Centers of graded fusion categories

Shlomo Gelaki, Deepak Naidu and Dmitri Nikshych

Centers of graded fusion categories

Shlomo Gelaki, Deepak Naidu and Dmitri Nikshych

Let $\mathscr{C}$ be a fusion category faithfully graded by a finite group G and let $\mathscr{D}$ be the trivial component of this grading. The center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$ is shown to be canonically equivalent to a G-equivariantization of the relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. We use this result to obtain a criterion for $\mathscr{C}$ to be group-theoretical and apply it to Tambara–Yamagami fusion categories. We also find several new series of modular categories by analyzing the centers of Tambara–Yamagami categories. Finally, we prove a general result about the existence of zeroes in S-matrices of weakly integral modular categories.

1. Introduction

Throughout this paper we work over an algebraically closed field k of characteristic 0. All categories considered in this paper are finite, abelian, semisimple, and k-linear. We freely use the language and basic theory of fusion categories, module categories over them, braided categories, and Frobenius–Perron dimensions [Bakalov and Kirillov 2001; Ostrik 2003; Etingof et al. 2005].

Let G be a finite group. A fusion category $\mathscr{C}$ is *G-graded* if there is a decomposition

$$\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g$$

of $\mathscr{C}$ into a direct sum of full abelian subcategories such that the tensor product of $\mathscr{C}$ maps $\mathscr{C}_g \times \mathscr{C}_h$ to $\mathscr{C}_{gh}$, for all $g, h \in G$. A *G-extension* of a fusion category $\mathscr{D}$ is a G-graded fusion category $\mathscr{C}$ whose trivial component $\mathscr{C}_e$, where e is the identity of G, is equivalent to $\mathscr{D}$.

Gradings and extensions play an important role in the study and classification of fusion categories. For example, *nilpotent* fusion categories (that is, those categories that can be obtained from the trivial category by a sequence of group extensions) were studied in [Gelaki and Nikshych 2008]. It was proved in [Etingof et al. 2005] that every fusion category of prime power dimension is nilpotent. Group-theoretical properties of such categories were studied in [Drinfeld et al. 2007]. Recently, fusion categories of dimension $p^n q^m$, where p, q are primes, were shown to be Morita equivalent to nilpotent categories [Etingof et al. 2009].

The main goal of this paper is to describe the center $\mathscr{Z}(\mathscr{C})$ of a G-graded fusion category $\mathscr{C}$ in terms of its trivial component $\mathscr{D}$ (Theorem 3.5) and apply this description to the study of structural properties of $\mathscr{C}$ and the construction of new examples of modular categories.

The organization of the paper is as follows. In Section 2 we recall some basic notions, results, and examples of fusion categories, notably the notions of the relative center of a bimodule category [Majid 1991], group action on a fusion category and crossed product [Tambara 2001], equivariantization and de-equivariantization theory [Arkhipov and Gaitsgory 2003; Bruguières 2000; Gaitsgory 2005; Kirillov 2002; Müger 2000; Drinfeld et al. 2009], and braided G-crossed fusion categories [Turaev 2000; 2008].

In Section 3 we study the center $\mathscr{Z}(\mathscr{C})$ of a G-graded fusion category $\mathscr{C}$. We show that if $\mathscr{D}$ is the trivial component of $\mathscr{C}$, then the relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ has a canonical structure of a braided G-crossed category and there is an equivalence of braided fusion categories $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})^G \cong \mathscr{Z}(\mathscr{C})$ (Theorem 3.5). Thus, the structure of $\mathscr{Z}(\mathscr{C})$ can be understood in terms of a smaller and more transparent category $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. In particular, there is a canonical braided action (studied in detail in [Etingof et al. 2009]) of G on $\mathscr{Z}(\mathscr{D})$. In Corollary 3.10 we use this action to prove that $\mathscr{C}$ is group-theoretical if and only if $\mathscr{Z}(\mathscr{D})$ contains a G-stable Lagrangian subcategory. As an illustration, we describe the center of a crossed product fusion category $\mathscr{C} = \mathscr{D} \rtimes G$.

We apply the results from Section 4 to the study of Tambara–Yamagami categories [Tambara and Yamagami 1998]. We obtain a convenient description of the centers of such categories as equivariantizations and compute their modular data, that is, S- and T-matrices. This computation was previously done in [Izumi 2001] using different techniques. We establish a criterion for a Tambara–Yamagami category to be group-theoretical (Theorem 4.6). We also extend the construction of non-group-theoretical semisimple Hopf algebras from Tambara–Yamagami categories given in [Nikshych 2008].

In Section 5 we construct a series of new modular categories as factors of the centers of Tambara–Yamagami categories. One associates a pair of such categories $\mathscr{E}(q, \pm)$ with any nondegenerate quadratic form q on an abelian group A of odd order. The categories $\mathscr{E}(q, \pm)$ have dimension $4|A|$. They are group-theoretical if

and only if A contains a Lagrangian subgroup with respect to q. We compute the S- and T-matrices of $\mathscr{E}(q, \pm)$ and write down several small examples explicitly.

Section 6 is independent from the rest of the paper and contains a general result about existence of zeroes in S-matrices of weakly integral modular categories (Theorem 6.1). This is a categorical analogue of a classical result of Burnside in character theory.

2. Preliminaries

2A. *Dual fusion categories and Morita equivalence.* Let $\mathscr{C}$ be a fusion category and let $\mathcal{M}$ be an indecomposable right $\mathscr{C}$-module category $\mathcal{M}$. The category $\mathscr{C}^*_{\mathcal{M}}$ of $\mathscr{C}$-module endofunctors of $\mathcal{M}$ is a fusion category, called the dual of $\mathscr{C}$ with respect to $\mathcal{M}$ [Etingof et al. 2005; Ostrik 2003].

Following [Müger 2003a], we say that two fusion categories $\mathscr{C}$ and $\mathscr{D}$ are *Morita equivalent* if $\mathscr{D}$ is equivalent to $\mathscr{C}^*_{\mathcal{M}}$, for some indecomposable right $\mathscr{C}$-module category $\mathcal{M}$. A fusion category is said to be *pointed* if all its simple objects are invertible (any such category is equivalent to the category Vec^ω_G of vector spaces graded by a finite group G with the associativity constraint given by a 3-cocycle $\omega \in Z^3(G, k^\times)$). A fusion category is called *group-theoretical* if it is Morita equivalent to a pointed fusion category. See [Ostrik 2003; Etingof et al. 2005; Nikshych 2008] for details of the theory of group-theoretical categories.
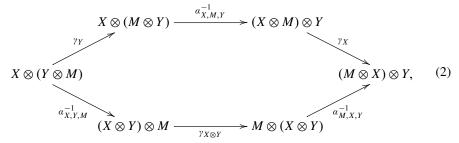
2B. *The center of a bimodule category and the relative center of a fusion category.* Let $\mathscr{C}$ be a fusion category with unit object $\mathbf{1}$ and associativity constraint $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and let $\mathcal{M}$ be a $\mathscr{C}$-bimodule category.

Definition 2.1. The *center* of $\mathcal{M}$ is the category $\mathscr{Z}_{\mathscr{C}}(\mathcal{M})$ of $\mathscr{C}$-bimodule functors from $\mathscr{C}$ to $\mathcal{M}$.

Explicitly, the objects of $\mathscr{Z}_{\mathscr{C}}(\mathcal{M})$ are pairs (M, γ), where M is an object of $\mathcal{M}$ and

$$\gamma = \{\gamma_X : X \otimes M \xrightarrow{\sim} M \otimes X\}_{X \in \mathscr{C}} \tag{1}$$

is a natural family of isomorphisms making the diagram

$$\begin{array}{ccc}
 & \xrightarrow{\;\alpha^{-1}_{X,M,Y}\;} & \\
X \otimes (M \otimes Y) & & (X \otimes M) \otimes Y \\
{\scriptstyle \gamma_Y}\nearrow & & \searrow{\scriptstyle \gamma_X} \\
X \otimes (Y \otimes M) & & (M \otimes X) \otimes Y, \\
{\scriptstyle \alpha^{-1}_{X,Y,M}}\searrow & & \nearrow{\scriptstyle \alpha^{-1}_{M,X,Y}} \\
(X \otimes Y) \otimes M & \xrightarrow[{\scriptstyle \gamma_{X \otimes Y}}]{} & M \otimes (X \otimes Y)
\end{array} \tag{2}$$

commutative, where the α's denote the associativity constraints in $\mathcal{M}$.

Indeed, a $\mathscr{C}$-bimodule functor $F : \mathscr{C} \to \mathcal{M}$ is completely determined by the pair $(F(\mathbf{1}), \{\gamma_X\}_{X \in \mathscr{C}})$, where $\gamma = \{\gamma_X\}_{X \in \mathscr{C}}$ is the collection of isomorphisms

$$\gamma_X : X \otimes F(\mathbf{1}) \xrightarrow{\sim} F(X) \xrightarrow{\sim} F(\mathbf{1}) \otimes X,$$

coming from the $\mathscr{C}$-bimodule structure on F.

We will call the natural family of isomorphisms (1) the *central structure* of an object $X \in \mathscr{Z}_\mathscr{C}(\mathcal{M})$.

Remark 2.2. (i) The definition of the center of a bimodule category is parallel to that of the center of a bimodule over a ring.

 (ii) We will often suppress the central structure while working with objects of $\mathscr{Z}_\mathscr{C}(\mathcal{M})$ and refer to (M, γ) simply as M.

(iii) $\mathscr{Z}_\mathscr{C}(\mathcal{M})$ is a semisimple abelian category. It has the obvious canonical structure of a $\mathscr{Z}(\mathscr{C})$-module category, where $\mathscr{Z}(\mathscr{C})$ is the center of $\mathscr{C}$ (see, for example, [Kassel 1995, Section XIII.4] for the definition of $\mathscr{Z}(\mathscr{C})$).

Here is an important special case of this construction. Let $\mathscr{C}$ be a fusion category and let $\mathscr{D} \subset \mathscr{C}$ be a fusion subcategory. Then $\mathscr{C}$ is a $\mathscr{D}$-bimodule category. We will call $\mathscr{Z}_\mathscr{D}(\mathscr{C})$ the *relative center* of $\mathscr{C}$.

Remark 2.3. The aforementioned construction of the relative center is a special case of a more general construction considered in [Majid 1991, Definition 3.2 and Theorem 3.3].

It is easy to see that $\mathscr{Z}_\mathscr{D}(\mathscr{C})$ is a tensor category with tensor product defined as follows. If (X, γ) and (X', γ') are objects in $\mathscr{Z}_\mathscr{D}(\mathscr{C})$ then

$$(X, \gamma) \otimes (X', \gamma') := (X \otimes X', \tilde{\gamma}),$$

where $\tilde{\gamma}_V : V \otimes (X \otimes X') \xrightarrow{\sim} (X \otimes X') \otimes V$, $V \in \mathscr{D}$, is defined by the diagram

$$
\begin{array}{ccccc}
V \otimes (X \otimes X') & \xrightarrow{\alpha_{V,X,X'}^{-1}} & (V \otimes X) \otimes X' & \xrightarrow{\gamma_V} & (X \otimes V) \otimes X' \\
\tilde{\gamma}_V \downarrow & & & & \downarrow \alpha_{X,V,X'} \\
(X \otimes X') \otimes V & \xleftarrow{\alpha_{X,X',V}^{-1}} & X \otimes (X' \otimes V) & \xleftarrow{\gamma'_V} & X \otimes (V \otimes X').
\end{array}
\qquad (3)
$$

The unit object of $\mathscr{Z}_\mathscr{D}(\mathscr{C})$ is $(\mathbf{1}, \mathrm{id})$. The dual of (X, γ) is $(X^*, \overline{\gamma})$, where $\overline{\gamma}_V := (\gamma_{*V})^*$.

Remark 2.4. Let $\mathscr{C}$ and $\mathscr{D}$ be as above.

 (i) $\mathscr{Z}_\mathscr{D}(\mathscr{C})$ is dual to the fusion category $\mathscr{D} \boxtimes \mathscr{C}^{\mathrm{rev}}$ (where $\mathscr{C}^{\mathrm{rev}}$ is the fusion category obtained from $\mathscr{C}$ by reversing the tensor product and $\boxtimes$ is Deligne's tensor product of fusion categories) with respect to its module category $\mathscr{C}$,

where $\mathscr{D}$ and $\mathscr{C}^{\mathrm{rev}}$ act on $\mathscr{C}$ via the right and left multiplication respectively. In particular, $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ is a fusion category.

(ii) $\mathrm{FPdim}(\mathscr{Z}_{\mathscr{D}}(\mathscr{C})) = \mathrm{FPdim}(\mathscr{C})\,\mathrm{FPdim}(\mathscr{D})$, where FPdim denotes the Frobenius–Perron dimension of a category.

(iii) $\mathscr{Z}_{\mathscr{C}}(\mathscr{C})$ coincides with the center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$. This category has a canonical braiding given by

$$c_{(X,\gamma),(X',\gamma')} = \gamma_{X'} : (X,\gamma) \otimes (X',\gamma') \xrightarrow{\sim} (X',\gamma') \otimes (X,\gamma). \qquad (4)$$

(iv) There is an obvious forgetful tensor functor:

$$\mathscr{Z}(\mathscr{C}) \mapsto \mathscr{Z}_{\mathscr{D}}(\mathscr{C}) : (X,\gamma) \mapsto (X, \gamma\,|_{\mathscr{D}}). \qquad (5)$$

2C. *Centralizers in braided fusion categories.* Let $\mathscr{C}$ be a braided fusion category with braiding c. Two objects X and Y of $\mathscr{C}$ are said to *centralize* each other [Müger 2003b] if $c_{Y,X}c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

For any fusion subcategory $\mathscr{D} \subseteq \mathscr{C}$ its *centralizer* $\mathscr{D}'$ is the full fusion subcategory of $\mathscr{C}$ consisting of all objects $X \in \mathscr{C}$ centralizing every object in $\mathscr{D}$. The category $\mathscr{C}$ is said to be *nondegenerate* if $\mathscr{C}' = \mathrm{Vec}$. In this case one has $\mathscr{D}'' = \mathscr{D}$ [Müger 2003b]. If $\mathscr{C}$ is a premodular category, that is, has a spherical structure, then it is nondegenerate if and only if it is modular.

A braided fusion category $\mathscr{E}$ is called *Tannakian* if it is equivalent to the representation category $\mathrm{Rep}(G)$ of a finite group G as a braided fusion category. Here $\mathrm{Rep}(G)$ is considered with its standard symmetric braiding. The group G is defined by $\mathscr{E}$ up to an isomorphism [Deligne 1990].

A fusion subcategory $\mathscr{L}$ of a braided fusion category is called *Lagrangian* if it is Tannakian and $\mathscr{L} = \mathscr{L}'$.

Theorem 2.5 [Drinfeld et al. 2007]. *A fusion category $\mathscr{C}$ is group-theoretical if and only if $\mathscr{Z}(\mathscr{C})$ contains a Lagrangian subcategory.*
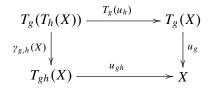
2D. *Group actions on fusion categories and equivariantization.* Let G be a finite group, and let $\underline{G}$ denote the monoidal category whose objects are elements of G, whose morphisms are identities, and whose tensor product is given by multiplication in G. Recall that an action of G on a fusion category $\mathscr{C}$ is a monoidal functor $\underline{G} \to \mathrm{Aut}_{\otimes}(\mathscr{C}) : g \mapsto T_g$. For any $g, h \in G$, let

$$\gamma_{g,h} = T_g \circ T_h \simeq T_{gh}$$

be the isomorphism defining the monoidal structure on the functor $\underline{G} \to \mathrm{Aut}_{\otimes}(\mathscr{C})$.

Definition 2.6. A *G-equivariant object* in $\mathscr{C}$ is a pair $(X, \{u_g\}_{g \in G})$ consisting of an object X of $\mathscr{C}$ together with a collection of isomorphisms $u_g : T_g(X) \simeq X$, $g \in G$,

such that the diagram

$$
\begin{array}{ccc}
T_g(T_h(X)) & \xrightarrow{\;\;T_g(u_h)\;\;} & T_g(X) \\[2pt]
{\scriptstyle \gamma_{g,h}(X)}\Big\downarrow & & \Big\downarrow{\scriptstyle u_g} \\[2pt]
T_{gh}(X) & \xrightarrow{\;\;u_{gh}\;\;} & X
\end{array}
$$

commutes for all $g, h \in G$. One defines morphisms of equivariant objects to be morphisms in $\mathscr{C}$ commuting with u_g, $g \in G$.

Equivariant objects in $\mathscr{C}$ form a fusion category, called the *equivariantization* of $\mathscr{C}$ and denoted by $\mathscr{C}^G$ [Tambara 2001; Arkhipov and Gaitsgory 2003; Gaitsgory 2005]. One has $\mathrm{FPdim}(\mathscr{C}^G) = |G|\,\mathrm{FPdim}(\mathscr{C})$.

There is another fusion category that comes from an action of G on $\mathscr{C}$. It is the *crossed product* category $\mathscr{C} \rtimes G$ defined as follows [Tambara 2001; Nikshych 2008]. As an abelian category, $\mathscr{C} \rtimes G := \mathscr{C} \boxtimes \mathrm{Vec}_G$, where Vec_G denotes the fusion category of G-graded vector spaces. The tensor product in $\mathscr{C} \rtimes G$ is given by

$$
(X \boxtimes g) \otimes (Y \boxtimes h) := (X \otimes T_g(Y)) \boxtimes gh, \qquad X, Y \in \mathscr{C}, \quad g, h \in G. \tag{6}
$$

The unit object is $\mathbf{1} \boxtimes e$ and the associativity and unit constraints come from those of $\mathscr{C}$. Clearly, $\mathscr{C} \rtimes G$ is faithfully G-graded with the trivial component $\mathscr{C}$.

As explained in [Nikshych 2008], $\mathscr{C}$ is a right $\mathscr{C} \rtimes G$-module category via

$$
Y \otimes (X \boxtimes g) := T_{g^{-1}}(Y \otimes X),
$$

and the corresponding dual category $(\mathscr{C} \rtimes G)^*_{\mathscr{C}}$ is equivalent to $\mathscr{C}^G$. It follows from [Müger 2003a] that there is an equivalence of braided fusion categories

$$
\mathscr{Z}(\mathscr{C} \rtimes G) \cong \mathscr{Z}(\mathscr{C}^G).
$$

Let G be a finite group. For any conjugacy class K of G fix a representative $a_K \in K$. Let G_K denote the centralizer of a_K in G.

Proposition 2.7. *Let* $\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g$ *be a* G-graded fusion category with an action $g \mapsto T_g$ of G on $\mathscr{C}$ such that T_g carries $\mathscr{C}_h$ to $\mathscr{C}_{ghg^{-1}}$. Let $H := \{g \in G \mid \mathscr{C}_g \neq 0\}$. There is a bijection between the set of isomorphism classes of simple objects of $\mathscr{C}^G$ and pairs (K, X), where $K \subset H$ is a conjugacy class of G and X is a simple G_K-equivariant object of $\mathscr{C}_{a_K}$.

Proof. A simple G-equivariant object of $\mathscr{C}$ must be supported on a single conjugacy class K. Let $Y = \oplus_{g \in K} Y_g$ be such an object. Then Y_{a_K} is a simple G_K-equivariant object.

Conversely, given a G_K-equivariant object X in $\mathscr{C}_{a_K}$ let

$$
Y = \bigoplus_h T_h(X),
$$

where the summation is taken over the set of representatives of cosets of G_K in G. It is easy to see that Y acquires the structure of a simple G-equivariant object.

Clearly, the two constructions are inverses of each other. $\qquad\square$

Remark 2.8. The Frobenius–Perron dimension of the simple object corresponding to a pair (K, X) in Proposition 2.7 is $|K|\operatorname{FPdim}(X)$.

2E. *De-equivariantization of fusion categories.* Let $\mathscr{C}$ be a fusion category. Let $\mathscr{E} = \operatorname{Rep}(G)$ be a Tannakian category along with a braided tensor functor $\mathscr{E} \to \mathscr{Z}(\mathscr{C})$ such that the composition $\mathscr{E} \to \mathscr{Z}(\mathscr{C}) \to \mathscr{C}$ (where the second arrow is the forgetful functor) is fully faithful. The following construction was introduced in [Bruguières 2000] and [Müger 2000]. Let $A := \operatorname{Fun}(G)$ be the algebra of functions on G. It is a commutative algebra in $\mathscr{E}$ and thus its image is a commutative algebra in $\mathscr{Z}(\mathscr{C})$. This fact allows us to view the category $\mathscr{C}_G$ of A-modules in $\mathscr{C}$ as a fusion category, called *de-equivariantization* of $\mathscr{C}$. There is a canonical surjective tensor functor

$$F : \mathscr{C} \to \mathscr{C}_G : X \mapsto A \otimes X. \tag{7}$$

It was explained in [Müger 2000; Drinfeld et al. 2009] that the group G acts on $\mathscr{C}_G$ by tensor autoequivalences (this action comes from the action of G on A by right translations). Furthermore, there is a bijection between subcategories of $\mathscr{C}$ containing the image of $\mathscr{E} = \operatorname{Rep}(G)$ and G-stable subcategories of $\mathscr{C}_G$. This bijection preserves Tannakian subcategories.

The procedures of equivariantization and de-equivariantization are inverses of each other: that is, there are canonical equivalences $(\mathscr{C}_G)^G \cong \mathscr{C}$ and $(\mathscr{C}^G)_G \cong \mathscr{C}$.

In particular, the construction above applies when $\mathscr{C}$ is a braided fusion category containing a Tannakian subcategory $\mathscr{E} = \operatorname{Rep}(G)$. In this case the braiding of $\mathscr{C}$ gives rise to an additional structure on the de-equivariantization functor (7). Namely, there is natural family of isomorphisms

$$X \otimes F(Y) \xrightarrow{\sim} F(Y) \otimes X, \qquad X \in \mathscr{C}_G, \ Y \in \mathscr{C}, \tag{8}$$

satisfying obvious compatibility conditions. In other words, F can be factored through a braided functor $\mathscr{C} \to \mathscr{Z}(\mathscr{C}_G)$, that is, F is a *central* functor.

If $\mathscr{E} \subset \mathscr{C}'$ then $\mathscr{C}_G$ is a braided fusion category with the braiding inherited from that of $\mathscr{C}$. If $\mathscr{E} = \mathscr{C}'$, the category $\mathscr{C}_G$ is nondegenerate. (In the presence of a spherical structure this category is called the *modularization* of $\mathscr{C}$ by $\mathscr{E}$ [Bruguières 2000; Müger 2000].)

Remark 2.9. The category $\mathscr{C}_G$ is not braided in general. However it does have an additional structure — it is a *braided G-crossed fusion category*. See next section (2F) for details.
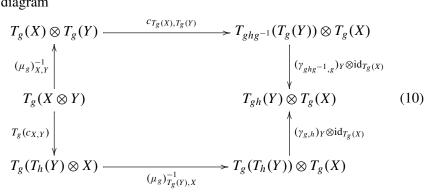
2F. *Braided G-crossed categories.* Let G be a finite group. Kirillov [2002] and Müger [2004] found a description of all braided fusion categories $\mathcal{D}$ containing $\mathrm{Rep}(G)$. Namely, they showed that the datum of a braided fusion category $\mathcal{D}$ containing $\mathrm{Rep}(G)$ is equivalent to the datum of a braided G-crossed category $\mathcal{C}$; see Theorem 2.12. The notion of a braided G-crossed category is due to Turaev [2000; 2008] and is recalled below.

Definition 2.10. A *braided G-crossed fusion category* is a fusion category $\mathcal{C}$ equipped with (i) a (not necessarily faithful) grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, (ii) an action $g \mapsto T_g$ of G on $\mathcal{C}$ such that $T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$, and (iii) a natural collection of isomorphisms

$$c_{X,Y} : X \otimes Y \simeq T_g(Y) \otimes X, \qquad X \in \mathcal{C}_g, \ g \in G \text{ and } Y \in \mathcal{C}, \qquad (9)$$

called the *G-braiding*. These structures are required to satisfy certain compatibility conditions, which we now state. Let $\gamma_{g,h} : T_g T_h \xrightarrow{\sim} T_{gh}$ denote the tensor structure of the functor $g \mapsto T_g$ and μ_g the tensor structure of T_g.
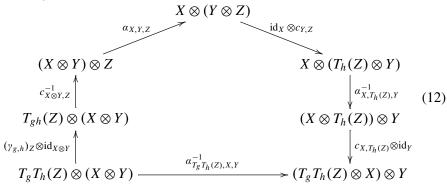
(a) The diagram

commutes for all $g, h \in G$ and objects $X \in \mathcal{C}_h$, $Y \in \mathcal{C}$.

(b) The diagram

commutes for all $g \in G$ and objects $X \in \mathcal{C}_g$, $Y, Z \in \mathcal{C}$.

(c) The diagram

$$
\begin{array}{c}
X \otimes (Y \otimes Z)
\end{array}
$$

with morphisms: $\alpha_{X,Y,Z}$ from $(X \otimes Y) \otimes Z$ to $X \otimes (Y \otimes Z)$; $\mathrm{id}_X \otimes c_{Y,Z}$ from $X \otimes (Y \otimes Z)$ to $X \otimes (T_h(Z) \otimes Y)$; $c^{-1}_{X \otimes Y, Z}$ from $T_{gh}(Z) \otimes (X \otimes Y)$ to $(X \otimes Y) \otimes Z$; $\alpha^{-1}_{X, T_h(Z), Y}$ from $X \otimes (T_h(Z) \otimes Y)$ to $(X \otimes T_h(Z)) \otimes Y$; $(\gamma_{g,h})_Z \otimes \mathrm{id}_{X \otimes Y}$ from $T_g T_h(Z) \otimes (X \otimes Y)$ to $T_{gh}(Z) \otimes (X \otimes Y)$; $c_{X, T_h(Z)} \otimes \mathrm{id}_Y$ from $(X \otimes T_h(Z)) \otimes Y$ to $(T_g T_h(Z) \otimes X) \otimes Y$; $\alpha^{-1}_{T_g T_h(Z), X, Y}$ from $T_g T_h(Z) \otimes (X \otimes Y)$ to $(T_g T_h(Z) \otimes X) \otimes Y$. $\qquad$ (12)

commutes for all $g, h \in G$ and objects $X \in \mathscr{C}_g$, $Y \in \mathscr{C}_h$, $Z \in \mathscr{C}$.

Remark 2.11. The trivial component $\mathscr{C}_e$ of a braided G-crossed fusion category $\mathscr{C}$ is a braided fusion category with the action of G by braided autoequivalences. This can be seen by taking $X, Y \in \mathscr{C}_e$ in diagrams (10)–(12).

Theorem 2.12 ([Kirillov 2002; Müger 2004]). *The equivariantization and de-equivariantization constructions establish a bijection between the set of equivalence classes of G-crossed braided fusion categories and the set of equivalence classes of braided fusion categories containing $\mathrm{Rep}(G)$ as a symmetric fusion subcategory.*

We shall now sketch the proof of this theorem. An alternative approach is given in [Drinfeld et al. 2009].

Suppose $\mathscr{C}$ is a braided G-crossed fusion category. We define a braiding $\tilde{c}$ on its equivariantization $\mathscr{C}^G$ as follows.

Let $(X, \{u_g\}_{g \in G})$ and $(Y, \{v_g\}_{g \in G})$ be objects in $\mathscr{C}^G$. Let $X = \oplus_{g \in G} X_g$ be a decomposition of X with respect to the grading of $\mathscr{C}$. Define an isomorphism

$$
\tilde{c}_{X,Y} : X \otimes Y = \bigoplus_{g \in G} X_g \otimes Y \xrightarrow{\oplus c_{X_g, Y}} \bigoplus_{g \in G} T_g(Y) \otimes X_g \xrightarrow{\oplus v_g \otimes \mathrm{id}_{X_g}} \bigoplus_{g \in G} Y \otimes X_g = Y \otimes X. \quad (13)
$$

It follows from condition (a) of Definition 2.10 that $\tilde{c}_{X,Y}$ respects the equivariant structures, that is, it is an isomorphism in $\mathscr{C}^G$. Its naturality is clear. The fact that $\tilde{c}$ is a braiding on $\mathscr{C}^G$ (that is, the hexagon axioms) follows from the commutativity of diagrams (11) and (12). It is easy to check that $\tilde{c}$ restricts to the standard braiding on $\mathrm{Rep}(G) = \mathrm{Vec}^G \subset \mathscr{C}^G$. Hence, $\mathscr{C}^G$ contains a Tannakian subcategory $\mathrm{Rep}(G)$.

Conversely, let $\mathscr{C}$ be a braided fusion category with braiding c containing a Tannakian subcategory $\mathrm{Rep}(G)$. The restriction of the de-equivariantization functor F from (7) on $\mathrm{Rep}(G)$ is isomorphic to the fiber functor $\mathrm{Rep}(G) \to \mathrm{Vec}$. Hence for any object X in $\mathscr{C}_G$ and any object V in $\mathrm{Rep}(G)$ we have an automorphism of

$F(V) \otimes X$ defined as the composition

$$F(V) \otimes X \xrightarrow{\sim} X \otimes F(V) \xrightarrow{\sim} F(V) \otimes X, \tag{14}$$

where the first isomorphism comes from the fact that $F(V) \in \text{Vec}$ and the second one is (8).

When X is simple we have an isomorphism $\text{Aut}_{\mathscr{C}}(F(V) \otimes X) \cong \text{Aut}_{\text{Vec}}(F(V))$, hence we obtain a tensor automorphism i_X of $F|_{\text{Rep}(G)}$. Since $\text{Aut}_\otimes(F|_{\text{Rep}(G)}) \cong G$ we have an assignment $X \mapsto i_X \in G$. The hexagon axiom of braiding implies that this assignment is multiplicative, that is, that $i_Z = i_X i_Y$ for any simple object Z contained in $X \otimes Y$. Thus, it defines a G-grading on $\mathscr{C}$:

$$\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g, \qquad \text{where } \mathbb{O}(\mathscr{C}_g) = \{X \in \mathbb{O}(\mathscr{C}) \mid i_X = g\}. \tag{15}$$

It is straightforward to check that $i_{T_g(X)} = ghg^{-1}$ whenever $i_X = h$.

Finally, to construct a G-crossed braiding on $\mathscr{C}$, observe that $\mathscr{C}$ and $\mathscr{C}^{\text{rev}}$ are embedded into the crossed product category $\mathscr{C} \rtimes G = (\mathscr{C}^G)^*_{\mathscr{C}}$ as subcategories $\mathscr{C}_{\text{left}}$ and $\mathscr{C}_{\text{right}}$, consisting, respectively, of functors of left and right multiplications by objects of $\mathscr{C}$. Clearly, there is a natural family of isomorphisms

$$X \otimes Y \xrightarrow{\sim} Y \otimes X, \qquad \text{with } X \in \mathscr{C}_{\text{left}} \text{ and } Y \in \mathscr{C}_{\text{right}}, \tag{16}$$

satisfying obvious compatibility conditions. Note that $\mathscr{C}_{\text{left}}$ is identified with the diagonal subcategory of $\mathscr{C} \rtimes G$ spanned by objects $X \boxtimes g$, $X \in \mathscr{C}_g$, $g \in G$, and $\mathscr{C}_{\text{right}}$ is identified with the trivial component subcategory $\mathscr{C} \boxtimes e$. Using (6) we conclude that isomorphisms (16) give rise to a G-crossed braiding on $\mathscr{C}$.

One can check that the two constructions above (from braided fusion categories containing $\text{Rep}(G)$ to braided G-crossed categories and vice versa) are inverses of each other; see [Kirillov 2002; Müger 2004; Drinfeld et al. 2009] for details.

Remark 2.13. Let $\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g$ be a braided G-crossed fusion category. It was shown in [Drinfeld et al. 2009] that the braided category $\mathscr{C}^G$ is nondegenerate if and only if $\mathscr{C}_e$ is nondegenerate and the G-grading of $\mathscr{C}$ is faithful.

3. The center of a graded fusion category

Let G be a finite group and let $\mathscr{D}$ be a fusion category. Throughout this section $\mathscr{C}$ will denote a fusion category with a faithful G-grading, whose trivial component is $\mathscr{D}$; that is, $\mathscr{C}$ is a G-extension of $\mathscr{D}$:

$$\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g, \qquad \mathscr{C}_e = \mathscr{D}. \tag{17}$$

In what follows we consider only *faithful* gradings: that is, those such that $\mathscr{C}_g \neq 0$ for all $g \in G$. An object of $\mathscr{C}$ contained in $\mathscr{C}_g$ will be called *homogeneous* of degree g.

Our goal is to describe the center $\mathscr{Z}(\mathscr{C})$ as an equivariantization of the relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ defined in Section 2B.

3A. *The relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ as a braided G-crossed category.* Let us define a canonical braided G-crossed category structure on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$.

First of all, there is an obvious faithful G-grading on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$:

$$\mathscr{Z}_{\mathscr{D}}(\mathscr{C}) = \bigoplus_{g \in G} \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g). \tag{18}$$

Indeed, it is clear that for every simple object X of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ the forgetful image of X in $\mathscr{C}$ must be homogeneous.

We now define the action of G on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. Take $g, h \in G$. Let $\mathrm{Fun}_{\mathscr{D} \boxtimes \mathscr{D}^{\mathrm{rev}}}(\mathscr{C}_g, \mathscr{C}_h)$ denote the category of $\mathscr{D}$-bimodule functors from $\mathscr{C}_g$ to $\mathscr{C}_h$. Clearly, it is a $\mathscr{Z}(\mathscr{D})$-bimodule category.

Proposition 3.1. *Let $g, h \in G$. The functors*

$$L_{g,h} : \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h) \xrightarrow{\sim} \mathrm{Fun}_{\mathscr{D} \boxtimes \mathscr{D}^{\mathrm{rev}}}(\mathscr{C}_g, \mathscr{C}_{hg}) : Z \mapsto Z \otimes ?, \tag{19}$$

$$R_{g,h} : \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h) \xrightarrow{\sim} \mathrm{Fun}_{\mathscr{D} \boxtimes \mathscr{D}^{\mathrm{rev}}}(\mathscr{C}_g, \mathscr{C}_{gh}) : Z \mapsto ? \otimes Z. \tag{20}$$

are equivalences of $\mathscr{Z}(\mathscr{D})$-bimodule categories.

Proof. We prove that (19) is an equivalence. Let $\mathrm{Fun}_{\mathscr{D}}(\mathscr{C}_g, \mathscr{C}_{hg})$ be the category of right $\mathscr{D}$-module functors from $\mathscr{C}_g$ to $\mathscr{C}_{hg}$. It suffices to prove that

$$M_{g,h} : \mathscr{C}_h \to \mathrm{Fun}_{\mathscr{D}}(\mathscr{C}_g, \mathscr{C}_{hg}) : X \mapsto X \otimes ? \tag{21}$$

is an equivalence. Indeed, $\mathscr{D}$-bimodule functor structures on $M_{g,h}(X)$ for $X \in \mathscr{C}_h$ are in bijection with central structures on X.

For every $g \in G$ choose a simple object $X_g \in \mathscr{C}_g$. Then $A_g := X_g \otimes X_g^*$ is an algebra in $\mathscr{D}$. It follows from [Ostrik 2003, Theorem 1] that the functor $Y \mapsto Y \otimes X_g^*$ is a left $\mathscr{C}$-module category equivalence between $\mathscr{C}$ and the category of right A_g-modules in $\mathscr{C}$. Since $Y \otimes X_g^*$ belongs to $\mathscr{D}$ if and only if Y is in $\mathscr{C}_g$ we see that the functor above restricts to a left $\mathscr{D}$-module category equivalence between $\mathscr{C}_g$ and the category of right A_g-modules in $\mathscr{D}$. There are also similar equivalences of right module categories.

It follows that for all $g, h \in G$ there is an equivalence

$$Y \mapsto X_g \otimes Y \otimes X_{hg}^* \tag{22}$$

between $\mathscr{C}$ and the category of $(A_g - A_{hg})$-bimodules in $\mathscr{C}$. The right-hand side of (22) belongs to $\mathscr{D}$ if and only if Y is in $\mathscr{C}_h$. Hence, (22) restricts to an equivalence

between $\mathscr{C}_h$ and the category of $(A_g - A_{hg})$-bimodules in $\mathscr{D}$. The latter category is identified with the category of right $\mathscr{D}$-module functors between the categories of right A_g-modules and A_{hg}-modules in $\mathscr{D}$, that is, with $\mathrm{Fun}_{\mathscr{D}}(\mathscr{C}_g, \mathscr{C}_{hg})$. It is easy to see that upon this identification the restriction of equivalence (22) to $\mathscr{C}_h$ coincides with (21).

The proof of the equivalence (20) is completely similar.   □

We define tensor functors

$$T_{g,h} := L^{-1}_{g,ghg^{-1}} R_{g,h} : \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h) \to \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_{ghg^{-1}}), \qquad g, h \in G, \qquad (23)$$

and set

$$T_g := \bigoplus_{h \in G} T_{g,h} : \mathscr{Z}_{\mathscr{D}}(\mathscr{C}) \to \mathscr{Z}_{\mathscr{D}}(\mathscr{C}). \qquad (24)$$

The definition of T_g along with Proposition 3.1 give rise to the following natural isomorphism of $\mathscr{D}$-bimodule functors from $\mathscr{C}_g$ to $\mathscr{C}$:

$$c_{-,Y} : ? \otimes Y \xrightarrow{\sim} T_g(Y) \otimes ?. \qquad (25)$$

It translates to a natural family of isomorphisms

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} T_g(Y) \otimes X, \qquad X \in \mathscr{C}_g, \ Y \in \mathscr{Z}_{\mathscr{D}}(\mathscr{C}), \ g \in G, \qquad (26)$$

satisfying natural compatibility conditions corresponding to the $\mathscr{D}$-bimodule structure on (25). Since the grading (18) is faithful, we have $T_g(\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h)) \subset \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_{ghg^{-1}})$.

Take $X_1 \in \mathscr{C}_{g_1}$, $X_2 \in \mathscr{C}_{g_2}$ and set $X = X_1 \otimes X_2$ in (26). We obtain a natural isomorphism

$$T_{g_1} T_{g_2}(Y) \otimes X_1 \otimes X_2 \xrightarrow{\sim} T_{g_1 g_2}(Y) \otimes X_1 \otimes X_2. \qquad (27)$$

Since every object $Z \in \mathscr{C}_{g_1 g_2}$ is contained in $X_1 \otimes X_2$ for some $X_1 \in \mathscr{C}_{g_1}$, $X_2 \in \mathscr{C}_{g_2}$, using naturality of (27) we obtain a natural isomorphism

$$T_{g_1} T_{g_2}(Y) \otimes Z \xrightarrow{\sim} T_{g_1 g_2}(Y) \otimes Z, \qquad Z \in \mathscr{C}_{g_1 g_2}, \qquad (28)$$

of $\mathscr{D}$-bimodule functors $T_{g_1} T_{g_2}(Y) \otimes ?$ and $T_{g_1 g_2}(Y) \otimes ?$. By Proposition 3.1 this gives an isomorphism $T_{g_1} T_{g_2}(Y) \xrightarrow{\sim} T_{g_1 g_2}(Y)$, $Y \in \mathscr{Z}_{\mathscr{D}}(\mathscr{C})$, that is, an isomorphism of functors $T_{g_1} T_{g_2} \xrightarrow{\sim} T_{g_1 g_2}$. Thus, the assignment $g \mapsto T_g$ is an action of G on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ by tensor autoequivalences.

Suppose that X is an object in $\mathscr{Z}(\mathscr{C}_g)$. Then both sides of (26) have structure of objects in $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ obtained by composing central structures of X and Y.

Lemma 3.2. *Isomorphisms* (26) *define a G-braiding on* $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$.

Proof. That isomorphisms (26) are indeed morphisms in $\mathfrak{L}_{\mathcal{D}}(\mathcal{C})$ follows from commutativity of the diagram

$$
\begin{array}{ccccc}
X \otimes Y \otimes V & \xrightarrow{\mathrm{id}_X \otimes \delta_V} & X \otimes V \otimes Y & \xrightarrow{\gamma_V \otimes \mathrm{id}_Y} & V \otimes X \otimes Y \\
{\scriptstyle c_{X,Y} \otimes \mathrm{id}_V} \downarrow & \nearrow {\scriptstyle c_{X \otimes V, Y}} & & \nwarrow {\scriptstyle c_{V \otimes X, Y}} & \downarrow {\scriptstyle \mathrm{id}_V \otimes c_{X,Y}} \quad (29) \\
T_g(Y) \otimes X \otimes V & \xrightarrow{\mathrm{id}_{T_g(Y)} \otimes \gamma_V} & T_g(Y) \otimes V \otimes X & \xrightarrow{T_g(\delta)_V \otimes \mathrm{id}_X} & V \otimes T_g(Y) \otimes X,
\end{array}
$$

where $(X, \gamma) \in \mathfrak{L}_{\mathcal{D}}(\mathcal{C}_g)$, $(Y, \delta) \in \mathfrak{L}_{\mathcal{D}}(\mathcal{C})$, and $V \in \mathcal{D}$. Indeed, the parallelogram in the middle commutes by naturality of c, and the two triangular faces commute since the natural isomorphism (25) is an isomorphism of $\mathcal{D}$-bimodule functors.

It is straightforward to check that isomorphisms $c_{X,Y}$ satisfy the compatibility conditions of Definition 2.10. $\qquad\square$

The constructions and arguments above prove the following theorem.

Theorem 3.3. *Let G be a finite group and let $\mathcal{C}$ be a fusion category with a faithful G-grading whose trivial component is $\mathcal{D}$. The relative center $\mathfrak{L}_{\mathcal{D}}(\mathcal{C})$ has a canonical structure of a braided G-crossed category.*

Remark 3.4. In particular, to every G-extension of a fusion category $\mathcal{D}$ we assigned an action of G by braided autoequivalences of $\mathfrak{L}(\mathcal{D})$. This assignment is studied in detail in [Etingof et al. 2009].

3B. *The center $\mathfrak{L}(\mathcal{C})$ as an equivariantization.* As before, let G be a finite group and let $\mathcal{C}$ be a fusion category with a faithful G-grading (17). Let $\mathfrak{L}_{\mathcal{D}}(\mathcal{C})$ be the braided G-crossed category constructed in Section 3A.

Theorem 3.5. *There is an equivalence of braided fusion categories*

$$
\mathfrak{L}_{\mathcal{D}}(\mathcal{C})^G \xrightarrow{\sim} \mathfrak{L}(\mathcal{C}). \tag{30}
$$

Proof. We see from (26) that a G-equivariant object in $\mathfrak{L}_{\mathcal{D}}(\mathcal{C})$ has a structure of a central object in $\mathcal{C}$ defined as in (13). It follows from definitions that the corresponding tensor functor $\mathfrak{L}_{\mathcal{D}}(\mathcal{C})^G \to \mathfrak{L}(\mathcal{C})$ is braided.

Conversely, given an object Y in $\mathfrak{L}(\mathcal{C})$, consider its forgetful image $\tilde{Y}$ in $\mathfrak{L}_{\mathcal{D}}(\mathcal{C})$. Combining the central structure of Y with isomorphism (26) we obtain a family of isomorphisms

$$
\tilde{Y} \otimes X \xrightarrow{\sim} T_g(\tilde{Y}) \otimes X, \qquad X \in \mathcal{C}_g, \ g \in G,
$$

which gives rise to the isomorphism of $\mathcal{D}$-bimodule functors $\tilde{Y} \otimes \ ? \xrightarrow{\sim} T_g(\tilde{Y}) \otimes \ ? :$ $\mathcal{C}_g \to \mathcal{C}$. By Proposition 3.1 we obtain a natural isomorphism $\tilde{Y} \xrightarrow{\sim} T_g(\tilde{Y})$ and, hence, a G-equivariant structure on $\tilde{Y}$. Thus, we have a tensor functor $\mathfrak{L}(\mathcal{C}) \to \mathfrak{L}_{\mathcal{D}}(\mathcal{C})^G$. It is clear that the two functors are quasiinverses of each other. $\qquad\square$

We describe the Tannakian subcategory $\mathscr{E} \cong \mathrm{Rep}(G) \subset \mathscr{Z}(\mathscr{C})$ corresponding to equivalence (30). For any representation $\pi : G \to GL(V)$ of the grading group G, consider an object I_π in $\mathscr{Z}(\mathscr{C})$ where $I_\pi = V \otimes \mathbf{1}$ as an object of $\mathscr{C}$ with the permutation isomorphism

$$c_{I_\pi, X} := \pi(g) \otimes \mathrm{id}_X : I_\pi \otimes X \cong X \otimes I_\pi, \qquad \text{when } X \in \mathscr{C}_g. \qquad (31)$$

Then $\mathscr{E}$ is the subcategory of $\mathscr{Z}(\mathscr{C})$ consisting of objects I_π, where π runs through all finite-dimensional representations of G.

Remark 3.6. Here is another description of the subcategory $\mathscr{E}$: it consists of all objects in $\mathscr{Z}(\mathscr{C})$ sent to Vec by the forgetful functor $\mathscr{Z}(\mathscr{C}) \to \mathscr{Z}_{\mathscr{D}}(\mathscr{C})$.

Corollary 3.7. *Let $\mathscr{C}$ be a faithfully G-graded fusion category with the trivial component $\mathscr{D}$. Let $\mathscr{E} = \mathrm{Rep}(G) \subset \mathscr{Z}(\mathscr{C})$ be the Tannakian subcategory constructed above. Then the de-equivariantization category $(\mathscr{E}')_G$ is braided tensor equivalent to $\mathscr{Z}(\mathscr{D})$.*

Proof. The statement follows from Theorem 3.5 since $(\mathscr{E}')_G$ is the trivial component of the grading of $\mathscr{Z}(\mathscr{C})_G = \mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. □

Remark 3.8. The assignment above

$$\{G\text{-extensions of } \mathscr{D}\} \mapsto \{\text{braided } G\text{-crossed extensions of } \mathscr{Z}(\mathscr{D})\} \qquad (32)$$

can be thought of as an analogue of the center construction for G-extensions.

Next, we describe simple objects of $\mathscr{Z}(\mathscr{C})$. For any conjugacy class K in G fix a representative $a_K \in K$. Let G_K denote the centralizer of a_K in G. Note that the action (24) of G on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ restricts to the action of G_K on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_{a_K})$.

Proposition 3.9. *There is a bijection between the set of isomorphism classes of simple objects of $\mathscr{Z}(\mathscr{C})$ and pairs (K, X), where K is a conjugacy class of G and X is a simple G_K-equivariant object of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_{a_K})$.*

Proof. By Theorem 3.5 we have $\mathscr{Z}(\mathscr{C}) \simeq \mathscr{Z}_{\mathscr{D}}(\mathscr{C})^G$, so the stated parameterization is immediate from the description of simple objects of the equivariantization category given in Proposition 2.7. □

3C. *A criterion for a graded fusion category to be group-theoretical.* We have seen in Corollary 3.7 that $\mathscr{Z}(\mathscr{C})$ contains a Tannakian subcategory $\mathscr{E} = \mathrm{Rep}(G)$ such that the de-equivariantization $(\mathscr{E}')_G$ is braided equivalent to $\mathscr{Z}(\mathscr{D})$, where $\mathscr{D}$ is the trivial component of $\mathscr{C}$. Furthermore, by Remark 2.11, there is a canonical action of G on $\mathscr{Z}(\mathscr{D})$, by braided autoequivalences. By [Drinfeld et al. 2009], Tannakian subcategories of $\mathscr{Z}(\mathscr{C})$ containing $\mathscr{E}$ bijectively correspond to G-stable Tannakian subcategories of $(\mathscr{E}')_G \simeq \mathscr{Z}(\mathscr{D})$. Combining this observation with Theorem 2.5(ii) we obtain the following criterion.

Corollary 3.10. *A graded fusion category* $\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g$, $\mathscr{C}_e = \mathscr{D}$, *is group-theoretical if and only if* $\mathfrak{Z}(\mathscr{D})$ *contains a G-stable Lagrangian subcategory.*

Corollary 3.10 will be useful in Section 4D, where we characterize group-theoretical Tambara–Yamagami categories.

We can specialize Corollary 3.10 to equivariantization categories. Let G be a finite group acting on a fusion category $\mathscr{C}$. The equivariantization $\mathscr{C}^G$ is Morita equivalent to the crossed product category $\mathscr{C} \rtimes G$ (see Section 2D). Therefore, $\mathfrak{Z}(\mathscr{C}^G) \cong \mathfrak{Z}(\mathscr{C} \rtimes G)$. Clearly, the trivial component of $\mathfrak{Z}(\mathscr{C} \rtimes G)_G$ is $\mathfrak{Z}(\mathscr{C})$ and the canonical action of G on $\mathfrak{Z}(\mathscr{C})$ is induced from the action of G on $\mathscr{C}$ in an obvious way.

Corollary 3.11. *The equivariantization $\mathscr{C}^G$ is group-theoretical if and only if there exists a G-stable Lagrangian subcategory of $\mathfrak{Z}(\mathscr{C})$.*

Remark 3.12. Let G act on $\mathscr{C}$ as before. One can check (independently from the results of this section) that the G-set of Lagrangian subcategories of $\mathfrak{Z}(\mathscr{C})$ is isomorphic to the G-set consisting of indecomposable $\mathscr{C}$-module categories $\mathscr{M}$ such that the dual category $\mathscr{C}^*_{\mathscr{M}}$ is pointed. This isomorphism is given by the map constructed in [Naidu and Nikshych 2008, Theorem 4.17]. Thus, the criterion in Corollary 3.11 is the same as [Nikshych 2008, Corollary 3.6].

3D. *Example: the relative center of a crossed product category.* Let G be a finite group and let $g \mapsto T_g$, $g \in G$, be an action of G on a fusion category $\mathscr{D}$. Let $\mathscr{C} := \mathscr{D} \rtimes G$ be the crossed product category defined in Section 2D. It has a natural grading

$$\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g, \quad \text{where } \mathscr{C}_g = \{Y \boxtimes g \mid Y \in \mathscr{D}\}.$$

We describe the braided G-crossed fusion category structure on the relative center

$$\mathfrak{Z}_{\mathscr{D}}(\mathscr{C}) = \bigoplus_{g \in G} \mathfrak{Z}_{\mathscr{D}}(\mathscr{C}_g).$$

By definition, the objects of $\mathfrak{Z}_{\mathscr{D}}(\mathscr{C}_g)$ are pairs $(Y \boxtimes g, \gamma)$, where $Y \in \mathscr{D}$ and

$$\gamma = \{\gamma_X : X \otimes Y \xrightarrow{\sim} Y \otimes T_g(X)\}_{X \in \mathscr{D}} \tag{33}$$

is a natural family of isomorphisms satisfying natural compatibility conditions. Thus, $\mathfrak{Z}_{\mathscr{D}}(\mathscr{C}_g)$ can be viewed as a "deformation" of $\mathfrak{Z}(\mathscr{D})$ by means of T_g.

The action of G on $\mathscr{D}$ induces an action $h \mapsto \tilde{T}_h$ on $\mathfrak{Z}_{\mathscr{D}}(\mathscr{C})$ defined as follows. Applying T_h, $h \in G$, to $\gamma_{T_{h^{-1}}(X)}$ in (33), we obtain an isomorphism

$$\tilde{\gamma}_X : X \otimes T_h(Y) \xrightarrow{\sim} T_h(Y) \otimes T_{hgh^{-1}}(X). \tag{34}$$

Set $\tilde{T}_h(Y \boxtimes g, \gamma) := (T_h(Y) \boxtimes hgh^{-1}, \tilde{\gamma})$. Thus, $\tilde{T}_h$ maps $\mathfrak{Z}_{\mathscr{D}}(\mathscr{C}_g)$ to $\mathfrak{Z}_{\mathscr{D}}(\mathscr{C}_{hgh^{-1}})$.

Finally, the G-braiding between objects $(X \boxtimes h) \in \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h)$ and $(Y \boxtimes g) \in \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g)$ comes from the isomorphism

$$
(X \boxtimes h) \otimes (Y \boxtimes g) = (X \otimes T_h(Y)) \boxtimes hg \xrightarrow{\tilde{\gamma}} (T_h(Y) \otimes T_{hgh^{-1}}(X)) \boxtimes hg
$$
$$
= (T_h(Y) \boxtimes hgh^{-1}) \otimes (X \boxtimes h)
$$
$$
= \tilde{T}_h(Y \boxtimes g) \otimes (X \boxtimes h).
$$

By Theorem 3.5, the category $\mathscr{Z}(\mathscr{D} \rtimes G) \cong \mathscr{Z}(\mathscr{D}^G)$ is equivalent to the equivariantization of the braided G-crossed category above.

4. The centers of Tambara–Yamagami categories

Our goal in this section is to apply techniques developed in Section 3 to Tambara–Yamagami categories introduced in [Tambara and Yamagami 1998] (see Section 4A below for the definition). Namely, using the techniques in Section 3 we establish a criterion for a Tambara–Yamagami category to be group-theoretical. We then use this criterion together with Corollary 3.11 to produce a series of non-group-theoretical semisimple Hopf algebras. In this section we assume that our ground field k is the field of complex numbers $\mathbb{C}$. We begin by recalling the definition of a Tambara–Yamagami category.

4A. *Definition of Tambara–Yamagami categories.* Let $\mathbb{Z}_2 = \langle \delta \mid \delta^2 = 1 \rangle$ be the cyclic group of order 2.

Tambara and Yamagami [1998] completely classified all $\mathbb{Z}_2$-graded fusion categories in which all but one simple objects are invertible and the noninvertible simple object has nontrivial graded degree.

They showed that any such category $\mathscr{T}\mathscr{Y}(A, \chi, \tau)$ is determined, up to an equivalence, by a finite abelian group A, a nondegenerate symmetric bilinear form $\chi : A \times A \to k^{\times}$, and a square root $\tau \in k$ of $|A|^{-1}$. The category $\mathscr{T}\mathscr{Y}(A, \chi, \tau)$ is described as follows. It is a skeletal category (that is, such that any two isomorphic objects are equal) with simple objects $\{a \mid a \in A\}$ and m, and tensor product

$$
a \otimes b = a + b, \quad a \otimes m = m, \quad m \otimes a = m, \quad m \otimes m = \bigoplus_{a \in A} a,
$$

for all $a, b \in A$, and the unit object $0 \in A$. The associativity constraints are given by

$$
\alpha_{a,b,c} = \mathrm{id}_{a+b+c}, \quad \alpha_{a,b,m} = \mathrm{id}_m, \quad \alpha_{a,m,b} = \chi(a, b)\, \mathrm{id}_m, \quad \alpha_{m,a,b} = \mathrm{id}_m,
$$
$$
\alpha_{a,m,m} = \bigoplus_{b \in A} \mathrm{id}_b, \quad \alpha_{m,a,m} = \bigoplus_{b \in A} \chi(a, b)\, \mathrm{id}_b,
$$
$$
\alpha_{m,m,a} = \bigoplus_{b \in A} \mathrm{id}_b, \quad \alpha_{m,m,m} = \bigoplus_{a,b \in A} \tau \chi(a, b)^{-1}\, \mathrm{id}_m \,.
$$

The unit constraints are the identity maps. The category $\mathcal{TY}(A, \chi, \tau)$ is rigid with $a^* = -a$ and $m^* = m$ (with obvious evaluation and coevaluation maps).

Let $n := |A|$. The dimensions of simple objects of $\mathcal{TY}(A, \chi, \tau)$ are $\mathrm{FPdim}(a) = 1$, $a \in A$, and $\mathrm{FPdim}(m) = \sqrt{n}$. We have $\mathrm{FPdim}(\mathcal{TY}(A, \chi, \tau)) = 2n$.

The $\mathbb{Z}_2$-grading on $\mathcal{TY}(A, \chi, \tau)$ is

$$\mathcal{TY}(A, \chi, \tau) = \mathcal{TY}(A, \chi, \tau)_1 \oplus \mathcal{TY}(A, \chi, \tau)_\delta,$$

where $\mathcal{TY}(A, \chi, \tau)_1$ is the full fusion subcategory generated by the invertible objects $a \in A$ and $\mathcal{TY}(A, \chi, \tau)_\delta$ is the full abelian subcategory generated by the object m.

Let $\mathscr{C} := \mathcal{TY}(A, \chi, \tau)$ and $\mathscr{D} := \mathcal{TY}(A, \chi, \tau)_1$.

4B. Braided $\mathbb{Z}_2$-crossed category $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. First, let us describe the simple objects of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}) = \mathscr{Z}(\mathscr{C}_1) \oplus \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_\delta)$. Let $\widehat{A} := \mathrm{Hom}(A, k^\times)$. Clearly, $\mathscr{Z}(\mathscr{C}_1) = \mathscr{Z}(\mathrm{Vec}_A)$, so its simple objects are parameterized by $(a, \phi) \in A \times \widehat{A}$. The object $X_{(a,\phi)}$ corresponding to such a pair is equal to a as an object of $\mathscr{C}$ and its central structure is given by

$$\phi(x)\, \mathrm{id}_{a+x} : x \otimes X_{(a,\phi)} \xrightarrow{\sim} X_{(a,\phi)} \otimes x. \tag{35}$$

Using Definition 2.1 we see that simple objects of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_\delta)$ are parameterized by functions $\rho : A \to k^\times$ satisfying

$$\rho(a + b) = \chi(a, b)^{-1} \rho(a)\rho(b), \quad a, b \in A \tag{36}$$

(clearly, such functions form a torsor over $\widehat{A}$). The corresponding object Z_ρ is equal to m as an object of $\mathscr{C}$ and has the relative central structure

$$\rho(x)\, \mathrm{id}_m : x \otimes Z_\rho \xrightarrow{\sim} Z_\rho \otimes x, \quad x \in A. \tag{37}$$

Let $A \to \widehat{A} : a \mapsto \widehat{a}$ be the homomorphism defined by $\widehat{a}(x) = \chi(x, a)$. Similarly, let $\widehat{A} \to A : \phi \mapsto \widehat{\phi}$ be the homomorphism defined by $\phi(x) = \chi(x, \widehat{\phi})$ (recall that χ is nondegenerate). Clearly, these two maps are inverses of each other.

The fusion rules of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ are computed using formula (3) :

$$X_{(a,\phi)} \otimes X_{(b,\psi)} = X_{(a+b,\phi+\psi)},$$
$$X_{(a,\phi)} \otimes Z_\rho = Z_{\rho\phi(-\widehat{a})},$$
$$Z_\rho \otimes X_{(a,\phi)} = Z_{\rho\phi(-\widehat{a})},$$
$$Z_{\rho'} \otimes Z_\rho = \bigoplus_{a \in A} X_{(a,\widehat{a}\rho'/\overline{\rho})}.$$

We have $X_{(a,\phi)}^* = X_{(-a,-\phi)}$ and $Z_\rho^* = Z_{\overline{\rho}}$, where $\overline{\rho}(x) = \rho(-x)$, $x \in A$.

Using the construction given in Section 3A we see that the action of $\mathbb{Z}_2$ on $\mathscr{L}_{\mathscr{D}}(\mathscr{C})$ is given by

$$T_1 = \mathrm{id}_{\mathscr{L}_{\mathscr{D}}(\mathscr{C})}; \qquad T_\delta(X_{(a,\phi)}) = X_{(-\widehat{\phi}, -\widehat{a})}, \qquad T_\delta(Z_\rho) = Z_\rho. \qquad (38)$$

The monoidal functor structure on $\mathbb{Z}_2 \to \mathrm{Aut}_\otimes(\mathscr{L}_{\mathscr{D}}(\mathscr{C}))$ is given by the natural isomorphism $\gamma := \gamma_{\delta,\delta} : T_\delta \circ T_\delta \xrightarrow{\sim} T_1$ defined by

$$\gamma_{X_{(a,\phi)}} = \phi(a)\,\mathrm{id}_{X_{(a,\phi)}}, \qquad \gamma_{Z_\rho} = \left(\tau \sum_{x \in A} \rho(x)^{-1}\right) \mathrm{id}_{Z_\rho}.$$

The crossed braiding morphisms on $\mathscr{L}_{\mathscr{D}}(\mathscr{C})$ are given by

$$c_{X_{(a,\phi)}, X_{(b,\psi)}} = \psi(a)\,\mathrm{id}_{a+b} : X_{(a,\phi)} \otimes X_{(b,\psi)} \xrightarrow{\sim} X_{(b,\psi)} \otimes X_{(a,\phi)},$$
$$c_{X_{(a,\phi)}, Z_\rho} = \rho(a)\,\mathrm{id}_m : X_{(a,\phi)} \otimes Z_\rho \xrightarrow{\sim} Z_\rho \otimes X_{(a,\phi)},$$
$$c_{Z_\rho, X_{(a,\phi)}} = \mathrm{id}_m : Z_\rho \otimes X_{(a,\phi)} \xrightarrow{\sim} X_{(-\widehat{\phi}, -\widehat{a})} \otimes Z_\rho,$$
$$c_{Z_{\rho'}, Z_\rho} = \oplus_{a \in A}\, \rho(-a)^{-1}\,\mathrm{id}_a : Z_{\rho'} \otimes Z_\rho \xrightarrow{\sim} Z_\rho \otimes Z_{\rho'}.$$

4C. *The equivariantization category $\mathscr{L}_{\mathscr{D}}(\mathscr{C})^{\mathbb{Z}_2}$.* A simple calculation of $\mathbb{Z}_2$-equivariant objects in $\mathscr{L}_{\mathscr{D}}(\mathscr{C})$ establishes the following.

Proposition 4.1. *The following is a complete list of simple objects of $\mathscr{L}_{\mathscr{D}}(\mathscr{C})^{\mathbb{Z}_2} \cong \mathscr{L}(\mathscr{TY}(A, \chi, \tau))$ up to an isomorphism:*

(1) *$2n$ invertible objects parameterized by pairs (a, ϵ), where $a \in A$ and $\epsilon^2 = \chi(a, a)^{-1}$. The corresponding object $X_{a,\epsilon}$ is equal to $X_{(a,-\widehat{a})}$ as an object of $\mathscr{L}_{\mathscr{D}}(\mathscr{C})$ and has $\mathbb{Z}_2$-equivariant structure*

$$\epsilon\,\mathrm{id}_{X_{(a,-\widehat{a})}} : T_\delta(X_{(a,-\widehat{a})}) \xrightarrow{\sim} X_{(a,-\widehat{a})};$$

(2) *$\frac{n(n-1)}{2}$ two-dimensional objects parameterized by unordered pairs (a, b) of distinct objects in A. The corresponding object $Y_{a,b}$ is equal to $X_{(a,-\widehat{b})} \oplus X_{(b,-\widehat{a})}$ as an object of $\mathscr{L}_{\mathscr{D}}(\mathscr{C})$ and has $\mathbb{Z}_2$-equivariant structure*

$$\left(\mathrm{id}_{X_{(a,-\widehat{b})}} \oplus \chi(a, b)^{-1}\,\mathrm{id}_{X_{(b,-\widehat{a})}}\right) : T_\delta(X_{(a,-\widehat{b})} \oplus X_{(b,-\widehat{a})}) \xrightarrow{\sim} X_{(a,-\widehat{b})} \oplus X_{(b,-\widehat{a})};$$

(3) *$2n \sqrt{n}$-dimensional objects parameterized by pairs (ρ, Δ), where $\rho : A \to k^\times$ satisfies (36) and $\Delta^2 = \tau \sum_{x \in A} \rho(x)^{-1}$. The corresponding object $Z_{\rho,\Delta}$ is equal to Z_ρ as an object of $\mathscr{L}_{\mathscr{D}}(\mathscr{C})$ and has $\mathbb{Z}_2$-equivariant structure*

$$\Delta\,\mathrm{id}_{Z_\rho} : T_\delta(Z_\rho) \xrightarrow{\sim} Z_\rho.$$

Recall from [Etingof et al. 2005] that in a braided fusion category of an integer Frobenius–Perron dimension there is a canonical choice of a twist θ such that the categorical dimensions of objects coincide with their Frobenius–Perron

dimensions. Namely, for any simple object X the scalar θ_X is defined in such a way that the composition

$$\mathbf{1} \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{\theta_X c_{X,X^*}} X^* \otimes X \xrightarrow{\text{ev}_X} \mathbf{1} \tag{39}$$

is equal to $\text{FPdim}(X)\,\text{id}_X$.

Let θ be the canonical twist on $\mathfrak{Z}(\mathscr{C})$. Using the previous observation, explicit formulas from Section 4B, and Section 2F, we immediately obtain the following.

$$\theta_{X_{a,\epsilon}} = \chi(a,a)^{-1}, \quad \theta_{Y_{a,b}} = \chi(a,b)^{-1}, \quad \theta_{Z_{\rho,\Delta}} = \Delta.$$

Using the fusion rules of $\mathfrak{Z}(\mathscr{C})$ (which may be computed using the explicit formulas in Section 4B), values of the twists above, and the well known formula

$$S_{X,Y} = \theta_X^{-1}\theta_Y^{-1} \sum_Z N_{X,Y}^Z \theta_Z d_Z, \tag{40}$$

we obtain the S- and T-matrices of $\mathfrak{Z}(\mathscr{C})$:

$$S_{X_{a,\epsilon},X_{a',\epsilon'}} = \chi(a,a')^2, \quad S_{X_{a,\epsilon},Y_{b,c}} = 2\chi(a,b+c),$$

$$S_{X_{a,\epsilon},Z_{\rho,\Delta}} = \epsilon\sqrt{n}\rho(a), \quad S_{Y_{a,b},Y_{c,d}} = 2\left(\chi(a,d)\chi(b,c) + \chi(a,c)\chi(b,d)\right),$$

$$S_{Y_{a,b},Z_{\rho,\Delta}} = 0, \quad S_{Z_{\rho,\Delta},Z_{\rho',\Delta'}} = \frac{1}{\Delta\,\Delta'}\sum_{a\in A}\chi(a,a)^2\rho(a)\rho'(a);$$

$$T_{X_{a,\epsilon}} = \chi(a,a)^{-1}, \quad T_{Y_{a,b}} = \chi(a,b)^{-1}, \quad T_{Z_{\rho,\Delta}} = \Delta.$$

Proposition 4.2. *The maximal pointed subcategory of $\mathfrak{Z}(\mathscr{C})$ is nondegenerate if and only if $|A|$ is odd.*

Proof. Let $a \in A$ be an element of order 2. Then $X_{a,\epsilon}$ centralizes every invertible object of $\mathfrak{Z}(\mathscr{C})$. $\square$

Remark 4.3. We note that simple objects and the S- and T-matrices of $\mathfrak{Z}(\mathscr{C})$ were described in [Izumi 2001] using very different methods.

4D. *A criterion for a Tambara–Yamagami category to be group-theoretical.* The group $A \times \widehat{A}$ is equipped with a canonical nondegenerate quadratic form $q : A \times \widehat{A} \to k^\times$ given by

$$q((a,\phi)) := \phi(a), \quad (a,\phi) \in A \times \widehat{A}.$$

We will call a subgroup $B \subset A \times \widehat{A}$ *Lagrangian* if $q|_B = 1$ and $B = B^\perp$ with respect to the bilinear form defined by q. Lagrangian subgroups of $A \times \widehat{A}$ correspond to Lagrangian subcategories of $\mathfrak{Z}(\text{Vec}_A) \cong \text{Vec}_{A\times\widehat{A}}$.

The braided tensor autoequivalence T_δ of $\mathfrak{Z}(\text{Vec}_A)$ defined in Section 4B determines an order 2 automorphism of $A \times \widehat{A}$, which we denote simply by δ:

$$\delta((a,\phi)) = (-\widehat{\phi}, -\widehat{a}), \quad (a,\phi) \in A \times \widehat{A}. \tag{41}$$

Definition 4.4. We will say that a subgroup $L \subset A$ is *Lagrangian (with respect to χ)* if $L = L^{\perp}$ with respect to the inner product on A given by χ. Equivalently, $|L|^2 = |A|$ and $\chi|_L = 1$.

Lemma 4.5. *Let A be an abelian 2-group such that $|A| = 2^{2n}$ and let χ be a nondegenerate symmetric bilinear form on A. Then A contains a Lagrangian subgroup.*

Proof. It suffices to show that A contains an isotropic element, that is, an element $x \in A$, $x \neq 0$, such that $\chi(x, x) = 1$. Then one can pass from A to $\langle x \rangle^{\perp}/\langle x \rangle$ and use induction.

Suppose that A is cyclic with a generator a. Then $2^{2n}a = 0$ and $\chi(a, a)$ is a (2^{2n})th root of unity, hence $\chi(2^n a, 2^n a) = \chi(a, a)^{2^{2n}} = 1$.

If A is not cyclic then it contains a subgroup $A_0 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let x_1, x_2 be distinct nonzero elements of A_0. Suppose $\chi(x_i, x_i) \neq 1$, $i = 1, 2$. Then $\chi(x_i, x_i) = -1$ and $\chi(x_1 + x_2, x_1 + x_2) = 1$, as desired. $\square$

Theorem 4.6. *Let $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$ be a Tambara–Yamagami fusion category. Then $\mathcal{C}$ is group-theoretical if and only if A contains a Lagrangian subgroup (with respect to χ).*

Proof. By Corollary 3.10, $\mathcal{C}$ is group-theoretical if and only if $\mathcal{Z}(\mathcal{D})$ contains a T_{δ}-stable Lagrangian subcategory. Equivalently, $\mathcal{C}$ is group-theoretical if and only if $A \times \widehat{A}$ contains a Lagrangian subgroup B stable under the action

$$(a, \phi) \mapsto (\widehat{\phi}, \widehat{a}). \tag{42}$$

This condition on B is the same as being stable under the action of δ from (41).

Let L be a Lagrangian (with respect to χ) subgroup of A and let $\widehat{L} := \{\widehat{a} \mid a \in L\}$. Then $L \times \widehat{L}$ is a Lagrangian subgroup of $A \times \widehat{A}$ stable under (42). Hence $\mathcal{C}$ is group-theoretical.

Conversely, suppose that $\mathcal{C}$ is group-theoretical. Let us write $A = A_{\text{even}} \oplus A_{\text{odd}}$, where A_{even} is the Sylow 2-subgroup of A and A_{odd} is the maximal odd order subgroup of A. Since $|A|$ must be a square, we conclude that $|A_{\text{even}}|$ is a square, and so A_{even} contains a Lagrangian subgroup with respect to $\chi|_{A_{\text{even}}}$ by Lemma 4.5.

So it remains to show that A_{odd} contains a Lagrangian subgroup with respect to $\chi|_{A_{\text{odd}}}$. For this end we may assume that $|A|$ is odd. Let $B \subset A \times \widehat{A}$ be a Lagrangian subgroup stable under (42). Then $B = B_+ \oplus B_-$, where

$$B_{\pm} := \{(a, \pm\widehat{a}) \mid (a, \pm\widehat{a}) \in B\}.$$

Let $L_{\pm} = B_{\pm} \cap (A \times \{1\})$. Then $|L_+||L_-| = |A|$, and $\chi|_{L_{\pm}} = 1$. Hence, $L_{\pm}$ are Lagrangian subgroups of A. $\square$

Remark 4.7. It was observed in [Etingof et al. 2005, Remark 8.48] that for an odd prime p and elliptic bicharacter χ on $A = (\mathbb{Z}/p\mathbb{Z})^2$, the category $\mathcal{TY}((\mathbb{Z}/p\mathbb{Z})^2, \chi, \tau)$ is not group-theoretical. The criterion from Theorem 4.6 extends this observation.

4E. *A series of non-group-theoretical semisimple Hopf algebras obtained from Tambara–Yamagami categories.* Here we apply Corollary 3.11 to produce a series of non-group-theoretical fusion categories admitting fiber functors (that is, representation categories of non-group-theoretical semisimple Hopf algebras), generalizing examples constructed in [Nikshych 2008]. We refer the reader to [Montgomery 1993] as a reference on Hopf algebra theory.

Let A be a finite abelian group with a nondegenerate bilinear form χ. Let $\text{Aut}(A, \chi)$ denote the group of automorphisms of A preserving χ.

The following proposition was proved in [Nikshych 2008, Proposition 2.10].

Proposition 4.8. *There is an action of* $\text{Aut}(A, \chi)$ *on* $\mathcal{TY}(A, \chi, \tau)$ *given by* $g \mapsto T_g$, *where*

$$T_g(A) = g(a), \quad T_g(m) = m, \qquad a \in A, \ g \in \text{Aut}(A, \chi),$$

with the tensor structure of T_g *given by identity morphisms.*

Corollary 4.9. *Let G be a subgroup of* $\text{Aut}(A, \chi)$. *Then the fusion category* $\mathcal{TY}(A, \chi, \tau)^G$ *is group-theoretical if and only if there is a Lagrangian subgroup of* (A, χ) *stable under the action of G.*

Proof. Combine Corollary 3.11 and Theorem 4.6. □

We will say that a nondegenerate symmetric bilinear form $\chi : A \times A \to k^\times$ is *hyperbolic* if there are Lagrangian subgroups $L, L' \subset A$ such that $A = L \oplus L'$. In this case L' is isomorphic to the group $\widehat{L} = \text{Hom}(L, k^\times)$ of characters of L and χ is identified with the canonical bilinear form on $L \oplus \widehat{L}$.

It was demonstrated in Tambara [2000] that when $n = |A|$ is odd the category $\mathcal{TY}(A, \chi, \tau)$ admits a fiber functor (that is, $\mathcal{TY}(A, \chi, \tau)$ is equivalent to the representation category of a semisimple Hopf algebra) if and only if τ^{-1} is a positive integer and χ is hyperbolic.

Corollary 4.10. *Let p be an odd prime, let $L = (\mathbb{Z}/p\mathbb{Z})^N$, $N \geq 1$, let $A = L \oplus \widehat{L}$, and let $\chi : A \times A \to k^\times$ be the canonical bilinear form defined by*

$$\chi((a, \phi), (b, \psi)) = \psi(a)\phi(b), \quad a, b \in A, \ \phi, \psi \in \widehat{A}.$$

Suppose that G is a subgroup of $\text{Aut}(A, \chi)$ *not contained in any conjugate of* $\text{Aut}(L) \subset \text{Aut}(A, \chi)$. *Then the equivariantization category* $\mathcal{TY}(A, \chi, p^{-N})^G$ *is a non-group-theoretical fusion category equivalent to the representation category of a semisimple Hopf algebra of dimension $2p^{2N}|G|$.*

Proof. Note that $\text{Aut}(A, \chi)$ acts transitively on the set of Lagrangian subgroups of (A, χ) and the stabilizer of L is $\text{Aut}(L)$. Apply Corollary 4.9. □

Remark 4.11. The series of fusion categories in Corollary 4.10 extends the one constructed in [Nikshych 2008], where the case of $N = 1$ and $G = \mathbb{Z}/2\mathbb{Z}$ was considered.

5. Examples of modular categories arising from quadratic forms

As before, let $\mathscr{C} := \mathscr{TY}(A, \chi, \tau)$ be a Tambara–Yamagami category and let $\mathscr{D} := \mathscr{TY}(A, \chi, \tau)_1$ be the trivial component of $\mathbb{Z}_2$-grading of $\mathscr{TY}(A, \chi, \tau)$. In this section we assume that our ground field k is the field of complex numbers $\mathbb{C}$.

Suppose that the symmetric bicharacter $\chi : A \times A \to k^\times$ comes from a quadratic form on A, that is, there is a function $q : A \to k^\times$ such that

$$q(a+b) = q(a)q(b)\chi(a, b), \quad a, b \in A \quad \text{and} \quad q(-a) = q(a).$$

From the description obtained in Section 4B we observe that $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ contains a fusion subcategory spanned by the simple objects $X_{(a,\widehat{a})}, a \in A$, and $Z_{q^{-1}}$. It is clear from the Tambara–Yamagami classification in Section 4A that this category is equivalent to $\mathscr{C}$.

Proposition 5.1. *Suppose that the symmetric bicharacter χ comes from a quadratic form on A. Then $\mathscr{C}$ admits a $\mathbb{Z}_2$-crossed braided category structure. The equivariantization $\mathscr{C}^{\mathbb{Z}_2}$ is nondegenerate if and only if $|A|$ is odd.*

Proof. Clearly, $\mathscr{C}$ inherits the $\mathbb{Z}_2$-crossed braided category structure from $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. The nondegeneracy claim follows from Proposition 4.2 and Remark 2.13. □

Let us assume that $n := |A|$ is odd. Then χ corresponds to a unique quadratic form q. Let $\mathscr{E}(q, \pm) := \mathscr{C}^{\mathbb{Z}_2}$ be the modular category constructed in Proposition 5.1 (the $\pm$ corresponding to $\tau = \pm\frac{1}{\sqrt{n}}$, respectively). In what follows we describe the fusion rules and S- and T-matrices of $\mathscr{E}(q, \pm)$.

5A. *Fusion rules of $\mathscr{E}$.* Clearly, $\mathscr{E}(q, \pm)$ is a fusion category of dimension $4n$. It has the following simple objects:

two invertible objects, $\mathbf{1} = X_+$ and X_-;

$\frac{n-1}{2}$ two-dimensional objects Y_a, $a \in A - \{0\}$ (with $Y_{-a} = Y_a$); and

two $\sqrt{n}$-dimensional objects Z_l, $l \in \mathbb{Z}/2\mathbb{Z}$.

Here we simplify the notation used in Section 4C and define

$$X_\pm := X_{0,\pm1}, \quad Y_a := Y_{a,-a}, \quad Z_l := Z_{q^{-1}, \Delta_l},$$

where $\Delta_l, l \in \mathbb{Z}/2\mathbb{Z}$, are distinct square roots of $\pm\frac{1}{\sqrt{n}} \sum_{a \in A} q(a)$.

The fusion rules of $\mathcal{E}(q, \pm)$ are given by

$$X_- \otimes X_- = X_+, \qquad X_\pm \otimes Y_a = Y_a, \qquad\qquad X_+ \otimes Z_l = Z_l,$$

$$X_- \otimes Z_l = Z_{l+1}, \qquad Y_a \otimes Y_b = Y_{a+b} \oplus Y_{a-b}, \qquad Y_a \otimes Y_a = X_+ \oplus X_- \oplus Y_{2a},$$

$$Y_a \otimes Z_l = Z_0 \oplus Z_1, \quad Z_l \otimes Z_l = X_+ \oplus (\oplus Y_a), \quad Z_l \otimes Z_{l+1} = X_- \oplus (\oplus Y_a),$$

where $a, b \in A$ ($a \neq b$) and $l \in \mathbb{Z}/2\mathbb{Z}$. All objects of $\mathcal{E}(q, \pm)$ are self-dual.

Remark 5.2. Note that the fusion rules of $\mathcal{E}(q, \pm)$ do not depend on the quadratic form q and the number τ. We show below that the S- and T-matrices of $\mathcal{E}(q, \pm)$ do depend on q and τ.

5B. *S- and T-matrices of* $\mathcal{E}$.

Lemma 5.3. *The Gauss sums corresponding to q and q^2 are equal up to a sign, that is,*

$$\frac{\sum_{a \in A} q(a)^2}{\sum_{a \in A} q(a)} \in \{\pm 1\}.$$

Proof. Consider the group $A \times A$ with a nondegenerate quadratic form $Q = q \times q$. The Gaussian sum for this form is

$$\tau(A \times A, Q) = \sum_{a,b \in A} q(a)q(b) = \tau(A, q)^2.$$

The restriction of Q on the diagonal subgroup $D := \{(a, a) \mid a \in A\}$ is nondegenerate since $|A|$ is odd. The restriction of Q on the orthogonal complement $D^\perp = \{(a, -a) \mid a \in A\}$ is nondegenerate as well. By the multiplicativity of Gaussian sums we have

$$\tau(A \times A, Q) = \tau(D, Q)\tau(D^\perp, Q) = \left(\sum_{a \in A} q(a)^2\right)^2,$$

which implies the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using the formulas for the S- and T- matrices of $\mathcal{Z}(\mathcal{C})$ given in Section 4C we can write down the S- and T- matrices of $\mathcal{E}(q, \pm)$:

$$S_{X_\pm, X_\pm} = 1, \qquad S_{X_\mp, X_\pm} = 1, \qquad S_{X_\pm, Y_a} = 2, \qquad S_{Y_a, Z_l} = 0,$$

$$S_{X_+, Z_l} = \sqrt{n}, \qquad S_{X_-, Z_l} = -\sqrt{n}, \qquad S_{Y_a, Y_b} = 2\left(\frac{q(a+b)^2}{q(a)^2 q(b)^2} + \frac{q(a)^2 q(b)^2}{q(a+b)^2}\right),$$

$$S_{Z_l, Z_l} = \begin{cases} \pm\sqrt{n} \text{ if the Gauss sums of } q \text{ and } q^2 \text{ coincide,} \\ \mp\sqrt{n} \text{ otherwise,} \end{cases}$$

$$S_{Z_l, Z_{l+1}} = \begin{cases} \mp\sqrt{n} \text{ if the Gauss sums of } q \text{ and } q^2 \text{ coincide,} \\ \pm\sqrt{n} \text{ otherwise.} \end{cases}$$

$$T_{X_\pm} = 1, \quad T_{Y_a} = q(a)^2, \quad T_{Z_l} = \Delta_l.$$

(Recall that $\Delta_l, l \in \mathbb{Z}/2\mathbb{Z}$, are distinct square roots of $\pm \dfrac{1}{\sqrt{n}} \sum_{a \in A} q(a)$.)

5C. *Example with $A = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.* Let p be an odd prime and let $A :=$ $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol modulo p, that is, $\left(\frac{a}{p}\right) = 1$ if $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ is a square modulo p and -1 otherwise.

Let $a, b \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $\xi := e^{2\pi i/p}$. Consider the following nondegenerate quadratic form q on A:

$$q(x_1, x_2) = \xi^{ax_1^2 - bx_2^2}.$$

It is hyperbolic if $\left(\frac{ab}{p}\right) = 1$ and elliptic if $\left(\frac{ab}{p}\right) = -1$.

Lemma 5.4. *For every $a, b \in A^\times$, we have*

$$\sum_{x \in \mathbb{Z}/p\mathbb{Z}} \xi^{ax^2} = \begin{cases} \left(\frac{a}{p}\right)\sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ \left(\frac{a}{p}\right)i\sqrt{p} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and

$$\sum_{(x_1,x_2) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}} \xi^{ax_1^2 - bx_2^2} = \left(\frac{ab}{p}\right)p.$$

Proof. The first assertion is well known; see, for example, [Ireland and Rosen 1990]. The second assertion is an easy consequence of the first. □

Using Lemma 5.4 we can explicitly write the S-matrix of $\mathscr{E}(q, \pm)$:

$$S_{X_\pm, X_\pm} = 1, \; S_{X_\mp, X_\pm} = 1, \qquad S_{X_\pm, Y_{(x_1,x_2)}} = 2,$$
$$S_{X_+, Z_l} = p, \; S_{X_-, Z_l} = -p, \quad S_{Y_{(x_1,x_2)}, Y_{(y_1,y_2)}} = 4\operatorname{Re}(\xi^{4ax_1y_1 - 4bx_2y_2}),$$
$$S_{Y_{(x_1,x_2)}, Z_l} = 0, \quad S_{Z_l, Z_l} = \pm p, \qquad S_{Z_l, Z_{l+1}} = \mp p,$$

and its T-matrix:

$$T_{X_\pm} = 1, \quad T_{Y_{(x_1,x_2)}} = \xi^{2ax_1^2 - 2bx_2^2}, \quad T_{Z_l} = \Delta_l,$$

where $\Delta_l, l \in \mathbb{Z}/2\mathbb{Z}$, are distinct square roots of $\pm\left(\frac{ab}{p}\right)$.

The central charge of the modular category $\mathscr{E}(q, \pm)$ is

$$\zeta(\mathscr{E}(q, \pm)) = \left(\frac{ab}{p}\right).$$

Below we give the S- and T-matrices of the modular category $\mathscr{E}(q, \pm)$ for $p = 3$. Order simple objects of $\mathscr{E}(q, \pm)$ as follows: $\mathbf{1}, X_-, Y_{(0,1)}, Y_{(1,0)}, Y_{(1,1)}, Y_{(1,2)}, Z_+, Z_-$. There are four modular categories $\mathscr{E}(q, \pm)$ of dimension 36 corresponding to the choices of hyperbolic/elliptic q and $\tau = \pm\frac{1}{3}$.

(a) When q is hyperbolic we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & 3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & \pm 3 & \mp 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & \mp 3 & \pm 3 \end{pmatrix},$$

$T = \mathrm{diag}\{1, 1, \xi^2, \xi, 1, 1, 1, -1\}$ when $\tau = \frac{1}{3}$,

$T = \mathrm{diag}\{1, 1, \xi^2, \xi, 1, 1, i, -i\}$ when $\tau = -\frac{1}{3}$.

Note that both the corresponding modular categories are group-theoretical with central charge 1; in fact the one with $\tau = \frac{1}{3}$ is equivalent to the representation category of the double $D(S_3)$ of the symmetric group S_3 and the one with $\tau = -\frac{1}{3}$ is equivalent to the twisted double of S_3.

(b) When q is elliptic we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & 3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & \pm 3 & \mp 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & \mp 3 & \pm 3 \end{pmatrix},$$

$T = \mathrm{diag}\{1, 1, \xi, \xi, \xi^2, \xi^2, i, -i\}$ when $\tau = \dfrac{1}{3}$,

$T = \mathrm{diag}\{1, 1, \xi, \xi, \xi^2, \xi^2, 1, -1\}$ when $\tau = -\dfrac{1}{3}$.

Both the corresponding modular categories are not group-theoretical. They both have central charge -1 and so are not equivalent to centers of fusion categories. In particular, they are not equivalent to representation categories of any twisted group doubles.

5D. *Example with $A = \mathbb{Z}/p\mathbb{Z}$.* Let p be an odd prime and let $A := \mathbb{Z}/p\mathbb{Z}$. Let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $\xi := e^{2\pi i/p}$. Up to isomorphism there are two nondegenerate quadratic forms q on A:

$$q(x) = \xi^{ax^2},$$

one corresponding to $\left(\frac{a}{p}\right) = 1$ and another to $\left(\frac{a}{p}\right) = -1$.

Using Lemma 5.4 we can explicitly write the S-matrix of $\mathcal{E}(q, \pm)$:

$$S_{X_\pm, X_\pm} = 1, \qquad S_{X_\mp, X_\pm} = 1, \qquad S_{X_\pm, Y_x} = 2,$$
$$S_{X_+, Z_l} = \sqrt{p}, \qquad S_{X_-, Z_l} = -\sqrt{p}, \qquad S_{Y_x, Y_y} = 4\,\mathrm{Re}(\xi^{4axy}),$$
$$S_{Y_a, Z_l} = 0, \qquad S_{Z_l, Z_l} = \pm\left(\frac{2}{p}\right)\sqrt{p}, \quad S_{Z_l, Z_{l+1}} = \mp\left(\frac{2}{p}\right)\sqrt{p}.$$

Further, we have

$$T_{X_\pm} = 1, \quad T_{Y_x} = \xi^{-2ax^2}, \quad T_{Z_l} = \Delta_l,$$

where

$$\Delta_l,\ l \in \mathbb{Z}/2\mathbb{Z},\ \text{ are distinct square roots of } \begin{cases} \pm\left(\frac{a}{p}\right) & \text{if } p \equiv 1 \,(\mathrm{mod}\,4), \\ \pm\left(\frac{a}{p}\right)i & \text{if } p \equiv 3 \,(\mathrm{mod}\,4). \end{cases}$$

The central charge of the modular category $\mathcal{E}(q, \pm)$ is

$$\zeta(\mathcal{E}(q, \pm)) = \begin{cases} \left(\frac{2a}{p}\right) & \text{if } p \equiv 1 \,(\mathrm{mod}\,4), \\ -\left(\frac{2a}{p}\right)i & \text{if } p \equiv 3 \,(\mathrm{mod}\,4). \end{cases}$$

Below we give the S- and T-matrices of the modular category $\mathcal{E}(q, \pm)$ for $p = 3$ and 5. For $p = 3$ we order the simple objects as $\mathbf{1}, X_-, Y_1, Z_0, Z_1$ and for $p = 5$ we order them as $\mathbf{1}, X_-, Y_1, Y_2, Z_0, Z_1$. (In (c) and (d) below, $\xi = e^{2\pi i/5}$.)

(a) When $p = 3$ and $a = 1$ we have

$$S = \begin{pmatrix} 1 & 1 & 2 & \sqrt{3} & \sqrt{3} \\ 1 & 1 & 2 & -\sqrt{3} & -\sqrt{3} \\ 2 & 2 & -2 & 0 & 0 \\ \sqrt{3} & -\sqrt{3} & 0 & \mp\sqrt{3} & \pm\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & \pm\sqrt{3} & \mp\sqrt{3} \end{pmatrix},$$

$$T = \mathrm{diag}\left\{1, 1, \frac{-1+i\sqrt{3}}{2}, \frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}\right\} \quad \text{when } \tau = \frac{1}{\sqrt{3}},$$

$$T = \mathrm{diag}\left\{1, 1, \frac{-1+i\sqrt{3}}{2}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}\right\} \quad \text{when } \tau = -\frac{1}{\sqrt{3}}.$$

The central charge of both the corresponding modular categories is i.

(b) When $p = 3$ and $a = 2$ we have

$$S = \text{ the } S\text{-matrix in (a)},$$

$$T = \text{diag}\left\{1, 1, \frac{-1 - i\sqrt{3}}{2}, \frac{1 - i}{\sqrt{2}}, \frac{-1 + i}{\sqrt{2}}\right\} \quad \text{when } \tau = \frac{1}{\sqrt{3}},$$

$$T = \text{diag}\left\{1, 1, \frac{-1 - i\sqrt{3}}{2}, \frac{1 + i}{\sqrt{2}}, \frac{-1 - i}{\sqrt{2}}\right\} \quad \text{when } \tau = \frac{1}{\sqrt{3}}.$$

The central charge of both the corresponding modular categories is $-i$.

(c) When $p = 5$ and $a = 1$ we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & \sqrt{5} & \sqrt{5} \\ 1 & 1 & 2 & 2 & -\sqrt{5} & -\sqrt{5} \\ 2 & 2 & \sqrt{5} - 1 & -\sqrt{5} - 1 & 0 & 0 \\ 2 & 2 & -\sqrt{5} - 1 & \sqrt{5} - 1 & 0 & 0 \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \mp\sqrt{5} & \pm\sqrt{5} \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \pm\sqrt{5} & \mp\sqrt{5} \end{pmatrix},$$

$$T = \text{diag}\left\{1, 1, \xi^3, \xi^2, 1, -1\right\} \quad \text{when } \tau = \frac{1}{\sqrt{5}},$$

$$T = \text{diag}\left\{1, 1, \xi^3, \xi^2, i, -i\right\} \quad \text{when } \tau = -\frac{1}{\sqrt{5}}.$$

The central charge of both the corresponding modular categories is -1.

(d) When $p = 5$ and $a = 2$ we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & \sqrt{5} & \sqrt{5} \\ 1 & 1 & 2 & 2 & -\sqrt{5} & -\sqrt{5} \\ 2 & 2 & -\sqrt{5} - 1 & \sqrt{5} - 1 & 0 & 0 \\ 2 & 2 & \sqrt{5} - 1 & -\sqrt{5} - 1 & 0 & 0 \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \mp\sqrt{5} & \pm\sqrt{5} \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \pm\sqrt{5} & \mp\sqrt{5} \end{pmatrix},$$

$$T = \text{diag}\left\{1, 1, \xi, \xi^4, i, -i\right\} \quad \text{when } \tau = \frac{1}{\sqrt{5}},$$

$$T = \text{diag}\left\{1, 1, \xi, \xi^4, 1, -1\right\} \quad \text{when } \tau = -\frac{1}{\sqrt{5}}.$$

The central charge of both the corresponding modular categories is 1.

6. Appendix: Zeroes in S-matrices

There is a classical result of Burnside in character theory saying that if χ is an irreducible character of a finite group G and $\chi(1) > 1$, then $\chi(g) = 0$ for some $g \in G$; see [Berkovich and Zhmud' 1999, Chapter 21].

In this appendix we establish a categorical analogue of this result for weakly integral modular categories. Recall from [Etingof et al. 2008] that a fusion category $\mathscr{C}$ is called *weakly integral* if its Frobenius–Perron dimension is an integer. In this case the Frobenius–Perron dimension of every simple object of $\mathscr{C}$ is the square root of an integer [Etingof et al. 2005].

Let $\mathscr{C}$ be a weakly integral modular category with the S-matrix S. Let $\mathbb{O}(\mathscr{C})$ denote the set of all (representatives of isomorphism classes of) simple objects of $\mathscr{C}$. Given $X \in \mathbb{O}(\mathscr{C})$ define the sets

$$T_X = \{Y \in \mathbb{O}(\mathscr{C}) \mid S_{X,Y} = 0\}, \quad D_X = \mathbb{O}(\mathscr{C}) - (T_X \cup \{\mathbf{1}\}).$$

Clearly, we have a partition $\mathbb{O}(\mathscr{C}) = T_X \cup D_X \cup \{\mathbf{1}\}$. Let $\mathscr{T}_X$ and $\mathscr{D}_X$ be full abelian subcategories of $\mathscr{C}$ generated by T_X and D_X, respectively.

Let K be the field extension of $\mathbb{Q}$ generated by the entries of S. It is known [de Boer and Goeree 1991; Coste and Gannon 1994] that there is a root of unity ξ such that $K \subset \mathbb{Q}(\xi)$. In particular, the operation of taking the square of an absolute value of an element of S is well defined. Let $G := \mathrm{Gal}(K/\mathbb{Q})$. Every element $\sigma \in G$ comes from a permutation σ of $\mathbb{O}(\mathscr{C})$ such that $\sigma(S_{X,Y}) = S_{X,\sigma(Y)}$ for all $X, Y \in \mathbb{O}(\mathscr{C})$.

Let $\mathscr{C}$ be a weakly integral modular category. It was shown in [Etingof et al. 2005] that there is a canonical spherical structure on $\mathscr{C}$ such that categorical dimensions in $\mathscr{C}$ coincide with Frobenius–Perron dimensions. Let us fix this structure for the remainder of this section. For any $X \in \mathbb{O}(\mathscr{C})$ let d_X denote the dimension of X. For any full abelian subcategory $\mathscr{A}$ of $\mathscr{C}$ let $\dim \mathscr{A}$ denote the sum of squares of dimensions of simple objects of $\mathscr{A}$.

Theorem 6.1. *Let $\mathscr{C}$ be a weakly integral modular category with the S-matrix S. Then T_X is not empty for every noninvertible simple object X of $\mathscr{C}$. That is, every row (column) of S corresponding to a noninvertible simple object contains at least one zero entry.*

Proof. Note that the statement of Proposition does not depend on the choice of spherical structure.

We have $\sum_{Y \in \mathbb{O}(\mathscr{C})} |S_{X,Y}|^2 = \dim \mathscr{C}$; hence,

$$1 = \frac{\dim \mathscr{C}}{d_X^2} - \sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 = \frac{1 + \dim \mathscr{T}_X}{d_X^2} - \left(\sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 - \frac{\dim \mathscr{D}_X}{d_X^2} \right), \quad (43)$$

where d_X denotes the dimension of X. It suffices to check that

$$\frac{1}{\dim \mathcal{D}_X} \sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 \geq \frac{1}{d_X^2}, \qquad (44)$$

since then (43) implies that $1 \leq (1 + \dim \mathcal{T}_X)/d_X^2$, whence

$$\dim \mathcal{T}_X \geq d_X^2 - 1. \qquad (45)$$

But X is noninvertible so $d_X > 1$ and $\mathcal{T}_X \neq 0$.

Rewriting the left hand side of (44) as the sum of $\dim \mathcal{D}_X$ terms and using the inequality of arithmetic and geometric means we obtain

$$\frac{1}{\dim \mathcal{D}_X} \sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 = \frac{1}{\dim \mathcal{D}_X} \sum_{Y \in D_X} d_Y^2 \left| \frac{S_{X,Y}}{d_X d_Y} \right|^2$$

$$\geq \frac{1}{d_X^2} \left(\prod_{Y \in D_X} \left| \frac{S_{X,Y}}{d_Y} \right|^{2d_Y^2} \right)^{1/\dim \mathcal{D}_X}.$$

The set D_X is clearly stable under all automorphisms in the Galois group, and hence so is the product $\prod_{Y \in D_X} \left| S_{X,Y}/d_Y \right|^{2d_Y^2}$. Therefore, this product belongs to $\mathbb{Q}$. Its factors are squares of absolute values of characters of $K_0(\mathcal{C})$ on X and hence are algebraic integers. Since all factors are positive, the product is ≥ 1, which implies (44). $\qquad \square$

For $X \in \mathcal{O}(\mathcal{C})$ define

$$U_X = \{ Y \in \mathcal{O}(\mathcal{C}) \mid |S_{X,Y}| = d_Y \}.$$

Let $\mathcal{U}_X$ be the full abelian subcategory of $\mathcal{C}$ generated by U_X.

Proposition 6.2. *Let $\mathcal{C}$ be a weakly integral modular category and let X be a simple noninvertible object in $\mathcal{C}$. Then*

$$3 \dim \mathcal{T}_X + \dim \mathcal{U}_X > \dim \mathcal{C}. \qquad (46)$$

Proof. We may assume $d_X \geq \sqrt{2}$.

We will use the following theorem of Siegel [1945] from number theory. Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Let α be a totally positive algebraic integer in K, $\alpha \neq 1$. Then

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma(\alpha) \geq \frac{3}{2}.$$

We apply this to the situation when K is the extension of $\mathbb{Q}$ generated by entries of S. We compute

$$\dim \mathscr{C} = \sum_{Y \in \mathscr{C}} |S_{X,Y}|^2 = d_X^2 + \sum_{Y \in U_X} d_Y^2 + \sum_{Y \in \mathbb{O}(\mathscr{C})-(T_X \cup U_X \cup \{\mathbf{1}\})} |S_{X,Y}|^2$$

$$= d_X^2 + \dim \mathscr{U}_X + \sum_{Y \in \mathbb{O}(\mathscr{C})-(T_X \cup U_X \cup \{\mathbf{1}\})} d_Y^2 \left(\frac{1}{|G|} \sum_{\sigma \in G} \sigma \left(\frac{|S_{X,Y}|^2}{d_Y^2} \right) \right)$$

$$\geq 2 + \dim \mathscr{U}_X + \tfrac{3}{2} (\dim \mathscr{C} - \dim \mathscr{T}_X - \dim \mathscr{U}_X - 1);$$

therefore $3 \dim \mathscr{T}_X + \dim \mathscr{U}_X \geq \dim \mathscr{C} + 1 > \dim \mathscr{C}$, as required. □

Remark 6.3. Our proofs of Theorem 6.1 and Proposition 6.2 imitate the corresponding proofs for group characters given in [Berkovich and Zhmud' 1999].

Acknowledgments

References

[Arkhipov and Gaitsgory 2003] S. Arkhipov and D. Gaitsgory, "Another realization of the category of modules over the small quantum group", *Adv. Math.* **173**:1 (2003), 114–143. MR 2004e:17010 Zbl 1025.17004

[Bakalov and Kirillov 2001] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series **21**, American Mathematical Society, Providence, RI, 2001. MR 2002d:18003 Zbl 0965.18002

[Berkovich and Zhmud' 1999] Y. G. Berkovich and E. M. Zhmud', *Characters of finite groups*, vol. 2, Translations of Mathematical Monographs **181**, American Mathematical Society, Providence, RI, 1999. MR 99j:20007 Zbl 0934.20009

[de Boer and Goeree 1991] J. de Boer and J. Goeree, "Markov traces and II_1 factors in conformal field theory", *Comm. Math. Phys.* **139**:2 (1991), 267–304. MR 93i:81211 Zbl 0760.57002

[Bruguières 2000] A. Bruguières, "Catégories prémodulaires, modularisations et invariants des variétés de dimension 3", *Math. Ann.* **316**:2 (2000), 215–236. MR 2001d:18009 Zbl 0943.18004

[Coste and Gannon 1994] A. Coste and T. Gannon, "Remarks on Galois symmetry in rational conformal field theories", *Phys. Lett. B* **323**:3-4 (1994), 316–321. MR 95h:81031

[Deligne 1990] P. Deligne, "Catégories tannakiennes", pp. 111–195 in *The Grothendieck Festschrift*, vol. 2, edited by P. Cartier et al., Progr. Math. **87**, Birkhäuser, Boston, MA, 1990. MR 92d:14002 Zbl 0727.14010

[Drinfeld et al. 2007] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, "Group-theoretical properties of nilpotent modular categories", preprint, 2007. arXiv 0704.0195

[Drinfeld et al. 2009] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, "On braided fusion categories I", preprint, 2009. arXiv 0906.0620

[Etingof et al. 2005] P. Etingof, D. Nikshych, and V. Ostrik, "On fusion categories", *Ann. of Math.* (2) **162**:2 (2005), 581–642. MR 2006m:16051 Zbl 1125.16025

[Etingof et al. 2008] P. Etingof, D. Nikshych, and V. Ostrik, "Weakly group-theoretical and solvable fusion categories", preprint, 2008. arXiv 0809.3031

[Etingof et al. 2009] P. Etingof, D. Nikshych, and V. Ostrik, "Fusion categories and homotopy theory", preprint, 2009. arXiv 0909.3140

[Gaitsgory 2005] D. Gaitsgory, "The notion of category over an algebraic stack", preprint, 2005. arXiv 0507192

[Gelaki and Nikshych 2008] S. Gelaki and D. Nikshych, "Nilpotent fusion categories", *Adv. Math.* **217**:3 (2008), 1053–1071. MR 2009b:18015 Zbl 1168.18004

[Ireland and Rosen 1990] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, 2nd ed., Graduate Texts in Mathematics **84**, Springer, New York, 1990. MR 92e:11001 Zbl 0712.11001

[Izumi 2001] M. Izumi, "The structure of sectors associated with Longo–Rehren inclusions, II: Examples", *Rev. Math. Phys.* **13**:5 (2001), 603–674. MR 2002k:46161

[Kassel 1995] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer, New York, 1995. MR 96e:17041 Zbl 0808.17003

[Kirillov 2002] A. Kirillov, Jr., "Modular categories and orbifold models", *Comm. Math. Phys.* **229**:2 (2002), 309–335. MR 2003m:17024 Zbl 1073.17011

[Majid 1991] S. Majid, "Representations, duals and quantum doubles of monoidal categories", *Rend. Circ. Mat. Palermo* (2) *Suppl.* **26** (1991), 197–206. MR 93c:18008 Zbl 0762.18005

[Montgomery 1993] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics **82**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. MR 94i:16019 Zbl 0793.16029

[Müger 2000] M. Müger, "Galois theory for braided tensor categories and the modular closure", *Adv. Math.* **150**:2 (2000), 151–201. MR 2001a:18008

[Müger 2003a] M. Müger, "From subfactors to categories and topology, I: Frobenius algebras in and Morita equivalence of tensor categories", *J. Pure Appl. Algebra* **180**:1-2 (2003), 81–157. MR 2004f:18013

[Müger 2003b] M. Müger, "On the structure of modular categories", *Proc. London Math. Soc.* (3) **87**:2 (2003), 291–308. MR 2004g:18009

[Müger 2004] M. Müger, "Galois extensions of braided tensor categories and braided crossed *G*-categories", *J. Algebra* **277**:1 (2004), 256–281. MR 2005b:18011

[Naidu and Nikshych 2008] D. Naidu and D. Nikshych, "Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups", *Comm. Math. Phys.* **279**:3 (2008), 845–872. MR 2009b:16092 Zbl 1139.16028

[Nikshych 2008] D. Nikshych, "Non-group-theoretical semisimple Hopf algebras from group actions on fusion categories", *Selecta Math.* (*N.S.*) **14**:1 (2008), 145–161. MR 2009k:16075 Zbl 05574000

[Ostrik 2003] V. Ostrik, "Module categories, weak Hopf algebras and modular invariants", *Transform. Groups* **8**:2 (2003), 177–206. MR 2004h:18006 Zbl 1044.18004

[Siegel 1945] C. L. Siegel, "The trace of totally positive and real algebraic integers", *Ann. of Math.* (2) **46** (1945), 302–312. MR 6,257a Zbl 0063.07009

[Tambara 2000] D. Tambara, "Representations of tensor categories with fusion rules of self-duality for abelian groups", *Israel J. Math.* **118** (2000), 29–60. MR 2001j:18015 Zbl 0969.18007

[Tambara 2001] D. Tambara, "Invariants and semi-direct products for finite group actions on tensor categories", *J. Math. Soc. Japan* **53**:2 (2001), 429–456. MR 2002e:18010 Zbl 0980.18003

[Tambara and Yamagami 1998] D. Tambara and S. Yamagami, "Tensor categories with fusion rules of self-duality for finite abelian groups", *J. Algebra* **209**:2 (1998), 692–707. MR 2000b:18013 Zbl 0923.46052

[Turaev 2000] V. Turaev, "Homotopy field theory in dimension 3 and crossed group-categories", preprint, 2000. arXiv 0005291

[Turaev 2008] V. Turaev, "Crossed group-categories", *Arab. J. Sci. Eng. Sect. C Theme Issues* **33**:2 (2008), 483–503. MR 2500054

gelaki@math.technion.ac.il   *Department of Mathematics,*
Technion-Israel Institute of Technology, 32000 Haifa, Israel

dnaidu@math.tamu.edu   *Department of Mathematics, Texas A&M University,*
College Station, TX 77843, United States

nikshych@math.unh.edu   *Department of Mathematics and Statistics, University of*
New Hampshire, Durham, NH 03824, United States

Algebra & Number Theory

www.jant.org

See inside back cover or www.jant.org for submission instructions.

Copyright ©2009 by Mathematical Sciences Publishers

Algebra & Number Theory

Volume 3   No. 8   2009