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CENTERED COMPLEXITY ONE HAMILTONIAN TORUS ACTIONS

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ABSTRACT. We consider symplectic manifolds with Hamiltonian torus actions which are "almost but not quite completely integrable": the dimension of the torus is one less than half the dimension of the manifold. We provide a complete set of invariants for such spaces when they are "centered" and the moment map is proper. In particular, this classifies the preimages under the moment map of all sufficiently small open sets, which is an important step towards global classification. As an application, we construct a full packing of each of the Grassmannians $\mathrm{Gr}^+(2,\mathbb{R}^5)$ and $\mathrm{Gr}^+(2,\mathbb{R}^6)$ by two equal symplectic balls.

1. Introduction

Let a torus $T \cong (S^1)^{\dim T}$ act effectively on a symplectic manifold (M,ω) by symplectic transformations with a moment map $\Phi\colon M \longrightarrow \mathfrak{t}^*$, that is,

$$\iota(\xi_M)\omega = -\mathrm{d}\,\langle \Phi, \xi\rangle \tag{1.1}$$

for every ξ in the Lie algebra $\mathfrak{t}$ of T, where ξ_M is the corresponding vector field on M. The dimension of the torus is at most half the dimension of the manifold. The difference $k = \frac{1}{2}\dim M - \dim T$ is half the dimension of the symplectic quotient $\Phi^{-1}(\alpha)/T$ at a regular value α in the moment image $\Phi(M)$. We call this number k the **complexity**.[1]

Compact symplectic manifolds with complexity zero Hamiltonian torus actions, also known as *symplectic toric manifolds* or *Delzant spaces*, are classified by their moment images [De1]. The first examples where the complexity is one are compact symplectic surfaces (with no action). By Moser [Mo], these are classified by their genus and total area. The next examples are compact symplectic four manifolds with Hamiltonian circle actions, which were classified by the first author [K2]; also see [AH, Au1, Au2]. In the algebraic category, complexity one actions (of possibly non-abelian groups) were recently classified by Timashëv [T1, T2]. Among other works on Lie group actions of complexity zero or one are [I, De2, W, GSj, Kn] in

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[1]We changed our earlier term *deficiency* to *complexity* in order to be consistent with the algebraic geometers' terminology.

the symplectic category; [KKMS, OW, R, FK, BB, LV] in the algebraic category; [F, OR] in the smooth category.

This is the first in a series of papers in which we study complexity one torus actions in arbitrary dimension. In this paper we study the basic building blocks: the preimages under the moment map of sufficiently small open subsets in $\mathfrak{t}^*$. We provide invariants which determine these spaces up to an equivariant symplectomorphism. Our techniques apply to "large" complexity one spaces, as long as they are centered (see Definition 1.4).

In this paper, because we wish to restrict to the preimages of open subsets of $\mathfrak{t}^*$, we do not insist that our manifolds be compact. Instead, we assume that the moment map is proper as a map to an open convex set $U \subset \mathfrak{t}^*$, that is, that the preimage of every compact subset of U is compact. For instance, if M is compact, Φ is proper.

Definition 1.2. Let T be a torus. A **proper Hamiltonian T-manifold** is a connected symplectic manifold (M, ω) together with an effective action of T, an open convex subset $U \subseteq \mathfrak{t}^*$, and a proper moment map $\Phi \colon M \longrightarrow U$. Here, $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ the dual space. For brevity, in this paper we call (M, ω, Φ, U) a **complexity k space**, where $k = \frac{1}{2}\dim M - \dim T$. An **isomorphism** between two such spaces over the same set U is an equivariant symplectomorphism that respects the moment maps.

Example 1.3. Let (M, ω, Φ, U) be a proper Hamiltonian T-manifold. For any open convex subset $V \subseteq U$, the preimage $\Phi^{-1}(V)$ is a proper Hamiltonian T-manifold over V. The fact that it is connected follows from the facts that the restriction $\Phi \colon \Phi^{-1}(V) \longrightarrow V$ is proper and its image and fibers are connected (see Theorem 2.3) by easy point-set topology.

Here are a few examples of complexity one spaces. The complete flags on $\mathbb{C}^3$ form a six dimensional compact symplectic manifold with a two dimensional Hamiltonian torus action. The Grassmannians $\mathrm{Gr}^+(2, \mathbb{R}^5)$ and $\mathrm{Gr}^+(2, \mathbb{R}^6)$ of oriented two-planes in $\mathbb{R}^5$ and $\mathbb{R}^6$ are also complexity one spaces; see section 14 for more details. Any symplectic toric manifold gives rise to complexity one spaces in several ways: one can either restrict the action to a codimension one subtorus, or take the product of the manifold with a surface. Finally, the example in [To], of a symplectic manifold with a Hamiltonian torus action with isolated fixed points that is not equivariantly Kähler, is a complexity one space.

We now describe invariants of a complexity one space.

The **Liouville measure** on a $2n$ dimensional symplectic manifold (M, ω) is given by integration of the volume form $\omega^n/n!$ with respect to the symplectic orientation. In the presence of a Hamiltonian action, the **Duistermaat-Heckman measure** is the push-forward of Liouville measure by the moment map. It is equal to Lebesgue measure on $\mathfrak{t}^*$ times the **Duistermaat-Heckman function**, which is piecewise linear.[2]

Assume M is connected. For any value $\alpha \in \Phi(M)$, if the symplectic quotient $\Phi^{-1}(\alpha)/T$ is not a single point, it is homeomorphic to a connected closed oriented surface (see Proposition 6.1). The genus of this surface does not depend on α (see Corollary 9.7); we call it the **genus** of the complexity one space.

[2]For a complexity k space it is piecewise polynomial of degree at most k.

The **stabilizer** of a point $x \in M$ is the closed subgroup $H = \{\lambda \in T \mid \lambda \cdot x = x\}$. The isotropy representation at x is the linear representation of H on the tangent space $T_x M$. Points in the same orbit have the same stabilizer, and their isotropy representations are linearly symplectically isomorphic; this isomorphism class is the **isotropy representation** of the orbit.

An orbit is **exceptional** if every nearby orbit in the same moment fiber has a strictly smaller stabilizer. In particular, if a moment fiber $\Phi^{-1}(\alpha)$ consists of a single orbit, that orbit is exceptional. Since each moment fiber is compact, it contains finitely many exceptional orbits. The **isotropy data** at $\alpha \in U$ is the unordered list of isotropy representations of the exceptional orbits in $\Phi^{-1}(\alpha)$.

With these definitions on hand, let us state our main theorem, which gives necessary and sufficient conditions for two complexity one spaces to be locally isomorphic.

Theorem 1 (Local Uniqueness)**.** *Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces. Assume that their Duistermaat-Heckman measures are the same and that their genus and isotropy data over $\alpha \in U$ are the same. Then there exists a neighborhood of α over which the spaces are isomorphic.*

Here is a simple proof of Theorem 1 in the case where the torus action on $\Phi^{-1}(\alpha)$ is free:

> The symplectic quotient $\Phi^{-1}(\alpha)/T$ is a symplectic surface. Its symplectic area is the value of the Duistermaat-Heckman function at α. Together with the genus, this determines the surface.
>
> The moment fiber $Z = \Phi^{-1}(\alpha)$ is a principal T-bundle over the symplectic quotient. Its Chern class is given by the slope of the Duistermaat-Heckman function at α [DH].
>
> The pullback to the moment fiber Z of the symplectic form on the symplectic quotient is the restriction $i_Z^* \omega$ of ω to Z. By the equivariant co-isotropic embedding theorem (see [W1, lecture 5]), a neighborhood of the moment fiber Z is determined up to equivariant symplectomorphism by $(Z, i_Z^* \omega)$. Since the moment map is proper, this neighborhood contains the preimage of a neighborhood of α.

This argument easily extends to the case that α is any regular value of the moment map. The main volume of this paper consists of carefully extending the argument to singular values of the moment map.

Recall that the **orbit type strata** are the connected components of the sets of points with the same stabilizer.

Definition 1.4. A proper Hamiltonian T-manifold (M, ω, Φ, U) is **centered** about a point $\alpha \in U$ if α is contained in the closure of the moment image of every orbit type stratum.

Example 1.5. A linear action on a symplectic vector space is centered.

Example 1.6. Every point in $\mathfrak{t}^*$ has a neighborhood whose preimage is centered. This follows from the local normal form theorem (see below) and the properness of the moment map. Conversely, a (non-trivial) compact symplectic manifold is never centered, because it has fixed points with different moment images.

Example 1.7. Consider the Grassmannian $\mathrm{Gr}^+(2,\mathbb{R}^5)$, as shown in Figure 1 in Section 14. The preimage of the open upper half-plane is centered; so is the preimage of the interior of the shaded diamond.

Theorem 2 (Centered Uniqueness)**.** *Let (M,ω,Φ,U) and (M',ω',Φ',U) be complexity one spaces that are centered about $\alpha \in U$. Assume that their Duistermaat-Heckman measures are the same and that their genus and isotropy data over $\alpha \in \mathfrak{t}^*$ are the same. Then the spaces are isomorphic.*

Finally, we present an application of our results to an interesting question in symplectic topology: to what extent can a symplectic manifold be filled by disjoint symplectic balls? Holomorphic techniques give obstructions to embedding balls, and, in dimension four, lead to existence theorems [Bi1], [Bi2]. Some explicit embeddings can be found in [McD, 2.7.1], [K1], [Tr]. Our construction uses equivariant techniques to solve this non-equivariant problem. In a future paper, we will extend these techniques to address this question more deeply. Here, we are content with a simple, but fairly representative, application:

Theorem 3. *Let M be the Grassmannian $\mathrm{Gr}^+(2,\mathbb{R}^5)$ or $\mathrm{Gr}^+(2,\mathbb{R}^6)$. There exists an equivariant symplectic embedding of a disjoint union of two open symplectic balls with linear actions and with equal radii into M such that the complement of the image has zero volume. A fortiori, each of these Grassmannians can be fully packed by two equal symplectic balls.*

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2. Background

We now set our notation and review some background material.

An effective linear action of a compact abelian group H on a symplectic vector space is isomorphic to the action of H on $\mathbb{C}^n$ through an inclusion

$$\rho = (\rho_1, \dots, \rho_n)\colon H \longrightarrow (S^1)^n.$$

The characters ρ_i are determined up to permutation. The differential of each $\rho_i\colon H \longrightarrow S^1$ is an element η_i of the dual space $\mathfrak{h}^*$; the η_i are called the weights.

Let a torus T act effectively on a symplectic manifold (M,ω). The symplectic slice at $x \in M$ is the symplectic vector space

$$(T_x\mathcal{O})^\omega \,/\, (T_x\mathcal{O} \cap (T_x\mathcal{O})^\omega)\,,$$

where $\mathcal{O}$ is the T-orbit of x in M. Let $H \subset T$ be the stabilizer of x. The isotropy representation of H on T_xM induces a representation on the symplectic slice, called the **slice representation**. The isotropy representation is the direct sum of the slice representation and a trivial representation. The weights of the slice representation are called the **isotropy weights**.

We fix an inner product on the Lie algebra $\mathfrak{t}$ of our torus T, once and for all. This determines an inclusion $\mathfrak{h}^* \hookrightarrow \mathfrak{t}^*$ for any subspace $\mathfrak{h} \subset \mathfrak{t}$. Throughout this paper, we identify $\mathfrak{h}^*$ with its image in $\mathfrak{t}^*$.

The Guillemin-Sternberg-Marle local normal form theorem describes the neighborhood of an orbit in a symplectic manifold with a Hamiltonian action of a compact Lie group [GS2], [M]. We state it for torus actions.

Theorem 2.1 (Local normal form)**.** *Let a closed subgroup H of a torus T act on $\mathbb{C}^n$ by an inclusion $\rho\colon H \longrightarrow (S^1)^n$ with weights $\eta_1, \dots, \eta_n$ and moment map*

$$\Phi_H(z) = \frac{1}{2}\sum_{j=1}^{n} |z_j|^2 \eta_j.$$

1. *Equip $T^*(T) \times \mathbb{C}^n$ with the standard symplectic form and the diagonal H action. Its symplectic quotient by H can be identified with the* ***local model***
$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0,$$
where $\mathfrak{h}^0$ denotes the annihilator of $\mathfrak{h}$ in $\mathfrak{t}^*$. Given $\alpha \in \mathfrak{t}^*$,
$$\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu$$
is a moment map for the left T action. Here, $T^*(T) = T \times \mathfrak{t}^*$ is the cotangent bundle of T
2. *Let the torus T act effectively on a symplectic manifold (M, ω) with a moment map $\Phi\colon M \longrightarrow \mathfrak{t}^*$. Given a point $x \in M$ with slice representation ρ, there exists a neighborhood of the orbit $T \cdot x$ that is equivariantly symplectomorphic to a neighborhood of the orbit $\{[t, 0, 0]\}$ in the model Y with $\alpha = \Phi(x)$. We call Y the* ***local model*** *for the orbit.*

The following immediate corollary of the local normal form theorem implies an important special case of Theorem 1:

Proposition 2.2. *Let (M, ω, Φ, U) and (M', ω', Φ', U) be proper Hamiltonian T-manifolds. Consider a value $\alpha \in U$ so that the moment fibers $\Phi^{-1}(\alpha)$ and $\Phi'^{-1}(\alpha)$ each consist of a single orbit. Suppose that these orbits have the same slice representation. Then there exists a neighborhood V of α over which M and M' are isomorphic.*

We will also use the following intimately related global properties:

Theorem 2.3. *Every proper Hamiltonian T-manifold (M, ω, Φ, U) has the following properties.*

Convexity: *The moment image, $\Phi(M)$, is convex.*
Connectedness: *The moment fiber, $\Phi^{-1}(\alpha)$, is connected for all $\alpha \in U$.*
Stability: *As a map to $\Phi(M)$, the moment map is open.*

For the compact case, see [At], [GS1], and [Sj, Theorem 6.5]. For proper moment maps to open convex sets and a brief history, see [LMTW].

3. Eliminating the symplectic form

Our first task is to free ourselves from the symplectic form. In this section we show that, instead of working with equivariant symplectomorphisms, it is enough to work with equivariant diffeomorphisms that respect the orientation and the moment map. These are much easier to work with, as one can apply techniques from differential topology.

Definition 3.1. Let a torus T act on oriented manifolds M and M' with T-invariant maps $\Phi\colon M \longrightarrow \mathfrak{t}^*$ and $\Phi'\colon M' \longrightarrow \mathfrak{t}^*$. A **$\Phi$–$T$–diffeomorphism** from (M, Φ) to (M, Φ') is an orientation preserving equivariant diffeomorphism $\Psi\colon M \longrightarrow M'$ that satisfies $\Psi^*(\Phi') = \Phi$.

In this section and the next one we use the following technical condition:

$$\text{The restriction map } H^2(M/T,\mathbb{Z}) \longrightarrow H^2(\Phi^{-1}(y)/T,\mathbb{Z}) \text{ is one-to-one for some regular value } y \text{ of } \Phi. \tag{3.2}$$

In a later paper we will prove that this condition is satisfied by all complexity one spaces.

Proposition 3.3. *Let (M,ω,Φ,U) and (M',ω',Φ',U) be complexity one spaces that satisfy Condition (3.2) and have the same Duistermaat-Heckman measure.*[3] *Then there exists an equivariant symplectomorphism from M to M' if and only if there exists a Φ–T–diffeomorphism from M to M'.*

Our proof of Proposition 3.3 uses Moser's method. The special case $T = \{e\}$ is Moser's theorem that two compact symplectic surfaces with the same area are symplectomorphic exactly if they are diffeomorphic. Before giving our proof, we need to recall the definition of basic forms and prove a few technical lemmas.

Let a compact Lie group G act on a manifold M, and let ξ_M, for $\xi \in \mathfrak{g}$, be the generating vector-fields. A differential form β on M is **basic** if it is G invariant and horizontal, that is, $\iota_{\xi_M}\beta = 0$ for all $\xi \in \mathfrak{g}$.

Remark 3.4. The basic differential forms on M constitute a differential complex whose cohomology coincides with the Čech cohomology of the topological quotient, M/G. See [Kl]. To see this, repeat the standard Čech-de Rham spectral-sequence argument, as in [BT]. It still works because, by the local normal form for smooth actions of compact Lie groups, every orbit in M has a neighborhood on which the complex of basic forms is acyclic.

Lemma 3.5. *Let (M,ω,Φ,U) and (M',ω',Φ',U) be complexity one spaces that satisfy Condition (3.2) and have the same Duistermaat-Heckman measure. Then for every Φ–T–diffeomorphism $g\colon M \longrightarrow M'$ there exists a basic one-form β on M such that $\mathrm{d}\beta = g^*\omega' - \omega$.*

Proof. Let $\Omega = g^*\omega' - \omega$. Since ω and $g^*\omega'$ are closed invariant symplectic forms on M with the same moment map, $\iota(\xi_M)\Omega = 0$ for all $\xi \in \mathfrak{t}$. Since Ω is also invariant, it is basic.

By Condition (3.2) it suffices to show that the restriction of Ω to the moment fiber $\Phi^{-1}(\alpha)$ is exact for some regular value α of Φ. Since this restriction is the pull-back of a differential form Ω_{red} on the orbifold $M_{\mathrm{red}} = \Phi^{-1}(\alpha)/T$, it is enough to show that Ω_{red} is exact. Since M_{red} is two dimensional, it is enough to show that the integral of Ω_{red} over it is zero, i.e., that the integrals of ω_{red} and $g^*\omega'_{\mathrm{red}}$ are equal. This follows from the fact that the Duistermaat-Heckman measures for M and M' are the same, because the density functions for these measures are given by the symplectic volumes of the symplectic quotients; see [DH, §3]. $\square$

Lemma 3.6. *Let an $(n-1)$-dimensional abelian group T act effectively on a $2n$ dimensional manifold M. Let ω_0 and ω_1 be invariant symplectic forms that induce the same orientation and have the same moment map. Then the two-form $\omega_t = (1-t)\omega_0 + t\omega_1$ is nondegenerate for all $0 \le t \le 1$.*

[3] In fact, we only need that the Duistermaat-Heckman functions agree at a point. Contrast with footnote 4.

Proof. Let $x \in M$ be a point with stabilizer H; let h be the dimension of H. By the local normal form theorem, a neighborhood of the orbit of x with the symplectic form ω_0 is equivariantly symplectomorphic to a neighborhood of the orbit $\{[t,0,0]\}$ in the model $T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$. The tangent space at x splits into $\mathfrak{t}/\mathfrak{h} \oplus \mathfrak{h}^0 \oplus \mathbb{C}^{h+1}$, where $\mathfrak{t}/\mathfrak{h}$ is the tangent space to the orbit. By the definition of the moment map, $\omega_t|_x$ is given by a block matrix of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \tilde{\omega}_t \end{pmatrix}$$

where I is the natural pairing between the vector space $\mathfrak{t}/\mathfrak{h}$ and its dual, $\mathfrak{h}^0$, and where $\tilde{\omega}_0$ and $\tilde{\omega}_1$ are linear symplectic forms on $\mathbb{C}^{h+1}$ with the same moment map and the same orientation. It suffices to show that $\tilde{\omega}_t$ is nondegenerate.

Case 1. Suppose that the stabilizer of x is trivial. Then $\tilde{\omega}_0$ and $\tilde{\omega}_1$ are non-zero two-forms on $\mathbb{C}$ that induce the same orientation. Hence, ω_t is nondegenerate.

Case 2. Suppose that the stabilizer of x is non-trivial. Because each $\tilde{\omega}_t$ is translation invariant, to prove that it is nondegenerate it is enough to prove it at any $v \in \mathbb{C}^{h+1}$. We choose $v \in \mathbb{C}^{h+1}$ whose stabilizer is trivial and apply Case 1 to the H action on $\mathbb{C}^{h+1}$. $\square$

Proof of Proposition 3.3. Let g be any Φ–T–diffeomorphism from M to M'. By Lemma 3.6, $\omega_t := (1-t)\omega + tg^*\omega'$ is nondegenerate for all $0 \leq t \leq 1$.

By Lemma 3.5, there exists a basic one-form β on M such that $\mathrm{d}\beta = g^*\omega' - \omega$. The time dependent vector field X_t determined by $i_{X_t}\omega_t = -\beta$ preserves the level sets of Φ, because for every $\xi \in \mathfrak{t}$, $\langle \mathrm{d}\Phi(X_t), \xi \rangle = -\omega_t(\xi_M, X_t) = -i_{\xi_M}\beta = 0$. Moreover, the vector field X_t is invariant because ω_t and β are invariant.

Since Φ is proper, X_t integrates to a flow, $F_t \colon M \longrightarrow M$, with F_0 the identity map. Let $g_t = g \circ F_t$. Then $\Phi' \circ g_t = \Phi$. Additionally, g_t is equivariant, and hence is a Φ–T–diffeomorphism. As in the standard Moser argument [Mo], the choice of X_t implies that $\frac{\mathrm{d}}{\mathrm{d}t}(F_t^*\omega_t) = 0$, hence $F_1^*\omega_1 = F_0^*\omega_0 = \omega_0$. Finally, g_1 is an (equivariant) symplectomorphism because $g_1^*\omega' = F_1^*(g^*\omega') = F_1^*(\omega_1) = \omega_0 = \omega$. $\square$

4. Passing to the quotient

In this section we show that, as long as two complexity one spaces have the same Duistermaat-Heckman measure, we can reduce the problem of finding a Φ–T–diffeomorphism between them to the easier problem of finding a Φ-diffeomorphism between their quotients.

Let a compact torus T act on a manifold N. The quotient N/T can be given the quotient topology and a natural differential structure, consisting of the sheaf of real-valued functions whose pullbacks to N are smooth. We say that a map $h \colon N/T \longrightarrow N'/T$ is **smooth** if it pulls back smooth functions to smooth functions; it is a **diffeomorphism** if it is smooth and has a smooth inverse. See [Sch2]. If N and N' are oriented, the choice of an orientation on T determines orientations on the smooth part of N/T and N'/T. Whether or not a diffeomorphism $f \colon N/T \longrightarrow N'/T$ preserves orientation is independent of this choice.

While this notion of diffeomorphism is natural, we also need another notion, which is stronger but less natural.

Definition 4.1. Let a torus T act on oriented manifolds M and M' with T-invariant maps $\Phi\colon M \longrightarrow \mathfrak{t}^*$ and $\Phi'\colon M' \longrightarrow \mathfrak{t}^*$. A **$\Phi$-diffeomorphism** from M/T to M'/T is an orientation preserving diffeomorphism $\Psi\colon M/T \longrightarrow M'/T$ such that

1. Ψ preserves the maps to $\mathfrak{t}^*$, i.e., $\Psi^*\Phi' = \Phi$.
2. Each of Ψ and Ψ^{-1} lifts to a Φ–T–diffeomorphism in a neighborhood of each exceptional orbit.

We now state the main result of this section.

Proposition 4.2. *Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces that satisfy Condition* (3.2) *and have the same Duistermaat-Heckman measure.*[4] *Then every Φ-diffeomorphism from M/T to M'/T lifts to Φ–T–diffeomorphism from M to M'.*

Example 4.3. Consider any closed surface Σ and integer k. Let $P \longrightarrow \Sigma$ be the principal circle bundle with Euler number k. Let $M = P \times_{S^1} S^2$ be the associated sphere bundle, with the fiberwise S^1 action. Let ω be an invariant symplectic form on M. (These do exist.) Let h and A denote the areas of the fiber and south pole section, respectively. Up to translation, the moment map is the height function on the sphere. The Duistermaat-Heckman function is $A + kx$ for $0 \leq x \leq h$, and is zero otherwise. Up to equivariant diffeomorphism, M is determined by k and the genus of Σ. The Duistermaat-Heckman measure determines k; the quotient $M/S^1 = \Sigma \times [0, h]$ determines the genus.

The first step in proving Proposition 4.2 is to show that on the non-exceptional orbits every Φ-diffeomorphism lifts locally to a Φ–T–diffeomorphism. We do this in the next three lemmas.

Lemma 4.4. *Every local model for a non-exceptional orbit in a complexity one space has the form*

$$Y = T \times_H \mathbb{C}^h \times \mathbb{C} \times \mathfrak{h}^0, \tag{4.5}$$

where $H \subseteq T$ is a closed h dimensional subgroup which acts on $\mathbb{C}^h$ through an isomorphism with $(S^1)^h$.

Proof. Let $Y := T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be the local model for a non-exceptional orbit in $\Phi^{-1}(\alpha)$, with $h = \dim H$. Inside the moment fiber $\Phi_Y^{-1}(\alpha)$, the set of points with stabilizer H is

$$T \times_H (\mathbb{C}^{h+1})^H \times \{0\}, \tag{4.6}$$

where $(\mathbb{C}^{h+1})^H$ is the subspace fixed by H. By the definition of exceptional orbit, this subspace is not trivial. Therefore, the local model becomes (4.5), where the group H acts trivially on $\mathbb{C}$ and acts on $\mathbb{C}^h$ through an inclusion into $(S^1)^h$. By a dimension count, this inclusion must be an isomorphism. ☐

The following lemma tells us that neighborhoods of non-exceptional orbits can be read off from the moment image.

[4] In fact, we only need their Duistermaat-Heckman functions to have the same slope; the functions may differ by a constant. Contrast with footnote 3.

Lemma 4.7. *Let (M, ω, Φ, U) be a complexity one space. Assume that the moment fiber $\Phi^{-1}(\alpha)$ contains a non-exceptional orbit. Then there exists a closed connected subgroup $H \subseteq T$ with Lie algebra $\mathfrak{h}$ and a basis $\{\eta_j\}$ for the weight lattice in $\mathfrak{h}^*$ so that*

1. *The group H is the stabilizer and the η_j are the non-zero isotropy weights of every non-exceptional orbit in $\Phi^{-1}(\alpha)$.*
2. *In a neighborhood of α, the image $\Phi(M)$ coincides with the Delzant cone $\alpha + \mathfrak{h}^0 + \sum_j \mathbb{R}_+\eta_j$.*

Proof. Consider the slice representation at any non-exceptional orbit. By Lemma 4.4 above, the stabilizer H is connected. Since the action is effective, the isotropy weights η_j generate the weight lattice. The stabilizer and these weights are determined by the image of the moment map for a local model; this image is the Delzant cone $\mathfrak{h}^0 + \sum_j \mathbb{R}_+\eta_j$. Finally, by the stability of the moment map, the image of the moment map is the same for every local model in $\Phi^{-1}(\alpha)$. □

Example 4.8. Suppose that, near $(0, 0)$, the moment image coincides with the positive quadrant in $\mathbb{R}^2$. Then, at every non-exceptional orbit in $\Phi^{-1}(0, 0)$, the space is locally isomorphic to $(S^1)^2$ acting on $\mathbb{C}^3$ by $(\alpha, \beta) \cdot (z_1, z_2, z_3) = (\alpha z_1, \beta z_2, z_3)$. In contrast, the $(S^1)^2$ action on $\mathbb{C}^3$ given by $(\alpha, \beta) \cdot (z_1, z_2, z_3) = (\alpha z_1, \beta z_2, \alpha\beta z_3)$ has the same moment image, but $(0, 0, 0)$ is an exceptional orbit.

Corollary 4.9. *Over the interior of the moment image, the non-exceptional orbits are precisely the free orbits.*

Lemma 4.10. *Let Y be a local model for a non-exceptional orbit with a moment map $\Phi_Y : Y \longrightarrow \mathfrak{t}^*$. Let W and W' be invariant open subsets of Y. Let $g: W/T \longrightarrow W'/T$ be a diffeomorphism which preserves the moment map. Then g lifts to an equivariant diffeomorphism from W to W'.*

Proof. Assume $W = W' = Y$; the general case is similar. Since, by Lemma 4.4, $Y = T \times_H \mathbb{C}^h \times \mathbb{C} \times \mathfrak{h}^0$, we can identify Y/T with $\mathfrak{h}^0 \times (\mathbb{C}^h/H) \times \mathbb{C}$. Since g preserves the moment map, it necessarily has the form

$$g(\nu, [z], \zeta) = (\nu, [z], \psi(\nu, [z], \zeta)),$$

for some $\psi: \mathfrak{h}^0 \times (\mathbb{C}^h/H) \times \mathbb{C} \longrightarrow \mathbb{C}$. Similarly, the inverse g^{-1} sends $(\nu, [z], \zeta)$ to $(\nu, [z], \gamma(\nu, [z], \zeta))$, where $\gamma: \mathfrak{h}^o \times (\mathbb{C}^h/H) \times \mathbb{C} \longrightarrow \mathbb{C}$. Since both g and its inverse are smooth, ψ and γ must themselves be smooth.

We define $\tilde{g}: Y \longrightarrow Y$ by $\tilde{g}([t, z, \zeta, \nu]) = [t, z, \psi(\nu, [z], \zeta), \nu]$. Then $\tilde{g}$ is a smooth equivariant lift of g, and it has a smooth inverse given by $[t, z, \zeta, \nu] \mapsto [t, z, \gamma(\nu, [z], \zeta), \nu]$. □

From Definition 4.1 and Lemma 4.10 we deduce that a Φ-diffeomorphism lifts to a Φ–T–diffeomorphism locally; we still need to show that, in the proper circumstances, a Φ-diffeomorphism that lifts locally also lifts globally. We do this in the lemma below; the basic idea is that the Duistermaat-Heckman measure determines the "fibration" $M \longrightarrow M/T$.

Lemma 4.11. *Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces that satisfy Condition* (3.2) *and have the same Duistermaat-Heckman measure. Then every homeomorphism from M/T to M'/T that locally lifts to a Φ–T–diffeomorphism also lifts globally to a Φ–T–diffeomorphism from M to M'.*

The proof uses some techniques adapted from Haefliger and Salem, and relies on the following theorem, which is based on a lemma of Schwarz [HS].

Theorem 4.12 ([HS]). *Let a torus T act on a manifold M. Let $h\colon M \longrightarrow M$ be an equivariant diffeomorphism that sends each orbit to itself. Then there exists a smooth invariant function $f\colon M \longrightarrow T$ such that $h(m) = f(m) \cdot m$ for all $m \in M$.*

Proof of Lemma 4.11. Let $T, \mathfrak{t}$, and ℓ denote the sheaves of smooth functions from M/T to T, $\mathfrak{t}$, and ℓ, respectively. Here, ℓ denotes the lattice in $\mathfrak{t}$. Let $\underline{\mathfrak{t}}$ denote the sheaf of locally constant function to $\mathfrak{t}$.

Fix a homeomorphism $\Psi\colon M/T \longrightarrow M'/T$ that lifts locally. Choose a cover $\mathcal{U}$ of M by open invariant sets, and on each $U_i \in \mathcal{U}$ a Φ–T–diffeomorphism $\Psi_i\colon U_i \longrightarrow M'$ that is a lift of Ψ. By Theorem 4.12, there exist smooth invariant functions $g_{ij}\colon U_i \cap U_j \longrightarrow T$ such that $g_{ij} \cdot \Psi_j = \Psi_i$ for all i and j. These functions form a Čech cocycle $g \in \check{\mathrm{C}}^1(\mathcal{U}, T)$. The map Ψ lifts to a global Φ–T–diffeomorphism exactly if the corresponding cohomology class $[g] \in \check{\mathrm{H}}^1(M/T, T)$ is trivial.

The short exact sequence $0 \longrightarrow \ell \longrightarrow \mathfrak{t} \longrightarrow T \longrightarrow 0$ induces a long exact sequence in cohomology. Since there exists a smooth partition of unity on M/T, the cohomology $\check{\mathrm{H}}^i(M/T, \mathfrak{t})$ vanishes for all $i > 0$. Therefore, $\check{\mathrm{H}}^1(M/T, T) = \check{\mathrm{H}}^2(M/T, \ell)$. Condition (3.2) implies that the restriction map $\check{\mathrm{H}}^2(M/T, \ell) \longrightarrow \check{\mathrm{H}}^2(\Sigma, \ell)$ is one-to-one, where $\Sigma = \Phi^{-1}(\alpha)/T$ is a regular symplectic quotient. Therefore, it is enough to show that the image of $[g]$ in $\check{\mathrm{H}}^2(\Sigma, \ell)$ is zero. Since $\check{\mathrm{H}}^2(\Sigma, \ell)$ is torsion free, it is enough to show that the image of $[g]$ in $\check{\mathrm{H}}^2(\Sigma, \underline{\mathfrak{t}})$ vanishes. The Čech-de Rham isomorphism for basic forms on $\Phi^{-1}(\alpha)$ (see Remark 3.4) takes this image to the cohomology class of the basic differential two-form whose restriction to each open set $U_i \cap \Phi^{-1}(\alpha)$ is

$$\pm \sum_j \mathrm{d}\lambda_j g_{ij}^{-1} \mathrm{d}g_{ij}, \tag{4.13}$$

(the sign depending on conventions), where $\{\lambda_i\}$ is a partition of unity subordinate to $\mathcal{U} \cap \Phi^{-1}(\alpha)$. We claim that this is exact as a basic form.

Let Θ' be a connection one-form on ${\Phi'}^{-1}(\alpha) \subset M'$, that is, a T-invariant $\mathfrak{t}$-valued one-form such that $\Theta'(\xi_{M'}) \equiv \xi$ for all $\xi \in \mathfrak{t}$. Then $\Theta = \sum \lambda_i {\Psi_i}^* \Theta'$ is a connection one-form on $\Phi^{-1}(\alpha) \subset M$. The curvature forms $\mathrm{d}\Theta$ and $\mathrm{d}\Theta'$ are basic. Their integrals over the symplectic quotients are equal to the slopes of the Duistermaat-Heckman function of M and of M' at α [DH]. Since these slopes are the same, and since $\Phi^{-1}(\alpha)/T$ is a two dimensional orbifold, the difference between $\mathrm{d}\Theta$ and $\Psi^* \mathrm{d}\Theta'$ is exact as a basic form. A simple computation shows that this difference is equal to (4.13). □

We are finally ready to prove our main proposition.

Proof of Proposition 4.2. By definition, a Φ-diffeomorphism g lifts near exceptional orbits. Let $\mathcal{O}$ be a non-exceptional orbit in M. By Definition 4.1, g sends $\mathcal{O}$ to a non-exceptional orbit $\mathcal{O}'$ in M'. By Lemma 4.7, the local models for $\mathcal{O}$ and $\mathcal{O}'$ are the same. By Lemma 4.10 and the local normal form theorem, g lifts to a Φ–T–diffeomorphism from a neighborhood of $\mathcal{O}$ to a neighborhood of $\mathcal{O}'$. By Lemma 4.11, g lifts globally. □

5. Symplectic representations

By Propositions 3.3 and 4.2, two complexity one spaces M and M' with the same Duistermaat-Heckman measure are equivariantly symplectomorphic if their quotients M/T and M'/T are Φ-diffeomorphic (and Condition (3.2) is satisfied). Therefore, to prove Theorem 1 it is enough to prove that the genus and isotropy data determine these quotients up to Φ-diffeomorphism (and that Condition (3.2) is satisfied), at least over small subsets of $\mathfrak{t}^*$. Sections 5 through 11 are dedicated to proving that this is true.

As a first step, in this section we analyze linear symplectic representations. The key ingredient, which we use repeatedly, is simply the formula for the moment map: let a compact abelian group H act on $\mathbb{C}^n$ as a subgroup of $(S^1)^n$ with weights $\eta_1, \dots, \eta_n$. Then one moment map is

$$\Phi_H(z) = \frac{1}{2} \sum_{j=1}^{n} |z_j|^2 \eta_j. \tag{5.1}$$

Lemma 5.2. *The moment map Φ_H is surjective if and only if there exist $\xi_j > 0$ so that $\sum \xi_j \eta_j = 0$.*

Proof. Suppose that the moment map Φ_H is surjective. Then every element of $\mathfrak{h}^*$ is in the non-negative span of the $\{\eta_j\}$. In particular, there exist $a_j \geq 0$ such that $\sum a_j \eta_j = \sum -\eta_j$, that is, $\sum (1 + a_j)\eta_j = 0$. Let $\xi_j = 1 + a_j$.

Conversely, suppose that there exist positive ξ_j's so that $\sum \xi_j \eta_j = 0$. Let $\alpha \in \mathfrak{h}^*$ be any element. Because the action is effective, the weights η_j span $\mathfrak{h}^*$, so there exist a_j, $j = 1, \dots, n$, such that $\alpha = \sum a_j \eta_j$. Because $\sum \xi_j \eta_j = 0$, we also have $\alpha = \sum (a_j + t\xi_j)\eta_j$ for any $t \in \mathbb{R}$. Because $\xi_j > 0$ for all j, if we take t large enough we get that α is in the positive span of the η_j. $\square$

Lemma 5.3. *The moment map Φ_H is not proper if and only if there exist $\xi_j \geq 0$, not all zero, such that $\sum \xi_j \eta_j = 0$.*

Proof. Suppose that $\sum \xi_j \eta_j = 0$ for some $\xi_j \geq 0$, not all zero. Since the moment fiber $\Phi_H^{-1}(0)$ contains the line $\mathbb{R} \cdot \left(\sqrt{\xi_1}, \dots, \sqrt{\xi_k}\right)$, the map is not proper.

Conversely, suppose that $\sum \xi_i \eta_i \neq 0$ whenever $\xi_i \geq 0$ are not all zero. Then $m = \min\{|\Phi_H(z)|\}_{|z|^2=1}$ is positive. Since Φ_H is quadratic, $|\Phi_H(z)| \geq m|z|^2$ for all z. Hence, Φ_H is proper. $\square$

This analysis distinguishes the two possibilities for non-empty moment fibers:

Lemma 5.4. *Let (M, ω, Φ, U) be a proper Hamiltonian T-manifold. If a moment fiber $\Phi^{-1}(\alpha)$ is not empty, it consists of either*

1. *a single orbit, which has a local model with a proper moment map, or*
2. *infinitely many orbits, each of which has a local model with an improper moment map, that is, a moment map that is not proper.*

If $\alpha \in$ interior$(\Phi(M))$, the second case occurs.

Proof. Consider a local model $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ with moment map

$$\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu. \tag{5.5}$$

By Lemma 5.3 and equations (5.1) and (5.5), the moment map Φ_Y is proper if and only if the moment fiber $\Phi_Y^{-1}(\alpha)$ consists of a single orbit, and otherwise $\Phi_Y^{-1}(\alpha)$

contains infinitely many orbits near $\{[t,0,0]\}$. The lemma now follows from the local normal form theorem and the connectedness of moment fibers. □

Definition 5.6. Let (M,ω,Φ,U) be a complexity one space. A non-empty moment fiber $\Phi^{-1}(\alpha)$, or the symplectic quotient $\Phi^{-1}(\alpha)/T$, is **short** if it consists of a single orbit; otherwise it is **tall**.

We will use the following corollary of Lemma 5.4.

Lemma 5.7. *Let (M,ω,Φ,U) be a proper Hamiltonian T-manifold, and let $\Phi^{-1}(\alpha)$ be a short moment fiber. Then every neighborhood of α contains a smaller neighborhood V such that the quotient $\Phi^{-1}(V)/T$ is contractable.*

Moreover, if the complexity is one, any regular non-empty symplectic quotient $\Phi^{-1}(y)/T$ is homeomorphic to a 2-sphere, for $y \in V$.

Proof. Let $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ be the local model at $x \in \Phi^{-1}(\alpha)$. By Lemma 5.4, the moment map Φ_Y is proper. By Proposition 2.2, the preimages in M and in Y of a sufficiently small neighborhood V of α are isomorphic. Thus, we may work purely inside Y. Choose a neighborhood of α of the form $V = V_1 \times V_2$, where $V_1 \subset \mathfrak{h}^*$ and $V_2 \subset \mathfrak{h}^0$ are convex, and where we identify $\mathfrak{t}^* = \mathfrak{h}^* \times \mathfrak{h}^0$. Because Φ_H is homogeneous, $\Phi_Y^{-1}(V)/T = (\Phi_H^{-1}(V_1)/T) \times V_2$ is contractable.

A regular non-empty symplectic quotient $\Phi^{-1}(y)/T$ is a symplectic orbifold. It can be identified with $\Phi_H^{-1}(y)/H$ and hence admits a residual Hamiltonian action of $(S^1)^n/H$. A two dimensional compact symplectic orbifold with a Hamiltonian circle action is homeomorphic to a 2-sphere by the classification of [LT]. □

For short moment fibers, Theorem 1 follows from Proposition 2.2. Therefore, we may focus on tall moment fibers.

So far, we have been allowing actions of any complexity, but we now restrict to complexity one to define a useful monomial:

Lemma 5.8. *Let an h dimensional compact abelian Lie group H act on $\mathbb{C}^{h+1}$ as a subgroup of $(S^1)^{h+1}$. There exist a (unique up to sign) weight $\xi = (\xi_0, \dots, \xi_h) \in \mathbb{Z}^{h+1}$ such that the following sequence is exact:*

$$1 \longrightarrow H \overset{\rho}{\hookrightarrow} (S^1)^{h+1} \overset{P}{\longrightarrow} S^1 \longrightarrow 1, \tag{5.9}$$

where

$$P(z) = \prod_{j=0}^{h} z_j^{\xi_j}. \tag{5.10}$$

We can choose the exponents ξ_j to all be non-negative if and only if the moment map is improper. We can choose the ξ_j to all be positive if and only if the moment map is surjective.

Proof. Because the quotient $(S^1)^{h+1}/H$ is a one dimensional compact connected Lie group, ξ exists, is non-zero, and is unique up to sign.

Let $\eta_j \in \mathfrak{h}^*$ denote the weights for the H-action on $\mathbb{C}^{h+1}$. Differentiating the identity $P \circ \rho = 1$ from (5.9), we get

$$\sum \xi_j \eta_j = 0. \tag{5.11}$$

The result follows from Lemmas 5.3 and 5.2 and the fact that, up to scalar multiplication, only one vector ξ satisfies (5.11). □

Definition 5.12. We call P the **defining monomial** of the representation of H on $\mathbb{C}^{h+1}$. We also use this name for the map $P\colon \mathbb{C}^{h+1} \longrightarrow \mathbb{C}$ given by the same formula, its extension to the local model $P\colon Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0 \longrightarrow \mathbb{C}$ given by $P([t, z, \nu]) = P(z)$, and the induced map $\overline{P}\colon Y/T \longrightarrow \mathbb{C}$. We trust that this will not cause confusion.

Example 5.13. Let S^1 act on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = \big(\lambda z_1, \lambda^{-1} z_2\big)$. The moment map $\frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)$ is not proper but is surjective. Both exponents of the defining monomial $z_1 z_2$ are positive.

The moment map for the action given by $(\lambda z_1, z_2)$ is $\frac{1}{2}|z_1|^2$, which is neither proper nor surjective, so only one exponent of the defining monomial z_2 is positive.

In contrast, the action given by $(\lambda z_1, \lambda z_2)$ with the proper moment map $\frac{1}{2}\left(|z_1|^2 + |z_2|^2\right)$ does not have a defining monomial with both exponents non-negative.

Let an h dimensional compact abelian Lie group H act on $\mathbb{C}^{h+1}$ as a subgroup of $(S^1)^{h+1}$ with an improper moment map, and let $P(z) = \prod {z_j}^{\xi_j}$ be the defining monomial. We may identify H with the subgroup of $(S^1)^{h+1}$ by which it acts. The stabilizer of $z \in \mathbb{C}^n$ then consists of those elements $\lambda \in (S^1)^{h+1}$ such that $\lambda_j = 1$ whenever $z_j \neq 0$ and such that $P(\lambda) = \prod \lambda_j^{\xi_j} = 1$.

Remark 5.14. This leads immediately to the following geometric interpretation of the defining monomial. For each j, ξ_j is the order of the stabilizer of the jth coordinate hyperplane in $\mathbb{C}^{h+1}$ if this stabilizer is finite, and $\xi_j = 0$ otherwise.

We can also derive a criterion for exceptional orbits.

Lemma 5.15. *Assume that the moment map Φ_H is surjective. The orbit of $z \in \mathbb{C}^{h+1}$ is exceptional unless*

1. *$z_j \neq 0$ for all j, or*
2. *there exists an index i such that $\xi_i = 1$ and $z_j \neq 0$ for all $j \neq i$.*

Proof. Since the moment map is onto, by Corollary 4.9 the H-orbit of z is non-exceptional if and only if the stabilizer of z in H is trivial. ☐

Example 5.16. Let S^1 act on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-2} z_2)$. The defining monomial is $z_1^2 z_2$. The orbit of (z_1, z_2) is exceptional exactly if $z_1 = 0$.

A complexity one linear representation splits into the direct sum of two representations, one whose moment map is surjective, and one which is toric.

Lemma 5.17. *Let an h dimensional compact abelian Lie group H act on $\mathbb{C}^{h+1}$ through an inclusion $\rho\colon H \hookrightarrow (S^1)^{h+1}$ with an improper moment map. After a permutation of the coordinates, there exist splittings*

$$H = H' \times H'' \qquad \textit{and} \qquad \mathbb{C}^{h+1} = \mathbb{C}^{h'+1} \times \mathbb{C}^{h''},$$

such that H' acts on $\mathbb{C}^{h'+1}$ as a subgroup of $(S^1)^{h'+1}$ with a surjective moment map, and H'' acts on $\mathbb{C}^{h''}$ through an isomorphism with $(S^1)^{h''}$. The defining monomial only depends on the $\mathbb{C}^{h'+1}$ coordinates.

Proof. Consider the defining monomial, $P(z) = \prod z_j^{\xi_j}$. We can assume that $\xi_j > 0$ for $0 \leq j \leq h'$ and $\xi_{h'+j} = 0$ for $1 \leq j \leq h''$, where $h' + h'' = h$. Then P defines a monomial $P' \colon (S^1)^{h'+1} \longrightarrow S^1$. Let us identify H with its image in $(S^1)^{h+1}$. Then

$$H = \ker P = \ker P' \times (S^1)^{h''}.$$

Let $H' = \ker P'$ and $H'' = (S^1)^{h''}$. By Lemma 5.8, the moment map for H' is onto. $\square$

6. The topology of the quotient

In this section we describe the topology of the quotient M/T. This will help us to show that two such quotients are Φ-diffeomorphic if they have the same genus and isotropy data. Our main result is the following proposition:

Proposition 6.1. *Let (M, ω, Φ, U) be a complexity one space. The subset of M/T consisting of all tall symplectic quotients is, topologically, a manifold with boundary. Each tall symplectic quotient is, topologically, a closed connected oriented surface.*

The proof uses the following result.

Lemma 6.2. *Let T be a torus. Let a closed h dimensional subgroup $H \subseteq T$ act on $\mathbb{C}^{h+1}$ as a subgroup of $(S^1)^{h+1}$ with an improper moment map and defining monomial $P(z) = \prod z_j^{\xi_j}$. Consider the model $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ and the map $\overline{\Phi}_Y \colon Y/T \longrightarrow \mathfrak{t}^*$ induced by the moment map. The map*

$$F := (\overline{\Phi}_Y, \overline{P}) \colon Y/T \longrightarrow \mathfrak{t}^* \times \mathbb{C}$$

is a homeomorphism of Y/T with its image, $(\operatorname{image} \Phi_Y) \times \mathbb{C}$.

Corollary 6.3. *The restriction of the defining monomial to the symplectic quotient, $\overline{P}_\alpha \colon \Phi_Y^{-1}(\alpha)/T \longrightarrow \mathbb{C}$, is a homeomorphism for all $\alpha \in \operatorname{image} \Phi_Y$.*

Definition 6.4. Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a local model with an improper moment map. The map F of Lemma 6.2 is called the **trivializing homeomorphism** of the model.

Proof of Proposition 6.1. By Lemma 5.4, the local normal form theorem, and Lemma 6.2, the subset of M/T of all tall symplectic quotients is locally homeomorphic to sets of the form $(\operatorname{image} \Phi_Y) \times \mathbb{C}$. By formulas (5.5) and (5.1), the image of Φ_Y is a convex polyhedral cone. Topologically, it is a manifold with boundary.

The fact that each tall symplectic quotient is topological surface follows immediately from Lemma 5.4, the local normal form theorem, and Corollary 6.3. This surface is closed because the moment map is proper. It is connected by the connectedness of moment fibers. The symplectic structure on the symplectic quotient induces an orientation on the complement of a discrete set of points (namely, the exceptional orbits) and hence on the symplectic quotient itself. $\square$

Proof of Lemma 6.2. To show that F is a homeomorphism, it is both necessary and sufficient to prove that the map $(\Phi_H, P) \colon \mathbb{C}^{h+1} \longrightarrow (\operatorname{image} \Phi_H) \times \mathbb{C}$ is proper and surjective and that its fibers are exactly the H-orbits.

We begin by assuming that the moment map Φ_H is onto $\mathfrak{h}^*$. By Lemma 5.8, this implies that the ξ_j's are positive.

Consider the commuting diagram

$$\begin{array}{ccc} \mathbb{C}^{h+1} & \xrightarrow{(\Phi_H, P)} & \mathfrak{h}^* \times \mathbb{C} \\ q_1 \downarrow & & \downarrow q_2 \\ \mathbb{R}_+^{h+1} & \xrightarrow{\overline{F}} & \mathfrak{h}^* \times \mathbb{R}_+, \end{array} \tag{6.5}$$

where

$$q_1(z_0,\dots,z_h)=(|z_0|^2,\dots,|z_h|^2),\qquad q_2(\alpha,\zeta)=(\alpha,|\zeta|^2),$$

and

$$\overline{F}(x_0,\dots,x_h)=(\frac{1}{2}\sum_{j=0}^{h}x_j\eta_j,\prod_{j=0}^{h}x_j^{\xi_j}).$$

Let W be the boundary of the positive orthant $\mathbb{R}_+^{h+1}$. Since $\xi_j>0$ for all j, the map $(x,t)\mapsto x+t\xi$ is a homeomorphism from the product $W\times\mathbb{R}_+$ to the orthant $\mathbb{R}_+^{h+1}$. The map $\overline{F}$, composed with this homeomorphism, becomes a map from $W\times\mathbb{R}_+$ to $\mathfrak{h}^*\times\mathbb{R}_+$, given by the formula

$$(x,t)\mapsto\left(\frac{1}{2}\sum_{j=0}^{h}x_j\eta_j,\prod_{j=0}^{h}(x_j+t\xi_j)^{\xi_j}\right),$$

where in the first coordinate we used the equality $\sum\eta_i(x_i+t\xi_i)=\sum\eta_ix_i$.

The function $x\mapsto\sum x_j\eta_j$ is a homeomorphism from W onto $\mathfrak{h}^*$. For each $x\in W$, the function $t\mapsto\prod(x_j+t\xi_j)^{\xi_j}$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ is strictly monotone, is zero when $t=0$, and approaches infinity uniformly in $x\in W$ as $t\longrightarrow\infty$. Therefore, $\overline{F}$ is one-to-one, onto, and proper.

The properness of (Φ_H,P) follows from that of $\overline{F}$ and q_1.

Let us now show that (Φ_H,P) is onto $\mathfrak{h}^*\times\mathbb{C}$. Since $\overline{F}$ is onto, for any $(\alpha,\zeta)\in\mathfrak{h}^*\times\mathbb{C}$, there exists $z\in\mathbb{C}^{h+1}$ such that $\Phi_H(z)=\alpha$ and $|P(z)|^2=|\zeta|^2$. Choose $b\in S^1$ so that $P(z)=b\zeta$. Since the map $P\colon(S^1)^{h+1}\longrightarrow S^1$ is onto, there exists $a\in(S^1)^{h+1}$ such that $P(a)=b^{-1}$. Then $(\Phi_H,P)(az)=(\alpha,\zeta)$.

Let us now show that the level sets of (Φ_H,P) are the orbits of H. Suppose that $\Phi_H(z)=\Phi_H(z')$ and $P(z)=P(z')$ for some z and z' in $\mathbb{C}^{h+1}$. Since $\overline{F}$ is one to one, there exists $\lambda\in(S^1)^{h+1}$ such that $z'=\lambda z$. We must show that λ can be chosen to be in H. If all the coordinates of z are non-zero, $P(\lambda z)=P(z)$ implies that $P(\lambda)=1$, which further implies that $\lambda\in H$, by (5.9).

If one of the coordinates of z, say z_0, is zero, then it is enough to show that the $(S^1)^{h+1}$-orbit of z coincides with the H-orbit of z. By a dimension count, it is enough to show that the $(S^1)^{h+1}$-stabilizer of z is not contained in H. Because $z_0=0$, the $(S^1)^{h+1}$-stabilizer of z contains the circle $(a_0,1,\dots,1)$. Since $\xi_0\neq0$, the monomial P is not constant on this circle. By exactness of (5.9), this circle is not contained in H.

For the general case, we may let

$$\mathbb{C}^{h+1}=\mathbb{C}^{h'+1}\times\mathbb{C}^{h''}\quad\text{and}\quad H=H'\times H''$$

be the splitting into a surjective part and a toric part, as described in Lemma 5.17. Then $\Phi_H(z,w)=(\Phi_{H'}(z),\Phi_{H''}(w))$. The map $z\mapsto(\Phi_{H'}(z),P(z))$ is proper, its fibers are the H' orbits, and it is onto $(\mathfrak{h}')^*\times\mathbb{C}$, as we have shown above. The map $w\mapsto\Phi_{H''}(w)$ is a moment map for a toric action, so it is proper and its level sets are H'' orbits. Thus, the map

$$(\Phi_H,P)\colon(z,w)\mapsto(\Phi_{H'}(z),\Phi_{H''}(w),P(z))$$

is proper, onto $(\operatorname{image}\Phi_H)\times\mathbb{C}$, and its level sets are the H-orbits. This is precisely what we needed in order to deduce that F is a homeomorphism. $\square$

7. The smooth structure on the quotient

In the previous section, we studied the topology of the quotient M/T near tall fibers. In this section, we study the smooth structure of the quotient.

More specifically, we proved that the trivializing homeomorphism F is in fact a homeomorphism, and concluded that the quotient M/T is a topological manifold with corners. We would like to prove that F is a diffeomorphism, and conclude that the quotient M/T is naturally a smooth manifold with corners.

Unfortunately, near the exceptional orbits, the trivializing homeomorphism F is not a diffeomorphism; however, it is a diffeomorphism on the complement of the exceptional orbits. Consequently, on this subset the quotient M/T is naturally a smooth manifold with corners.

Lemma 7.1. *Let T be a torus. Let a closed h dimensional subgroup H of T act on $\mathbb{C}^{h+1}$ as a subgroup of $(S^1)^{h+1}$ with an improper moment map. Consider the model $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$. Let $E \subset Y$ be the union of the exceptional orbits. Then the restriction of the trivializing homeomorphism*

$$F := (\overline{\Phi}_Y, \overline{P})\colon (Y \smallsetminus E)/T \longrightarrow \mathfrak{t}^* \times \mathbb{C}$$

pulls back the sheaf of smooth functions on $\mathfrak{t}^* \times \mathbb{C}$ onto the sheaf of smooth functions on the quotient.

A similar statement holds for the symplectic quotients:

Corollary 7.2. *The restriction of the defining monomial*

$$\overline{P}_\alpha\colon (\Phi_Y^{-1}(\alpha) \cap (Y \smallsetminus E))/T \longrightarrow \mathbb{C}$$

is a diffeomorphism onto its image.

Before proving Lemma 7.1 in its full generality, we prove the following variant, which implies the lemma for the case that the moment map is onto.

Lemma 7.3. *Let a compact abelian group H act on $\mathbb{C}^{h+1}$ as a codimension one subgroup of $(S^1)^{h+1}$ with a surjective moment map Φ_H. Denote by U the union of the non-exceptional orbits in $\mathbb{C}^{h+1}$. For any manifold N, the map*

$$(id, \overline{\Phi}_H, \overline{P})\colon N \times (U/H) \longrightarrow N \times \mathfrak{h}^* \times \mathbb{C}, \tag{7.4}$$

given by

$$(n, [z]) \mapsto (n, \Phi_H(z), P(z)),$$

is a diffeomorphism with its image, i.e., it pulls back the sheaf of smooth functions on $N \times \mathfrak{h}^* \times \mathbb{C}$ onto the sheaf of smooth functions on the quotient.

Proof. Since H acts freely on U, the quotient U/H is naturally a smooth manifold. The map (7.4) is smooth, and, by Lemma 6.2, it is a homeomorphism onto its image. Since a smooth homeomorphism between two smooth manifolds of the same dimension is a diffeomorphism exactly at those points where it is a submersion, it is enough to show that $(\mathrm{d}\Phi_H, \mathrm{d}P)|_z$ is onto for all non-exceptional $z \in \mathbb{C}^{h+1}$.

To show that $(\mathrm{d}\Phi_H, \mathrm{d}P)|_z$ is onto, it is enough to find $\zeta \in T_z\mathbb{C}^n = \mathbb{C}^n$ such that $\mathrm{d}\Phi_H|_z(\zeta) = \mathrm{d}\Phi_H(\sqrt{-1}\zeta) = 0$ and $\mathrm{d}P|_z(\zeta) \neq 0$. To see this, note that, since H acts freely, $\mathrm{d}\Phi_H|_z$ is onto $\mathfrak{h}^*$. Additionally, since P is holomorphic, $\mathrm{d}P|_z(\zeta)$ and $\mathrm{d}P|_z(\sqrt{-1}\zeta) = \sqrt{-1}\mathrm{d}P|_z(\zeta)$ form a real basis to $\mathbb{C}$.

Recall that $P(z)=\prod z_j^{\xi_j}$ and $\Phi_H(z)=\frac{1}{2}\sum_j \eta_j z_j\overline{z}_j$. Hence

$$\mathrm{d}\Phi_H|_z(\zeta)=\frac{1}{2}\sum_j \eta_j(z_j\overline{\zeta}_j+\zeta_j\overline{z}_j).$$

By Lemma 5.15, we only need to consider the following two subcases.

Subcase A: *all the coordinates of z are non-zero.* In this case,

$$\mathrm{d}P|_z\,(\zeta)=P(z)\sum_j \frac{\xi_j}{z_j}\zeta_j.$$

Let $\zeta_j=\dfrac{\xi_j}{\overline{z}_j}$ for $1\le j\le n$. Then

$$\begin{aligned}\mathrm{d}\Phi_H|_z(\zeta) &= \frac{1}{2}\sum \eta_j\left(z_j\frac{\xi_j}{z_j}+\frac{\xi_j}{\overline{z}_j}\overline{z}_j\right)\\ &= \frac{1}{2}\sum \eta_j\left(\xi_j+\xi_j\right)=0\end{aligned}$$

by (5.11), and

$$\begin{aligned}\mathrm{d}\Phi_H(\sqrt{-1}\zeta) &= \frac{1}{2}\sum \eta_j\left(z_j\left(-\sqrt{-1}\ \frac{\xi_j}{z_j}\right)+\sqrt{-1}\ \frac{\xi_j}{\overline{z}_j}\overline{z}_j\right)\\ &= \frac{1}{2}\sum \eta_j(-\sqrt{-1}\ \xi_j+\sqrt{-1}\ \xi_j)=0,\end{aligned}$$

whereas

$$\mathrm{d}P|_z(\zeta)=P(z)\sum_j \frac{\xi_j}{z_j}\frac{\xi_j}{\overline{z}_j}\neq 0.$$

Subcase B: *one of the coordinates of z, say, z_1, is zero, $\xi_1=1$, and $z_j\neq 0$ for all $j\neq 1$.* In this case,

$$\mathrm{d}P|_z(\zeta)=\left(\prod_{j\neq 1} z_j^{\xi_j}\right)\zeta_1.$$

Let $\zeta_1=1$ and $\zeta_j=0$ for all $j\neq 1$. Then

$$\mathrm{d}\Phi_H|_z(\zeta)=\frac{1}{2}\eta_1(z_1+\overline{z}_1)=0,$$

and

$$\mathrm{d}\Phi_H|_z(\sqrt{-1}\zeta)=\frac{1}{2}\eta_1(-\sqrt{-1}\ z_1+\sqrt{-1}\ \overline{z}_1)=0,$$

whereas

$$\mathrm{d}P|_z(\zeta)=\left(\prod_{j\neq 1} z_j^{\xi_j}\right)\neq 0.$$

□

Proof of Lemma 7.1. Let $\mathbb{C}^{h+1}=\mathbb{C}^{h'+1}\times\mathbb{C}^{h''}$ and $H=H'\times H''$ be the splitting into a surjective part and a toric part, as described in Lemma 5.17. With this splitting, the local model is

$$Y=T\times_{H'}\mathbb{C}^{h'+1}\times_{H''}\mathbb{C}^{h''}\times\mathfrak{h}^0,$$

and its quotient is

$$Y/T = (\mathbb{C}^{h'+1}/H') \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0.$$

The union of the non-exceptional orbits in this quotient is

$$(U'/H') \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0, \tag{7.5}$$

where U' is the union of the free orbits in $\mathbb{C}^{h'+1}$. Under the identification $\mathfrak{t}^* = (\mathfrak{h}')^* \times (\mathfrak{h}'')^* \times \mathfrak{h}^0$, the trivializing homeomorphism F on (7.5) is

$$F([z'], [z''], \nu) = (\Phi_{H'}(z'), \Phi_{H''}(z''), \nu, P(z')),$$

where P is the defining monomial.

Lemma 7.3 implies that the map

$$([z'], [z''], \nu) \mapsto (\Phi_{H'}(z'), [z''], \nu, P(z'))$$

pulls back the sheaf of smooth functions on $(\mathfrak{h}')^* \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0 \times \mathbb{C}$ onto the sheaf of smooth functions on $(U'/H') \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0$. Therefore, it is enough to show that the map

$$(\alpha, [z''], \nu, \zeta) \mapsto (\alpha, \Phi_{H''}(z''), \nu, \zeta) \tag{7.6}$$

pulls back the sheaf of smooth functions on $(\mathfrak{h}')^* \times (\mathfrak{h}'')^* \times \mathfrak{h}^0 \times \mathbb{C}$ onto the sheaf of smooth functions on $(\mathfrak{h}')^* \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0 \times \mathbb{C}$.

By a theorem of Schwartz [Sch1], any invariant smooth function can be expressed as a smooth function of real invariant polynomials. Since H'' acts on $\mathbb{C}^{h''}$ through an isomorphism with $(S^1)^{h''}$, the ring of H''-invariant polynomials in $(\alpha, z'', \nu, \zeta)$ is generated by the coordinates of α and ν, the real and imaginary parts of ζ, and $|z_1|^2, \dots, |z_{h''}|^2$. Finally, note that

$$\Phi_{H''}(z_1, \dots, z_{h''}) = A(|z_1|^2, \dots, |z_{h''}|^2),$$

where $A \colon \mathbb{R}^{h''} \longrightarrow (\mathfrak{h}'')^*$ is a linear isomorphism. Hence, every smooth invariant function is the pullback via (7.6) of a smooth function. □

8. Grommets

All we have left to show is that the genus and isotropy data determine M/T up to Φ-diffeomorphism. Roughly speaking, if the defining homeomorphism were a diffeomorphism, we could finish the proof in four paragraphs:

> Assume (usually falsely) that the defining homeomorphism is a diffeomorphism. Then M/T is a manifold with corners and the map $\overline{\Phi}$ induced by the moment map is a submersion, in the sense that M/T is locally diffeomorphic to $\Phi(M) \times \mathbb{C}$.
>
> Because $\overline{\Phi}$ is a proper submersion, there is a diffeomorphism from $\Phi^{-1}(V)/T$ to $(\Phi(M) \cap V) \times (\Phi^{-1}(\alpha)/T)$ for some neighborhood V of α. This diffeomorphism can be chosen to take exceptional orbits to exceptional orbits.
>
> The symplectic quotient $\Sigma = \Phi^{-1}(\alpha)/T$ is an oriented 2-manifold, with marked points given by the exceptional orbits. The genus and isotropy data determine Σ up to a diffeomorphism which takes marked points to marked points with the same isotropy data.
>
> Therefore, they determine $\Phi^{-1}(V)/T$ up to Φ-diffeomorphism.

Unfortunately, the above argument fails near the exceptional orbits. Nevertheless, on the complement of the exceptional orbits we will follow this argument precisely. Near the exceptional orbits, we will follow the same basic pattern, but we will not demand that our maps be diffeomorphisms. Instead, we will fix an identification with our local model near each exceptional orbit, and use this to restrict to a well behaved class of maps.

Definition 8.1. Let (M, ω, Φ, U) be a complexity one space. A **grommet** is a Φ–T–diffeomorphism $\psi\colon D \longrightarrow M$ from an invariant open subset D of a local model $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ onto an open subset of M.[5]

We think of this as attaching a grommet to the fabric of the manifold at every point where the orbit is exceptional. For an ordinary grommet,[6] the fabric can flow however it wants away from the grommet, but at the grommet it can only spin. Grommets are designed to allow all the necessary freedom of movement but prevent the fabric from ripping at the points of stress.

Similarly, in Sections 9, 10, 11, and subsequent papers we consider maps that are arbitrary diffeomorphisms away from the grommets but are "rigid" near the grommets. This gives enough freedom so that we can approximate any map, but we don't need to really understand what happens at the difficult points where the orbits are exceptional.

We will need grommets that are sufficiently large, in the following sense:

Definition 8.2. Let Y be a local model with an improper moment map. The **exceptional sheet** is the subset

$$S = \left\{ [t, z, \nu] \in T \times_H \mathbb{C}^n \times \mathfrak{h}^0 \mid P(z) = 0 \right\}.$$

Not every orbit in S is exceptional. However, by Lemmas 5.15 and 5.17, every exceptional orbit is contained in S.

Definition 8.3. Let (M, ω, Φ, U) be a complexity one space. Let $\psi\colon D \longrightarrow M$ be a grommet whose domain D is a subset of a local model Y with an improper moment map Φ_Y. The grommet is **wide** if D contains that part of the exceptional sheet that lies over U, i.e., $(\Phi_Y^{-1}(U) \cap S) \subset D$.

Locally, we will always be able to find such grommets:

Lemma 8.4. *Let (M, ω, Φ, U) be a complexity one space, and let $\Phi^{-1}(\alpha)$ be a tall moment fiber. Denote the exceptional orbits in $\Phi^{-1}(\alpha)$ by $\{E_j\}$.*

After replacing M by the preimage of some neighborhood of α in U, there exist wide grommets $\psi_j : D_j \longrightarrow M$ such that $\psi_j(\{[t, 0, 0]\}) = E_j$ and the images $\psi_j(D_j)$ have pairwise disjoint closures.

Proof. By Lemma 5.4, for every exceptional orbit E_j over α, the corresponding local model has an improper moment map, $\Phi_j : Y_j \longrightarrow \mathfrak{t}^*$. By the local normal form theorem, we may choose a grommet $\psi_j : D_j \longrightarrow M$ such that $\psi_j(\{[t, 0, 0]\}) = E_j$.

By Lemma 6.2 and Definition 8.2, the moment map Φ_j restricts to a homeomorphism of the exceptional sheet $S_j \subset Y_j$ with the image of Φ_j. Hence there exists a neighborhood W_j of α such that $S_j \cap D_j = S_j \cap \Phi_j^{-1}(W_j)$.

[5] The domain D need not contain the orbit $\{[t, 0, 0]\}$.

[6] Grommet: **1:** a flexible loop that serves as a fastening, support, or reinforcement. **2:** an eyelet of firm material to strengthen or protect an opening or to insulate or protect something passed through it. (See [MW].)

Let $W = \bigcap W_j$, and replace M by $M \cap \Phi^{-1}(W)$ and D_j by $D_j \cap \Phi_j^{-1}(W)$. Then, the grommets ψ_j are wide and the exceptional sheets $\psi_j(S_j \cap D_j)$ are closed.

For $i \neq j$, the intersection $\psi_i(S_i \cap D_i) \cap \psi_j(S_j \cap D_j)$ is a closed subset of M which does not meet the moment fiber $\Phi^{-1}(\alpha)$. Since the moment map is proper, there exists a neighborhood $V \subset W$ of α which does not meet the image under the moment map of any of these intersections.

We now replace M by $M \cap \Phi^{-1}(V)$ and D_j by $D_j \cap \Phi_j^{-1}(V)$. The grommets ψ_j are still wide. Also, the exceptional sheets $\psi_j(S_j \cap D_j)$ are then closed and disjoint, so we can shrink each D_j to a smaller neighborhood of $S_j \cap D_j$ to obtain wide grommets whose images have pairwise disjoint closures. □

9. Flattening the quotient

Following the pattern explained in the beginning of the previous section, the ideal next step would be to find a diffeomorphism from the quotient $\Phi^{-1}(V)/T$ to the product $(\Phi(M) \cap V) \times (\Phi^{-1}(\alpha)/T)$ for some neighborhood V of α. Instead, we now find a *homeomorphism* between these spaces. Away from the exceptional orbits, this homeomorphism will be a diffeomorphism; near the exceptional orbits, it will be determined by the grommets. Such a homeomorphism is called a flattening; a precise definition is given below. The main result of this section is that flattenings always exist locally. Our proof is a modification of the standard proof that a proper submersion is a fiber bundle.

We begin by defining a flattening for a local model. Let T be a torus, and let a closed subgroup $H \subseteq T$ act on $\mathbb{C}^n$ as a codimension one subgroup of $(S^1)^n$ with an improper moment map. Consider the model $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$, with moment map $\Phi_Y : Y \longrightarrow \mathfrak{t}^*$.

Definition 9.1. The **standard flattening** of Y is the map

$$\delta \colon Y/T \longrightarrow (\operatorname{image} \Phi_Y) \times \left(\Phi_Y^{-1}(\alpha)/T\right)$$

given by

$$\delta := \left(\overline{\Phi}_Y, \left(\overline{P}_\alpha\right)^{-1} \circ \overline{P}\right).$$

The standard flattening δ is well-defined and is a homeomorphism by Lemma 6.2 and Corollary 6.3. By Lemma 7.1 and Corollary 7.2, it is also a diffeomorphism of $(Y \smallsetminus S)/T$ with its image. Here S is the exceptional sheet; see Definition 8.2.

We are now ready to define the flattening of a complexity one space.

Definition 9.2. Let (M, ω, Φ, U) be a complexity one space, and let $\Phi^{-1}(\alpha)$ be a tall moment fiber. A **flattening** of the space about α consists of the following data:

1. A homeomorphism

$$\delta \colon M/T \longrightarrow (\operatorname{image} \Phi) \times (\Phi^{-1}(\alpha)/T) \tag{9.3}$$

   whose first coordinate is induced by the moment map.
2. For each exceptional orbit E_j in $\Phi^{-1}(\alpha)$, a wide grommet $\psi_j \colon D_j \longrightarrow M$ such that $\psi_j(\{[t, 0, 0]\}) = E_j$.

We require that the following two conditions be satisfied:

1. The restriction of δ to the complement of the exceptional sheets,
$$\delta\colon M/T \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j)/T \longrightarrow (\operatorname{image}\Phi) \times (\Phi^{-1}(\alpha) \smallsetminus \bigsqcup_j E_j)/T,$$
must be a diffeomorphism, in the sense discussed in the second paragraph of section 4.
2. Additionally, near the exceptional sheets δ must be given by the standard flattenings of the local models. More precisely, the following diagram must commute:

$$\begin{array}{ccc} D_j/T & \xrightarrow{\delta_j} & \mathfrak{t}^* \times ((\Phi_j^{-1}(\alpha) \cap D_j)/T) \\ \downarrow \overline{\psi}_j & & \downarrow (\mathrm{id}, \overline{\psi}_j) \\ M/T & \xrightarrow{\delta} & \mathfrak{t}^* \times (\Phi^{-1}(\alpha)/T), \end{array} \tag{9.4}$$

where Φ_j is the moment map on the corresponding local model $Y_j \supset D_j$, where $\delta_j\colon Y_j/T \longrightarrow (\operatorname{image}\Phi_j) \times (\Phi_j^{-1}(\alpha)/T)$ denotes the standard flattening of Y_j, and where $\overline{\psi}_j\colon D_j/T \longrightarrow M/T$ is induced by the grommet. In particular, we require that the diagram be well defined, i.e., that the image $\delta_j(D_j/T)$ be contained in $\mathfrak{t}^* \times ((\Phi_j^{-1}(\alpha) \cap D_j)/T)$.

The main result of this section is that flattenings always exist locally:

Proposition 9.5. *Let (M, ω, Φ, U) be a complexity one space, and let $\Phi^{-1}(\alpha)$ be a tall moment fiber. Then there exists a neighborhood V of α contained in U whose preimage, $\Phi^{-1}(V)$, admits a flattening about α.*

We will prove this proposition after giving a few corollaries.

Corollary 9.6. *Let (M, ω, Φ, U) be a complexity one space. Let M_0 be the union of the tall moment fibers and $\Delta_0 = \Phi(M_0)$. Then the map $M_0/T \longrightarrow \Delta_0$ induced by the moment map is, topologically, a surface bundle: each $\alpha \in \Delta_0$ has a neighborhood V whose preimage is homeomorphic to $(V \cap \Delta_0) \times \Sigma$, where Σ is a surface, in a manner that respects the moment map.*

This surface bundle plays an important role in the global classification of complexity one spaces, which will be given in subsequent papers.

Since the set Δ_0 of points in U whose moment fiber is tall is connected by Lemma 5.4, we have the following result:

Corollary 9.7. *Let (M, ω, Φ, U) be a complexity one space. Then all the tall symplectic quotients $\Phi^{-1}(\alpha)/T$ have the same genus. Thus, the genus of a complexity one space (see section 1) is well-defined.*

Remark 9.8. This corollary was first proved by Dusa McDuff [McD], for $\dim M = 4$. It follows from the fact that, as one crosses a singular value of the moment map, the symplectic quotients change by blowups and blowdowns: in real dimension 2, blowups and blowdowns don't change the topology. See [GS2] for the case of quasi-free actions. A generalization of Corollary 9.7, to arithmetic genus, appears in [MS].

Since $\Phi(M)$ is convex, we also have the following result.

Corollary 9.9. *Let (M,ω,Φ,U) be a complexity one space, and let $\Phi^{-1}(\alpha)$ be a tall moment fiber. Then for every sufficiently small convex neighborhood V of α, the restriction map*

$$H^*(\Phi^{-1}(V)/T) \longrightarrow H^*(\Phi^{-1}(y)/T)$$

is an isomorphism for all $y \in V \cap \Phi(M)$. In particular, $\Phi^{-1}(V)$ satisfies Condition (3.2).

Proof of Proposition 9.5. Let $\psi_j \colon D_j \longrightarrow M$ be wide grommets such that

$$\psi_j(\{[t,0,0]\}) = E_j$$

are the exceptional orbits in the moment fiber $\Phi^{-1}(\alpha)$ and such that the images $\psi_j(D_j)$ have disjoint closures in M. These grommets exist by Lemma 8.4. This is not ruined when we further restrict to a smaller neighborhood of α.

Recall that the standard flattening of the local model Y_j is

$$\delta_j = (\overline{\Phi}_j, g_j) \colon Y_j/T \longrightarrow (\operatorname{image}\Phi_j) \times (\Phi_j^{-1}(\alpha)/T)$$

where

$$g_j = (\overline{P}_{j,\alpha})^{-1} \circ \overline{P}_j.$$

Replace D_j/T by its intersection with $g_j^{-1}((\Phi_j^{-1}(\alpha) \cap D_j)/T)$. Then the restriction

$$\delta_j \colon D_j/T \longrightarrow (\operatorname{image}\Phi_j) \times ((\Phi_j^{-1}(\alpha) \cap D_j)/T)$$

is well defined. After this, the grommets determine a unique map δ on the images of D_j/T in M/T such that the following diagram commutes:

$$\begin{array}{ccc} D_j/T & \xrightarrow{\delta_j} & \mathfrak{t}^* \times ((\Phi_j^{-1}(\alpha) \cap D_j)/T) \\ \downarrow \overline{\psi}_j & & \downarrow (\mathrm{id}, \overline{\psi}_j) \\ \bigsqcup_j \overline{\psi}_j(D_j/T) & \xrightarrow{\delta} & \mathfrak{t}^* \times (\Phi^{-1}(\alpha)/T). \end{array} \tag{9.10}$$

We need to extend δ to the rest of M/T, perhaps after shrinking the D_j's to smaller neighborhoods of $S_j \cap \Phi_j^{-1}(U)$.

Using the stability of the moment map, Lemma 7.1 implies that on the complement of the exceptional sheets in the quotient M/T, the map

$$\overline{\Phi} \colon M/T \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j)/T \longrightarrow (\operatorname{image}\Phi)$$

induced by the moment map is a submersion. Namely, for each point $[m]$ in the domain of this map there exists a neighborhood W of $\Phi(m)$ in $\mathfrak{t}^*$ such that a neighborhood of $[m]$ is diffeomorphic to the product of a disk with $W \cap (\operatorname{image}\Phi)$ with the map $\overline{\Phi}$ being the projection map.

The partial flattening (9.10) determines an Ehresmann connection for this submersion, defined on the open subset $\bigsqcup_j \psi_j(D_j \smallsetminus S_j)/T$: we declare the horizontal tangent vectors to be those whose push-forward by δ is tangent to the sheets $\{q\} \times \mathfrak{t}^*$ for $q \in \Phi^{-1}(\alpha)/T$.

We extend this to an Ehresmann connection on the entire complement of the exceptional sheets, $M/T \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j)/T$, perhaps after shrinking the D_j's; this is easily done with a partition of unity. Then for a point p in

$$M/T \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j)/T,$$

any path γ in U which starts at $\overline{\Phi}(p)$ can be lifted to a horizontal path in $M/T \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j)/T$.

We proceed as in the proof of Ehresmann's lemma. Let us assume that $\alpha = 0$ and that U is a ball centered at 0. We can choose coordinates on $\mathfrak{t}^*$ such that the image of Φ becomes

$$(\operatorname{image}\Phi) = U \cap (\mathbb{R}^k \times \mathbb{R}^l_+) \ , \qquad k + l = m = \dim \mathfrak{t}^*.$$

(This is possible by Lemma 4.7.) Denote by $v_1, \dots, v_m$ the standard vector fields on $\mathfrak{t}^*$ that are parallel to the coordinate axes, let $\tilde{v}_1, \dots, \tilde{v}_m$ be their horizontal liftings to $M/T \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j)/T$, and let f^t_j, for $i = 1, \dots, m$ and $t \in \mathbb{R}$, be the flows which the $\tilde{v}_i$ generate. For $p \in M/T$, define

$$\delta(p) = (\overline{\Phi}(p), g(p))$$

where, if $(t_1, \dots, t_m)$ are the coordinates of p, then $g(p) \in \Phi^{-1}(\alpha)/T$ is given by $g(p) = f_1^{-t_1} \cdots f_m^{-t_m}(p)$. □

10. The associated surface

Let us recall the next step from the pattern that we outlined in the beginning of section 8. We would like to claim that the symplectic quotient $\Sigma = \Phi^{-1}(\alpha)/T$ is a smooth compact oriented surface, and hence argue that it is determined up to diffeomorphism by its genus and isotropy data. On the complement of the exceptional orbits, Σ is naturally a smooth oriented surface. Near the exceptional orbits, it is not. However, we can give it a smooth structure using the grommets on M. We can then show that it is still determined by its genus and isotropy data, up to a map which is a diffeomorphism away from the marked points, and strictly determined by the grommets near them.

On a surface, a grommet is simply a choice of a coordinate chart for a marked point. We give it this name to remind ourselves that we are going to fix it once and for all and then use it to restrict to maps which behave nicely near the marked point.

Definition 10.1. Let Σ be a smooth oriented two dimensional manifold. A **grommet** at a point $q \in \Sigma$ is a diffeomorphism $\varphi\colon B \longrightarrow \Sigma$ from a neighborhood B of the origin in $\mathbb{C}$ onto an open subset of Σ, such that φ sends the origin 0 to the point q.

Definition 10.2. Let Σ and Σ' be closed oriented surfaces with labeled marked points and with grommets at these points. (See Definition 10.1.) An orientation preserving diffeomorphism $g\colon \Sigma \longrightarrow \Sigma'$ is **rigid** if

- it induces a bijection between the marked points in Σ and those in Σ', and sends each marked point to a marked point with the same label;
- for each marked point $q_j \in \Sigma$ and $q'_j \in \Sigma'$ and corresponding grommets φ_j and φ'_j, the composition ${\varphi'_j}^{-1} \circ g \circ \varphi_j$ coincides with a rotation of $\mathbb{C}$ on some neighborhood of 0.

The following result is standard in differential topology; see, e.g., [Ko, II, 5.2].

Lemma 10.3. *Let Σ and Σ' be closed oriented surfaces with labeled marked points and with grommets at these points. Suppose that Σ and Σ' have the same genus. Then any bijection from the marked points in Σ to the marked points in Σ' which* *sends each marked point to a marked point with the same label extends to a rigid map from Σ to Σ'.*

Remark 10.4. Let (M, ω, Φ, U) be a complexity one space, and let $\Phi^{-1}(\alpha)$ be a tall moment fiber. Any grommet $\psi\colon D \longrightarrow M$ with $\psi(\{[t,0,0]\}) = \mathcal{O} \subset \Phi^{-1}(\alpha)$ induces a homeomorphism φ from a subset $B \subset \mathbb{C}$ into the symplectic quotient $\Phi^{-1}(\alpha)/T$, such that $\varphi(0) = \mathcal{O}/T$. Explicitly, the map $\overline{P}_\alpha\colon (D \cap \Phi_Y^{-1}(\alpha))/T \longrightarrow \mathbb{C}$ given by the defining monomial is a homeomorphism onto its image B, and $\varphi := \overline{\psi} \circ \overline{P}_\alpha^{-1}\colon B \longrightarrow \Phi^{-1}(\alpha)/T$ is a homeomorphism onto its image, where $\overline{\psi}$ is induced from ψ.

Definition 10.5. Let (M, ω, Φ, U) be a complexity one space, and let $\Phi^{-1}(\alpha)$ be a tall moment fiber. For each exceptional orbit E_j in $\Phi^{-1}(\alpha)$, let $\psi_j\colon D_j \longrightarrow M$ be a grommet such that $\psi_j(\{[t,0,0]\}) = E_j$. The **associated marked surface** consists of the following data:

1. The connected oriented topological surface $\Sigma = \Phi^{-1}(\alpha)/T$.
2. The set of marked points $\{q_j\}$ in Σ that corresponds to the set of exceptional orbits $\{E_j\}$ in $\Phi^{-1}(\alpha)$.
3. The smooth manifold structure on Σ that is given by the following coordinate charts. For each exceptional orbit E_j in $\Phi^{-1}(\alpha)$, take the given grommet. For each non-exceptional orbit $\mathcal{O}$ in $\Phi^{-1}(\alpha)$, choose an arbitrary grommet with $\psi(\{[t,0,0]\}) = \mathcal{O}$. For each grommet, take the induced coordinate chart on Σ as described in Remark 10.4.
4. At each marked point q_j, the grommet on Σ that is given by the coordinate chart of item (3).
5. For each marked point, a label consisting of the isotropy representation at the corresponding exceptional orbit.

The image of the grommet ψ_j does not contain any of the other exceptional orbits, E_i, $i \neq j$; this follows from the fact that the model contains at most one exceptional orbit in each moment fiber (see Lemma 5.15). The fact that the charts in item (3) give a well defined smooth structure on M/T follows from this and from the fact that the smooth structures coincide on this complement (see Corollary 7.2).

Example 10.6. For the complexity one space given in Example 4.3, the associated marked surface is the original surface Σ.

Example 10.7. Let S^1 act on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^2 z_2)$. The associated marked surface is S^2 with one point labeled by the standard non-trivial action of $\mathbb{Z}^2$ on $\mathbb{C}$, and a coordinate chart around that point.

11. Diffeomorphism between quotients

In this section, we give the last step in our argument. In the pattern outlined at the beginning of section 8, we argued that $\Phi^{-1}(V)/T$ is Φ-diffeomorphic to $\Sigma \times (\Phi(M) \cap V)$, and Σ is determined up to diffeomorphism by its genus and isotropy data, therefore, $\Phi^{-1}(V)/T$ must itself be determined up to Φ-diffeomorphism by the same data. We must adapt this argument to the fact that our maps are diffeomorphisms only away from the marked points; near the marked points, they behave well with respect to the grommets. More precisely, we show the following result.

Proposition 11.1. *Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces equipped with flattenings about a point $\alpha \in U$. Let Σ and Σ' be the associated* *marked surfaces. Then any rigid map* $h\colon \Sigma \longrightarrow \Sigma'$ *extends to a* Φ*-diffeomorphism* $g\colon M/T \longrightarrow M'/T$.

Proof. If there exists a rigid map $h\colon \Sigma \longrightarrow \Sigma'$, the labels on Σ and on Σ' are the same, hence, the isotropy data over α are the same in M and M'. See Definitions 10.2 and 10.5. By Lemma 4.7, $\operatorname{image}\Phi = \operatorname{image}\Phi'$.

Identifying the symplectic quotients $\Phi^{-1}(\alpha)/T$ and ${\Phi'}^{-1}(\alpha)/T$ with Σ and Σ', respectively, the maps given in the flattenings become

$$\delta\colon M/T \longrightarrow (\operatorname{image}\Phi) \times \Sigma$$

and

$$\delta'\colon M'/T \longrightarrow (\operatorname{image}\Phi) \times \Sigma'.$$

We will show that the map $g\colon M/T \longrightarrow M'/T$ defined by

$$g := {\delta'}^{-1} \circ (\mathrm{id}, h) \circ \delta$$

is a Φ-diffeomorphism.

The diffeomorphism h determines an identification between exceptional orbits in $\Phi^{-1}(\alpha)$ and ${\Phi'}^{-1}(\alpha)$ with the same isotropy representation. Thus, we can unequivocally denote by $\{Y_j\}$ the local models for the exceptional orbits over α in both M and M'. Let

$$\psi_j\colon D_j \longrightarrow M \quad \text{and} \quad \psi'_j\colon D'_j \longrightarrow M'$$

denote the grommets, with $D_j \subseteq Y_j$ and $D'_j \subseteq Y_j$, and let E_j and E'_j denote the exceptional orbits in $\Phi^{-1}(\alpha)$ and in ${\Phi'}^{-1}(\alpha)$.

Our first claim is that the restriction

$$g\colon \left(M \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j) \right) /T \longrightarrow \left(M' \smallsetminus \bigsqcup_j \psi'_j(S_j \cap D_j) \right) /T$$

is a Φ-diffeomorphism. This is easy: by the definition of flattening, the restrictions

$$\delta\colon \left(M \smallsetminus \bigsqcup_j \psi_j(S_j \cap D_j) \right) /T \longrightarrow (\operatorname{image}\Phi) \times \left(\Phi^{-1}(\alpha) \smallsetminus \bigsqcup_j E_j \right) /T$$

and

$$\delta'\colon \left(M' \smallsetminus \bigsqcup_j \psi'_j(S_j \cap D_j) \right) /T \longrightarrow (\operatorname{image}\Phi') \times \left({\Phi'}^{-1}(\alpha) \smallsetminus \bigsqcup_j E'_j \right) /T$$

are both diffeomorphisms. Moreover, the map

$$\left(\Phi^{-1}(\alpha) \smallsetminus \bigsqcup_j E_j \right) /T \longrightarrow \left({\Phi'}^{-1}(\alpha) \smallsetminus \bigsqcup_j E'_j \right) /T$$

induced by h is a diffeomorphism, since the smooth structures on $\Phi^{-1}(\alpha)/T$ and Σ agree for the exceptional orbits.

It remains to show that g is a Φ-diffeomorphism in a neighborhood of each exceptional sheet $\psi_j(S_j \cap D_j)/T$.

Let $\varphi_j \colon B_j \longrightarrow \Sigma$ and $\varphi'_j \colon B'_j \longrightarrow \Sigma'$ denote the grommets of the associated surfaces. Since h is rigid, there exist $a_j \in S^1$ such that $\varphi_j^{-1} \circ h \circ \varphi_j$ is given by rotation by $a_j \in S^1$ on some neighborhood of the origin in $\mathbb{C}$.

Let $P_j \colon (S^1)^{n_j} \longrightarrow S^1$ be the defining monomial for the exceptional orbit E_j. Since P_j is surjective, we may choose $\lambda_j \in (S^1)^{n_j}$ so that $P_j(\lambda_j) = a_j$. This defines an equivariant symplectomorphism from the local model $Y_j = T \times_{H_j} \mathbb{C}^{n_j} \times \mathfrak{h}_j^0$ to itself as follows:

$$\lambda_j \cdot ([t, z, \nu]) = [t, \lambda_j \cdot z, \nu]. \tag{11.2}$$

This map induces a Φ-diffeomorphism on the quotient, $g_j \colon Y_j/T \longrightarrow Y_j/T$. It remains to show only that the g_j and g agree in some neighborhood of $\psi_j(D_j \cap S_j)$. Indeed, when we use the trivializing homeomorphism F_j to identify Y_j with $(\operatorname{image} \Phi_j) \times \mathbb{C}$, the map g_j sends (β, z) to $(\beta, a_j z)$. □

12. Proof of the Local Uniqueness Theorem

We now have all the ingredients to prove Theorem 1. We recall the statement:

Theorem 1. *Let (M, Φ, ω, U) and (M', Φ', ω', U) be complexity one spaces. Assume that their Duistermaat-Heckman measures are the same, and that their genus and isotropy data over a point $\alpha \in \mathfrak{t}^*$ are the same. Then there exists a neighborhood of the point α over which the spaces are isomorphic.*

Proof. Since the case that the moment fiber $\Phi^{-1}(\alpha)$ is short is covered by Proposition 2.2, we may assume that the moment fiber is tall.

By Proposition 9.5, after possibly restricting to the preimage of a small neighborhood of α, we may assume that M and M' are equipped with flattenings. By assumption, the spaces M and M' have the same genus and isotropy data, hence the associated marked surfaces, Σ and Σ', have the same genus and labels. By Lemma 10.3, there exists a rigid map $h \colon \Sigma \longrightarrow \Sigma'$. By Proposition 11.1, there exists a Φ-diffeomorphism $g \colon M/T \longrightarrow M'/T$.

Since the spaces have flattenings, Condition (3.2) is satisfied. (See Corollary 9.9.) By assumption, the Duistermaat-Heckman measures of M and M' are the same. Hence, we can apply Propositions 3.3 and 4.2. The first implies that the map g lifts to a Φ–T–diffeomorphism from M to M'. The second then guarantees that there exists Φ–T–symplectomorphism from M to M'. □

13. Proof of uniqueness for centered spaces

In this section we prove that the invariants described in Theorem 2 also separate centered spaces.

We recall Definition 1.4: a proper Hamiltonian T-manifold (M, ω, Φ, U) is **centered** about a point α if α is contained in the closure of the moment image of every orbit type stratum in M.

We recall the statement of the theorem.

Theorem 2 (Centered Uniqueness). *Let (M, Φ, ω, U) and (M', Φ', ω', U) be complexity one spaces that are centered about $\alpha \in U$. Assume that their Duistermaat-Heckman measures are the same and that their genus and isotropy data over $\alpha \in \mathfrak{t}^*$ are the same. Then the spaces are isomorphic.*

The proof of Theorem 2 relies on the following "stretching lemma", which tells us that a centered space retracts onto a neighborhood of its central fiber.

Lemma 13.1. *Let $\mathfrak{t}^*$ be the dual of the Lie algebra of a torus T, $U \subset \mathfrak{t}^*$ an open convex neighborhood of a point $\alpha \in \mathfrak{t}^*$, and $V \subset U$ any sub-neighborhood. Then there exists a convex neighborhood W of α contained in V, and a diffeomorphism $f \colon U \longrightarrow W$ with the following property: for any proper Hamiltonian T-manifold (M, ω, Φ, U) that is centered about α, there exists a smooth equivariant orientation preserving diffeomorphism $F \colon M \longrightarrow \Phi^{-1}(W)$ such that $\Phi \circ F = f \circ \Phi$.*

Proof of Theorem 2.

Case I: the moment fiber is short. By Proposition 2.2, there exists a convex sub-neighborhood $V \subset U$ of α and a Φ–T–diffeomorphism (in fact, symplectomorphism) from $\Phi^{-1}(V)$ to ${\Phi'}^{-1}(V)$. By Lemma 5.7 we can choose V so that $\Phi^{-1}(V)$ and ${\Phi'}^{-1}(V)$ satisfy Condition (3.2).

By Lemma 13.1, this implies that there exists a Φ–T–diffeomorphism from (M, ω, Φ) to (M', ω', Φ'), and that M and M' themselves also satisfy Condition (3.2). The Duistermaat-Heckman measures coincide; hence we may apply Proposition 3.3, which completes the proof.

Case II: the moment fiber is tall. By Proposition 9.5, there exists a convex sub-neighborhood $V \subset U$ of α so that $\Phi^{-1}(V)$ and ${\Phi'}^{-1}(V)$ are equipped with flattenings. By assumption, the spaces M and M' have the same genus and isotropy data. Therefore, by Proposition 11.1, there is a Φ-diffeomorphism $g \colon \Phi^{-1}(V) \longrightarrow {\Phi'}^{-1}(V)$. Since these spaces have flattenings, Condition (3.2) is satisfied. (See Corollary 9.9.) By assumption, the Duistermaat-Heckman measures of M and M' are the same. Hence, Proposition 4.2 implies that the map g lifts to a Φ–T–diffeomorphism from $\Phi^{-1}(V)$ to ${\Phi'}^{-1}(V)$. Proposition 3.3 completes the proof. $\square$

Recall that the **Euler vector field** on a vector space V is $X = \sum x_i \frac{\partial}{\partial x_i}$, where x_i are linear coordinates. This vector field is the generator of the flow $x \mapsto e^t x$, thus it is independent of the choice of coordinates.

Lemma 13.2. *Let (M, ω, Φ, U) be a proper Hamiltonian T-manifold. Suppose that U contains the origin 0 of $\mathfrak{t}^*$, and that the space is centered about the origin. Then the Euler vector field X on $\mathfrak{t}^*$ lifts to a smooth invariant vector field $\tilde{X}$ on M, that is, $\Phi_*(\tilde{X}) = X$.*

Proof. By the local normal form theorem, it is enough to construct the vector field $\tilde{X}$ on the local models. We can then patch together the pieces by an invariant partition of unity.

Notice that if a map $\Phi \colon V \longrightarrow W$ between vector spaces is homogeneous of degree m, then $\Phi_* X_V = m X_W$ where X_V and X_W are the Euler vector fields on V and W; this follows from the equality $\Phi(e^t v) = e^{mt}\Phi(v)$. In particular, the Euler vector field on W lifts to a vector field on V. Similarly, if $\Phi_i \colon V_i \longrightarrow W_i$, $i = 1, 2$, are homogeneous (possibly of different degrees), and $\Phi = \Phi_1 \times \Phi_2 \colon V_1 \times V_2 \longrightarrow W_1 \times W_2$, then the Euler vector field on $W_1 \times W_2$ lifts to a vector field on $V_1 \times V_2$.

Consider a local model in M, namely, $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$, with a moment map $\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu$. The stratum fixed by H is $T \times_H (\mathbb{C}^n)^H \times \mathfrak{h}^0$. Because the space is centered about 0, we must have that $\alpha \in \mathfrak{h}^0$. Without loss of generality, we may assume that $\alpha = 0$. To lift the Euler vector field on $\mathfrak{t}^*$ to a T-invariant vector field on Y, it is enough to lift it to an H-invariant vector field on $\mathbb{C}^n \times \mathfrak{h}^0$. This is possible by the previous paragraph, because $\Phi|_{\mathbb{C}^n \times \mathfrak{h}^0}$ is bihomogeneous. $\square$

Proof of Lemma 13.1. Without loss of generality, we may assume that $\alpha = 0$. Choose $\epsilon > 0$ so that an ϵ-ball about 0 is contained in V. Let $g_t \colon [0, \infty) \longrightarrow [0, \infty)$ for $0 \leq t \leq 1$ be an isotopy such that g_0 is the identity map, the image of g_1 is contained in $[0, \epsilon)$, $g_t(x) \leq x$ for all x and all t, and $g_t(x) = x$ for all x near zero and all t.

Take $f_t(v) = g_t(|v|)\frac{v}{|v|}$ for all $v \in U \smallsetminus \{0\}$ and $f_t(0) = 0$; let $f = f_1$.

Let ξ_t be the vector field on V which generates this isotopy: $\frac{df_t}{dt} = \xi_t \circ f_t$. Since ξ_t vanishes near $v = 0$, we can write $\xi_t = \psi_t \cdot X$, where $\psi_t \colon \mathfrak{t}^* \longrightarrow \mathbb{R}$ is a smooth function, and X is the Euler vector field on $\mathfrak{t}^*$. By Lemma 13.2, there exists a smooth invariant vector field $\tilde{X}$ on M such that $\psi_*(\tilde{X}) = X$. So $\tilde{\xi}_t = (\psi_t \circ \Phi) \cdot \tilde{X}$ is a smooth invariant vector field on M which is a lifting of ξ_t. Because Φ is proper, the vector field $\tilde{\xi}_t$ generates an isotopy, F_t. Take $F = F_1$. □

14. Application to packings of Grassmannians

We are now ready to present our application. First, we recall a definition from symplectic topology:

Definition 14.1. A symplectic manifold M admits a **full packing by k equal balls** if for any $\epsilon > 0$ there exists a symplectic embedding into M of a disjoint union of k symplectic balls with equal radii such that the complement of the image has volume less than ϵ.

Let $\mathrm{Gr}^+(2, \mathbb{R}^n)$ denote the Grassmannian of all oriented real 2-planes in $\mathbb{R}^n$, together with an $\mathrm{SO}(n)$-invariant symplectic structure (which is unique up to scalar), and with the $\lfloor \frac{n}{2} \rfloor$ dimensional torus action given by restricting the standard action of $\mathrm{SO}(n)$.

Theorem 3. *Let M be the Grassmannian $\mathrm{Gr}^+(2, \mathbb{R}^5)$ or $\mathrm{Gr}^+(2, \mathbb{R}^6)$. There exists an equivariant symplectic embedding of a disjoint union of two open symplectic balls with linear actions and with equal radii into M such that the complement of the image has zero volume. A fortiori, these Grassmannians can be fully packed by two equal balls.*

The following tool is useful:

Lemma 14.2. *Let (M, ω, Φ) be a complexity one space over $\mathfrak{t}^*$. Let $p \in M$ be an isolated fixed point with isotropy weights $\eta_1, \ldots, \eta_n$. Assume that the differences $\eta_i - \eta_j$ span a codimension one subspace, H, of $\mathfrak{t}^*$. Assume, moreover, that p is the only fixed point whose moment map image lies on one open side, H_+, of H.*

Then the preimage $\Phi^{-1}(H_+)$ is equivariantly symplectomorphic to a ball with a linear T-action.

Example 14.3. Let $(S^1)^2$ act on $\mathbb{C}^3$ with weights $\eta_1 = (-1, -1)$, $\eta_2 = (0, -1)$, and $\eta_3 = (1, -1)$, and moment map $\Phi(z) = (0, 1) + \frac{1}{2} \sum \eta_j |z_j|^2$. The preimage of the upper half-plane is the ball of radius $\sqrt{2}$.

Proof of Lemma 14.2. First, we show that $\Phi^{-1}(H_+)$ is centered. The closure N of an orbit type stratum in M is itself a compact symplectic manifold with the restricted T action and moment map. By the convexity theorem, its moment image is the convex hull of the moment images of its fixed points. Either N contains p, or its moment image is contained in $\mathrm{conv}(M^T \smallsetminus p)$, and therefore disjoint from H_+.

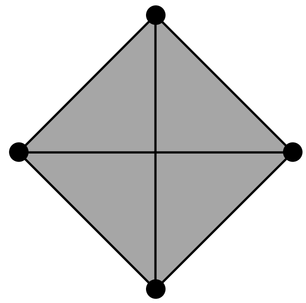

FIGURE 1. Moment images of orbit type strata in the Grassmannian $\mathrm{Gr}^+(2,\mathbb{R}^5)$

Let T act on $\mathbb{C}^n$ with weights $\eta_1,\ldots,\eta_n$ and with the moment map that sends $(z_1,\ldots,z_n)$ to $\Phi(p)+\sum\frac{1}{2}\eta_i|z_i|^2$. Because the differences $\eta_i-\eta_j$ span H, the moment preimage of H_+ in $\mathbb{C}^n$ is a ball. Hence, this ball is also a centered complexity one space over H_+.

Since both spaces are centered about $a=\Phi(p)$, and the preimages of a are both single orbits with the same isotropy data, by Theorem 2 the spaces are equivariantly symplectomorphic. □

Now consider any semi-simple compact Lie group G, and let T be a maximal torus. Use the Killing form to identify $\mathfrak{t}$ and $\mathfrak{t}^*$ and embed $\mathfrak{t}^*$ in $\mathfrak{g}^*$. Recall that the coadjoint orbit in $\mathfrak{g}^*$ through an element x of $\mathfrak{t}^*$ is a symplectic manifold, and the projection to $\mathfrak{t}^*$ is a moment map for the T-action. The fixed points for the T-action are exactly the Weyl group orbit of x in $\mathfrak{t}^*$. The isotropy weights at a fixed point $y\in\mathfrak{t}^*$ are exactly those roots $\alpha\in\mathfrak{t}^*$ for which $\langle\alpha,y\rangle<0$.

Proof of Theorem 3 for $\mathrm{Gr}^+(2,\mathbb{R}^5)$. The Lie algebra of the maximal torus of SO(5) can be identified with $\mathbb{R}^2$ with the standard metric. The roots are $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$. The Weyl group acts by permuting the coordinates and by flipping their signs.

The orbit through the point $(1,0)$ is naturally identified with the Grassmannian $\mathrm{Gr}^+(2,\mathbb{R}^5)=\mathrm{SO}(5)/\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(3))$. The Weyl group orbit of this point consists of the points $(1,0)$, $(-1,0)$ $(0,1)$, and $(0,-1)$. The moment image is a diamond; see Figure 1. The isotropy weights at $(1,0)$ are $(-1,1)$, $(-1,0)$, and $(-1,-1)$. By Lemma 14.2, the preimage of the half space $\{(x,y)\mid x>0\}$ is a ball as required. A similar argument shows that the preimage of the opposite half space is again a ball. □

Proof of Theorem 3 for $\mathrm{Gr}^+(2,\mathbb{R}^6)$. The Lie algebra of the maximal torus of SO(6) can be identified with $\mathbb{R}^3$ with the standard metric. The Weyl group acts by permuting the coordinates and by flipping the signs of two coordinates at a time. The roots are $(\pm1,\pm1,\pm1)$.

The orbit through the point $(1,0,0)$ is naturally identified with the Grassmannian $\mathrm{Gr}^+(2,\mathbb{R}^6)=\mathrm{SO}(6)/\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(4))$. The Weyl group orbit of this point consists of the points $(\pm1,0,0)$, $(0,\pm1,0)$, and $(0,0,\pm1)$. The moment image is an octahedron. The isotropy weights at $(1,0,0)$ are $(-1,\pm1,\pm1)$. By Lemma 14.2, the preimage of the half space $\{(x,y,z)\mid x>0\}$ is a ball as required. A similar argument shows that the preimage of the opposite half space is again a ball. □

References

[AH] K. Ahara and A. Hattori, 4-*dimensional symplectic* S^1*-manifolds admitting moment map*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **38** (1991), 251–298. MR **93b:**58048

[At] M. Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15. MR **83e:**53037

[Au1] M. Audin, *Hamiltoniens périodiques sur les variétés symplectiques compactes de dimension* 4, Géométrie symplectique et mécanique, Proceedings 1988, C. Albert ed., Springer Lecture Notes in Math. **1416** (1990). MR **91f:**57013

[Au2] M. Audin, *The topology of torus actions on symplectic manifolds*, Progress in Mathematics **93**, Birkhäuser Verlag, Basel, 1991. MR **92m:**57046

[BB] A. Bialynicki-Birula, *Remarks on the action of an algebraic torus on* k^n*. II*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **15**, (1967), 123–125. MR **35:**6666

[Bi1] P. Biran, *A stability property of symplectic packing*, Invent. Math. **136** (1999), 123–155. MR **2000b:**57039

[Bi2] P. Biran, *Symplectic packing in dimension* 4, GAFA, Geom. Fun. Anal. **7** (1997), 420–437. MR **98i:**57057

[BT] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Springer–Verlag, 1982. MR **83i:**57016

[De1] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339. MR **90b:**58069

[De2] T. Delzant, *Classification des actions hamiltoniennes complèment intégrables de rang deux*, Ann. Global Anal. Geom. **8** (1990), 87–112. MR **92f:**58078

[DH] J. J. Duistermaat and G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69**, 259–268 (1982); Addendum, Invent. Math. **72** (1983), 153–158. MR **84h:**58051a MR **84h:**58051b

[FK] K.-H. Fieseler and L. Kaup, *On the geometry of affine algebraic* $\mathbb{C}^*$*-surfaces*, in: *Problems in the theory of surfaces and their classification (Cortona, 1988)*, 111–140, Sympos. Math., XXXII, Academic Press, London, 1991. MR **95c:**14065

[F] R. Fintushel, *Circle actions on simply connected 4-manifolds*, Trans. Amer. Math. Soc. **230** (1977), 147–171, and *Classification of circle actions on 4-manifolds*, Trans. Amer. Math. Soc. **242** (1978), 377–390. MR **56:**16659; MR **81e:**57036

[GSj] V. Guillemin and R. Sjamaar, Lecture at Mass. Inst. Tech., 1995.

[GS1] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping, I*, Invent. Math. **67** (1982), 491–513. MR **83m:**58037

[GS2] V. Guillemin and S. Sternberg, *A normal form for the moment map*, Differential geometric methods in mathematical physics, (S. Sternberg, Ed.) Reidel, Dordrecht, Holland, 1984. MR **86d:**58033

[GS2] V. Guillemin and S. Sternberg, *Birational equivalence in the symplectic category*, Invent. Math. **97** (1989), 485–522. MR **90f:**58060

[HS] A. Haefliger and E. Salem, *Actions of tori on orbifolds*, Ann. Global Anal. Geom. **9** (1991), 37–59. MR **92f:**57047

[I] P. Iglesias, *Les* SO(3)*-variétés symplectiques et leurs classification en dimension 4*, Bull. Soc. Math. France **119** (1991), 371–396. MR **92i:**57023

[K1] Y. Karshon, *Appendix* to: *Symplectic packings and algebraic geometry* by D. McDuff and L. Polterovich, Invent. Math. **115**, 431–434 (1994). MR **95a:**58042

[K2] Y. Karshon, *Periodic Hamiltonian flows on four-dimensional manifolds*, Mem. Amer. Math. Soc. **672** (1999). MR **2000c:**53113

[Kn] F. Knop, private communication.

[Ko] A. Kosinski, *Differential manifolds*, Pure and Applied Mathematics **138**, Academic Press, Inc., Boston, MA (1993). MR **95b:**57001

[Kl] J. L. Koszul, *Sur certains groupes de transformations de Lie*, Colloque international du Centre National de la Recherche Scientifique **52** (1953), 137–141. MR **15:**600g

[LMTW] E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, *Nonabelian convexity by symplectic cuts*, Topology **37** (1998), 245–259. MR **99a:**58069

[LT] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc., **349** (1997), no. 10, 4201–4230. MR **98a:**57043

[LV] D. Luna and Th. Vust, *Plongements d'espaces homogènes*, Comment. Math. Helv. **58** (1983), 186–245. MR **85a:**14035

[M] C. M. Marle, Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique, *Rendiconti del Seminario Matematico* **43** (1985), 227–251. MR **88a:**58075

[McD] D. McDuff, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. **5** (1988), 149–160. MR **91c:**58042

[McD] D. McDuff, *Remarks on the uniqueness of symplectic blowing up*, in: *Symplectic Geometry*, London Math. Soc. Lecture Note Ser. **192**, 157–167, Cambridge Univ. Press, Cambridge, 1993. MR **95h:**57028

[MS] E. Meinrenken and R. Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), 699–762. MR **2000f:**53114

[Mo] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294. MR **32:**409

[KKMS] G. Kempf, F. F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal embeddings. I.* Lecture Notes in Math. **339**, Springer-Verlag, Berlin-New York, 1973. MR **49:**299

[OR] P. Orlik and F. Raymond, *Actions of the torus on* 4*-manifolds.* I and II, Trans. Amer. Math. Soc. **152** (1970), 531–559, and Topology **13** (1974), 89–12. MR **42:**3808; MR **50:**1274

[OW] P. Orlik and P. Wagreich, *Algebraic surfaces with* k^**-actions*, Acta Math. **138** (1977), 43–81. MR **57:**336

[R] J. Rynes, *Nonsingular affine* k^**-surfaces*, Trans. Amer. Math. Soc. **332** (1992), 889–921. MR **92j:**14061

[Sch1] G. W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68. MR **51:**6870

[Sch2] G. W. Schwarz, *Lifting smooth homotopies of orbit spaces*, I.H.E.S. Publ. Math. **51** (1980), 37–135. MR **81h:**57024

[Sj] R. Sjamaar, *Convexity properties of the moment mapping re-examined*, Adv. Math. **138** (1998), 46–91. MR **2000a:**53148

[T1] D. A. Timashëv, *G-manifolds of complexity* 1, (Russian) Uspekhi Mat. Nauk **51** (1996), no. 3 (309), 213–214; English translation: Russian Math. Surveys **51** (1996), 567–568. CMP 96:17

[T2] D. A. Timashëv, *Classification of G-manifolds of complexity* 1, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), 127–162; English translation: Izv. Math. **61** (1997), 363–397. MR **98h:**14058

[To] S. Tolman, *Examples of non-Kähler Hamiltonian torus actions*, Invent. Math., **131** (1998), 299–310. MR **2000d:**53129

[Tr] L. Traynor *Symplectic packing constructions*, J. Diff. Geom. **42** (1995), 411–429. MR **96k:**53046

[W1] A. Weinstein, *Lectures on Symplectic Manifolds*, CBMS Reg. Conf. Ser. in Math. **29**, 1977. MR **57:**4244; Corrected reprint MR **82b:**58039

[MW] Merriam-Webster's Collegiate Dictionary, `http://www.m-w.com/cgi-bin/dictionary`

[W] C. Woodward, *The classification of transversal multiplicity-free group actions*, Ann. Global Anal. Geom. **14** (1996), 3–42. MR **97a:**58067

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