

## Cellular Statistical Models of Broken Cloud Fields. Part III: Markovian properties

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### ABSTRACT

In the third part of our “Cellular Statistical Models of Broken Cloud Fields” series the cloud statistics formalism developed in the first two parts is interpreted in terms of the theory of Markov processes. The master matrix introduced in this study is a unifying generalization of both the cloud fraction probability distribution function (PDF) and the Markovian transition probability matrix. To illustrate the new concept we use the master matrix for computation of the moments of the cloud fraction PDF, in particular, the variance, which until now has not been analytically derived in the framework of our previous work. This paper also serves as a bridge to our future studies of the effects of sampling and averaging on satellite-based cloud masks.

### 1. Introduction

Clouds are among the major contributors of the uncertainty to estimates of the Earth’s energy budget (e.g., Boucher et al. 2013; Flato et al. 2013). The combination of observations, theory and models is essential for the understanding of how clouds contribute and respond to climate change (Stephens 2005). To this end, studies of cloud cover and changes of its structure on the global scale are very important. For example, the inability of a model to properly capture the stratocumulus-to-cumulus cloud transition in the tropics can lead to significant errors in radiative fluxes at the ground (de Roode et al. 1996). Current climate models’ disagreement on the change of the subtropical low-cloud amount under a global warming scenario results in considerable uncertainties in global-mean temperature predictions (Bony and Dufresne 2005; Webb et al. 2013; Tsushima et al. 2016).

An essential role in the understanding of the cloud cover structure and development is played by physically-based dynamical models such as large-eddy simulations (LES). At the same time, computationally inexpensive, stochastic cloud models have been used to generate cloud fields resembling observations (e.g., Evans and Wiscombe 2004; Hogan and Kew 2005; Venema et al. 2006; Prigarin and Marshak 2009). Such models can include the internal

cloud structure (e.g., Schertzer and Lovejoy 1987; Cahalan 1994; Marshak et al. 1994; Schmidt et al. 2007) or simply describe cloud fields as binary mixtures of cloudy and clear areas (e.g., Su and Pomraning 1994; Zuev and Titov 1995; Prigarin et al. 2002).

This study continues the series of Alexandrov et al. (2010a,b) (hereinafter referred to as Part I and Part II, respectively) devoted to statistical parameterization and modeling of the cloud cover and structure (characterized by sizes of clouds and gaps between them). The approach adopted in this series is based on cloud-mask statistics of 2D broken cloud fields derived from observations made along linear transects (chords). Such observations consist of the lengths of cloudy and clear intervals in each transect. In distinction to, e.g., area-based characterization, this approach works equally well for cumulus and stratocumulus cloud fields with a smooth transition between these types. In Part II the analytical expressions derived in Part I were demonstrated to adequately describe the statistics of shallow, broken cloud fields generated using a realistic LES model.

While the statistical framework of Part I was built by generalizing a discrete lattice model to the continuous case, it also can be equivalently formulated using the language of the Markov processes theory. In this formulation each transect consisting of subsequent cloudy and clear segments is considered as a realization of a binary Markov process, which can take on only two values: occupied (cloudy, “•”, “1”) or empty (clear, “o”, “0”) (see, e.g., Kulkarni 2011; Ibe 2013). Thus, the algorithm used

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in Part I for analytical computations and the construction of examples is essentially a differential binary Markov model based on probabilities of the transition between close points.

Binary Markovian mixtures in their own right were subjects of statistical studies (Sanchez et al. 1993; Astin and Di Girolamo 1999), in particular those focused on cloud-field properties (Su and Pomraning 1994; Astin and Latter 1998; Astin et al. 2001; van de Poll et al. 2006). At the same time Markovian cloud models are extensively used in the stochastic radiative transfer theory and simulations (Levermore et al. 1988; Titov 1990; Zuev and Titov 1995; Su and Pomraning 1995; Pomraning 1989, 1996, 1998; Malvagi et al. 1993; Lane et al. 2002; Kassianov 2003; Byrne 2005; Kassianov and Veron 2011). For example, our algorithm outlined in Part I was used for the generation of simulated broken cloud fields in a recent series of stochastic radiative transfer studies (Doicu et al. 2013, 2014a,b; Efremenko et al. 2016). (We should note here that radiative transfer in Markovian clouds should be distinguished from the approach in which the propagation of light itself is described as a Markov process (e.g., Xu et al. 2012, 2016).) The Markovian approach to cloud fraction has been also the basis for an analysis of ground-based measurements of sunshine duration and vertical visibility (see, e.g., Morf (2011) and references therein).

In our series we focus on potential satellite remote sensing applications, in particular, on the effects of the measurement's finite footprint and resolution on the observed cloud statistics. Thus, we were interested in cloud fraction and cloud/gap lengths distributions in an ensemble of finite-size samples. In the current study we establish the connection between the “cellular” and Markovian formalisms for the description of broken cloud fields by including into consideration the statistical relationship between the states of end-points of such samples. In order to do this we introduce the notion of a “master matrix”, which is a unifying generalization of both the probability distribution function (PDF) of the observed cloud fraction and the transition matrix of a binary Markov process. This provides a new theoretical framework for the investigation of resolution and scale effects on cloud statistics. We also explore the use of integrals of the master matrix with respect to the cloud fraction as a convenient computational tool, by using which the variance of the cloud-fraction PDF is derived analytically directly from the results of Part I.

## 2. Binary Markov models

We define the binary Markov model (BMM) as a statistical ensemble of functions on the real line  $\mathbb{R}$  which can take only two (generally non-numeric) values (states): “•” (occupied, cloudy) or “◦” (empty, clear). The states of the

points are statistically related; however, the states with coordinates  $x > x_0$  (where  $x_0 \in \mathbb{R}$  corresponds to some “initial” point) depend only on the state at  $x_0$  (and not on the states with  $x < x_0$ ). This constitutes the Markovian property of the model. The properties of the BMM are governed by the transition matrix of the form

$$\mathbf{P} = \begin{pmatrix} P_{\bullet\bullet} & P_{\bullet\circ} \\ P_{\circ\bullet} & P_{\circ\circ} \end{pmatrix}, \quad (1)$$

where  $P_{ij}$  is the probability of transition from the state  $i$  at  $x_0$  into state  $j$  at  $x > x_0$  ( $i$  and  $j$  can be either “•” or “◦”). Each row of  $\mathbf{P}$  sums to unity:

$$P_{\bullet\bullet} + P_{\bullet\circ} = 1, \quad (2)$$

$$P_{\circ\bullet} + P_{\circ\circ} = 1, \quad (3)$$

since the probability to get from, e.g., “•” to “either • or ◦” equals one. We assume the model to be spatially homogeneous, so the transition matrix depends only on the distance  $L = x - x_0$ , rather than on  $x$  and  $x_0$  themselves:  $\mathbf{P} = \mathbf{P}(L)$ . We also assume single-layer cloud fields (see Kassianov (2003); Kassianov and Veron (2011) for generalization to multi-layer cases). The transition matrices for two consequent intervals of the lengths  $L_1$  and  $L_2$  obey the group property (Chapman-Kolmogorov equation):

$$\mathbf{P}(L_1 + L_2) = \mathbf{P}(L_1) \mathbf{P}(L_2), \quad (4)$$

and  $\mathbf{P}(L)$  becomes the identity matrix when  $L \rightarrow 0$ :

$$\mathbf{P}(0) = \mathbf{I}. \quad (5)$$

The realizations of a binary Markov model on  $\mathbb{R}$  are infinite patterns of interchanging clear and cloudy intervals of finite lengths. The statistical distributions of these cloudy and clear lengths appear to be exponential (Levermore et al. 1988; Pomraning 1989; Alexandrov et al. 2010a) with the respective means  $L_c = L_\bullet$  and  $L_g = L_\circ$  (in the notation of Parts I and II “c” stands for “clouds” and “g” – for “gaps”). The pair of numbers  $(L_c, L_g)$  provides complete parameterization of the model in the infinite space. Often  $\lambda_i = 1/L_i$  representing the rate at which the system leaves state  $i$  are used instead of  $L_i$  (see e.g., Ibe 2013). We should note that while the above definition of the BMM implies specification of the positive direction on  $\mathbb{R}$ , the parameters of the model and other statistics expressed in their terms do not depend on the choice of this direction. This makes BMMs applicable to characterization of cloud fields in 1D (and even in 2D, see Part II).

## 3. State matrices

The state of the initial point of the interval can be also considered as a random variable taking value “•” with the probability  $u$  and “◦” – with the probability  $v = 1 - u$ . It is

convenient to describe such state by a “state matrix” with identical rows:

$$\mathbf{U} = \begin{pmatrix} u & v \\ u & v \end{pmatrix}. \quad (6)$$

If the probabilities of a state at  $x_0$  to be cloudy or clear are specified by  $\mathbf{U}(x_0)$ , the corresponding probabilities for the state at  $x_0 + L$  will be specified by the matrix

$$\mathbf{U}(x_0 + L) = \mathbf{U}(x_0) \mathbf{P}(L). \quad (7)$$

The state matrices have the following properties:

$$\det \mathbf{U} = 0, \quad \text{Tr } \mathbf{U} = 1, \quad (8)$$

$$\mathbf{U}' \mathbf{U} = \mathbf{U}, \quad \text{thus, } \mathbf{U}^2 = \mathbf{U}. \quad (9)$$

(here  $\mathbf{U}'$  is another state matrix).

The state matrices for definitely known (“pure”) cloudy and clear states of a point are

$$\mathbf{U}^\bullet = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and } \mathbf{U}^\circ = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad (10)$$

respectively. Another important example of state matrix is

$$\mathbf{C} = \begin{pmatrix} \bar{c} & \bar{s} \\ \bar{c} & \bar{s} \end{pmatrix}, \quad (11)$$

where  $\bar{c}$  is the cloud fraction in the infinite sample and  $\bar{s} = 1 - \bar{c}$ . This matrix describes probabilities to be cloudy or clear for a point randomly selected from such sample. We will refer to the state described by  $\mathbf{C}$  as to the “random state”. The matrix  $\mathbf{C}$  can be viewed as a transition matrix from an infinitely distant state ( $L \rightarrow \infty$ ), since the dependence on the initial state is expected to disappear with the distance (see Section 5 below).

In this study state matrices will be used for characterization of the initial state of the sample interval, thus, characterizing the sampling process.

#### 4. Statistics of the ensemble of finite intervals

##### a. Model parameterization and sampling procedure

In Part I we considered the ensemble of finite-length samples extracted from infinite-length realizations of binary Markov model. It was demonstrated in Part II that the same analysis is valid for finite 1D transects extracted from a 2D field (in that case LES-derived 2D cloud masks). The sampling procedure is specified by the length  $L$  of the sample and the probabilities of its initial point to be cloudy or clear (specified by a state matrix of the form Eq. (6)). If the samples are chosen at random (which is natural), the initial state is described by the random state matrix  $\mathbf{C}$  (Eq. (11)). However, in general, we can allow

for a “biased” sampling with a generic choice of the initial state matrix  $\mathbf{U}$ . For example, in the case of  $\mathbf{U} = \mathbf{U}^\bullet$  the selection is restricted to the samples starting with a cloudy point.

The samples can be classified according to the states of their initial and end points, as well as their cloud fraction  $c$ . Below we derive cloud fraction PDFs separately for the four combination of the initial and final states of the samples. Their combination represents both statistics of cloud fraction and transition probabilities of the binary Markov process.

##### b. Even and odd diagrams

In order to derive cloud fraction PDFs conditioned by the end-states of the sample intervals we recall the computation of the statistical sum in Appendix C of Part I. This computation was based on four types of sample structures classified by the states (clear or cloudy) of their beginning and end points. These types were schematically represented by the diagrams presented below. In our convention each of these diagrams is associated with transition from its initial state (at the left end) into its final state (at the right end). The specified left-to-right direction reflects the difference between the initial state (which is pre-selected regardless of the fraction of such states in the dataset) and the final state (which probability is conditioned upon the initial state). This means, for example, that two diagrams looking like mirror images of each other may correspond to different values. The computation in Part I included two even diagrams

$$\bullet - \bullet - \circ - \circ \dots \bullet - \bullet - \circ - \circ, \quad (12)$$

$$\circ - \circ - \bullet - \bullet \dots \circ - \circ - \bullet - \bullet, \quad (13)$$

corresponding to the two respective terms in the statistical sum

$$F_{\bullet\circ}(c) = a_c e^{-(a_c c + a_g s)} I_0(Z), \quad (14)$$

$$F_{\circ\bullet}(c) = a_g e^{-(a_c c + a_g s)} I_0(Z), \quad (15)$$

and two odd diagrams

$$\bullet - \bullet - \circ - \circ \dots \circ - \circ - \bullet - \bullet, \quad (16)$$

$$\circ - \circ - \bullet - \bullet \dots \bullet - \bullet - \circ - \circ, \quad (17)$$

corresponding to the terms

$$F_{\bullet\bullet}(c) = 2c e^{-(a_c c + a_g s)} a_c a_g \frac{I_1(Z)}{Z} + e^{-a_c} \delta(s), \quad (18)$$

$$F_{\circ\circ}(c) = 2s e^{-(a_c c + a_g s)} a_c a_g \frac{I_1(Z)}{Z} + e^{-a_g} \delta(c), \quad (19)$$

respectively. The  $\delta$ -function term in Eq. (18) correspond to completely overcast samples, while that in Eq. (19) – to completely clear ones. The notations in Eqs. (14), (15), (18), and (19) are the following:  $I_0$  and  $I_1$  are the modified Bessel functions;  $c$  is the cloud fraction in the sample (which is a stochastic variable),  $s = 1 - c$ ;

$$a_c = \frac{L}{L_c}, \quad a_g = \frac{L}{L_g}, \quad (20)$$

where  $L$  is the sample length; and

$$Z = 2\sqrt{a_c a_g c s}. \quad (21)$$

We remind that  $L_c$  and  $L_g$  are the mean lengths of respectively cloudy and clear (gap) intervals in infinite space (or as  $L \rightarrow \infty$ ). The corresponding parameters in an ensemble of finite samples are different and can be expressed in terms of  $L_c$ ,  $L_g$ , and also  $L$ . The PDFs of cloudy and clear interval lengths in such ensembles are no longer exponential and, as Eqs. (18) and (19), include  $\delta$ -function terms corresponding to completely clear and overcast samples (see Part I for details).

#### *c. $(\bar{c}, L_*)$ parameterization*

In Markovian framework it is convenient to chose another model parameterization, which is equivalent to  $(L_c, L_g)$  and has the two independent parameters:

$$\bar{c} = \frac{L_c}{L_c + L_g} \quad \text{and} \quad L_* = \frac{L_c L_g}{L_c + L_g}. \quad (22)$$

Here  $\bar{c}$  is the mean cloud fraction (which is independent of  $L$  for an ensemble of randomly selected samples) and  $L_*$  is the double of the geometric mean of  $L_c$  and  $L_g$ :

$$\frac{1}{L_*} = \frac{1}{L_c} + \frac{1}{L_g}. \quad (23)$$

$L_*$  can be considered as a universal scale length of the cloud field. This quantity is also often called “autocorrelation length” (e.g., Levermore et al. 1988; Pomraning 1989). While it enters the exponent in the corresponding autocorrelation function (Morf 1998, 2011), we should note that the symbols “•” and “o” are not real numbers, but only elements of a set with no algebraic structure defined on it. Thus, in order to define autocorrelation function one has to assign real numerical values (e.g., 0 and 1) to cloudy and clear states (cf. Supplement I to Alexandrov et al. (2016)). Note that  $L_* < \min(L_c, L_g)$ , so it can be considered as a characteristic inhomogeneity size in the cloud field. For example, in a sparse cumulus field with  $L_c \ll L_g$  the value of  $L_* \approx L_c$  is determined by the cloud size and is practically independent of the distance between clouds. A scale transformation of the cloud field when all distances are multiplied by the same number  $\alpha$

(so  $L_c \rightarrow \alpha L_c$  and  $L_g \rightarrow \alpha L_g$ ) results in rescaling of  $L_*$  to  $\alpha L_*$ , while  $\bar{c}$  remains intact.

In the  $(\bar{c}, L_*)$  parameterization

$$a_c = \bar{s} r \quad \text{and} \quad a_g = \bar{c} r, \quad (24)$$

where  $\bar{s} = 1 - \bar{c}$  and

$$r = \frac{L}{L_*} \quad (25)$$

is independent from the cloud fraction  $\bar{c}$ . In this notation

$$a_c + a_g = r, \quad a_c a_g = \bar{c} \bar{s} r^2, \quad (26)$$

$$a_c c + a_g s = (\bar{s} c + \bar{c} s) r, \quad Z = 2r \sqrt{\bar{c} \bar{s} c s}, \quad (27)$$

and the Eqs. (14), (15), (18), and (19) take the following more compact forms:

$$F_{\bullet\circ}(c) = \bar{s} r I_0(Z) e^{-(\bar{s}c + \bar{c}s)r}, \quad (28)$$

$$F_{\circ\bullet}(c) = \bar{c} r I_0(Z) e^{-(\bar{s}c + \bar{c}s)r}, \quad (29)$$

$$\begin{aligned} F_{\bullet\bullet}(c) &= 2c \bar{c} \bar{s} r^2 \frac{I_1(Z)}{Z} e^{-(\bar{s}c + \bar{c}s)r} + \delta(s) e^{-\bar{s}r} \\ &= \frac{Z}{2s} I_1(Z) e^{-(\bar{s}c + \bar{c}s)r} + \delta(s) e^{-\bar{s}r}, \end{aligned} \quad (30)$$

$$\begin{aligned} F_{\circ\circ}(c) &= 2s \bar{c} \bar{s} r^2 \frac{I_1(Z)}{Z} e^{-(\bar{s}c + \bar{c}s)r} + \delta(c) e^{-\bar{c}r} \\ &= \frac{Z}{2c} I_1(Z) e^{-(\bar{s}c + \bar{c}s)r} + \delta(c) e^{-\bar{c}r}. \end{aligned} \quad (31)$$

## 5. Master matrix

### *a. Definition*

The expressions Eqs. (28) – (31) can be combined into the “master” matrix:

$$\mathbf{F}(c) = \begin{pmatrix} F_{\bullet\bullet}(c) & F_{\bullet\circ}(c) \\ F_{\circ\bullet}(c) & F_{\circ\circ}(c) \end{pmatrix}, \quad (32)$$

which in  $(\bar{c}, L_*)$  parameterization has the form:

$$\begin{aligned} \mathbf{F}(c) &= \begin{pmatrix} \frac{Z}{2s} I_1(Z) & \bar{s} r I_0(Z) \\ \bar{c} r I_0(Z) & \frac{Z}{2c} I_1(Z) \end{pmatrix} e^{-(\bar{s}c + \bar{c}s)r} \\ &+ \begin{pmatrix} \delta(s) e^{-\bar{s}r} & 0 \\ 0 & \delta(c) e^{-\bar{c}r} \end{pmatrix}. \end{aligned} \quad (33)$$

We show in Appendix A that in the limit case of short sample ( $r \rightarrow 0$ ,  $L \ll L_*$ ) the master matrix has the form

$$\mathbf{F}^0(c) = \begin{pmatrix} \delta(s) & 0 \\ 0 & \delta(c) \end{pmatrix}. \quad (34)$$

This expression indicates the absence of partially cloudy samples in this limit case (all samples are either overcast or all-clear). In the opposite case of long sample ( $r \rightarrow \infty$ ,  $L \gg L_*$ )

$$\mathbf{F}^\infty(c) = \mathbf{C} \delta(c - \bar{c}) \quad (35)$$

(see Appendix A for derivation of this expression). This means that all long samples have cloud fraction equal to  $\bar{c}$ .

### b. Relations to cloud cover PDF and transition matrix

Each matrix element  $F_{ij}(c)$  is the cloud cover probability density conditional by having the first point of the interval in the state  $i$ , and the last point in the state  $j$ . This means that given an initial state described by a matrix  $\mathbf{U}$  of the form Eq. (6) (corresponding to the initial state probability  $u$  to be cloudy) we can compute the cloud fraction PDF as

$$F_{cf}(c) = uF_{\bullet\bullet}(c) + uF_{\bullet\circ}(c) + vF_{\circ\bullet}(c) + vF_{\circ\circ}(c), \quad (36)$$

which can be conveniently written in matrix form:

$$F_{cf}(c) = \text{Tr} [\mathbf{U} \mathbf{F}(c)] \quad (37)$$

(thus, justifying introduction of state matrices). If the ensemble consists of randomly chosen samples ( $\mathbf{U} = \mathbf{C}$ ), then Eq. (37) leads to the PDF computed in Part I, which in  $(\bar{c}, L_*)$  parameterization has the following form:

$$\begin{aligned} F_{cf}(c) &= 2\bar{c}\bar{s}r e^{-(\bar{s}c+\bar{c}s)r} \\ &\times \left[ I_0(Z) + (\bar{c}c + \bar{s}s) r \frac{I_1(Z)}{Z} \right] \\ &+ \bar{s} e^{-\bar{c}r} \delta(c) + \bar{c} e^{-\bar{s}r} \delta(s). \end{aligned} \quad (38)$$

In the limit cases of short and long samples this expression transforms into

$$F_{cf}^0(c) = \bar{s} \delta(c) + \bar{c} \delta(s) \quad \text{and} \quad F_{cf}^\infty(c) = \delta(c - \bar{c}), \quad (39)$$

respectively.

The master matrix elements  $F_{ij}(c)$  can be also considered as a transition probabilities between the sample's end-states  $i$  and  $j$  conditioned by the cloud fraction in the sample. This means that the transition matrix  $\mathbf{P}$  of the Markov process can be derived from the master matrix by integrating out the cloud fraction dependence:

$$\mathbf{P} = \int_0^1 \mathbf{F}(c) dc. \quad (40)$$

The expressions for  $\mathbf{P}$  in the limit cases of  $L \ll L_*$  and  $L \gg L_*$  immediately follow from Eqs. (34), (35), and (40):

$$\mathbf{P}^0 = \mathbf{I} \quad \text{and} \quad \mathbf{P}^\infty = \mathbf{C} \quad (41)$$

(since integrals of  $\delta$ -functions are equal to unity).

Note that normalization condition for  $F_{cf}(c)$  follows from Eqs. (37) and (40):

$$\begin{aligned} \int_0^1 F_{cf}(c) dc &= \int_0^1 \text{Tr} [\mathbf{U} \mathbf{F}(c)] dc = \text{Tr} \left[ \mathbf{U} \int_0^1 \mathbf{F}(c) dc \right] \\ &= \text{Tr} [\mathbf{U} \mathbf{P}] = \text{Tr} \mathbf{U}' = 1. \end{aligned} \quad (42)$$

Here  $\mathbf{U}'$  is some other state matrix of the form Eq. (6) and we used Eqs. (7) and (8).

Note that since the master matrix is a precursor of both the PDF (Eq. (38)) and the transition matrix (Eq. (40)), the sample length  $L$  also has a dual meaning being the sample length for cloud fraction statistics and also the lag length between its initial and final points in the Markov formalism.

## 6. Transition probability matrix

Explicit computation of the transition matrix according to Eq. (40) is performed in Appendix B resulting in the following expression

$$\mathbf{P} = \begin{pmatrix} \bar{c} + \bar{s} e^{-r} & \bar{s} - \bar{s} e^{-r} \\ \bar{c} - \bar{c} e^{-r} & \bar{s} + \bar{c} e^{-r} \end{pmatrix}. \quad (43)$$

This expression can be also derived in a simpler way using the differential form of the Chapman-Kolmogorov equation (see e.g., Ibe 2013; Morf 1998; Kassianov 2003; Kassianov and Veron 2011) or the uniformization method (Kulkarni 2011). Thus, our computation provides a closure demonstrating that the same result can be also obtained using the master matrix (Eq. (32)) and verifying Eq. (40). Eq. (43) can be also written as

$$\mathbf{P} = e^{-r} \mathbf{I} + (1 - e^{-r}) \mathbf{C}, \quad (44)$$

where  $\mathbf{I}$  is the identity matrix, while  $\mathbf{C}$  is the random state matrix defined by Eq. (11). The group properties Eqs. (4) and (5) of transition matrices can be verified using this expression and noticing that  $\mathbf{C}^2 = \mathbf{C}$ . The factor  $e^{-r} < 1$  balances the fractions of  $\mathbf{I}$  and  $\mathbf{C}$  in the transition matrix. In the limit case of very short interval ( $L \ll L_*$ ,  $r \ll 1$ ) we have  $\mathbf{P} \approx \mathbf{I}$ , i.e., state change between close points is improbable. In the opposite limit case of very long interval ( $L \gg L_*$ ,  $r \gg 1$ ) we see that  $\mathbf{P} \approx \mathbf{C}$ , thus, the transition probability no longer depends on the state of the initial point and is governed by the overall cloud fraction in the infinite space. This is in agreement with Eq. (41). The weakening of the coupling between points with increase of the lag  $L$  can be qualitatively evaluated by

$$\det \mathbf{P}(L) = e^{-r}, \quad (45)$$

which tends to zero as  $L \rightarrow \infty$ , reflecting increasing degeneracy of the transition matrix.

It follows from Eq. (44) that an arbitrary state  $\mathbf{U}$  (including the pure states defined by Eq. (10)) is transformed by the transition matrix into a mixture of itself and the random state:

$$\mathbf{U} \mathbf{P} = e^{-r} \mathbf{U} + (1 - e^{-r}) \mathbf{C}. \quad (46)$$

The fraction of the random state in this mixture increases with the distance from the initial point. It follows from Eq. (46) that the random state  $\mathbf{U} = \mathbf{C}$  remains random after the transition:  $\mathbf{C} \mathbf{P} = \mathbf{C}$ .

## 7. Matrix integrals

The transition matrix computed according to Eq. (40) can be considered as “zeroth moment” of the master matrix. We would like to extend this notion to integrals of a general form

$$\bar{\mathbf{F}}^{(f)} = \int_0^1 f(c) \mathbf{F}(c) dc, \quad (47)$$

where  $f(c)$  is an arbitrary function of cloud fraction defined on  $[0, 1]$ . They are related to the corresponding integrals of cloud fraction PDF as

$$\bar{f} = \int_0^1 f(c) F_{cf}(c) dc = \text{Tr} \left[ \mathbf{U} \bar{\mathbf{F}}^{(f)} \right], \quad (48)$$

where  $\mathbf{U}$  describes the initial state of the sample. In this notation  $\mathbf{P} = \bar{\mathbf{F}}^{(1)}$  corresponding to  $f(c) \equiv 1$ , and Eq. (48) yields  $\bar{f} = 1$ . Matrix integrals present a convenient tool for computation of averages (such as the moments of the cloud fraction PDF) since integration is performed separately for different matrix elements. The results of such integration are linear combinations of functions of  $r$  with matrix coefficients depending on  $\bar{c}$  (like  $\mathbf{C}$ ), so subsequent application of Eq. (48) is relatively simple.

As an example, we computed in Appendix C the first moment of the master matrix

$$\bar{\mathbf{F}}^{(c)} = \int_0^1 c \mathbf{F}(c) dc, \quad (49)$$

which took the form

$$\bar{\mathbf{F}}^{(c)} = \bar{c} \mathbf{C} + \bar{s} (\mathbf{I} - \mathbf{C}) e^{-r} + (2\bar{c}\bar{s} \mathbf{K} + \mathbf{L}) \frac{1 - e^{-r}}{r}, \quad (50)$$

where

$$\mathbf{K} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 0 & \bar{s} \\ -\bar{c} & 0 \end{pmatrix}. \quad (51)$$

We verified that application of Eq. (48) to Eq. (50) leads to the expression for the mean cloud fraction

$$\bar{c}(L) = \int_0^1 c F_{cf}(c) dc = \bar{c} + (u\bar{s} - v\bar{c}) \frac{1 - e^{-r}}{r} \quad (52)$$

derived in Part I without matrix formalism. Note that in our definition  $\bar{c}$  denotes mean cloud fraction in infinite space ( $L, r \rightarrow \infty$ ). The mean cloud fraction in finite sample becomes  $L$ -independent (and equal to  $\bar{c}$ ) only for unbiased random sampling corresponding to  $\mathbf{U} = \mathbf{C}$ .

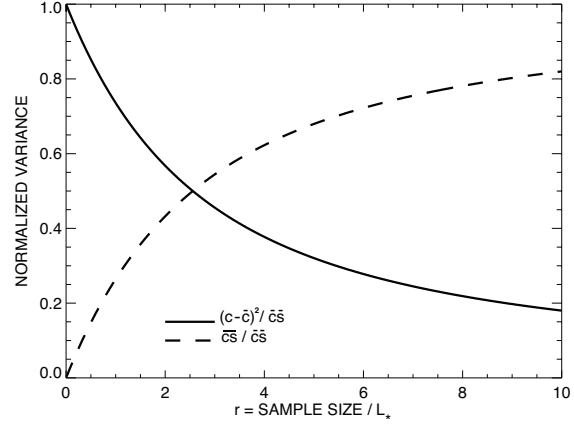


FIG. 1. Dependence of the cloud fraction variance (Eq. (58), solid curve) and  $\bar{c}\bar{s}$  (Eq. (57), dashed curve) on the parameter  $r = L/L_*$ . Both functions are normalized by  $\bar{c}\bar{s}$ .

In Appendix D we take the matrix integral

$$\bar{\mathbf{F}}^{(cs)} = \int_0^1 cs \mathbf{F}(c) dc, \quad (53)$$

obtaining the following expression:

$$\begin{aligned} \bar{\mathbf{F}}^{(cs)} &= \bar{c}\bar{s} \left[ \mathbf{P} - (\mathbf{B} - 6\mathbf{C} + 4\mathbf{I}) g(r) \right. \\ &\quad \left. + 2(\mathbf{B} - 6\mathbf{C} + 3\mathbf{I}) \frac{1 - g(r)}{r} \right], \end{aligned} \quad (54)$$

where  $\mathbf{P}$  is the transition matrix (Eq. (44)),

$$\mathbf{B} = \frac{1}{\bar{c}\bar{s}} \begin{pmatrix} 0 & \bar{s} \\ \bar{c} & 0 \end{pmatrix}, \quad (55)$$

and

$$g(r) = \frac{1 - e^{-r}}{r}. \quad (56)$$

Eq. (54) was used for derivation of the mean of the product  $cs$ , which in the random sample case has the form

$$\bar{c}\bar{s} = \bar{c}\bar{s} \left[ 1 - \frac{2}{r} + \frac{2}{r^2} (1 - e^{-r}) \right]. \quad (57)$$

This expression was used for computation of the variance

$$D = \overline{(c - \bar{c})^2} = \bar{c}\bar{s} - \bar{c}\bar{s} = \frac{2\bar{c}\bar{s}}{r} \left[ 1 - \frac{1 - e^{-r}}{r} \right]. \quad (58)$$

of the cloud fraction PDF  $F_{cf}(c)$  (Eq. (38)). Eq. (58) has not been previously derived analytically from the results

of Part I, however, it was obtained (in slightly different notations) by Morf (1998) using properties of the auto-correlation function. It is evident from Eq. (58) that the cloud fraction variance tends to zero as the sample length increases ( $r \rightarrow \infty$ ) and the cloud cover distribution narrows (see Eq. (39)). In the short-sample case ( $r \rightarrow 0$ ) the variance converges to the constant value  $\bar{c}\bar{s}$  (this can also be directly derived using Eq. (39)). Both  $\bar{c}\bar{s}$  and  $D$  (normalized by  $\bar{c}\bar{s}$ ) are plotted as functions of  $r$  in Fig. 1.

## 8. Concluding remarks

In Part III of our ‘‘Cellular Statistical Models of Broken Cloud Fields’’ series we further advance the approach to cloud statistics introduced in Part I and applied to realistic simulated cloud fields in Part II. This approach is focused on statistics of cloud properties in an ensemble of finite linear transects crossing a broken cloud field. Such an ensemble can be considered as an idealization of satellite or airborne datasets. In this study we interpret our previous results in terms of the theory of Markov processes and introduce the notion of the master matrix which unifies the statistics of the cloud fraction and the Markovian properties of the cloud field.

The explicit expression, Eq. (32), for the master matrix was derived based on the computations made in Part I. The cloud fraction PDF and the transition probability matrix of the Markov process were obtained using Eqs. (37) and (40), respectively. We have generalized the latter equation to define matrix integrals by Eq. (47) which appear to be a convenient tool for the computation of the moments of the cloud fraction PDF. We used the matrix integrals for the derivation of the mean cloud cover in a sample (obtaining the result of Part I in a different way), and also of the variance of  $F_{cf}(c)$ . The latter computation has been performed for the first time analytically based directly on the results of Part I.

Besides suggesting new ways to compute broken cloud field statistics, this study also sets up a framework for a quantitative estimate of the effects of coarse resolution of observations on the retrieved cloud masks. We plan to address this subject in the next part of this series. In that forthcoming study, an idealized satellite observation system will be characterized by a state attribution function (SAF) having the meaning of the probability that a sample (satellite pixel) with a certain CF is declared ‘‘cloudy’’ in the cloud mask. Such function can be chosen to be deterministic (e.g., equal to 1 for CF larger than 50% and to 0 otherwise) or stochastic (e.g., proportional to the CF). Physically, the stochasticity of SAF can be caused by external factors affecting the cloud-clear attribution (such as solar-viewing geometry, detector sensitivity to low light, etc.). We will use the SAF-based formalism to find the connection between binary Markov cloud fields (characterized by the master matrix) and binary cloud masks,

which also will be considered Markov chains. Our goal will be to establish a quantitative relationship between the ‘‘real’’ and the ‘‘observed’’ cloud-field statistics and to evaluate the possibilities to restore the former from the latter. This research will improve our understanding of how the presence of sub-pixel clouds/gaps in satellite observations affects the retrieved cloud field statistics and, subsequently, physical and radiative properties of the clouds.

Another anticipated direction of our future studies is the application of our techniques to global cloud-mask datasets derived from satellite observations made by, e.g., the Moderate Resolution Imaging Spectroradiometer (MODIS) or the Cloud-Aerosol Lidar with Orthogonal Polarization (CALIOP). In these studies we expect to deal with not only ‘‘narrow’’ sampling datasets (outlined in Part I and generalized in this study), but also with ‘‘diverse’’ sampling (described in Part II). Narrow sampling represents cloud populations with uniform statistical properties (thus, limited to smaller geographical scales) and having exponential chord-length distributions for both clouds and gaps. Diverse sampling, on the contrary, is characteristic of datasets with extensive geographic coverage (see, e.g., Cahalan and Joseph 1989; Koren et al. 2008), which may include numerous narrow-sampled subsets. In Part II we showed that the chord-lengths distributions for such datasets have a power-law form. While diverse datasets can present substantial challenges (especially for analytical studies) caused by additional averaging over various cloud populations, we look forward to explore the possibilities for advancing in this direction.

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## APPENDIX A

### Master matrix in asymptotic cases

In the short-sample limit ( $r \rightarrow 0, L \ll L_*$ )

$$I_0(Z) \rightarrow 1, \text{ and } \frac{I_1(Z)}{Z} \rightarrow \frac{1}{2}, \quad (\text{A1})$$

thus, all non-singular components of  $\mathbf{F}$  in Eq. (32) vanish, yielding

$$\mathbf{F}^0(c) = \begin{pmatrix} \delta(s) & 0 \\ 0 & \delta(c) \end{pmatrix}. \quad (\text{A2})$$

In the long-sample limit ( $r \rightarrow \infty, L \gg L_*$ ) the  $\delta$ -function term in Eq. (32) vanishes reflecting small probability of all-clear or overcast samples. To find the asymptotic shape of the non-singular term in Eq. (32) we use the

index-independent asymptotic representation of the modified Bessel functions (Gradshteyn and Ryzhik 1965):

$$I_\nu(x) \approx \frac{e^x}{\sqrt{2\pi x}}. \quad (\text{A3})$$

Thus,

$$I_0(Z) \approx I_1(Z) \approx \frac{\exp(Z)}{\sqrt{2\pi Z}}, \quad (\text{A4})$$

and the long-sample master matrix takes the following form:

$$\mathbf{F}^\infty(c) = \sqrt{\frac{Z}{2\pi}} \begin{pmatrix} \frac{1}{2s} & \frac{\bar{s}r}{Z} \\ \frac{\bar{c}r}{Z} & \frac{1}{2c} \end{pmatrix} e^{-(\bar{s}c + \bar{c}s - 2\sqrt{cs\bar{c}\bar{s}})r}, \quad (\text{A5})$$

where we used the definition of  $Z$  from Eq. (27):

$$Z = 2r \sqrt{cs\bar{c}\bar{s}}. \quad (\text{A6})$$

Note that the expression in the exponent can be written as

$$\bar{s}c + \bar{c}s - 2\sqrt{cs\bar{c}\bar{s}} = w^2, \quad (\text{A7})$$

where

$$w = \sqrt{\bar{s}c} - \sqrt{\bar{c}s}. \quad (\text{A8})$$

We then use the Gaussian approximation for  $\delta$ -function

$$\delta(w) \approx \frac{\sqrt{r}}{\sqrt{\pi}} e^{-w^2 r}, \quad (\text{A9})$$

as  $r \rightarrow \infty$ . The function  $\delta(w)$  can be also written as

$$\delta(w) = \frac{\delta(c - \bar{c})}{|w'(\bar{c})|}, \quad (\text{A10})$$

where  $w'(\bar{c})$  is the derivative of  $w$  with respect to  $c$  at  $c = \bar{c}$ :

$$\begin{aligned} w'(\bar{c}) &= \frac{d}{dc} \left[ \sqrt{\bar{s}c} - \sqrt{\bar{c}(1-c)} \right]_{c=\bar{c}} \\ &= \frac{1}{2} \left[ \sqrt{\frac{\bar{s}}{\bar{c}}} + \sqrt{\frac{\bar{c}}{\bar{s}}} \right]. \end{aligned} \quad (\text{A11})$$

The fact that  $\mathbf{F}^\infty(c) \propto \delta(c - \bar{c})$  allows us to set  $c = \bar{c}$  and  $s = \bar{s}$  in Eq. (A5) (note that  $Z = 2\bar{c}\bar{s}r$  in this case). After these substitutions and some simple algebra the asymptotic expression for the master matrix takes the following form:

$$\mathbf{F}^\infty(c) = \begin{pmatrix} \sqrt{\frac{\bar{c}}{\bar{s}}} & \sqrt{\frac{\bar{s}}{\bar{c}}} \\ \sqrt{\frac{\bar{c}}{\bar{s}}} & \sqrt{\frac{\bar{s}}{\bar{c}}} \end{pmatrix} \frac{\delta(c - \bar{c})}{\sqrt{\frac{\bar{s}}{\bar{c}}} + \sqrt{\frac{\bar{c}}{\bar{s}}}}. \quad (\text{A12})$$

This can be written as

$$\mathbf{F}^\infty(c) = \mathbf{U} \delta(c - \bar{c}), \quad (\text{A13})$$

where the state matrix  $\mathbf{U}$  of the form Eq. (6) corresponds to

$$u = \frac{\sqrt{\bar{c}/\bar{s}}}{\sqrt{\bar{c}/\bar{s}} + \sqrt{\bar{s}/\bar{c}}} = \frac{\bar{c}}{\bar{c} + \bar{s}} = \bar{c}, \quad (\text{A14})$$

thus,  $\mathbf{U} = \mathbf{C}$ , and finally we have

$$\mathbf{F}^\infty(c) = \mathbf{C} \delta(c - \bar{c}). \quad (\text{A15})$$

## APPENDIX B

### Derivation of the transition matrix

For derivation of the transition matrix directly from the master matrix (Eqs. (32), (33)) according to Eq. (40) we can use the results of computations made in Part I for verification of proper normalization of the cloud fraction PDF. According to Eq. (36) the singular density  $F_{cf}(c)$  can be explicitly written in terms of the master matrix elements:

$$F_{cf}(c) = uF_{\bullet\circ}(c) + vF_{\circ\bullet}(c) + uF_{\bullet\bullet}(c) + vF_{\circ\circ}(c), \quad (\text{B1})$$

where  $u$  and  $v = 1 - u$  are the elements of the initial state matrix  $\mathbf{U}$  from Eq. (6). They have the meaning of the probabilities of the initial state (first point of the sample) to be respectively cloudy or clear. Before computing the norm of  $F_{cf}(c)$  in Part I we took the integrals of the even part (first two terms) and the odd part (last two terms) of Eq. (B1). The integral of the even part is

$$\begin{aligned} \mathcal{N}_0 &= \int_0^1 [uF_{\bullet\circ}(c) + vF_{\circ\bullet}(c)] dc \\ &= 2 \frac{a_c u + a_g v}{a_c + a_g} \exp\left(-\frac{a_c + a_g}{2}\right) \sinh\left(\frac{a_c + a_g}{2}\right) \\ &= 2(u\bar{s} + v\bar{c}) e^{-r/2} \sinh\left(\frac{r}{2}\right) \\ &= (u\bar{s} + v\bar{c})(1 - e^{-r}) \end{aligned} \quad (\text{B2})$$

From here we can derive the integrals of  $F_{\bullet\circ}(c)$  and  $F_{\circ\bullet}(c)$  as coefficients at  $u$  and  $v$  respectively. Note that while  $u$  and  $v$  are related ( $u + v = 1$ ) they were treated in Part I as effectively independent variables (i.e., one was never expressed through the other). These integrals according to Eq. (40) are equal to the off-diagonal elements of the transition matrix  $\mathbf{P}$ :

$$P_{\bullet\circ}(c) = \int_0^1 F_{\bullet\circ}(c) dc = \bar{s}(1 - e^{-r}), \quad (\text{B3})$$

$$P_{\circ\bullet}(c) = \int_0^1 F_{\circ\bullet}(c) dc = \bar{c}(1 - e^{-r}). \quad (\text{B4})$$



The expression for the integral of the odd part of Eq. (B1) follows from the normalization condition

$$\int_0^1 F_{cf}(c) dc = 1 \quad (\text{B5})$$

(verified in Part I and holding regardless of the value of  $u$ ) and Eq. (B2):

$$\begin{aligned} 1 - \mathcal{N}_0 &= \int_0^1 [uF_{\bullet\bullet}(c) + vF_{\circ\circ}(c)] dc \\ &= u + v - (u\bar{s} + v\bar{c})(1 - e^{-r}) \\ &= u(\bar{c} + \bar{s}e^{-r}) + v(\bar{s} + \bar{c}e^{-r}) \quad (\text{B6}) \end{aligned}$$

(here we used the relation  $u + v = 1$ ). The expressions for the diagonal elements of the transition matrix follow immediately from this relation:

$$P_{\bullet\bullet}(c) = \int_0^1 F_{\bullet\bullet}(c) dc = \bar{c} + \bar{s}e^{-r}, \quad (\text{B7})$$

$$P_{\circ\circ}(c) = \int_0^1 F_{\circ\circ}(c) dc = \bar{s} + \bar{c}e^{-r}, \quad (\text{B8})$$

and the entire matrix takes the form from Eq. (43).

## APPENDIX C

### First moment of the master matrix

In order to compute the first moment of the master matrix representing the mean cloud fraction in the interval we again rely on computations performed in Part I. In Appendix D of Part I the mean cloud fraction was computed:

$$\begin{aligned} \bar{c}(L) &= \int_0^1 c F_{cf}(c) dc \\ &= \frac{a_g}{a_c + a_g} + \frac{a_c u - a_g v}{(a_c + a_g)^2} [1 - e^{-(a_c + a_g)}] \\ &= \bar{c} + (u\bar{s} - v\bar{c}) \frac{1 - e^{-r}}{r}. \quad (\text{C1}) \end{aligned}$$

It is clear from this expression that  $\bar{c}(L \rightarrow \infty) = \bar{c}$ . We also note that for random (unbiased) sampling with  $\mathbf{U} = \mathbf{C}$  (thus,  $u = \bar{c}$ ,  $v = \bar{s}$ ) the second term in Eq. (C1) vanishes so  $\bar{c}(L)$  no longer depends on  $L$  and is always equal to  $\bar{c}$ . The contribution of the even part of Eq. (B1) to  $\bar{c}(L)$  was computed in Part I:

$$\mathcal{M}_0 = \int_0^1 c [uF_{\bullet\circ}(c) + vF_{\circ\bullet}(c)] dc = \frac{1}{2} \mathcal{N}_0 - \mathcal{R}_0, \quad (\text{C2})$$

where  $\mathcal{N}_0$  is defined by Eq. (B2):

$$\mathcal{N}_0 = (u\bar{s} + v\bar{c})(1 - e^{-r}) \quad (\text{C3})$$

and

$$\begin{aligned} \mathcal{R}_0 &= \frac{(a_c u + a_g v)(a_c - a_g)}{(a_c + a_g)^2} \exp\left(-\frac{a_c + a_g}{2}\right) \\ &\times \left[ \cosh\left(\frac{a_c + a_g}{2}\right) - \frac{2}{a_c + a_g} \sinh\left(\frac{a_c + a_g}{2}\right) \right] \\ &= (u\bar{s} + v\bar{c})(\bar{s} - \bar{c}) e^{-r/2} \left[ \cosh\left(\frac{r}{2}\right) - \frac{2}{r} \sinh\left(\frac{r}{2}\right) \right] \\ &= (u\bar{s} + v\bar{c})(\bar{s} - \bar{c}) \left( \frac{1 + e^{-r}}{2} - \frac{1 - e^{-r}}{r} \right), \quad (\text{C4}) \end{aligned}$$

thus

$$\begin{aligned} \mathcal{M}_0 &= (u\bar{s} + v\bar{c}) \\ &\times \left[ \frac{1 - e^{-r}}{2} - (\bar{s} - \bar{c}) \left( \frac{1 + e^{-r}}{2} - \frac{1 - e^{-r}}{r} \right) \right] \\ &= (u\bar{s} + v\bar{c}) [\bar{c} - \bar{s}e^{-r} + (\bar{s} - \bar{c})g(r)], \quad (\text{C5}) \end{aligned}$$

where

$$g(r) = \frac{1 - e^{-r}}{r} \quad (\text{C6})$$

Eqs. (C1) and (C5) can be combined in order to determine contribution of the odd part of Eq. (B1):

$$\begin{aligned} \bar{c}(L) - \mathcal{M}_0 &= \bar{c} + (u\bar{s} - v\bar{c})g(r) \\ &- (u\bar{s} + v\bar{c})[\bar{c} - \bar{s}e^{-r} + (\bar{s} - \bar{c})g(r)] \\ &= (u + v)\bar{c} - u\bar{s}[\bar{c} - \bar{s}e^{-r} - 2\bar{c}g(r)] \\ &- v\bar{c}[\bar{c} - \bar{s}e^{-r} + 2\bar{s}g(r)] \\ &= u\bar{c} - u\bar{s}[\bar{c} - \bar{s}e^{-r} - 2\bar{c}g(r)] \\ &+ v\bar{c}\bar{s}[1 + e^{-r} - 2g(r)] \quad (\text{C7}) \end{aligned}$$

Collecting coefficients at  $u$  and  $v$  in Eqs. (C5) and (C7) we find the elements of the first moment matrix Eq. (49):

$$\bar{F}_{\bullet\bullet}^{(c)} = \bar{s}[\bar{c} - \bar{s}e^{-r} + (\bar{s} - \bar{c})g(r)], \quad (\text{C8})$$

$$\bar{F}_{\circ\bullet}^{(c)} = \bar{c}[\bar{c} - \bar{s}e^{-r} + (\bar{s} - \bar{c})g(r)], \quad (\text{C9})$$

$$\bar{F}_{\bullet\circ}^{(c)} = \bar{c} - \bar{s}[\bar{c} - \bar{s}e^{-r} - 2\bar{c}g(r)], \quad (\text{C10})$$

$$\bar{F}_{\circ\circ}^{(c)} = \bar{c}\bar{s}[1 + e^{-r} - 2g(r)]. \quad (\text{C11})$$

The matrix itself can be written in the following form:

$$\begin{aligned} \bar{\mathbf{F}}^{(c)} &= \bar{c} \begin{pmatrix} \bar{c} & \bar{s} \\ \bar{c} & \bar{s} \end{pmatrix} + \bar{s} \begin{pmatrix} \bar{s} & -\bar{s} \\ -\bar{c} & \bar{c} \end{pmatrix} e^{-r} \\ &+ \begin{pmatrix} 2\bar{c}\bar{s} & \bar{s}^2 - \bar{c}\bar{s} \\ \bar{c}\bar{s} - \bar{c}^2 & -2\bar{c}\bar{s} \end{pmatrix} \frac{1 - e^{-r}}{r}, \quad (\text{C12}) \end{aligned}$$

or

$$\bar{\mathbf{F}}^{(c)} = \bar{c} \mathbf{C} + \bar{s} (\mathbf{I} - \mathbf{C}) e^{-r} + (2\bar{c}\bar{s} \mathbf{K} + \mathbf{L}) \frac{1 - e^{-r}}{r}, \quad (\text{C13})$$

where

$$\mathbf{K} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 0 & \bar{s} \\ -\bar{c} & 0 \end{pmatrix}. \quad (\text{C14})$$

These expressions can be used to verify that Eq. (C1) can be obtained from Eq. (C13) by means of Eq. (48). Indeed,

$$\mathbf{U} \bar{\mathbf{F}}^{(c)} = \bar{c} \mathbf{C} + \bar{s} (\mathbf{U} - \mathbf{C}) e^{-r} + (2\bar{c}\bar{s} \mathbf{K} + \mathbf{UL}) g(r), \quad (\text{C15})$$

Here we implied that

$$\mathbf{UC} = \mathbf{C}^2 = \mathbf{C}, \text{ and } \mathbf{UK} = \mathbf{K} \quad (\text{C16})$$

for any state matrix  $\mathbf{U}$ . Noting that

$$\text{Tr } \mathbf{U} = \text{Tr } \mathbf{C} = 1, \quad \text{Tr } \mathbf{K} = 0 \quad (\text{C17})$$

and

$$\mathbf{UL} = \begin{pmatrix} -v\bar{c} & u\bar{s} \\ -v\bar{c} & u\bar{s} \end{pmatrix}, \text{ thus, } \text{Tr} [\mathbf{UL}] = u\bar{s} - v\bar{c}, \quad (\text{C18})$$

we obtain (C1). In the case of unbiased sampling, when  $\mathbf{U} = \mathbf{C}$ , Eq. (C15) takes simpler form

$$\mathbf{C} \bar{\mathbf{F}}^{(c)} = \bar{c} \mathbf{C} + \bar{c}\bar{s} \mathbf{K} g(r), \quad (\text{C19})$$

since

$$\mathbf{CL} = -\bar{c}\bar{s} \mathbf{K}, \quad (\text{C20})$$

and  $\text{Tr} [\mathbf{C} \bar{\mathbf{F}}^{(c)}] = \bar{c}$ .

## APPENDIX D

### Variance of cloud fraction distribution

The second moment of the cloud fraction PDF was not computed in Part I, thus, we perform this computation here. It is convenient first to compute the matrix

$$\mathbf{Y} = \bar{\mathbf{F}}^{(cs)} = \int_0^1 cs \mathbf{F}(c) dc, \quad (\text{D1})$$

which has the following elements:

$$Y_{\bullet\circ} = \bar{s}r \int_0^1 cs I_0(Z) e^{-(\bar{s}c + \bar{c}s)r} dc, \quad (\text{D2})$$

$$Y_{\circ\bullet} = \bar{c}r \int_0^1 cs I_0(Z) e^{-(\bar{s}c + \bar{c}s)r} dc, \quad (\text{D3})$$

$$Y_{\bullet\bullet} = 2\bar{c}\bar{s}r^2 \int_0^1 c^2 s \frac{I_1(Z)}{Z} e^{-(\bar{s}c + \bar{c}s)r} dc, \quad (\text{D4})$$

$$Y_{\circ\circ} = 2\bar{c}\bar{s}r^2 \int_0^1 cs^2 \frac{I_1(Z)}{Z} e^{-(\bar{s}c + \bar{c}s)r} dc \quad (\text{D5})$$

(note that  $\delta$ -function terms vanish as a result of the integration). Thus, we need to take only two different integrals:

$$\mathcal{Y}_0 = \int_0^1 cs I_0(Z) e^{-(\bar{s}c + \bar{c}s)r} dc, \quad (\text{D6})$$

$$\mathcal{Y}_1 = \int_0^1 c^2 s \frac{I_1(Z)}{Z} e^{-(\bar{s}c + \bar{c}s)r} dc. \quad (\text{D7})$$

Note that the integral in Eq. (D5) can be obtained from Eq. (D7) by interchanging  $\bar{c}$  and  $\bar{s}$ . It is convenient to use the following substitution

$$t = 2\sqrt{cs} = 2\sqrt{c(1-c)}, \quad (\text{D8})$$

which runs from 0 to 1 when  $c \in [0, 1/2]$  and back from 1 to 0 when  $c \in [1/2, 1]$ . Thus,  $c$  can be expressed through  $t$  as

$$c = \frac{1}{2} \mp \frac{1}{2} \sqrt{1-t^2}, \quad (\text{D9})$$

where the minus sign is used when  $c < 1/2$ , and the plus sign is used when  $c > 1/2$ , while the integral over  $t$  includes both terms. In this notation

$$s = \frac{1}{2} \pm \frac{1}{2} \sqrt{1-t^2}, \text{ thus, } cs = \frac{t^2}{4}, \quad (\text{D10})$$

$$\bar{s}c + \bar{c}s = \frac{1}{2} \mp \frac{\bar{s} - \bar{c}}{2} \sqrt{1-t^2} \quad (\text{D11})$$

$$Z = 2r \sqrt{\bar{c}\bar{s}cs} = r \sqrt{\bar{c}\bar{s}} t \quad (\text{D12})$$

$$dc = \pm \frac{t dt}{2\sqrt{1-t^2}}. \quad (\text{D13})$$

Substitution of the above expressions into Eqs. (D6) and (D7) yields

$$\begin{aligned} \mathcal{Y}_0 &= \frac{e^{-r/2}}{4} \int_0^1 \frac{t^3 dt}{\sqrt{1-t^2}} \\ &\times \cosh\left(\frac{\bar{s}-\bar{c}}{2} r \sqrt{1-t^2}\right) I_0(\sqrt{\bar{c}\bar{s}} rt), \end{aligned} \quad (\text{D14})$$

while

$$\mathcal{Y}_1 = \mathcal{Y}_{1a} - \mathcal{Y}_{1b}, \quad (\text{D15})$$

where

$$\begin{aligned} \mathcal{Y}_{1a} &= \frac{e^{-r/2}}{8\sqrt{\bar{c}\bar{s}}} \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}} \\ &\times \cosh\left(\frac{\bar{s}-\bar{c}}{2} r \sqrt{1-t^2}\right) I_1(\sqrt{\bar{c}\bar{s}} rt), \end{aligned} \quad (\text{D16})$$

$$\begin{aligned} \mathcal{Y}_{1b} &= \frac{e^{-r/2}}{8\sqrt{\bar{c}\bar{s}}} \int_0^1 t^2 dt \\ &\times \sinh\left(\frac{\bar{s}-\bar{c}}{2} r \sqrt{1-t^2}\right) I_1(\sqrt{\bar{c}\bar{s}} rt). \end{aligned} \quad (\text{D17})$$

Here we have integrals of the following types:

$$\mathcal{Y}_0 = \int_0^1 x^3 \frac{\cosh(\beta\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(\gamma x) dx \quad (\text{D18})$$

$$\mathcal{Y}_{1a} = \int_0^1 x^2 \frac{\cosh(\beta\sqrt{1-x^2})}{\sqrt{1-x^2}} I_1(\gamma x) dx. \quad (\text{D19})$$

$$\mathcal{Y}_{1b} = \int_0^1 x^2 \sinh(\beta\sqrt{1-x^2}) I_1(\gamma x) dx \quad (\text{D20})$$

To take the integrals Eqs. (D18) – (D20) we will use the formula derived in Part I:

$$\begin{aligned} \mathcal{J}_0 &= \int_0^1 x \frac{\cosh(\beta\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(\gamma x) dx \\ &= \frac{\sinh(\sqrt{\beta^2+\gamma^2})}{\sqrt{\beta^2+\gamma^2}}, \end{aligned} \quad (\text{D21})$$

and its derivatives

$$\begin{aligned} \frac{\partial \mathcal{J}_0}{\partial \beta} &= \int_0^1 x \sinh(\beta\sqrt{1-x^2}) I_0(\gamma x) dx \\ &= \frac{\beta}{\beta^2+\gamma^2} \\ &\times \left[ \cosh(\sqrt{\beta^2+\gamma^2}) - \frac{\sinh(\sqrt{\beta^2+\gamma^2})}{\sqrt{\beta^2+\gamma^2}} \right], \end{aligned} \quad (\text{D22})$$

$$\begin{aligned} \frac{\partial \mathcal{J}_0}{\partial \gamma} &= \int_0^1 x^2 \frac{\cosh(\beta\sqrt{1-x^2})}{\sqrt{1-x^2}} I_1(\gamma x) dx \\ &= \frac{\gamma}{\beta^2+\gamma^2} \\ &\times \left[ \cosh(\sqrt{\beta^2+\gamma^2}) - \frac{\sinh(\sqrt{\beta^2+\gamma^2})}{\sqrt{\beta^2+\gamma^2}} \right]. \end{aligned} \quad (\text{D23})$$

We immediately notice that

$$\mathcal{Y}_{1a} = \frac{\partial \mathcal{J}_0}{\partial \gamma} \quad (\text{D24})$$

and use the relation

$$I_1(x) = \frac{dI_0(x)}{dx} \quad (\text{D25})$$

to see that

$$\mathcal{Y}_{1b} = \frac{\partial^2 \mathcal{J}_0}{\partial \beta \partial \gamma}. \quad (\text{D26})$$

Also, using the recurrent relation for modified Bessel functions

$$\frac{dI_\nu(x)}{dx} = I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x), \quad (\text{D27})$$

in particular

$$\frac{dI_1(x)}{dx} = I_0(x) - \frac{I_1(x)}{x}, \quad (\text{D28})$$

we have that

$$\frac{dI_1(\gamma x)}{d\gamma} = x I_0(\gamma x) - \frac{I_1(\gamma x)}{\gamma}. \quad (\text{D29})$$

This means that

$$\mathcal{Y}_0 = \frac{\partial^2 \mathcal{J}_0}{\partial^2 \gamma} + \frac{1}{\gamma} \frac{\partial \mathcal{J}_0}{\partial \gamma}. \quad (\text{D30})$$

Before starting to compute the derivatives, we notice that

$$\frac{\partial \mathcal{J}_0}{\partial \beta} = \beta f(\alpha), \quad \text{and} \quad \frac{\partial \mathcal{J}_0}{\partial \gamma} = \gamma f(\alpha), \quad (\text{D31})$$

where

$$\alpha = \sqrt{\beta^2 + \gamma^2}, \text{ so that } \frac{\partial \alpha}{\partial \gamma} = \frac{\gamma}{\sqrt{\beta^2 + \gamma^2}} = \frac{\gamma}{\alpha}, \quad (\text{D32})$$

and

$$f(\alpha) = \frac{\cosh \alpha}{\alpha^2} - \frac{\sinh \alpha}{\alpha^3}. \quad (\text{D33})$$

The derivative of  $f$  with respect to its argument is

$$\frac{df(\alpha)}{d\alpha} = \alpha^{-2} \sinh \alpha - 3\alpha^{-1} f(\alpha), \quad (\text{D34})$$

and

$$\frac{\partial f(\alpha)}{\partial \gamma} = \gamma[(\alpha^{-3} + 3\alpha^{-5}) \sinh \alpha - 3\alpha^{-4} \cosh \alpha],$$

or

$$f'_\gamma = \frac{\gamma}{\alpha} \left( \frac{\sinh \alpha}{\alpha^2} - \frac{3f(\alpha)}{\alpha} \right). \quad (\text{D35})$$

Thus,

$$\mathcal{Y}_0 = 2f + \gamma f'_\gamma, \quad (\text{D36})$$

$$\mathcal{Y}_{1a} = \gamma f, \quad \mathcal{Y}_{1b} = \beta f'_\gamma. \quad (\text{D37})$$

In the notations

$$\alpha = \frac{r}{2}, \quad \beta = \frac{\bar{s} - \bar{c}}{2} r, \quad \gamma = \sqrt{\bar{c}\bar{s}} r, \quad (\text{D38})$$

the matrix  $\mathbf{Y}$  can be expressed in terms of these integrals as

$$\mathbf{Y} = \frac{r e^{-r/2}}{4} \begin{pmatrix} \sqrt{\bar{c}\bar{s}} (\mathcal{Y}_{1a} - \mathcal{Y}_{1b}) & \bar{s} \mathcal{Y}_0 \\ \bar{c} \mathcal{Y}_0 & \sqrt{\bar{c}\bar{s}} (\mathcal{Y}_{1a} + \mathcal{Y}_{1b}) \end{pmatrix}, \quad (\text{D39})$$

or

$$\mathbf{Y} = \frac{r e^{-r/2}}{4} \begin{pmatrix} \sqrt{\bar{c}\bar{s}} (\gamma f - \beta f'_\gamma) & \bar{s} (2f + \gamma f'_\gamma) \\ \bar{c} (2f + \gamma f'_\gamma) & \sqrt{\bar{c}\bar{s}} (\gamma f + \beta f'_\gamma) \end{pmatrix}. \quad (\text{D40})$$

Introducing notations

$$f_1(r) = \frac{r e^{-r/2}}{4} f = \frac{1}{2r} [1 + e^{-r} - 2g(r)], \quad (\text{D41})$$

where

$$g(r) = \frac{1 - e^{-r}}{r}, \quad (\text{D42})$$

and

$$f_2(r) = \frac{r e^{-r/2}}{4} f'_\gamma = \sqrt{\bar{c}\bar{s}} \left[ g(r) - \frac{12}{r} f_1(r) \right], \quad (\text{D43})$$

This equation can be rewritten as

$$\mathbf{Y} = \begin{pmatrix} \bar{c}\bar{s} r & 2\bar{s} \\ 2\bar{c} & \bar{c}\bar{s} r \end{pmatrix} f_1(r) + \frac{\sqrt{\bar{c}\bar{s}} r}{2} \begin{pmatrix} \bar{c} - \bar{s} & 2\bar{s} \\ 2\bar{c} & \bar{s} - \bar{c} \end{pmatrix} f_2(r). \quad (\text{D44})$$

This expression can be written as

$$\mathbf{Y} = \bar{c}\bar{s} [(r \mathbf{I} + 2\mathbf{B}) f_1(r) + (2\mathbf{C} - \mathbf{I}) f_3(r)], \quad (\text{D45})$$

where

$$\mathbf{B} = \frac{1}{\bar{c}\bar{s}} \begin{pmatrix} 0 & \bar{s} \\ \bar{c} & 0 \end{pmatrix}, \quad (\text{D46})$$

and

$$f_3(r) = \frac{r}{2} g(r) - 6 f_1(r). \quad (\text{D47})$$

Further expansion yields

$$\begin{aligned} \frac{1}{\bar{c}\bar{s}} \mathbf{Y} &= [r \mathbf{I} + 2\mathbf{B} - 6(2\mathbf{C} - \mathbf{I})] f_1 + (2\mathbf{C} - \mathbf{I}) \frac{1 - e^{-r}}{2} \\ &= \frac{1}{2} (1 + e^{-r} - 2g - 1 + e^{-r}) \mathbf{I} + (1 - e^{-r}) \mathbf{C} \\ &+ [\mathbf{B} - 3(2\mathbf{C} - \mathbf{I})] r^{-1} (1 + e^{-r} - 2g) \\ &= \mathbf{P} - g \mathbf{I} + [\mathbf{B} - 3(2\mathbf{C} - \mathbf{I})] r^{-1} (2 - rg - 2g), \end{aligned} \quad (\text{D48})$$

where we used the expression Eq. (44) for the transition matrix  $\mathbf{P}$ . Finally,

$$\mathbf{Y} = \bar{c}\bar{s} \left[ \mathbf{P} - (\mathbf{B} - 6\mathbf{C} + 4\mathbf{I}) g + 2(\mathbf{B} - 6\mathbf{C} + 3\mathbf{I}) \frac{1 - g}{r} \right]. \quad (\text{D49})$$

For an arbitrary initial state  $\mathbf{U}$  we have

$$\mathbf{U} \mathbf{B} = \frac{1}{\bar{c}\bar{s}} \begin{pmatrix} v\bar{c} & u\bar{s} \\ v\bar{c} & u\bar{s} \end{pmatrix}, \text{ so } \text{Tr}(\mathbf{U} \mathbf{B}) = \frac{u\bar{s} + v\bar{c}}{\bar{c}\bar{s}}, \quad (\text{D50})$$

(D40) while for random initial state  $\mathbf{U} = \mathbf{C}$

$$\mathbf{C} \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ so } \text{Tr}(\mathbf{C} \mathbf{B}) = 2. \quad (\text{D51})$$

Taking into account these equations and also that  $\text{Tr} \mathbf{U} \mathbf{P} = 1$ , we obtain the expression for  $\bar{c}\bar{s}$  in general case:

$$\begin{aligned} \bar{c}\bar{s} &= \text{Tr}(\mathbf{U} \mathbf{Y}) \\ &= \bar{c}\bar{s} - (u\bar{s} + v\bar{c} - 2\bar{c}\bar{s}) g + 2(u\bar{s} + v\bar{c} - 3\bar{c}\bar{s}) \frac{1 - g}{r}. \end{aligned} \quad (\text{D52})$$

In the random sample case ( $u = \bar{c}$ ,  $v = \bar{s}$ ) the second term vanishes and the expression notably simplifies:

$$\overline{c\bar{s}} = \text{Tr}(\mathbf{C}\mathbf{Y}) = \bar{c}\bar{s} \left[ 1 - 2 \frac{1-g(r)}{r} \right], \quad (\text{D53})$$

or explicitly

$$\overline{c\bar{s}} = \bar{c}\bar{s} \left[ 1 - \frac{2}{r} + \frac{2}{r^2}(1 - e^{-r}) \right], \quad (\text{D54})$$

Note that in the limit case of long sample ( $r \gg 1$ ,  $L \gg L_*$ ) this function converges to  $\bar{c}\bar{s}$ , because the cloud fraction PDF becomes very narrow ( $F_{\text{cf}}(c) \approx \delta(c - \bar{c})$ ). In the opposite case of very short sample ( $r \ll 1$ ,  $L \ll L_*$ ) expansion of the exponent into Taylor series yields that  $\overline{c\bar{s}} \propto \bar{c}\bar{s} r \rightarrow 0$ .

Using Eq. (D54) we can derive an expression for the variance of the cloud fraction distribution

$$D = \overline{(c - \bar{c})^2} = \bar{c}^2 - (\bar{c})^2. \quad (\text{D55})$$

Indeed,

$$\overline{c(1-c)} = \bar{c} - \bar{c}^2 = \bar{c} - D - (\bar{c})^2, \quad (\text{D56})$$

thus,

$$D = \bar{c}(1 - \bar{c}) - \overline{c(1-c)} = \bar{c}\bar{s} - \overline{c\bar{s}}. \quad (\text{D57})$$

Eq. (D54) allows to write this expression explicitly as

$$D = \frac{2\bar{c}\bar{s}}{r} \left[ 1 - \frac{1}{r}(1 - e^{-r}) \right]. \quad (\text{D58})$$

$D \rightarrow 0$  as the sample length increases ( $r \rightarrow \infty$ ) and the cloud cover distribution narrows. On the other hand, when sample is short ( $r \rightarrow 0$ ), the variance converges to a constant value:  $D \rightarrow \bar{c}\bar{s}$ . Plots of both  $\overline{c\bar{s}}$  and  $D$  as functions of  $r$  are presented in Fig. 1.

The expression Eq. (D58) coincides with that derived (in a simpler way) by Morf (1998) using properties of the autocorrelation function of a binary Markov process. However, here we presented the first-time analytical derivation of the cloud fraction variance directly from the results of Part I (numerical verification has been reported by Morf (2014)).

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