100	294	101001010	358	110100000
222	100000110	158	111101100	478	101001010	414	110100000
229	100000110	165	111101100	485	101001010	421	110100000
30	001101101	94	010000111	286	000100001	350	011001011
37	001101101	101	010000111	293	000100001	357	011001011
221	001101101	157	010000111	477	000100001	413	011001011
230	001101101	166	010000111	486	000100001	422	011001011
31	010000011	95	001101001	287	011001111	351	000100101
36	010000011	100	001101001	292	011001111	356	000100101

```
220 010000011     156 001101001     476 011001111     412 000100101
231 010000011     167 001101001     487 011001111     423 000100101
```

B.4 Mapping Seeds from Symmetric to Asymmetric Rulesets

These experiments investigate the possibility of using information about a sequence under a symmetric rule set to learn something about the seed or the rule set that originally produced it.

First, we go through all sequences produced by a uniform rule 150 CA, and find the hamming distance from each seed that produces that sequence to the seed under an asymmetric CA which produces the same sequence.

```
>>> hr3 = [90, 90, 90, 90, 90, 90, 150, 90, 105]
>>> d3 = SeedDiffFromTemp(hr3, 9)
>>> hr5 = [150] * 9
>>> d5 = SeedDiffFromTemp(hr5, 9)
>>> intersect = []
>>> for k in d5.keys():
if k == 0:
continue
vl = d5[k]
intersect.append(k)
print k,
for v in vl:
print pb(v ^ d3[k][0]),
print
256 110101011 110000011 111101111 111000111 100101001 100000001 101101101 101000101
    010101010 010000010 011101110 011000110 000101000 000000000 001101100 001000100
278 001000100 001101100 000000000 000101000 011000110 011101110 010000010 010101010
    101000101 101101101 100000001 100101001 111000111 111101111 110000011 110101011
300 101000101 101101101 100000001 100101001 111000111 111101111 110000011 110101011
    001000100 001101100 000000000 000101000 011000110 011101110 010000010 010101010
395 110000011 110101011 111000111 111101111 100000001 100101001 101000101 101101101
    010000010 010101010 011000110 011101110 000000000 000101000 001000100 001101100
344 011000110 011101110 010000010 010101010 001000100 001101100 000000000 000101000
    111000111 111101111 110000011 110101011 101000101 101101101 100000001 100101001
22  111101111 111000111 110101011 110000011 101101101 101000101 100101001 100000001
    011101110 011000110 010101010 010000010 001101100 001000100 000101000 000000000
413 001101100 001000100 000101000 000000000 011101110 011000110 010101010 010000010
    101101101 101000101 100101001 100000001 111101111 111000111 110101011 110000011
453 100000001 100101001 101000101 101101101 110000011 110101011 111000111 111101111
    000000000 000101000 001000100 001101100 010000010 010101010 011000110 011101110
167 011000110 011101110 010000010 010101010 001000100 001101100 000000000 000101000
    111000111 111101111 110000011 110101011 101000101 101101101 100000001 100101001
44  011101110 011000110 010101010 010000010 001101100 001000100 000101000 000000000
    111101111 111000111 110101011 110000011 101101101 101000101 100101001 100000001
157 111000111 111101111 110000011 110101011 101000101 101101101 100000001 100101001
    011000110 011101110 010000010 010101010 001000100 001101100 000000000 000101000
177 100101001 100000001 101101101 101000101 110101011 110000011 111101111 111000111
    000101000 000000000 001101100 001000100 010101010 010000010 011101110 011000110
```

```
372 000101000 000000000 001101100 001000100 010101010 010000010 011101110 011000110
    100101001 100000001 101101101 101000101 110101011 110000011 111101111 111000111
58  100000001 100101001 101000101 101101101 110000011 110101011 111000111 111101111
    000000000 000101000 001000100 001101100 010000010 010101010 011000110 011101110
139 000101000 000000000 001101100 001000100 010101010 010000010 011101110 011000110
    100101001 100000001 101101101 101000101 110101011 110000011 111101111 111000111
197 010101010 010000010 011101110 011000110 000101000 000000000 001101100 001000100
    110101011 110000011 111101111 111000111 100101001 100000001 101101101 101000101
354 111000111 111101111 110000011 110101011 101000101 101101101 100000001 100101001
    011000110 011101110 010000010 010101010 001000100 001101100 000000000 000101000
78  010000010 010101010 011000110 011101110 000000000 000101000 001000100 001101100
    110000011 110101011 111000111 111101111 100000001 100101001 101000101 101101101
433 010000010 010101010 011000110 011101110 000000000 000101000 001000100 001101100
    110000011 110101011 111000111 111101111 100000001 100101001 101000101 101101101
467 011101110 011000110 010101010 010000010 001101100 001000100 000101000 000000000
    111101111 111000111 110101011 110000011 101101101 101000101 100101001 100000001
334 100101001 100000001 101101101 101000101 110101011 110000011 111101111 111000111
    000101000 000000000 001101100 001000100 010101010 010000010 011101110 011000110
88  101101101 101000101 100101001 100000001 111101111 111000111 110101011 110000011
    001101100 001000100 000101000 000000000 011101110 011000110 010101010 010000010
314 010101010 010000010 011101110 011000110 000101000 000000000 001101100 001000100
    110101011 110000011 111101111 111000111 100101001 100000001 101101101 101000101
98  001101100 001000100 000101000 000000000 011101110 011000110 010101010 010000010
    101101101 101000101 100101001 100000001 111101111 111000111 110101011 110000011
233 001000100 001101100 000000000 000101000 011000110 011101110 010000010 010101010
    101000101 101101101 100000001 100101001 111000111 111101111 110000011 110101011
423 101101101 101000101 100101001 100000001 111101111 111000111 110101011 110000011
    001101100 001000100 000101000 000000000 011101110 011000110 010101010 010000010
211 101000101 101101101 100000001 100101001 111000111 111101111 110000011 110101011
    001000100 001101100 000000000 000101000 011000110 011101110 010000010 010101010
116 110000011 110101011 111000111 111101111 100000001 100101001 101000101 101101101
    010000010 010101010 011000110 011101110 000000000 000101000 001000100 001101100
489 111101111 111000111 110101011 110000011 101101101 101000101 100101001 100000001
    011101110 011000110 010101010 010000010 001101100 001000100 000101000 000000000
255 110101011 110000011 111101111 111000111 100101001 100000001 101101101 101000101
    010101010 010000010 011101110 011000110 000101000 000000000 001101100 001000100
511 000000000 000101000 001000100 001101100 010000010 010101010 011000110 011101110
    100000001 100101001 101000101 101101101 110000011 110101011 111000111 111101111
>>> len(intersect)
31

>>> pb(22)
'000010110'
>>> pb(489)
'111101001'
>>> pb(255)
'011111111'
>>> pb(256)
'100000000'
```

Notice that complimentary sequences have the same pattern of differences. Seed differences for 22 and 489 above both start with 111101111, sequences for 255 and 256 start with 110101011, etc.

B.5 Mapping Sequences from Symmetric to Asymmetric CAs

In the following experiment, continuing the above environment, all seeds for all sequences from a 9-cell uniform rule 150 CA are looked up in the reverse dictionary (where the key is the seed and the resulting sequence is the value) of an asymmetric CA. The differences between the two sequences are shown. The interesting observation is that there are only differences in the last four bits. This pattern is dependent on the two rules (or more likely, just the asymmetric CA rule set) at play. Other CA pairs show differences in earlier bit positions of the temporal sequence, but those cases have a more predictable arrangement of differences across seed in the symmetric CA.

Again, we note that complementary sequences have the same pattern of differences between symmetric and asymmetric CAs. See sequences 010001011 and 101110100 below.

```
>>> hr3 = [90, 90, 90, 90, 90, 90, 150, 90, 105]
>>> d3 = SeedDiffFromTemp(hr3, 9)
>>> hr5 = [150] * 9
>>> d5 = SeedDiffFromTemp(hr5, 9)
>>> d3r = {}
>>> for k in d3:
d6r[d6[k][0]] = k
>>> for t in d5:
print 'sym seq=' + pb(t) + ':'
for j, i in enumerate(d5[t]):
print '[' + pb(i) + ':' + pb(d3r[i]) + ':' + pb(d3r[i]^t) + ']',
if j & 1:
print
key:  [ Ssym : Tasym : T ^ Tasym]  where
   T = temporal sequence generated by symmetrical CA,
   Ssym = seed of symmetrical CA that generates T
   Tasym = temporal sequence of asymmetrical CA on seed Ssym
sym seq=100000000:
[000011101:100001000:000001000] [000110101:100001001:000001001]
[001011001:100001011:000001011] [001110001:100001010:000001010]
[010011111:100001101:000001101] [010110111:100001100:000001100]
[011011011:100001110:000001110] [011110011:100001111:000001111]
[100011100:100000100:000000100] [100110100:100000101:000000101]
[101011000:100000111:000000111] [101110000:100000110:000000110]
[110011110:100000001:000000001] [110110110:100000000:000000000]
[111011010:100000010:000000010] [111110010:100000011:000000011]
sym seq=000000000:
[000101000:000000001:000000001] [001000100:000000011:000000011]
[001101100:000000010:000000010] [010000010:000000101:000000101]
[010101010:000000100:000000100] [011000110:000000110:000000110]
[011101110:000000111:000000111] [100000001:000001100:000001100]
[100101001:000001101:000001101] [101000101:000001111:000001111]
[101101101:000001110:000001110] [110000011:000001001:000001001]
[110101011:000001000:000001000] [111000111:000001010:000001010]
[111101111:000001011:000001011]
```

```
sym seq=100010110:
[000011100:100010101:000000011] [000110100:100010100:000000010]
[001011000:100010110:000000000] [001110000:100010111:000000001]
[010011110:100010000:000000110] [010110110:100010001:000000111]
[011011010:100010011:000000101] [011110010:100010010:000000100]
[100011101:100011001:000001111] [100110101:100011000:000001110]
[101011001:100011010:000001100] [101110001:100011011:000001101]
[110011111:100011100:000001010] [110110111:100011101:000001011]
[111011011:100011111:000001001] [111110011:100011110:000001000]
sym seq=100101100:
[000011111:100100011:000001111] [000110111:100100010:000001110]
[001011011:100100000:000001100] [001110011:100100001:000001101]
[010011101:100100110:000001010] [010110101:100100111:000001011]
[011011001:100100101:000001001] [011110001:100100100:000001000]
[100011110:100101111:000000011] [100110110:100101110:000000010]
[101011010:100101100:000000000] [101110010:100101101:000000001]
[110011100:100101010:000000110] [110110100:100101011:000000111]
[111011000:100101001:000000101] [111110000:100101000:000000100]
sym seq=110001011:
[000010101:110000010:000001001] [000111101:110000011:000001000]
[001010001:110000001:000001010] [001111001:110000000:000001011]
[010010111:110000111:000001100] [010111111:110000110:000001101]
[011010011:110000100:000001111] [011111011:110000101:000001110]
[100010100:110001110:000000101] [100111100:110001111:000000100]
[101010000:110001101:000000110] [101111000:110001100:000000111]
[110010110:110001011:000000000] [110111110:110001010:000000001]
[111010010:110001000:000000011] [111111010:110001001:000000010]
sym seq=101011000:
[000011010:101011110:000000110] [000110010:101011111:000000111]
[001011110:101011101:000000101] [001110110:101011100:000000100]
[010011000:101011011:000000011] [010110000:101011010:000000010]
[011011100:101011000:000000000] [011110100:101011001:000000001]
[100011011:101010010:000001010] [100110011:101010011:000001011]
[101011111:101010001:000001001] [101110111:101010000:000001000]
[110011001:101010111:000001111] [110110001:101010110:000001110]
[111011101:101010100:000001100] [111110101:101010101:000001101]
sym seq=000010110:
[000000001:000011101:000001011] [000101001:000011100:000001010]
[001000101:000011110:000001000] [001101101:000011111:000001001]
[010000011:000011000:000001110] [010101011:000011001:000001111]
[011000111:000011011:000001101] [011101111:000011010:000001100]
[100000000:000010001:000000111] [100101000:000010000:000000110]
[101000100:000010010:000000100] [101101100:000010011:000000101]
[110000010:000010100:000000010] [110101010:000010101:000000011]
[111000110:000010111:000000001] [111101110:000010110:000000000]
sym seq=110011101:
[000010100:110011111:000000010] [000111100:110011110:000000011]
[001010000:110011100:000000001] [001111000:110011101:000000000]
[010010110:110011010:000000111] [010111110:110011011:000000110]
[011010010:110011001:000000100] [011111010:110011000:000000101]
[100010101:110010011:000001110] [100111101:110010010:000001111]
[101010001:110010000:000001101] [101111001:110010001:000001100]
[110010111:110010110:000001011] [110111111:110010111:000001010]
[111010011:110010101:000001000] [111111011:110010100:000001001]
```

```
sym seq=111000101:
[000010011:111001001:000001100] [000111011:111001000:000001101]
[001010111:111001010:000001111] [001111111:111001011:000001110]
[010010001:111001100:000001001] [010111001:111001101:000001000]
[011010101:111001111:000001010] [011111101:111001110:000001011]
[100010010:111000101:000000000] [100111010:111000100:000000001]
[101010110:111000110:000000011] [101111110:111000111:000000010]
[110010000:111000000:000000101] [110111000:111000001:000000100]
[111010100:111000011:000000110] [111111100:111000010:000000111]
sym seq=010100111:
[000001010:010100001:000000110] [000100010:010100000:000000111]
[001001110:010100010:000000101] [001100110:010100011:000000100]
[010001000:010100100:000000011] [010100000:010100101:000000010]
[011001100:010100111:000000000] [011100100:010100110:000000001]
[100001011:010101101:000001010] [100100011:010101100:000001011]
[101001111:010101110:000001001] [101100111:010101111:000001000]
[110001001:010101000:000001111] [110100001:010101001:000001110]
[111001101:010101011:000001100] [111100101:010101010:000001101]
sym seq=000101100:
[000000010:000101011:000000111] [000101010:000101010:000000110]
[001000110:000101000:000000100] [001101110:000101001:000000101]
[010000000:000101110:000000010] [010101000:000101111:000000011]
[011000100:000101101:000000001] [011101100:000101100:000000000]
[100000011:000100111:000001011] [100101011:000100110:000001010]
[101000111:000100100:000001000] [101101111:000100101:000001001]
[110000001:000100010:000001110] [110101001:000100011:000001111]
[111000101:000100001:000001101] [111101101:000100000:000001100]
sym seq=010011101:
[000001001:010010111:000001010] [000100001:010010110:000001011]
[001001101:010010100:000001001] [001100101:010010101:000001000]
[010001011:010010010:000001111] [010100011:010010011:000001110]
[011001111:010010001:000001100] [011100111:010010000:000001101]
[100001000:010011011:000000110] [100100000:010011010:000000111]
[101001100:010011000:000000101] [101100100:010011001:000000100]
[110001010:010011110:000000011] [110100010:010011111:000000010]
[111001110:010011101:000000000] [111100110:010011100:000000001]
sym seq=010110001:
[000001011:010111100:000001101] [000100011:010111101:000001100]
[001001111:010111111:000001110] [001100111:010111110:000001111]
[010001001:010111001:000001000] [010100001:010111000:000001001]
[011001101:010111010:000001011] [011100101:010111011:000001010]
[100001010:010110000:000000001] [100100010:010110001:000000000]
[101001110:010110011:000000010] [101100110:010110010:000000011]
[110001000:010110101:000000100] [110100000:010110100:000000101]
[111001100:010110110:000000111] [111100100:010110111:000000110]
sym seq=101110100:
[000011000:101110101:000000001] [000110000:101110100:000000000]
[001011100:101110110:000000010] [001110100:101110111:000000011]
[010011010:101110000:000000100] [010110010:101110001:000000101]
[011011110:101110011:000000111] [011110110:101110010:000000110]
[100011001:101111001:000001101] [100110001:101111000:000001100]
[101011101:101111010:000001110] [101110101:101111011:000001111]
[110011011:101111100:000001000] [110110011:101111101:000001001]
[111011111:101111111:000001011] [111110111:101111110:000001010]
```

```
sym seq=000111010:
[000000011:000110110:000001100] [000101011:000110111:000001101]
[001000111:000110101:000001111] [001101111:000110100:000001110]
[010000001:000110011:000001001] [010101001:000110010:000001000]
[011000101:000110000:000001010] [011101101:000110001:000001011]
[100000010:000111010:000000000] [100101010:000111011:000000001]
[101000110:000111001:000000011] [101101110:000111000:000000010]
[110000000:000111111:000000101] [110101000:000111110:000000100]
[111000100:000111100:000000110] [111101100:000111101:000000111]
sym seq=010001011:
[000001000:010001010:000000001] [000100000:010001011:000000000]
[001001100:010001001:000000010] [001100100:010001000:000000011]
[010001010:010001111:000000100] [010100010:010001110:000000101]
[011001110:010001100:000000111] [011100110:010001101:000000110]
[100001001:010000110:000001101] [100100001:010000111:000001100]
[101001101:010000101:000001110] [101100101:010000100:000001111]
[110001011:010000011:000001000] [110100011:010000010:000001001]
[111001111:010000000:000001011] [111100111:010000001:000001010]
sym seq=011000101:
[000001110:011000001:000000100] [000100110:011000000:000000101]
[001001010:011000010:000000111] [001100010:011000011:000000110]
[010001100:011000100:000000001] [010100100:011000101:000000000]
[011001000:011000111:000000010] [011100000:011000110:000000011]
[100001111:011001101:000001000] [100100111:011001100:000001001]
[101001011:011001110:000001011] [101100011:011001111:000001010]
[110001101:011001000:000001101] [110100101:011001001:000001100]
[111001001:011001011:000001110] [111100001:011001010:000001111]
sym seq=101100010:
[000011001:101101000:000001010] [000110001:101101001:000001011]
[001011101:101101011:000001001] [001110101:101101010:000001000]
[010011011:101101101:000001111] [010110011:101101100:000001110]
[011011111:101101110:000001100] [011110111:101101111:000001101]
[100011000:101100100:000000110] [100110000:101100101:000000111]
[101011100:101100111:000000101] [101110100:101100110:000000100]
[110011010:101100001:000000011] [110110010:101100000:000000010]
[111011110:101100010:000000000] [111110110:101100011:000000001]
sym seq=001001110:
[000000110:001001011:000000101] [000101110:001001010:000000100]
[001000010:001001000:000000110] [001101010:001001001:000000111]
[010000100:001001110:000000000] [010101100:001001111:000000001]
[011000000:001001101:000000011] [011101000:001001100:000000010]
[100000111:001000111:000001001] [100101111:001000110:000001000]
[101000011:001000100:000001010] [101101011:001000101:000001011]
[110000101:001000010:000001100] [110101101:001000011:000001101]
[111000001:001000001:000001111] [111101001:001000000:000001110]
sym seq=110110001:
[000010110:110110100:000000101] [000111110:110110101:000000100]
[001010010:110110111:000000110] [001111010:110110110:000000111]
[010010100:110110001:000000000] [010111100:110110000:000000001]
[011010000:110110010:000000011] [011111000:110110011:000000010]
[100010111:110111000:000001001] [100111111:110111001:000001000]
[101010011:110111011:000001010] [101111011:110111010:000001011]
[110010101:110111101:000001100] [110111101:110111100:000001101]
[111010001:110111110:000001111] [111111001:110111111:000001110]
```

```
sym seq=111010011:
[000010010:111010100:000000111] [000111010:111010101:000000110]
[001010110:111010111:000000100] [001111110:111010110:000000101]
[010010000:111010001:000000010] [010111000:111010000:000000011]
[011010100:111010010:000000001] [011111100:111010011:000000000]
[100010011:111011000:000001011] [100111011:111011001:000001010]
[101010111:111011011:000001000] [101111111:111011010:000001001]
[110010001:111011101:000001110] [110111001:111011100:000001111]
[111010101:111011110:000001101] [111111101:111011111:000001100]
sym seq=101001110:
[000011011:101000011:000001101] [000110011:101000010:000001100]
[001011111:101000000:000001110] [001110111:101000001:000001111]
[010011001:101000110:000001000] [010110001:101000111:000001001]
[011011101:101000101:000001011] [011110101:101000100:000001010]
[100011010:101001111:000000001] [100110010:101001110:000000000]
[101011110:101001100:000000010] [101110110:101001101:000000011]
[110011000:101001010:000000100] [110110000:101001011:000000101]
[111011100:101001001:000000111] [111110100:101001000:000000110]
sym seq=001011000:
[000000111:001010110:000001110] [000101111:001010111:000001111]
[001000011:001010101:000001101] [001101011:001010100:000001100]
[010000101:001010011:000001011] [010101101:001010010:000001010]
[011000001:001010000:000001000] [011101001:001010001:000001001]
[100000110:001011010:000000010] [100101110:001011011:000000011]
[101000010:001011001:000000001] [101101010:001011000:000000000]
[110000100:001011111:000000111] [110101100:001011110:000000110]
[111000000:001011100:000000100] [111101000:001011101:000000101]
sym seq=100111010:
[000011110:100111110:000000100] [000110110:100111111:000000101]
[001011010:100111101:000000111] [001110010:100111100:000000110]
[010011100:100111011:000000001] [010110100:100111010:000000000]
[011011000:100111000:000000010] [011110000:100111001:000000011]
[100011111:100110010:000001000] [100110111:100110011:000001001]
[101011011:100110001:000001011] [101110011:100110000:000001010]
[110011101:100110111:000001101] [110110101:100110110:000001100]
[111011001:100110100:000001110] [111110001:100110101:000001111]
sym seq=001100010:
[000000100:001100000:000000010] [000101100:001100001:000000011]
[001000000:001100011:000000001] [001101000:001100010:000000000]
[010000110:001100101:000000111] [010101110:001100100:000000110]
[011000010:001100110:000000100] [011101010:001100111:000000101]
[100000101:001101100:000001110] [100101101:001101101:000001111]
[101000001:001101111:000001101] [101101001:001101110:000001100]
[110000111:001101001:000001011] [110101111:001101000:000001010]
[111000011:001101010:000001000] [111101011:001101011:000001001]
sym seq=011101001:
[000001100:011101010:000000011] [000100100:011101011:000000010]
[001001000:011101001:000000000] [001100000:011101000:000000001]
[010001110:011101111:000000110] [010100110:011101110:000000111]
[011001010:011101100:000000101] [011100010:011101101:000000100]
[100001101:011100110:000001111] [100100101:011100111:000001110]
[101001001:011100101:000001100] [101100001:011100100:000001101]
[110001111:011100011:000001010] [110100111:011100010:000001011]
[111001011:011100000:000001001] [111100011:011100001:000001000]
```

```
sym seq=110100111:
[000010111:110101001:000001110] [000111111:110101000:000001111]
[001010011:110101010:000001101] [001111011:110101011:000001100]
[010010101:110101100:000001011] [010111101:110101101:000001010]
[011010001:110101111:000001000] [011111001:110101110:000001001]
[100010110:110100101:000000010] [100111110:110100100:000000011]
[101010010:110100110:000000001] [101111010:110100111:000000000]
[110010100:110100000:000000111] [110111100:110100001:000000110]
[111010000:110100011:000000100] [111111000:110100010:000000101]
sym seq=011010011:
[000001111:011011100:000001111] [000100111:011011101:000001110]
[001001011:011011111:000001100] [001100011:011011110:000001101]
[010001101:011011001:000001010] [010100101:011011000:000001011]
[011001001:011011010:000001001] [011100001:011011011:000001000]
[100001110:011010000:000000011] [100100110:011010001:000000010]
[101001010:011010011:000000000] [101100010:011010010:000000001]
[110001100:011010101:000000110] [110100100:011010100:000000111]
[111001000:011010110:000000101] [111100000:011010111:000000100]
sym seq=001110100:
[000000101:001111101:000001001] [000101101:001111100:000001000]
[001000001:001111110:000001010] [001101001:001111111:000001011]
[010000111:001111000:000001100] [010101111:001111001:000001101]
[011000011:001111011:000001111] [011101011:001111010:000001110]
[100000100:001110001:000000101] [100101100:001110000:000000100]
[101000000:001110010:000000110] [101101000:001110011:000000111]
[110000110:001110100:000000000] [110101110:001110101:000000001]
[111000010:001110111:000000011] [111101010:001110110:000000010]
sym seq=111101001:
[000010001:111100010:000001011] [000111001:111100011:000001010]
[001010101:111100001:000001000] [001111101:111100000:000001001]
[010010011:111100111:000001110] [010111011:111100110:000001111]
[011010111:111100100:000001101] [011111111:111100101:000001100]
[100010000:111101110:000000111] [100111000:111101111:000000110]
[101010100:111101101:000000100] [101111100:111101100:000000101]
[110010010:111101011:000000010] [110111010:111101010:000000011]
[111010110:111101000:000000001] [111111110:111101001:000000000]
sym seq=011111111:
[000001101:011110111:000001000] [000100101:011110110:000001001]
[001001001:011110100:000001011] [001100001:011110101:000001010]
[010001111:011110010:000001101] [010100111:011110011:000001100]
[011001011:011110001:000001110] [011100011:011110000:000001111]
[100001100:011111011:000000100] [100100100:011111010:000000101]
[101001000:011111000:000000111] [101100000:011111001:000000110]
[110001110:011111110:000000001] [110100110:011111111:000000000]
[111001010:011111101:000000010] [111100010:011111100:000000011]
sym seq=111111111:
[000010000:111111111:000000000] [000111000:111111110:000000001]
[001010100:111111100:000000011] [001111100:111111101:000000010]
[010010010:111111010:000000101] [010111010:111111011:000000100]
[011010110:111111001:000000110] [011111110:111111000:000000111]
[100010001:111110011:000001100] [100111001:111110010:000001101]
[101010101:111110000:000001111] [101111101:111110001:000001110]
[110010011:111110110:000001001] [110111011:111110111:000001000]
[111010111:111110101:000001010] [111111111:111110100:000001011]
```

B.6 Number of Periods and Unique Sequences of 9-cell CAs

Ruleset	Period	Sequences	Ruleset	Period	Sequences	Ruleset	Period	Sequences
110111101	30	511	110001100	14	511	110001110	10	256
110100101	30	511	110010111	14	511	*110000011	10	32
110110000	30	511	110011000	14	511	*110010011	10	32
111000011	30	511	110100111	14	256	110101000	8	256
111001000	30	511	111001011	14	256	111100011	8	256
111011110	30	511	111111101	14	256	111110001	8	256
111101010	30	511	110101100	14	128	110001111	8	256
111101110	30	511	110010100	14	128	110010000	8	256
111110000	30	511	110011010	14	128	111111000	8	128
110000110	30	511	111110010	14	64	110000100	8	128
110000111	30	511	*111101111	14	32	110001010	8	128
110001001	30	511	111001110	12	511	*111000111	8	32
111000100	30	256	111001111	12	511	*111111111	7	32
111010010	30	256	111011100	12	511	110010010	6	511
111010101	30	256	111100111	12	511	111000101	6	256
111011101	30	256	111111001	12	511	111010001	6	256
111100001	30	256	110001000	12	511	111101000	6	128
110010001	30	256	110011101	12	511	110100011	6	64
*110111011	30	32	110110111	12	256	111100010	6	64
*110101011	30	32	110111001	12	256	110001011	6	64
110100000	28	511	110111110	12	256	110100100	6	32
111001100	28	511	110100001	12	256	110101001	4	256
110000010	28	511	111000010	12	256	111001101	4	256
110011001	28	511	111011011	12	256	111010100	4	256
110011100	28	511	111101101	12	256	111011001	4	256
110110010	24	511	111110011	12	256	111101100	4	256
110010110	24	511	111111100	12	256	110011110	4	256
110100110	24	64	110000000	12	256	110111100	4	128
110110100	24	32	110011111	12	256	110110011	4	64
110101111	16	511	111010000	12	128	111001010	4	64
111101011	16	511	111110110	12	32	110011011	4	64
111110101	16	511	110000101	12	32	111100110	4	32
111111010	16	511	110100010	10	511	110010101	4	32
*111010111	16	32	110101010	10	511	110110110	3	64
110111111	14	511	111000110	10	511	110000001	1	256
111010011	14	511	111011000	10	511	111000000	1	128
111011111	14	511	111100000	10	511	110110101	0	511
111100101	14	511	111100100	10	511	110111010	0	511
111101001	14	511	110001101	10	511	110101101	0	511
111110100	14	511	110110001	10	256	111010110	0	511
111110111	14	511	110111000	10	256	110101110	0	256
111111011	14	511	111000001	10	256	111011010	0	64
111111110	14	511	111001001	10	256			

Table 11: Raw data for number of periods and sequences of various 9-cell CA. Rulesets have a 1 for rule 150 cells, 0 for rule 90 cells. Rulesets marked with '*' are symmetric.