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Central configurations of the N-body problem via equivariant Morse theory

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CENTRAL CONFIGURATIONS OF THE N-BODY PROBLEM VIA THE EQUIVARIANT MORSE THEORY

BY

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CENTRAL CONFIGURATIONS OF THE N-BODY PROBLEM VIA THE EQUIVARIANT MORSE THEORY

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ABSTRACT

In this paper we use the equivariant Morse theory to give an estimate of the minimal number of central configurations in the N-body problem in $\mathbb{R}^3$.

In the case of equal masses we prove that the planar central configurations are saddle point for the potential energy.

From this we deduce the presence of nonplanar central configurations, for every $N \geqslant 4$.

The principal difficulty in applying Morse theory is that the potential function is defined on a manifold on which the group O(3) does not act freely. This suggests using the equivariant cohomology functor in order to obtain the Morse inequalities.

Introduction

It is known [17] that if $q_1, \ldots, q_N$ denote the positions of N bodies with masses $m_1, \ldots, m_N$ respectively, the problem of finding central configurations is equivalent to looking for the critical points of the potential energy

$$V(q_1, \ldots, q_N) = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$$

restricted to a particular manifold in the configuration space.

The collinear problem (that is, when the bodies are on the same line) has been studied by F.R. Moulton, who proved the existence of $N!/2$ collinear central configurations for every $m = (m_1, \ldots, m_N) \in \mathbb{R}^N_+$.

Regarding the planar problem, many interesting results have been achieved by J. Palmore [9 - 12] applying Morse theory to a "quotient" manifold obtained by considering the symmetries acting on the problem.

In fact, while the collinear problem admits a $\mathbb{Z}_2$-symmetry, the planar problem is symmetric under the action of the group S^1. However, since this action is free, to find the homology of the quotient space does not present much difficulty.

Using Morse theory, J. Palmore is able to give an estimate of the minimal number of planar central configurations whenever $m \in \mathbb{R}^N_+$ is such that the corresponding potential energy has only nondegenerate critical points.

In this paper we consider the general situation when the bodies are not constrained to move in the same plane. In this case the use of Morse theory presents some difficulties because the action of the orthogonal group $O(3)$ on the manifold M on which V is defined is not free. Thus the quotient space $M/O(3)$ fails to be a manifold.

This is not necessarily a problem since we could use the generalization of Morse theory due to C. Conley [3, 4], which applies to topological spaces. But in our case we do not know the cohomology of the quotient space; for this reason we will use the "equivariant" Morse theory [1, 2, 8].

The main steps to obtain the "equivariant Morse relations" (Theorem 2.1) are to compute the equivariant cohomology of the manifold M and take account of the different isotropy groups of the critical points of V.

Then we use some estimates of the Morse indices to obtain finer "Morse inequalities" (3.21). In fact, it is possible to compute the index of the collinear central configurations by proving that the hessian of $V|_M$ is positive definite in the normal direction to the submanifold of M corresponding to the collinear problem.

We conjecture that a similar result should be true also for the planar central configurations. For these types of configurations, in the case of equal masses, we are able to prove that the hessian of $V|_M$ is positive definite on a subspace of dimension greater than or equal to 1.

From this result we deduce the presence of nonplanar central configurations, for every $N \geqslant 4$.

In Section 4 we examine the case of 4 equal masses, providing the precise form of the Morse inequalities.

1. Preliminary

Let $q_1, \ldots, q_N \in \mathbb{R}^3$ denote the positions of N bodies with masses $m_1, \ldots, m_N$ respectively. Their motion is described by the following equations:

$$(1.1) \qquad m_i \ddot{q}_i = -\sum_{i \neq j} m_i m_j \frac{(q_i - q_j)}{|q_i - q_j|^3} = -\nabla V(q)$$

$$q = (q_1, \ldots, q_N), \qquad 1 \leqslant i, \ j \leqslant N$$

where

$$V(q) = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$$

is the potential energy.

The kinetic energy is

$$E(q) = \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{q}_i|^2 = \frac{1}{2}<\dot{q}, M\dot{q}> = \frac{1}{2}<p, M^{-1}p>$$

where

$$M = \begin{pmatrix} m_1 I_3 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & m_N I_3 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\dot{q} = \frac{dq}{dt}$ and $p = M\dot{q}$.

Denoting by $H(q, p)$ the total energy, that is,

$$H(q,p) = \frac{1}{2}<p, M^{-1}p> + V(q) , \tag{1.2}$$

the equations (1.1) can be written as a Hamiltonian system:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}(q,p) = M^{-1}p \\ \dot{p} = -\dfrac{\partial H}{\partial q}(q,p) = -\nabla V(q) . \end{cases} \tag{1.3}$$

From (1.3) it follows that $\sum_{i=1}^{N} \dot{p}_i$ is equal to zero on solutions. Therefore $\sum_{i=1}^{N} p_i$ is constant and can be assumed equal to zero. Then $\sum_{i=1}^{N} m_i \dot{q}_i = 0$ and this implies that $\sum_{i=1}^{N} m_i q_i$ is also constant on solutions; we suppose that this last constant is zero which is equivalent to fix the center of mass, $\sum_{i=1}^{N} m_i q_i$, in the origin of $\mathbb{R}^3$.

Thus, considering the 3N - 3 dimensional linear space

$$X = \left\{(q_1, \ldots, q_N) \in \mathbb{R}^{3N} \middle| \sum_{i=1}^{N} m_i q_i = 0\right\}$$

the configuration space is $X \setminus \Delta$ where $\Delta = \bigcup_{i<j} \Delta_{ij} \subset X$ is the set of the "diagonals": $\Delta_{ij} = \{(q_1, \ldots, q_N) \in X \mid q_i = q_j\}$, $1 \leqslant i, j \leqslant N$.

DEFINITION 1.1. <u>A point</u> $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_N) \in X \setminus \Delta$ <u>is a central configuration if there exists a scalar-valued function</u> $\phi(t)$ <u>such that a solution of</u> (1.3) <u>is in the form</u> $\phi(t)\bar{q}$.

It follows immediately from the definition that if $\bar{q}$ is a central configuration then $C\bar{q}$ is also, for any $C \in \mathbb{R}$. Therefore to study the central configurations we may restrict our attention to the "mass ellipsoid" $\mathcal{E} = \{q \in X : \langle q, Mq \rangle = 1\}$. Now suppose that $\bar{q}$ is a central configuration. From (1.3) and the homogeneity of $V(q)$ we have

$$\ddot{\phi} M\bar{q} = -\phi^{-2} \nabla V(\bar{q}) \;, \tag{1.4}$$

that is,

$$\lambda M\bar{q} = \nabla V(\bar{q}) \;, \qquad \lambda = -\ddot{\phi}\phi^2 = -V(\bar{q}) \;. \tag{1.5}$$

Conversely, if we suppose that $\lambda M\bar{q} = \nabla V(\bar{q})$ and $\langle \bar{q}, M\bar{q} \rangle = 1$ then a solution $\phi(t)$ of the equation: $\ddot{\phi} = \phi^{-2} V(\bar{q})$ provides a solution of (1.4), hence $\bar{q}$ is a central configuration.

This proves that $\bar{q}$ is a central configuration on $\mathcal{E}$ iff $\bar{q}$ is a critical point of V restricted to $\mathcal{E}$. This is the reason why, in order to study the central configurations, we will investigate the critical points of $V(q)$ on $\mathcal{E} \setminus \Delta$.

2. <u>The equivariant Morse inequalities</u>

The aim of this section is to write the equivariant version of the Morse inequalities for the potential $V(q)$ defined on $\mathcal{E} \setminus \Delta$.

First of all we observe that the manifold $\mathcal{E}\setminus\Delta$ is not compact and since we want to apply the equivariant Morse theory [2, 3, 8] we need some compactness condition. This comes from the behavior of $V(q)$ near Δ, namely $V(q) \to -\infty$ for $q \to \bar{q} \in \Delta$, and the existence of a neighbourhood of Δ in $\mathcal{E}$ in which there are no critical points of $V|_{\mathcal{E}}$ (for example, see [14]). This implies that in order to compute the critical points of $V(q)$ we can consider the function $V(q)$ defined in a compact K contained in $\mathcal{E}\setminus\Delta$. Then we observe that, if $N \geqslant 4$, the manifold $\mathcal{E}$ is invariant under the diagonal action of the group $O(3) = \{\text{matrices } \theta \mid \theta^T\theta = id\}$.(1)

The diagonal Δ is also invariant under this action and the potential $V(q)$ as well.

It is obvious that, as $O(3)$-manifold, $\mathcal{E}$ is homotopically equivalent to the unitary sphere S^{3N-4} with the same diagonal action of $O(3)$. Of course also $\mathcal{E}\setminus\Delta$ and $S^{3N-4}\setminus\Delta$ are homotopically equivalent, as $O(3)$-manifolds, and the compact set $K \subset \mathcal{E}\setminus\Delta$ can be taken homotopically equivalent to $S^{3N-4}\setminus\Delta$ by an equivariant homotopy.

It is easy to see that the action of $O(3)$ on $S^{3N-4}\setminus\Delta$ is not free(2) but there are no fixed points. In fact the isotropy groups(3) are: $O(2)$, if q represents a collinear configuration, and the identity otherwise.

Actually if there are some equal masses there are further symmetries; but since we are not interested in distinguishing the cases of equal or different masses we will not consider these symmetries.

---

(1) Diagonal action means:

$$\theta q = (\theta q_1, \dots, \theta q_N) \qquad q \in \mathcal{E} \quad , \quad \theta \in O(3) \quad .$$

(2) The action of a group G on a space X is fre iff

$$g \in G \text{ and } g \neq 1 \Rightarrow gx \neq x \ , \quad \text{for every } x \in X \ .$$

(3) The isotropy group of a point $x \in X$ is the set:

$$G_x = \{g \in G \mid gx = x\} \quad .$$

In order to avoid some technical difficulties in using the equivariant theory we would like to have only connected isotropy groups, while O(2) is not. For this reason we prefer to consider only the action of SO(3)[4] on $S^{3N-4} \setminus \Delta$, and to use the invariance under O(3), which is a double covering of SO(3), when we write the Morse inequalities. In this way, all the isotropy groups, different from the identity, are isomorphic to S^1 which is connected. From now on we denote by G the group SO(3) and M the manifold $\mathcal{E} \setminus \Delta$.

The equivariant Morse inequalities are [8]:

$$(2.1) \qquad M_t^G(V) = \sum_{j=1}^{n} P_t\big(h_G(M_j)\big) = P_t^G(M) + (1+t)Q_t(V)$$

where:

(i) $\{M_j\}$, $j = 1, \ldots, n$, is a Morse decomposition of M given by G-invariant Morse sets, with respect to the gradient flow $\dot{q} = \nabla V|_M$

(ii) $h_G(M_j)$ is the equivariant-homotopy index of M_j.

(iii) $P_t^G(M)$ is the Poincaré series which represents the equivariant cohomology of M with rational coefficients.

(iv) $Q_t(V)$ is a series with nonnegative coefficients.

We recall that the equivariant cohomology of M is the cohomology of $(M \times E)/SO(3)$ where E is a contractible space on which the action of SO(3) is free and which is unique, up to homotopy.

Our aim is to make explicit the relation (2.1) in order to have an estimate of the number of critical orbits of $V|_M$.

---

(4) $SO(3) = \{\theta \in O(3) \mid \det \theta = 1\}$.

As a first step we will compute $P_t^G(M)$.

The universal contractible space E associated to SO(3) is the space $V_{\infty,3} = \bigcup_{k\geq 3} V_{k,3}$, given by the union of the (orthonormal) 3-frames in $\mathbb{R}^k$ $(k \geq 3)$.

The manifold M is homotopically equivalent to the space $\mathbb{F}_N(\mathbb{R}^3) = \{(q_1,\ldots,q_N) \in \mathbb{R}^{3N},\ q_i \neq q_j \text{ for } i \neq j\} = \mathbb{R}^{3N} \setminus \Delta$. In fact, we can construct a fibration:

$$\begin{array}{c} \mathbb{R}^3 \\ \downarrow \\ \mathbb{R}^{3N} \setminus \Delta \\ \downarrow \pi \\ \left\{(q_1,\ldots,q_N) \in \mathbb{R}^{3N} \setminus \Delta : \sum_{i=1}^{N} m_i q_i = 0\right\} = X \setminus \Delta \end{array}$$

where π is the map which translates the center of mass to the origin. Since the fiber is contractible $\mathbb{R}^{3N} \setminus \Delta$ is homotopically equivalent to $X \setminus \Delta$ and hence to $M = S^{3N-4} \setminus \Delta$. The space $\mathbb{F}_N(\mathbb{R}^3)$, $N \geq 3$, has been studied in [5] and its cohomology with any coefficients is the following:

$$H^*(\mathbb{F}_N(\mathbb{R}^3)) = \bigotimes_{k=1}^{N-1} H^*(\underbrace{S^2 \vee \ldots \vee S^2}_{k \text{ times}})$$

where $\otimes$ is the tensor product and $\vee$ the wedge sum.

Therefore the Poincaré polynomial of $\mathbb{F}_N(\mathbb{R}^3)$ is

$$P_t(\mathbb{F}_N(\mathbb{R}^3)) = P_t(M) = (1+t^2)(1+2t^2)\ldots(1+(N-1)t^2)\ . \tag{2.2}$$

To compute the G-equivariant cohomology of M we will consider the fibration [2, 8]:

$$
\begin{array}{c}
M \\
\downarrow \\
\dfrac{M \times V_{\infty,3}}{SO(3)} \\
\Big\downarrow p \\
SG_{\infty,3} = \dfrac{V_{\infty,3}}{SO(3)}
\end{array}
$$

where p is the projection, $SG_{\infty,3} = \bigcup_{k \geqslant 3} SG_{k,3}$ is the union of the Grassmann varieties $SG_{k,3} = \frac{V_{k,3}}{SO(3)}$ of oriented 3-dimensional subspaces of $\mathbb{R}^k (k \geqslant 3)$. $SG_{\infty,3}$ is called the classifying space of $SO(3)$ and is usually denoted by $BSO(3)$. The cohomology of $BSO(3)$, [7],with rational coefficients is given by the series

$$
(2.3) \qquad P_t\big(BSO(3)\big) = \sum_{m=0}^{\infty} t^{4m} = \frac{1}{1-t^4} \quad .
$$

Finally, from (2.2), (2.3) and the spectral sequence associated to the previous fibration we obtain

$$
(2.4) \qquad P_t^G(M) = \frac{(1+t^2)(1+2t^2)\dots(1+(N-1)t^2)}{1-t^4} \quad .
$$

Now, if we suppose that each critical orbit of $V(q)$ is nondegenerate [5] we can write $M_t^G(V)$ in a more explicit way. In fact, in this case, a Morse decomposition of M is given by the critical orbits of V which, by virtue of the nondegeneracy hypothesis, are G-invariant isolated invariant sets and are a finite number.

---

[5] An orbit $Z \in M/G$ is a nondegenerate critical orbit of $V(q)$ iff:

(i) each point in Z is a critical point of $V(q)$.

(ii) the hessian of V is nondegenerate in the normal direction to Z (see [2, 8] for more details).

Let us denote by Z_j, $j = 1, \dots, n$, the critical orbits of $V(q)$. Then, since all the isotropy groups are connected(6), [8], $M_t^G(V)$ has this simple expression:

$$M_t^G(V) = \sum_{j=1}^{n} t^{\lambda_j} P_t(BG_j) \tag{2.5}$$

where

a) G_j is the isotropy group of each point of Z_j and BG_j is its classifying space.

b) λ_j is the number of the positive eigenvalues of the hessian HV in the normal direction to Z_j at each point of Z_j.

For simplicity we will call λ_j the index of each point of Z_j.

From (2.4) and (2.5) we deduce the following

THEOREM 2.1. (Equivariant Morse inequalities). For each system of N bodies, $N \geqslant 4$, with masses $m_1, \dots, m_N$ such that the corresponding potential energy $V(q)$ has only nondegenerate critical orbits we have:

$$\sum_{j=1}^{n} t^{\lambda_j} P_t(BG_j) = \frac{(1+t^2)(1+2t^2)\dots(1+(N-1)t^2)}{1-t^4} + (1+t)Q_t(V) \quad . \tag{2.6}$$

3. Main results

In this section we will use the Morse inequalities (2.6) to obtain some estimates of the number of the central configurations, whenever the potential $V(q)$ has only nondegenerate critical orbits.

---

(6) If the isotropy group of some critical point was not connected then local coefficients might be needed in the computation of the cohomology of BG_j. This is the reason for considering the action of SO(3) instead of that of O(3).

We begin by observing that from Moulton's theorem [17] we already know that there are exactly $N!/2$ critical orbits of V given by collinear configurations. For each of these configurations the isotropy group is S^1, hence the total contribution of the corresponding critical orbits in (2.5) is the following:

$$\sum_{i=1}^{N!/2} t^{\lambda_i} P_t(BS^1) = \sum_{i=1}^{N!/2} \frac{t^{\lambda_i}}{1-t^2}, \qquad N \geqslant 4 \tag{3.1}$$

where BS^1 is the classifying space of S^1 whose cohomology is expressed by the series $1/(1-t^2)$ and λ_i indicates the index of the critical points of each orbit.

Note that each critical orbit coming from a collinear configuration is a 2-dimensional manifold isomorphic to $SO(3)/S^1$.

On the other hand, each point of a critical orbit of $V(q)$ different from these has the identity as isotropy group. Hence the contribution, in (2.5), of each of these other critical orbits is given by a monomial t^{λ_j}.

In the next theorems we will give an estimate of the index of the collinear and planar configurations. Before doing this, let us observe that if $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_N)$ is a central configuration, from (1.5) we have

$$\lambda m_k \bar{q}_k = \sum_{j \neq k} \frac{m_j m_k}{|\bar{q}_k - \bar{q}_j|^3} (\bar{q}_k - \bar{q}_j) \qquad k = 1, 2, \ldots, N. \tag{3.2}$$

Hence, for $h \in \{1, \ldots, N\}$, $h \neq k$, we obtain

$$\lambda(\bar{q}_k - \bar{q}_h) = \sum_{j \neq h,k} m_j \left(\frac{(\bar{q}_k - \bar{q}_j)}{|\bar{q}_k - \bar{q}_j|^3} - \frac{(\bar{q}_h - \bar{q}_j)}{|\bar{q}_h - \bar{q}_j|^3} \right) + \frac{m_h + m_k}{|\bar{q}_k - \bar{q}_h|^3} (\bar{q}_k - \bar{q}_h). \tag{3.3}$$

Then, taking the inner product with $(\bar{q}_k - \bar{q}_h)$ and dividing by $|\bar{q}_k - \bar{q}_h|^2$, we get

$$\lambda = \sum_{j \neq h,k} \frac{m_j}{|\bar{q}_k - \bar{q}_h|^2} \left(\frac{\langle \bar{q}_k - \bar{q}_j, \bar{q}_k - \bar{q}_h \rangle}{|\bar{q}_k - \bar{q}_j|^3} - \frac{\langle \bar{q}_h - \bar{q}_j, \bar{q}_k - \bar{q}_h \rangle}{|\bar{q}_h - \bar{q}_j|^3} \right) + \frac{m_h + m_k}{|\bar{q}_k - \bar{q}_h|^3}. \tag{3.4}$$

We denote by Y the submanifold of M given by the planar configurations and by T the one defined by the collinear configurations.

THEOREM 3.1. If $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_N)$, $N \geqslant 3$, is a critical point of V_M corresponding to a collinear configuration, then the hessian $HV(\bar{q})$ is positive definite in the normal direction to T and to the orbit of $\bar{q}$.

A result like this is stated and used in many papers of J. Palmore [9 - 13] but we were not able to find the complete proof anywhere. The following proof is due to C. Conley. Before stating the proof we will make some remarks.

Given any matrix A with real eigenvalues, the differential equation

$$\dot{x} = Ax$$

induces a flow on the space of lines through the origin. The rest points of this flow are the lines corresponding to eigenvectors of A; in fact, if $A\xi = \mu\xi$, then $\dot{\xi} = \mu\xi$ and this implies that the line determined by ξ does not change with the time.

Suppose the smallest eigenvalue of A is simple. Then the corresponding rest point is a repeller in the sense that all solutions through nearly points tend to this point in backward time. This is obvious from the expression of the solutions:

$$x(t) = \sum_j c_j e^{\lambda_j t} x_j$$

where c_j are constants, x_j the eigenvectors and λ_j the eigenvalues ordered so that $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.

If $c_1 \neq 0$ then, in backward time the ratios $c_1 e^{\lambda_1 t} / c_k e^{\lambda_k t}$ goes to ∞ if $k \neq 1$, i.e., the line $x(t)$ tends to the line x_1.

We will use this idea in the proof which follows with A the matrix with entries:

$$A_{ij} = -\frac{m_j}{|\bar{q}_i - \bar{q}_j|^3}, \qquad A_{ii} = \sum_{j \neq i} \frac{m_j}{|q_i - q_j|^3}$$

and restricting the differential equation to the hyperplane normal to the eigenvector $(1,1,\ldots,1)$. We use a standard argument to show that the line corresponding to the eigenvector for λ is a repeller, namely we construct a repeller neighbourhood, i.e., one such that every boundary point leaves in forward time.

Proof. The quadratic form associated to the hessian of $V|_M$, at the point $\bar{q}$ is

$$(3.5) \qquad D^2V(\bar{q})(v,v) = \sum_{i<j} \frac{m_i m_j}{|\bar{q}_i - \bar{q}_j|^3} \left[|v_i - v_j|^2 - \frac{3 < \bar{q}_i - \bar{q}_j, v_i - v_j >^2}{|\bar{q}_i - \bar{q}_j|^2} \right] - \lambda \sum_i m_i |v_i|^2$$

$v = (v_1, \ldots, v_N)$ being a vector in the space tangent to M. In particular, if we take v normal to T (3.5) reduces to

$$(3.6) \qquad D^2V(\bar{q})(v,v) = \sum_{i<j} \frac{m_i m_j}{|\bar{q}_i - \bar{q}_j|^3} |v_i - v_j|^2 - \lambda \sum_i m_i |v_i|^2 \quad .$$

Equation (3.6) can be written as

$$(3.7) \qquad D^2V(\bar{q})(v,v) = (v, Av) - \lambda(v, Mv)$$

where A is the $N \times N$ matrix with elements

$$A_{ij} = - \frac{m_j}{|\bar{q}_i - \bar{q}_j|^3} \qquad A_{ii} = \sum_{j \neq i} \frac{m_j}{|q_i - q_j|^3}$$

and M is the diagonal matrix

$$M = \begin{pmatrix} m_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & m_N \end{pmatrix} \quad .$$

The eigenvalue equations for the matrix A are

(3.8)
$$\sum_{j\neq k} \frac{m_j}{|\xi_k - \xi_j|^3} (\xi_k - \xi_j) = \mu\xi_k \quad .$$

In particular (3.2) shows that λ is an eigenvalue of A with corresponding eigenvector $\bar{q}$.

Then, to prove that (3.7) is positive for the vectors v satisfying $\sum_i m_i v_i = 0$ is equivalent to proving that all the positive eigenvalues of the matrix A, different from λ, are greater than λ.

Note that the minimum eigenvalue of A is 0; this corresponds to the eigenvector $(1, 1, \ldots, 1)$.

The subspace orthogonal to this vector is the space of the vectors v satisfying the condition $\sum_i m_i v_i = 0$. We are considering A restricted to this subspace. We order the masses $m_1, \ldots, m_N$ so that if $i < j$ then $\bar{q}_i < \bar{q}_j$. Since the configuration is collinear this is always possible.

Then we consider the flow

(3.9)
$$\dot{v} = Av$$

but as a flow on the space of lines through the origin and define the set $S = \{v \mid i < j \Rightarrow v_i < v_j\}$. Of course $\bar{q}$ is in S. We will prove that $\bar{q}$ is the only eigenvector of A which belongs to S.

In fact, since A is symmetric in the sense that $m_i A_{ij} = m_j A_{ji}$, all the eigenvectors ξ corresponding to some eigenvalue μ different from λ have to be orthogonal to $\bar{q}$, that is,

(3.10)
$$\sum_i m_i \xi_i \bar{q}_i = 0$$

But since $\sum_i m_i \xi_i = 0$ and $\sum_i m_i \bar{q}_i = 0$, we have

$$(3.11) \qquad \left(\sum_i m_i\right) \sum_i m_i < \xi_i, \bar{q}_i > \; = \; \sum_{i<j} m_i m_j < \xi_i - \xi_j, \bar{q}_i - \bar{q}_j > \; .$$

Then (3.10) is equivalent to $\sum_{i<j} m_i m_j < \xi_i - \xi_j, \bar{q}_i - \bar{q}_j > \; = 0$ which implies that there exist i and j, $i < j$ such that $\xi_i > \xi_j$, otherwise each scalar product $< \xi_i - \xi_j, \bar{q}_i - \bar{q}_j >$ would be positive and consequently (3.10) could not be verified.

We claim that S is a repeller neighborhood of $\bar{q}$. In fact, from (3.9) we have:

$$(3.12) \qquad m_k \dot{v}_k = \sum_{j \neq k} \frac{m_j m_k}{|\bar{q}_k - \bar{q}_j|^3} (v_k - v_j) \; .$$

Then, as in (3.3), for $h \neq k$ we obtain:

$$(3.13) \qquad \frac{d}{dt}(v_k - v_h) = \sum_{j \neq h,k} m_j \left(\frac{v_k - v_j}{|\bar{q}_k - \bar{q}_j|^3} - \frac{v_h - v_j}{|\bar{q}_h - \bar{q}_j|^3} \right) + \frac{m_h + m_k}{|\bar{q}_k - \bar{q}_h|^3} (v_k - v_h)$$

$$= \frac{m_h + m_k}{|\bar{q}_k - q_h|^3} (v_k - v_h) + \frac{(v_k - v_h)}{2} \sum_{j \neq h,k} m_j \left(\frac{1}{|\bar{q}_k - \bar{q}_j|^3} + \frac{1}{|\bar{q}_h - \bar{q}_j|^3} \right)$$

$$+ \frac{1}{2} \sum_{j \neq h,k} m_j (v_h + v_k - 2v_j) \left(\frac{1}{|\bar{q}_k - \bar{q}_j|^3} - \frac{1}{|\bar{q}_h - \bar{q}_j|^3} \right)$$

If v is a boundary point of S, there is an index k such that $v_k = v_{k+1}$. Then from (3.13) we have for the solution through v_k:

$$\frac{d}{dt}(v_k - v_{k+1}) = \sum_{j \neq k,\, k+1} m_j (v_k - v_j) \left(\frac{1}{|\bar{q}_k - \bar{q}_j|^3} - \frac{1}{|\bar{q}_{k+1} - \bar{q}_j|^3} \right) > 0$$

which implies that the vector v leaves S.

But, if $\bar{q}$ is an eigenvector of A and is a repeller, then from what we have previously observed we deduce that the other positive eigenvalues must be greater than λ and hence the assertion is proved. ∎

THEOREM 3.2. If all the masses are equal and $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_N)$, $N \geqslant 4$, is a critical point of $V|_M$ corresponding to a planar configuration, then the subspace of the space normal to Y and to the orbit of $\bar{q}$ on which the hessian $HV(\bar{q})$ is positive definite is at least 1-dimensional.

Proof. We choose a vector v normal to Y given by $v_h = (0,0,1)$, $v_k = (0,0,-1)$, $v_j = (0,0,0)$ for $j \neq h,k$. Supposing that each mass is equal to 1, from (3.6) we have

$$(3.14) \qquad D^2V(\bar{q})(v,v) = \sum_{j\neq h,k}\left(\frac{1}{|\bar{q}_k-\bar{q}_j|^3} + \frac{1}{|\bar{q}_h-\bar{q}_j|^3}\right) + \frac{4}{|\bar{q}_k-\bar{q}_h|^3} - 2\lambda\ .$$

Since $\bar{q}$ is a central configuration from (3.4) we get

$$(3.15) \qquad D^2V(\bar{q})(v,v) = \sum_{j\neq h,k}\left[\frac{1}{|\bar{q}_k-\bar{q}_j|^3} + \frac{1}{|\bar{q}_h-\bar{q}_j|^3} + \frac{2}{|\bar{q}_k-\bar{q}_h|^2}\left(\frac{\langle \bar{q}_h-\bar{q}_j, \bar{q}_k-\bar{q}_h\rangle}{|\bar{q}_h-\bar{q}_j|^3} - \frac{\langle \bar{q}_k-\bar{q}_j, \bar{q}_k-\bar{q}_h\rangle}{|\bar{q}_k-\bar{q}_j|^3}\right)\right].$$

For each $j \in \{1,\ldots,N\}$, $j \neq h,k$, let us consider the triangle formed by $\bar{q}_j, \bar{q}_h, \bar{q}_k$ and call the sides $|\bar{q}_k-\bar{q}_h| = A$, $|\bar{q}_h-\bar{q}_j| = B_j$, $|\bar{q}_k-\bar{q}_j| = C_j$ and the angles $\widehat{BC}_j = \alpha_j$, $\widehat{AC}_j = \beta_j$, $\widehat{AB}_j = \gamma_j$. From (3.15) we have:

$$(3.16) \qquad D^2V(\bar{q})(v,v) = \sum_{j\neq h,k} \frac{AB_j^3 + AC_j^3 - 2B_jC_j^3\cos\gamma_j - 2B_j^3C_j\cos\beta_j}{AB_j^3C_j^3}\ .$$

Since $A = C_j\cos\beta_j + B_j\cos\gamma_j$, (3.16) reduces to

(3.17) $$D^2V(\bar{q})(v,v) = \sum_{j\neq h,k} \frac{(C_j^3 - B_j^3)(C_j\cos\beta_j - B_j\cos\gamma_j)}{AB_j^3C_j^3}$$

where each addendum is nonnegative because $C_j\cos\beta_j$ and $B_j\cos\gamma_j$ represent, respectively, the projections of C_j and B_j on the side A.

Thus $D^2V(\bar{q})(v,v) > 0$ unless $C_j = B_j$, for every $j \neq h,k$, that is, unless all the triangles formed by $\bar{q}_j$, $\bar{q}_h$, $\bar{q}_k$ are isosceles on the base A. But if this happens, the vector v does not give a direction normal to the orbit of $\bar{q}$. Consequently, the subspace of the space normal to Y on which $HV(\bar{q})$ is positive definite is at least 1-dimensional. ∎

REMARK 3.1. If $\bar{q}$ is a collinear configuration then $HV(\bar{q})$ never vanishes in the direction tangent to the submanifold T [9]. Thus from Theorem 3.1 it follows that this type of configuration gives nondegenerate critical orbits of the potential $V|_M$.

REMARK 3.1'. In [10] J. Palmore proves that the planar configurations of 4 equal masses are nondegenerate in the direction tangent to Y. Since, for N = 4 the space normal to Y and to the orbit of a planar central configuration is 1-dimensional, Theorem 3.2 proves that these critical orbits are nondegenerate also for the potential $V|_M$.

THEOREM 3.3. <u>If</u> $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_N)$, $N \geqslant 4$, <u>is a critical point of</u> $V|_M$ <u>corresponding to a collinear configuration, then its index is</u> $2N - 4$.

<u>Proof.</u> The submanifold T defined by the collinear configurations has dimension $N - 2$ and is invariant under the diagonal action of $\mathbb{Z}_2$.(7)

Also, if $\bar{q}$ is a critical point of V on T then $\bar{q}$ is a nondegenerate local maximum of $V|_T$; this implies that the hessian of V has N-2 negative eigenvalues

(7) If N bodies are on the same line their positions are described by N real numbers. Then, fixing the center of mass in the origin, the configuration space is N-1 dimensional. By the restriction to the ellipsoid we get the manifold T. It is obvious that the only symmetry acting on T is the $\mathbb{Z}_2$-action so that each orbit in T contains 2 points.

which correspond to the dimension of the space tangent to T in $\bar{q}$. On the other hand, from Theorem 3.1 we know that the hessian of $V|_M$ is positive definite in the normal direction to T and to the orbit of $\bar{q}$. Thus, since M is 3N-4 dimensional and the orbit of $\bar{q}$, under SO(3), is a 2-dimensional manifold in M, the index of $\bar{q}$ is 2N-4. ∎

The planar central configurations are also called relative equilibria. The indices of these configurations are not all the same, but using the results of [9] and [12] we can obtain some bounds.

THEOREM 3.4. If $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_N)$, $N \geq 4$, is a critical point of $V|_M$ corresponding to a planar configuration, then:

$$0 \leq \lambda(\bar{q}) \leq 2N-5 \ . \tag{3.18}$$

Moreover, if all the masses are equal:

$$1 \leq \lambda(\bar{q}) \leq 2N-5 \ . \tag{3.19}$$

Proof. Having fixed the center of mass, every planar configuration is represented by an element of $\mathbb{R}^{2N-2}$. Consequently the submanifold Y given by the planar configurations has dimension 2N-3. This submanifold is invariant with respect to the diagonal action of S^1, and this action is free. So, to every planar central configuration there corresponds a 1-dimensional critical manifold Z' in Y.

On the other hand, to each planar central configuration there corresponds also a 3-dimensional critical orbit Z'' in M. The space tangent to V and normal to Z' on which the hessian HV is negative definite is at least N-2-dimensional [5]. Hence $\lambda(\bar{q}) \leq 3N-4-3-(N-2) = 2N-5$. When all the masses are equal, from Theorem 3.2 we know that $HV(\bar{q})$ is positive definite on a 1-dimensional subspace normal to Y and Z''. Hence we have $\lambda(\bar{q}) \geq 1$. ∎

COROLLARY 3.1. If $N \geq 4$ and all the masses are equal, then every critical point corresponding to a planar configuration is a saddle point of $V|_M$.

Regarding the spatial configurations we have the following:

THEOREM 3.5. If $\bar{q}$ is a critical point of V corresponding to a spatial configuration then :

(3.20) $$0 \leqslant \lambda(\bar{q}) \leqslant 2N-5\ , \qquad N \geqslant 4.$$

Proof. (3.20) follows from [12], where it is shown that at any spatial central configuration the hessian of $V|_M$ is negative definite on a subspace of dimension greater than or equal to $N-2$. Since M is $3N-4$ dimensional and the critical orbit corresponding to a spatial configuration is 3-dimensional, we get the assertion. ∎

Using (2.6), (3.1) and the previous three theorems we obtain:

(3.21) $$\sum_{\lambda=0}^{2N-5} (\alpha_\lambda + \beta_\lambda) t^\lambda + \frac{N!}{2} \frac{t^{2N-4}}{1-t^2} = \frac{(1+t^2)(1+2t^2)\dots(1+(N-1)t^2)}{1-t^4} + (1+t)Q_t(V)$$

$$= \frac{\sum_{0 \leqslant i \leqslant N-2} \gamma_{2i} t^{2i}}{1-t^2} + (1+t)Q_t(V)$$

where α_λ is the number of spatial critical orbits whose points have index λ, β_λ is the number of planar critical orbits whose points have index λ and γ_{2i} are the Betti numbers[(8)].

REMARK 3.2. As observed in the previous section, the manifold M and the potential V are invariant not only for the actions of $SO(3)$ but also for that of $O(3)$. The group $O(3)$ is obtained by adding to $SO(3)$ the reflections with respect to the planes intersecting the origin in $\mathbb{R}^3$. This implies that, as far as the planar or collinear configurations are concerned, nothing changes (i.e., $O(2) \subset SO(3)$), but since $O(3)$ is the double covering of $SO(3)$, each spatial central configuration gives rise to two orbits of critical points. Thus the numbers α_λ are even numbers.

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(8) The numbers γ_{2i} can be computed by this formula:

$$\gamma_{2i} = (-1)^i \sum_{k=0}^{i} S_N^{N-k}$$

where $(-1)^{p-q} S_p^q$ is the number of permutations of p elements with q cycles. The numbers S_p^q are called Stirling numbers of the first kind.

REMARK 3.3. $\sum_{i=0}^{N-2} \gamma_{2i} = N!/2$ which is similar to the result found by J. Palmore for the planar problem.

From (3.21) we deduce the following:

THEOREM 3.6. If $m_1, \ldots, m_N$ are N masses $(N \geqslant 4)$, such that the corresponding potential energy $V(q)$ has only nondegenerate critical orbits, then we have:

$$\alpha_{2j} + \beta_{2j} \geqslant \gamma_0 + \gamma_2 + \ldots + \gamma_{2j} \qquad 0 \leqslant \gamma j \leqslant 2N-5 \, . \tag{3.22}$$

Proof. From (3.21) we have

$$(1-t^2) \sum_{\lambda=0}^{2N-5} (\alpha_\lambda + \beta_\lambda) t^\lambda + \frac{N!}{2} t^{2N-4} = \sum_{0 \leqslant i \leqslant N-2} \gamma_{2i} t^{2i} + (1+t)(1-t^2) \sum_{j=0}^{2N-6} A_j t^j \, . \tag{3.23}$$

Equating the coefficients of the even powers we get:

$$\alpha_{2j} + \beta_{2j} - \alpha_{2j-2} - \beta_{2j-2} = \gamma_{2j} + A_{2j} + A_{2j-1} - A_{2j-2} - A_{2j-3} \tag{3.24}$$

where we have used the convention that if some subscript is negative then the corresponding number is 0. We will prove that:

$$\alpha_{2j} + \beta_{2j} = \gamma_0 + \gamma_2 + \ldots + \gamma_{2j} + A_{2j} + A_{2j-1} \qquad 0 \leqslant 2j \leqslant 2N-5 \tag{3.25}$$

from which (3.22) follows. (3.25) is true for $j = 0$. Then supposing, by induction, that $\alpha_{2j-2} + \beta_{2j-2} = \gamma_0 + \ldots + \gamma_{2j-2} + A_{2j-2} + A_{2j-3}$, from (3.24) we obtain (3.25). ∎

COROLLARY 3.2. If all the masses are equal then $\alpha_0 \geqslant 2$.

Proof. (3.19) implies that $\beta_0 = 0$. Thus from (3.22) and Remark 3.2 we obtain $\alpha_0 \geqslant 2$. ∎

The previous corollary shows that every time the potential energy is nondegenerate, in the case of equal masses, there exist at least two SO(3)-orbits of critical points of V which correspond to spatial central configurations and which are nondegenerate maxima of V on M.

Since each pair of SO(3)-orbits corresponds to one O(3)-orbit in M, this means that, up to symmetry, there is at least one spatial central configuration which gives a nondegenerate maximum of V on M. In the case $N = 4$ this spatial central configuration corresponds to a regular tetrahedron and it is easy to see [17] that it is the only nonplanar central configuration.

4. The case of 4 equal masses.

In this last section we examine the case of 4 equal masses.

In [10] it is shown that there are exactly 146 S^1-orbits of planar and collinear configurations. They are:

a) 12 collinear.

b) 6 corresponding to square configurations.

c) 8 corresponding to configurations which have the shape of an equilateral triangle with a mass at each vertex and the fourth mass in the center.

d) 24 given by isosceles configurations with a mass at each vertex and another one on the axis of symmetry.

e) 96 given by two pairs of scalene configurations with a mass at each vertex and one in the interior.

The indices of these configurations in the submanifold $Y \subset M$ are, respectively, 2, 0, 2, 1 and 0, 1 for the two pairs of scalene configurations.

From Theorem 3.3 the collinear configurations give rise to 12 SO(3)-orbits with index $2N-4 = 4$ in M. Because of the SO(3) symmetry on the planar configurations we have three SO(3) orbits of b) type, four of c), 12 of d) and 48 of e).

On the other hand, from Theorem 3.2 and Remark 3.1', we know that on the space normal to Y and to the critical orbits the hessian of V is positive definite at the critical points. Then in order to compute the indices of the planar configurations, as belonging to SO(3)-orbits in M, we have to add to each of the previous indices the dimension of this space. It is easy to see that this number is $3N-6-(2N+3) = 1$. Hence, from the planar configurations we obtain exactly: 27 SO(3) orbits with index 1, 36 with index 2, and 4 with index 3.

With this information we have that the Morse inequalities (3.21) in the case of 4 equal masses are exactly:

$$2+27t+36t^2+4t^3+\frac{12t^2}{1-t^2} = \frac{1+5t^2+6t^4}{1-t^2} + (1+t)(1+26t+4t^2)$$

where the number 2 indicates the two SO(3) orbits coming from the unique spatial configuration given by a regular tetrahedron.

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