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CENTER OF A COMPLETE LATTICE

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1. INTRODUCTION

Let L be a complete lattice. We denote by $C = C(L)$ the center of L . It is well-known that $C(L)$ is a sublattice of L . If L is infinitely distributive, then $C(L)$ is a closed sublattice of L [2]. In this Note we show (Thm. 2) that $C(L)$ is a closed sublattice of L if and only if the following weakened infinite distributive law is valid in L :

For any $x, y \in L$, $x \geq y$, and any subset $\{a_i\} \subset C(L)$,

$$(1) \quad y \vee (x \wedge (\bigwedge a_i)) = \bigwedge (y \vee (x \wedge a_i)),$$

$$(2) \quad x \wedge (y \vee (\bigvee a_i)) = \bigvee (x \wedge (y \vee a_i)).$$

In [1] there is proposed the problem whether the center of any complete lattice is a closed sublattice (p. 131, Problem 34). In §4 below there is described a complete distributive lattice L and a subset $\{a_i\} \subset C(L)$ such that the element $\bigwedge a_i$ has no complement in L ; thus $C(L)$ is not a closed sublattice of L .

In §5 there are investigated relative centers and direct factors of a conditionally complete lattice L . There are found necessary and sufficient conditions under which each nonempty intersection of direct factors of L is a direct factor of L (Thm. 3). As a corollary, we obtain the assertion: If for each interval $[u, v] \subset L$ the center $C([u, v])$ is a closed sublattice of $[u, v]$, then each nonempty intersection of direct factors of L is a direct factor of L . Let us remark that the first condition has a local character (concerning intervals of L) while the second one has a global character.

Let us recall some basic notions and denotations (cf. [1]). The lattice operations will be denoted by \wedge, \vee (unless otherwise stated). The direct product $A \times B$ of two lattices A, B is the set of all pairs (a, b) with $a \in A, b \in B$, the lattice operations in $A \times B$ being defined componentwise. If a lattice L has the least element (the greatest element), then we denote this element by $0(L)$ or $1(L)$, respectively, and analogously for other lattices. Let $c, 0(L), 1(L) \in L$ and assume that there are lattices A, B and an isomorphism φ of L onto $A \times B$ such that $\varphi(c) = (c_1, c_2)$ with $c_1 = 1(A), c_2 = 0(B)$.

Then c is said to be a central element of L and the set $C(L)$ of all central elements of L is the center of L . The set $C(L)$ is a sublattice of L and $C(L)$ is a Boolean algebra. Each element $c \in C(L)$ has a unique complement in L . This complement will be always denoted by c' ; this element also belongs to $C(L)$. Each element $a \in C(L)$ is neutral, i.e., if $x, y \in L$, then the sublattice of L generated by the elements a, x, y is distributive.

2. WEAKENED INFINITE DISTRIBUTIVITY

In §2-4 we assume that L is a complete lattice. Let $\{a_i\} \subset C(L)$. If there exists the least upper bound of $\{a_i\}$ in $C(L)$, then we denote it by $\bigvee^* a_i$ and analogously for the greatest lower bound in $C(L)$. Since $C(L)$ is a sublattice of L , for a finite set $\{a_i\}$ we have $\bigvee^* a_i = \bigvee a_i$, and dually. If $\bigvee^* a_i$ exists, then clearly $\bigvee^* a_i \geq \bigvee a_i$, and dually.

Lemma 1. *Let $\emptyset \neq \{a_i\}_{i \in I} \subset C(L)$, $a = \bigwedge a_i \in C(L)$, $b = \bigvee a'_i \in C(L)$. Then $a \wedge b = 0(L)$, $a \vee b = 1(L)$.*

Proof. Since $\bigwedge a_i \in C$, $\bigvee a'_i \in C$, we have

$$\bigwedge a_i = \bigwedge^* a_i, \quad \bigvee a'_i = \bigvee^* a'_i.$$

Any Boolean algebra is infinitely distributive, therefore

$$a \wedge (\bigvee^* a'_i) = \bigvee^*(a \wedge a'_i).$$

Hence

$$a \wedge b = a \wedge (\bigvee a'_i) = a \wedge (\bigvee^* a'_i) = \bigvee^*(a \wedge a'_i).$$

Further we have $a \wedge a'_i \leq a_i \wedge a'_i = 0(L)$ for each $i \in I$, thus $a \wedge b = 0(L)$. In a dual way we prove that $a \vee b = 1(L)$.

Lemma 2. *Let $x \in L$, $c \in C(L)$. Then $x \wedge c \in C([0(L), x])$, $x \vee c \in C([x, 1(L)])$.*

Proof. There exist lattices A, B and an isomorphism φ of L onto $A \times B$ such that $\varphi(c) = (1(A), 0(B))$. Denote $L_1 = [0(L), x]$ and let φ^* be the corresponding partial mapping of the set L_1 into $A \times B$, $\varphi^*(x) = (x_1, x_2)$. Then φ^* is an isomorphism of L_1 onto $[0(A), x_1] \times [0(B), x_2]$. In fact, if $y \in L_1$, then $\varphi^*(y) = (y_1, y_2) \in [0(A), x_1] \times [0(B), x_2]$. Let $(z_1, z_2) \in [0(A), x_1] \times [0(B), x_2]$, $z \in L$, $\varphi(z) = (z_1, z_2)$; then $z \in [0(L), x]$. We have $x \wedge c \in L_1$ and $\varphi^*(x \wedge c) = (x_1, 0(B))$. Therefore $x \wedge c \in C([0(L), x])$. The second assertion can be proved dually.

Lemma 3. *Let $x, z \in L$, $\emptyset \neq \{a_i\} \subset C(L)$, $a = \bigwedge a_i \in C(L)$, $b = \bigvee a'_i \in C(L)$. Then $x \wedge b = \bigvee(x \wedge a'_i)$, $z \vee a = \bigwedge(z \vee a_i)$.*

Proof. According to lemma 2, the elements $x \wedge a_i$, $x \wedge a'_i$ belong to the center of the lattice $[0(L), x]$. Further from the fact that $a_i \in C(L)$ we infer that a_i is neutral,

hence the sublattice of L generated by the elements a_i, a'_i, x is distributive. Therefore $x \wedge a'_i$ is a relative complement of $x \wedge a_i$ in the interval $[0(L), x]$. Denote $a_0 = \bigwedge(x \wedge a_i), b_0 = \bigvee(x \wedge a'_i)$. From Lemma 1 applied to the lattice $[0(L), x]$ we obtain $a_0 \wedge b_0 = 0(L), a_0 \vee b_0 = x$. Put $v = x \wedge b = x \wedge (\bigvee a'_i)$. Clearly $a_0 = \bigwedge(x \wedge a_i) = x \wedge (\bigwedge a_i) = x \wedge a$, hence according to Lemma 1, $a_0 \wedge v = 0(L)$. Further, since $b \in C(L)$ is neutral, we have $a_0 \vee v = (x \wedge a) \vee (x \wedge b) = x \wedge (a \vee b) = x$. Therefore both elements b_0 and v are relative complements of a_0 in the interval $[0(L), x]$. Since (by Lemma 2) a_0 belongs to the center of $[0(L), x]$ we get $b_0 = v$. Thus we have $x \wedge b = \bigvee(x \wedge a'_i)$. By a dual method we verify the second assertion.

Lemma 4. *Let $x, y \in L, y \leq x, \emptyset \neq \{a_i\} \subset C(L), a = \bigwedge a_i \in C(L), b = \bigvee a'_i \in C(L)$. Then*

$$(3) \quad y \vee (x \wedge (\bigwedge a_i)) = \bigwedge(y \vee (x \wedge a_i)),$$

$$(4) \quad x \wedge (y \vee (\bigvee a'_i)) = \bigvee(x \wedge (y \vee a'_i)).$$

Proof. We have $\bigwedge(x \wedge a_i) = x \wedge (\bigwedge a_i) = x \wedge a \in C([0(L), x])$ by Lemma 2. Now according to Lemma 3 (applied to the lattice $[0(L), x]$ instead of L) we obtain

$$y \vee (x \wedge a) = \bigwedge(y \vee (x \wedge a_i)).$$

Therefore (3) is valid. By a dual method we verify (4).

From Lemma 4 we obtain as a corollary:

Lemma 5. *Assume that $C(L)$ is a closed sublattice of L . Then (1) and (2) are valid for each subset $\emptyset \neq \{a_i\} \subset C(L)$ and each $x, y \in L, x \geq y$.*

3. SUFFICIENT CONDITION FOR THE CENTER TO BE CLOSED

In this section we show that the validity of (3), (4) for each $x, y \in L, y \leq x$ is sufficient in order that the elements $\bigwedge a_i, \bigvee a'_i$ belong to the center $C(L)$ of a complete lattice L .

Let us remark that by putting $x = 1(L)$ we get from (3)

$$(3') \quad y \wedge (\bigvee a_i) = \bigvee(y \wedge a_i),$$

and by putting $y = 0(L)$ we obtain from (4)

$$(4') \quad x \wedge (\bigvee a'_i) = \bigvee(x \wedge a'_i).$$

Let $\emptyset \neq \{a_i\}$ be a fixed subset of $C(L), \bigwedge a_i = a, \bigvee a'_i = b$.

Lemma 6. *Assume that (4') holds for each $x \in L$. Then $a \wedge b = 0(L)$.*

Proof. We have

$$a \wedge b = a \wedge (\bigvee a'_i) = \bigvee (a \wedge a'_i).$$

Since $a \leq a_i$ and $a_i \wedge a'_i = 0(L)$, we obtain $a \wedge b = 0(L)$.

In a dual way we get:

Lemma 6'. Assume that (3') is valid for each $y \in L$. Then $a \vee b = 1(L)$.

Lemma 7. Assume that (3) and (4) are valid for each pair of elements $x, y \in L$ with $y \leq x$. Then $x \wedge a$ is a complement of $x \wedge b$ in the lattice $[0(L), x]$ for each $x \in L$.

Proof. By Lemma 6, $a \wedge b = 0(L)$, whence $(x \wedge a) \wedge (x \wedge b) = 0(L)$. Denote $z = x \wedge (\bigvee a'_j)$ ($j \in I$). According to (3) we have $z \vee (x \wedge (\bigwedge a_i)) = \bigwedge (z \vee (x \wedge a_i))$. Further from (4') we obtain (by using the neutrality of a_i)

$$\begin{aligned} (x \wedge a_i) \vee z &= (x \wedge a_i) \vee (x \wedge (\bigvee_j a'_j)) = \bigvee_j ((x \wedge a_i) \vee (x \wedge a'_j)) = \\ &= \bigvee_j (x \wedge (a_i \vee a'_j)). \end{aligned}$$

Since $a_i \vee a'_i = 1(L)$, we get

$$(x \wedge a_i) \vee z = x \quad \text{for each } i \in I.$$

Thus $z \vee (x \wedge (\bigwedge a_i)) = x$. The proof is complete.

Analogously we verify (by using Lemma 6'):

Lemma 7'. Assume that (3) and (4) are valid for each pair of elements $x, y \in L$ with $y \leq x$. Then $x \vee a$ is a complement of $x \vee b$ in the interval $[x, 1(L)]$ for each $x \in L$.

Lemma 8. Let the same assumptions as in Lemma 7 be valid. Let $x \in L$ and denote $x \wedge a = u_1$, $x \wedge b = u_2$. Let $v_1, v_2 \in L$, $v_1 \leq u_1$, $v_2 \leq u_2$, $v_1 \vee v_2 = x$. Then $v_i = u_i$ ($i = 1, 2$).

Proof. According to Lemma 7' we have

$$v_1 = (v_1 \vee a) \wedge (v_1 \vee b).$$

Since $v_1 \vee a = a$, $v_1 \vee b = v_1 \vee v_2 \vee b = x \vee b$, we obtain $v_1 = a \wedge (x \vee b) \geq \geq a \wedge x = u_1$. This shows that $u_1 = v_1$. Analogously we verify that $u_2 = v_2$.

Now consider the mapping

$$\psi : x \rightarrow (x \wedge a, x \wedge b)$$

of the lattice L into the direct product $[0(L), a] \times [0(L), b]$.

Lemma 9. *Let the assumptions as in Lemma 7 be fulfilled. Then the mapping ψ is an isomorphism of the lattice L onto $[0(L), a] \times [0(L), b]$ and $\psi(a) = (a, 0(L))$, $\psi(b) = (0(L), b)$.*

Proof. Let $x, y \in L$. The mapping ψ is monotone and by Lemma 7, $\psi(x) \leq \psi(y)$ implies that $x \leq y$. Let $a \geq u \in L$, $b \geq v \in L$, $x = u \vee v$. Under the denotations as above we have $u \leq u_1$, $v \leq u_2$, hence according to Lemma 8, $u = u_1$, $v = u_2$. Therefore the mapping ψ is onto and thus ψ is an isomorphism. By Lemma 6 we have $\psi(a) = (a, 0(L))$, $\psi(b) = (0(L), b)$.

Theorem 1. *Let L be a complete lattice and let $\{a_i\} \neq \emptyset$ be a subset of the center $C(L)$ of the lattice L . The following conditions are equivalent:*

- (i) *The elements $\bigwedge a_i$ and $\bigvee a_i$ belong to $C(L)$.*
- (ii) *If $x, y \in L$, $x \geq y$, then (3) and (4) are valid.*

Proof. By Lemma 4, (i) \Rightarrow (ii). From Lemma 9 it follows that (ii) \Rightarrow (i).

As an immediate consequence we obtain:

Theorem 2. *Let L be a complete lattice. Then the following conditions are equivalent:*

- (i) *The center $C(L)$ is a closed sublattice of L .*
- (ii) *If $\emptyset \neq \{a_i\} \subset C(L)$, $x \in L$, $y \in L$ and $x \geq y$, then (1) and (2) are valid.*

Corollary. (Cf. [2].) *If L is an infinitely distributive complete lattice, then $C(L)$ is a closed sublattice of L .*

4. AN EXAMPLE

Now we describe an example showing that the center of a complete distributive lattice L need not be a closed sublattice of L .

Let L_0 be the lattice of all real functions defined on the interval $[0, 1] = X$ with the natural partial order. The lattice operations in L_0 are denoted \wedge^* , \vee^* . Let L be the subset of L_0 consisting of all functions f that satisfy the following conditions:

- (i) If $0 \leq x < 1$, then $f(x) \in \{0, 2\}$.
- (ii) $f(1) \in \{0, 1, 2\}$.
- (iii) $f(1) = 2$ if and only if the set $s(f) = \{x : 0 \leq x < 1, f(x) = 2\}$ is infinite.

The set L is partially ordered by the induced order. The least and the greatest element of L will be denoted by f_0 and f_1 , respectively. Let $f \in L_0$, $f(x) \in \{0, 1, 2\}$ for

each $x \in X$. We define $f^-, f^+ \in L$ as follows. If $s(f)$ is finite and $f(1) = 2$, then we put $f^-(x) = f(x)$ for each $x \in X, x \neq 1$, and $f^-(1) = 1$; otherwise we put $f^- = f$. If $s(f)$ is infinite and $f(1) \neq 2$, we set $f^+(x) = f(x)$ for each $x \in X, x \neq 1$, and $f^+(1) = 2$; otherwise we put $f^+ = f$. If $f \in L$, then $f^- = f = f^+$.

Lemma 10. *The partially ordered set L is a complete lattice.*

Proof. Let $\emptyset \neq \{f_i\} (i \in I) \subset L$. Denote $\bigwedge^* f_i = f, \bigvee^* f_i = g$. The functions f, g satisfy the conditions (i) and (ii). If $f \in L (g \in L)$, then clearly $f = \inf \{f_i\} (g = \sup \{f_i\})$ in L .

Assume that $f \notin L$. Suppose that $s(f)$ is finite. Then $f(1) = 2, f_i > f^-$ for each $i \in I$ and $g_1 \leq f^-$ whenever $g_1 \in L, g_1 \leq f_i$ for each $i \in I$. Thus $f^- = \inf \{f_i\}$ in L . Assume that $s(f)$ is infinite. Then each $s(f_i)$ is infinite, whence $f_i(1) = 2$ for each $i \in I$ and therefore $f(1) = 2$. Thus $f \in L$, a contradiction.

Assume that $g \notin L$. If $s(g)$ is finite, then each $s(f_i)$ is finite, hence $f_i(1) < 2$ for each $i \in I$, thus $g(1) \leq 1$ and so $g \in L$, a contradiction. Therefore $s(g)$ is infinite and from $g \notin L$ we obtain $g(1) < 2$. Then $g^+ \geq f_i$ for each $i \in I$. Moreover $g_1 \in L, g_1 \geq f_i$ for each $i \in I$ implies $g_1 \geq g^+$. Thus $g^+ = \sup \{f_i\}$ in L . The proof is complete.

The lattice operations in L will be denoted by \wedge, \vee . We have shown that $\bigwedge f_i = (\bigwedge^* f_i)^-, \bigvee f_i = (\bigvee^* f_i)^+$ for any subset $\emptyset \neq \{f_i\} \subset L$. From this it follows that for each $f, g \in L$ and each $x \in X, x \neq 1$ we have

$$(5) \quad (f \wedge g)(x) = f(x) \wedge g(x), \quad (f \vee g)(x) = f(x) \vee g(x).$$

From (5) we obtain that

$$(6) \quad s(f \wedge g) = s(f) \cap s(g), \quad s(f \vee g) = s(f) \cup s(g).$$

Lemma 11. *The lattice L is distributive.*

Proof. Let $f, g, h \in L$ and denote

$$(f \wedge g) \vee h = F, \quad (f \vee h) \wedge (g \vee h) = G.$$

Obviously $G \geq F$ and according to (5), $F(x) = G(x)$ for each $x \in X, x \neq 1$. Hence we have to verify that $F(1) = G(1)$.

According to (6) we have

$$\begin{aligned} s(G) &= s(f \vee h) \cap s(g \vee h) = (s(f) \cup s(h)) \cap (s(g) \cup s(h)) = \\ &= s(f) \cap s(g) \cup s(h) = s(F), \end{aligned}$$

hence either both $G(1), F(1)$ are less than 2, or $G(1) = F(1) = 2$. Thus if $F(1) \geq 1$, then $F \geq G$. Assume that $F(1) = 0$. Then $h(1) = 0$, and either $f(1) = 0$ or $g(1) = 0$.

Therefore either $(f \vee h)(1) = 0$ or $(g \vee h)(1) = 0$. From this we get $G(1) = 0$. Hence $F(1) = G(1)$. The proof is complete.

For each $y \in X, y \neq 1$ we define the functions $f_y, \bar{f}_y \in L$ by the rule

$$\begin{aligned} f_y(y) &= 2, \quad \bar{f}_y(y) = 0, \\ f_y(x) &= 0, \quad \bar{f}_y(x) = 2 \quad \text{for each } x \in X, \quad x \neq y. \end{aligned}$$

Further let $g_0 \in L$ be such that $g_0(x) = 0$ for each $x \in X, x \neq 1$ and $g_0(1) = 1$. Then we have

$$f_y \wedge \bar{f}_y = f_0, \quad f_y \vee \bar{f}_y = f_1$$

for each $y \in X, y \neq 1$. Since L is distributive, each element of L is neutral. Therefore an element $f \in L$ belongs to the center of L if and only if f has a complement. Thus all elements f_y belong to the center of L . We have

$$(7) \quad \bigwedge_y \bar{f}_y = g_0.$$

Let $h \in L, h \wedge g_0 = f_0$. Then $h(1) = 0$, thus $s(h)$ is finite. Hence $f_1 \neq h \vee^* g_0 \in L$ and so $h \vee^* g_0 = h \vee g_0 \neq f_1$. Therefore the element g_0 has no complement in L . This implies that g_0 does not belong to $C(L)$. In view of (7), the center of L is not a closed sublattice of L .

On the other hand we have $\forall f_y = f_1 \in C(L)$. Thus if $\{a_i\}$ is a subset of the center of a complete lattice L and if $\forall a_i'$ belongs to $C(L)$, then $\bigwedge a_i$ need not belong to $C(L)$.

5. DIRECT FACTORS IN A CONDITIONALLY COMPLETE LATTICE

In this paragraph we assume that L is a conditionally complete lattice.

Let φ be an isomorphism of L onto a direct product $A \times B, x_0 \in L, \varphi(x_0) = (a_0, b_0)$. Put

$$\begin{aligned} A(x_0) &= \{y \in L: \varphi(y) = (a, b_0), a \in A\}, \\ B(x_0) &= \{y \in L: \varphi(y) = (a_0, b), b \in B\}. \end{aligned}$$

For each $z \in L$ with $\varphi(z) = (a_1, b_1)$ let

$$z_1 = \varphi^{-1}((a_1, b_0)), \quad z_2 = \varphi^{-1}((a_0, b_1)).$$

We denote by $\varphi'[x_0]$ the mapping of L onto $A(x_0) \times B(x_0)$ defined by the rule

$$\varphi'[x_0](z) = (z_1, z_2)$$

for each $z \in L$. It is easy to verify that $\varphi'[x_0]$ is an isomorphism of L onto $A(x_0) \times B(x_0)$. If the element x_0 is fixed we write φ' instead of $\varphi'[x_0]$.

All lattices $A(x_0)$ constructed in this way will be called direct factors of L with respect to x_0 and the system of all direct factors of L with respect to x_0 will be denoted

by $F(x_0)$. (Cf. [3], [4].) Each lattice $A \in \bigcup F(x_0)$ ($x_0 \in L$) will be called a direct factor of L .

Let φ be as above, $u, v \in L$, $u \leq v$, $\varphi(u) = (u_1, u_2)$, $\varphi(v) = (v_1, v_2)$, $c = \varphi^{-1}((v_1, u_2))$. Then c is said to be a relative central element of L with respect to the interval $[u, v]$. The set $C'([u, v])$ of all relative central elements with respect to $[u, v]$ will be called the relative center of L with respect to $[u, v]$. Let us consider the following condition on L :

(*) For each $x_0 \in L$ and each set $\emptyset \neq \{A_i(x_0)\}$ of direct factors of L with respect to x_0 the intersection $\bigcap A_i(x_0)$ is a direct factor of L with respect to x_0 .

If $A(x_0)$ is a direct factor of L and $x_1 \in A(x_0)$, then $A(x_1) = A(x_0)$; therefore the condition (*) is equivalent with the condition:

(**) Each nonempty intersection of direct factors of L is a direct factor of L .

The following lemma shows the relation between the condition (*) and the properties of the center of L in the case when L has the greatest and the least element:

Lemma 12. Let $0(L), 1(L) \in L$. Then L satisfies (*) if and only if the center $C(L)$ is a closed sublattice of L .

At first we prove the following lemma:

Lemma 12.1. Let φ be an isomorphism of L onto $A \times B$, $x, x_0 \in L$, $0(L), 1(L) \in L$, $a = \varphi^{-1}((1(A), 0(B)))$. Then $x \in A(x_0)$ if and only if $a \vee x = a \vee x_0$.

Proof. Let $\varphi(x_0) = (a_0, b_0)$, $\varphi(x) = (a_1, b_1)$. We have

$$\begin{aligned} (a_1, b_1) \vee (1(A), 0(B)) &= (1(A), b_1), \\ (a_0, b_0) \vee (1(A), 0(B)) &= (1(A), b_0). \end{aligned}$$

The element x belongs to $A(x_0)$ if and only if $b_1 = b_0$; since φ is an isomorphism, this is true if and only if $a \vee x = a \vee x_0$.

Proof of Lemma 12:

(a) Assume that $C(L)$ is a closed sublattice of L and let $x_0 \in L$, $\emptyset \neq \{A_i(x_0)\}$ ($i \in I$) $\subset F(x_0)$. For each $i \in I$ there exist lattices A_i, B_i and an isomorphism φ_i of L onto $A_i \times B_i$. Under the analogous denotations as above let φ'_i be the corresponding isomorphism of L onto $A_i(x_0) \times B_i(x_0)$. Since $1(L) \in L$, there exists a greatest element c_i in $A_i(x_0)$ and a least element d_i in $B_i(x_0)$. The element $a_i = (\varphi'_i)^{-1}((c_i, d_i))$ belongs to the center of L , hence

$$(\varphi'_i)^{-1}((c_i, d_i) \vee (x_0, x_0)) = (\varphi'_i)^{-1}((c_i, x_0)) = c_i$$

belongs to the center of the lattice $[x_0, 1(L)]$ (cf. Lemma 2). Put $c = \bigwedge c_i$, $a = \bigwedge a_i$.

Then, since $C(L)$ is a closed sublattice of L , $a \in C(L)$ and according to Thm. 2 we have

$$c = \bigwedge c_i = \bigwedge (x_0 \vee a_i) = x_0 \vee (\bigwedge a_i) = x_0 \vee a$$

and $c \in C([x_0, 1(L)])$ by Lemma 2. There exist lattices A, B and an isomorphism φ of L onto $A \times B$ such that $a = \varphi^{-1}((1(A), 0(B)))$. Consider the direct factor $A(x_0) \in F(x_0)$. Let $z \in A(x_0)$. By Lemma 12.1 we have $z \vee a = x_0 \vee a$, thus for each a_i ,

$$z \vee a_i = z \vee (a \vee a_i) = (z \vee a) \vee a_i = (x_0 \vee a) \vee a_i = x_0 \vee a_i,$$

hence $z \in A_i(x_0)$. Conversely, let $z \in A_i(x_0)$. Then

$$z \vee a_i = x_0 \vee a_i \quad \text{for each } i \in I.$$

Since the center of L is a closed sublattice, we have

$$z \vee a = z \vee (\bigwedge a_i) = \bigwedge (z \vee a_i) = \bigwedge (x_0 \vee a_i) = x_0 \vee (\bigwedge a_i) = x_0 \vee a,$$

therefore $z \in A(x_0)$. Thus $\bigcap A_i(x_0) = A(x_0) \in F(x_0)$.

(b) Let (*) be valid and let $\{c_i\} (i \in I) \subset C(L)$. For each $i \in I$ there are lattices A_i, B_i and an isomorphism φ_i of L onto $A_i \times B_i$ such that $c_i = \varphi_i^{-1}((1(A_i), 0(B_i)))$. Put $x_0 = 0(L)$. Then c_i is the greatest element of $A_i(x_0)$. According to the assumption, there exist lattices A, B and an isomorphism φ of L onto $A \times B$ such that $A(x_0) = \bigcap A_i(x_0)$. Thus $\bigcap A_i(x_0)$ has a greatest element c and $c \in C(L)$. Obviously $c = \bigwedge c_i (i \in I)$, hence $\bigwedge c_i \in C(L)$. Further consider the lattices $B_i(y_0)$ for $y_0 = 1(L)$. The element c_i is the least element of $B_i(y_0)$. According to (*), $\bigcap B_i(y_0)$ belongs to $F(y_0)$, therefore $\bigcap B_i(y_0)$ has a least element d and $d \in C(L)$. Clearly $d = \bigvee a_i$. The proof is complete.

Our purpose is to prove the following assertion:

Theorem 3. *Let L be a conditionally complete lattices. Then the following conditions are equivalent:*

(a) = (*).

(b) *For each interval $[u, v] \subset L$, the relative center $C'([u, v])$ is a closed sublattice of L .*

(c) *If $x, y, u, v \in L$, $u \leq y \leq x \leq v$ and $\{a_i\} \subset C'([u, v])$, then the relations (1) and (2) are valid.*

At first we introduce some auxiliary notions and prove some lemmas. Let us remark that for any $u, v \in L$, $C'([u, v])$ is a closed sublattice of L if and only if $C'([u, v])$ is a closed sublattice of $[u, v]$. Let L be a lattice, $x_0 \in L$. For each subset $\emptyset \neq X \subset L$ we denote by $X^\delta(x_0)$ the set of all $y \in L$ satisfying

$$(8) \quad (x \vee x_0) \wedge (y \vee x_0) = x_0 = (x \wedge x_0) \vee (y \wedge x_0)$$

for each $x \in X$. Let A, B be lattices and let φ be an isomorphism of L onto $A \times B$. Let $p, q \in L$. If $p \in A(q)$ we write $p \equiv q(R(A))$. Analogously we define the relation $p \equiv q(R(B))$. Then $R(A), R(B)$ are permutable congruence relations on L , $R(A) \wedge R(B)$ is the least congruence relation on L and $R(A) \vee R(B)$ is the greatest congruence relation on L . (Cf. [1].)

Lemma 13. *Let $z \in L$, $\varphi'(z) = (z_1, z_2)$. Then*

$$z_1 \in A(x_0) \cap B(z), \quad z_2 \in B(x_0) \cap A(z).$$

This is an immediate consequence of the definition of the sets $A(x)$ and $B(x)$ for $x \in L$.

Lemma 14. *Let $x_0, z \in [u, v] \subset L$, $\varphi'(z) = (z_1, z_2)$. Then $z_1, z_2 \in [u, v]$.*

Proof. According to Lemma 13 we have

$$x_0 \equiv z_1(R(A)), \quad z_1 \equiv z(R(B)),$$

and hence (because $R(A), R(B)$ are congruence relations on L)

$$x_0 \equiv (z_1 \vee u) \wedge v(R(A)), \quad (z_1 \vee u) \wedge v \equiv z(R(B)).$$

From this we infer that

$$z_1 \equiv (z_1 \vee u) \wedge v(R(A) \wedge R(B)).$$

Since $R(A) \wedge R(B)$ is the least congruence on L , we obtain $z_1 = (z_1 \vee u) \wedge v$. Thus $z_1 \in [u, v]$.

Lemma 15. $B(x_0) = (A(x_0))^\delta(x_0)$.

Proof. Let $y \in L$, $\varphi(y) = (a, b)$, $\varphi(x_0) = (a_0, b_0)$, $x \in A(x_0)$, $\varphi(x) = (a_1, b_0)$. If $y \in B(x_0)$, then $a = a_0$, thus

$$(\varphi(x) \vee \varphi(x_0)) \wedge (\varphi(y) \vee \varphi(x_0)) = (a_1 \vee a_0, b_0) \wedge (a_0, b \vee b_0) = (a_0, b_0),$$

therefore $(x \vee x_0) \wedge (y \vee x_0) = x_0$. Dually, $(x \wedge x_0) \vee (y \wedge x_0) = x_0$, hence $y \in (A(x_0))^\delta(x_0)$.

Let $x_0 \leq y \in (A(x_0))^\delta(x_0)$. Then $(a, b_0) \in A(x_0)$, $(a, b_0) \wedge (a, b) = (a, b_0)$ and hence by the definition of the set $(A(x_0))^\delta(x_0)$ we obtain $(a, b_0) = (a_0, b_0)$, therefore $y \in B(x_0)$. Similarly, if $x_0 \geq y \in (A(x_0))^\delta(x_0)$, then $y \in B(x_0)$. Now let y be any element of the set $(A(x_0))^\delta(x_0)$ and denote $y_1 = y \vee x_0$, $y_2 = y \wedge x_0$. Then y_1 and y_2 fulfil (8), hence $y_1, y_2 \in (A(x_0))^\delta(x_0)$, thus $y_1, y_2 \in B(x_0)$. Since $B(x_0)$ is a convex subset of L , we obtain $y \in B(x_0)$.

Lemma 16. Let $x_0 \in [u, v] \subset [u_1, v_1] \subset L$. Assume that φ is an isomorphism of $[u, v]$ onto $A \times B$ and that φ_1 is an isomorphism of $[u_1, v_1]$ onto $A_1 \times B_1$ such that $A(x_0) = A_1(x_0) \cap [u, v]$. Then $B(x_0) = B_1(x_0) \cap [u, v]$.

Proof. Let $y \in B_1(x_0) \cap [u, v]$, $x \in A(x_0)$. Then $x \in A_1(x_0)$ and hence according to Lemma 15 the relation (8) is valid. Thus $y \in B(x_0)$. Conversely, let $y \in B(x_0)$ and let $x \in A_1(x_0)$. Then since $A_1(x_0)$ is a convex sublattice of L we have $x_0 \leq (x \vee x_0) \wedge v \in A(x_0)$ and therefore by Lemma 15

$$\begin{aligned} (x \vee x_0) \wedge (y \vee x_0) &= (x \vee x_0) \wedge [v \wedge (y \vee x_0)] = \\ &= [(x \vee x_0) \wedge v] \vee (y \vee x_0) = x_0. \end{aligned}$$

Dually we obtain $(x \wedge x_0) \vee (y \wedge x_0) = x_0$, thus by Lemma 15, $y \in B_1(x_0) \cap [u, v]$.

Under the same assumptions as in Lemma 16 the following two lemmas are valid:

Lemma 17. Let $z \in [u, v]$. Then

$$A(z) = A_1(z) \cap [u, v], \quad B(z) = B_1(z) \cap [u, v].$$

Proof. According to Lemma 13 there exist $z_1 \in A(x_0)$, $z_2 \in B(x_0)$ such that

$$z_1 \in B(z), \quad z_2 \in A(z).$$

Thus $A(z_1) = A(z_0)$ and so according to the assumption we have $A(z_1) = A_1(z_1) \cap [u, v]$. Hence by Lemma 16, $B(z_1) = B_1(z_1) \cap [u, v]$. Therefore $z \in B_1(z_1)$ and thus $B(z) = B_1(z) \cap [u, v]$. From this and from Lemma 16 we infer that $A(z) = A_1(z) \cap [u, v]$.

Lemma 18. Let $z \in [u, v]$. Then $\varphi'(z) = \varphi'_1(z)$.

Proof. Let z_1, z_2 be as in the proof of Lemma 17. We have

$$z_1 \in A(x_0) \cap B(z), \quad z_2 \in B(x_0) \cap A(z)$$

and hence according to Lemma 17,

$$z_1 \in A_1(x_0) \cap B_1(z), \quad z_2 \in B_1(x_0) \cap A_1(z).$$

Therefore from Lemma 13 and from the fact that $R(A_1) \wedge R(B_1)$ is the least congruence relation on $[u_1, v_1]$ we get $\varphi'(z) = (z_1, z_2) = \varphi'_1(z)$.

Lemma 19. Let $L = [u, v]$ and let

$$\varphi : L \rightarrow A \times B, \quad \varphi_i : L \rightarrow A_i \times B_i$$

be isomorphisms of L onto $A \times B$ and $A_i \times B_i$, respectively ($i \in I$). Denote $a_i = \varphi_i^{-1}((1(A_i), 0(B_i)))$ and assume that

$$(c_1) \quad t \vee (\bigwedge a_i) = \bigwedge (t \wedge a_i) \text{ for each } t \in L$$

is valid. Let $x_0 \in L$, $\bigcap A_i(u) = A(u)$. Then $\bigcap A_i(x_0) = A(x_0)$.

Proof. There exists $a \in C([u, v])$ such that $a = 1(A(u))$. Clearly $a_i = 1(A_i(u))$. From $\bigcap A_i(u) = A(u)$ it follows $\bigwedge a_i = a$. Now, by using (c_1) and by the same method as in the part (a) of the proof of Lemma 12 we obtain that $\bigcap A_i(x_0) = A(x_0)$.

Lemma 20. $(c) \Rightarrow (*)$ for each conditionally complete lattice L .

Proof. Assume that L satisfies (c) and let $\{A_i(x_0)\}$ ($i \in I$) be a nonempty subset of $F(x_0)$ for some $x_0 \in L$. Let $z \in L$. Choose $u, v \in L$ such that $u \leq z$, $[x_0 \wedge z, x_0 \vee z] \subset [u, v]$.

For each $i \in I$ there is a lattice B_i and an isomorphism φ_i of L onto $A_i \times B_i$. Let $\varphi_i(u) = (u_1^i, u_2^i)$, $\varphi_i(v) = (v_1^i, v_2^i)$ and let $\bar{\varphi}_i$ be the corresponding partial mapping of the interval $[u, v]$ into $A_i \times B_i$. Then $\bar{\varphi}_i$ is an isomorphism of $[u, v]$ onto

$$[u_1^i, v_1^i] \times [u_2^i, v_2^i] = \bar{A}_i \times \bar{B}_i.$$

Let $a_i = \bar{\varphi}_i^{-1}(v_1^i, u_2^i)$, $a'_i = \bar{\varphi}_i^{-1}(u_1^i, v_2^i)$. The elements a_i, a'_i belong to the relative center $C'([u, v]) \subset C([u, v])$ and a_i is the complement of a'_i in the interval $[u, v]$. According to the assumption the condition (c) is valid and thus by Thm. 1 the elements $a = \bigwedge a_i$, $b = \bigvee a'_i$ belong to the center of the lattice $[u, v]$. Hence there are lattices X and Y and an isomorphism $\bar{\varphi}$ of $[u, v]$ onto $X \leq Y$ such that $\bar{\varphi}(a) = (1(X), 0(Y))$, $\bar{\varphi}(b) = (0(X), 1(Y))$. Clearly

$$X(u) = [u, a], \quad \bar{A}_i(u) = [u, a_i]$$

and therefore

$$X(u) = \bigcap \bar{A}_i(u) \quad (i \in I).$$

Hence by Lemma 19 (the condition (c_1) of this lemma is valid because of (c)), we have

$$X(x_0) = \bigcap \bar{A}_i(x_0) \subset \bigcap A_i(x_0).$$

Denote

$$A = \bigcap A_i(x_0), \quad B = A^\delta(x_0).$$

Let $x \in A$, $y \in Y(x_0)$ and denote $(x \vee x_0) \wedge v = z$. Then $z \in X(x_0)$ and hence according to Lemma 15,

$$z \wedge (y \vee x_0) = (z \vee x_0) \wedge (y \vee x_0) = x_0.$$

Therefore

$$\begin{aligned} (x \vee x_0) \wedge (y \vee x_0) &= (x \vee x_0) \wedge [v \wedge (y \vee x_0)] = \\ &= [(x \vee x_0) \wedge v] \wedge (y \vee x_0) = z \wedge (y \vee x_0) = x_0 \end{aligned}$$

and dually we obtain

$$(x \wedge x_0) \vee (y \wedge x_0) = x_0.$$

Thus $y \in A^\delta(x_0)$ and hence $Y(x_0) \subset A^\delta(x_0)$. Let $\varphi'(z) = (z_1, z_2)$. Then $z_1 \in A$, $z_2 \in B$. From Lemma 18 it follows that the elements z_1, z_2 do not depend from the particular choice of elements u, v . We write

$$z_1 = z[A], \quad z_2 = z[B].$$

If $t \in L$, we may choose $u, v \in L$ such that $\{x_0, z, t\} \subset [u, v]$ and then we obtain that

$$(z \wedge t)[A] = z[A] \wedge t[A], \quad (z \vee t)[A] = z[A] \vee t[A]$$

and analogously for B . Further $z \neq t$ implies $(z_1, z_2) \neq (t_1, t_2)$. Hence the mapping $\varphi : z \rightarrow (z_1, z_2)$ is an isomorphism of L into $A \times B$.

Let $p \in A$, $q \in B$ and choose $u, v \in L$ such that $\{x_0, p, q\} \subset [u, v]$. Then we have (by the same notations as above) $p \in X(x_0)$. From Lemma 15 it follows $q \in Y(x_0)$. Thus there is $z \in [u, v]$ such that $\bar{\varphi}'(z) = (p, q)$. Hence we obtain $p = z_1$, $q = z_2$. Therefore the mapping φ is onto. We have $\varphi(x_0) = (x_0, x_0)$ and if $z \in A(z \in B)$, then $\varphi(z) = (z, x_0)$ ($\varphi(z) = (x_0, z)$). Thus $A(x_0) = A$, $B(x_0) = B$. We have proved that $\bigcap A_i(x_0) = A$ belongs to $F(x_0)$.

Proof of Thm. 3.

(a) \Rightarrow (b). Let (a) be valid. Let $[u, v] \subset L$, $\emptyset \neq \{c_i\} (i \in I) \subset C'([u, v])$. For each $i \in I$ there is an isomorphism φ_i of L onto $A_i \times B_i$ such that the condition from the definition of $C'([u, v])$ is fulfilled. Put $x_0 = u$. According to (a), there are lattices A, B and an isomorphism φ of L onto $A \times B$ such that $A(x_0) = \bigcap A_i(x_0)$. The lattice $X = [u, v]$ is isomorphic with the direct product $(X \cap A(x_0)) \times (X \cap B(x_0))$, and $X \cap A(x_0) = \bigcap (X \cap A_i(x_0))$. Then the lattice $X \cap A(x_0)$ has a greatest element c and $c \in C'([u, v])$. The element c_i is the greatest element of $X \cap A_i(x_0)$, hence $\bigwedge c_i = c$ and so $\bigwedge c_i \in C'([u, v])$. By a dual method we can prove that $\bigvee c_i \in C'([u, v])$.

(b) \Rightarrow (c). Assume that (b) holds. Let $x, y, u, v \in L$, $u \leq y \leq x \leq v$, $\{a_i\} \subset C'([u, v])$. Let a'_i be the relative complement of a_i with respect to the interval $[u, v]$. Then $a'_i \in C'([u, v])$ and hence according to (b) we have $a = \bigwedge a_i \in C'([u, v])$, $b = \bigvee a'_i \in C'([u, v])$. Thus the elements a_i, a, b belong to $C([u, v])$ and therefore from Lemma 4 we infer that the relations (3) and (4) are valid. Thus (1) and (2) hold whenever the assumptions of (c) are fulfilled.

The implication (c) \Rightarrow (a) was proved in Lemma 20.

Corollary 1. *Let L be a complete lattice. Then the following conditions are equivalent:*

- (a) The center of L is a closed sublattice of L .
- (b) Each relative center of L is a closed sublattice of L .

Proof. Since the center of L is a relative center of L , (b) \Rightarrow (a). From Lemma 12 and Thm. 3 it follows that (a) implies (b).

Corollary 2. Let L be a conditionally complete lattice, $x_0 \in L$. If for each interval $[u, v]$ of L the center $C([u, v])$ is a closed sublattice of $[u, v]$, then for each set $\emptyset \neq \{A_i(x_0)\}$ of direct factors of L with respect to x_0 the intersection $\bigcap A_i(x_0)$ is a direct factor of L with respect to x_0 .

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