CENTRAL EXTENSIONS OF REDUCTIVE GROUPS BY K_2

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CONTENTS

0. Introduction

If a group G is equal to its commutator subgroup (G, G), it has a universal central extension

(0.1) $\pi \to E_G \to G$

(C. Moore (1968) no. 1). Suppose that G is the group $G(k)$, for G a split simple simply-connected algebraic group over a field k, and let us forget about some small groups over small finite fields. Then R. Steinberg (1962) describes the central extension E_G by generators and relations, while H. Matsumoto (1969) describes the kernel π by generators and relations. Except for G of type C_n ($n \geqslant 1$), π is $k^* \otimes k^* / \langle u \otimes v | u + v = 1 \rangle$, the group we now call $K_2(k)$, following J. Milnor (1971). For G of any type, including C_n, the following remains true: the adjoint group $G^{ad}(k)$ acts on $G(k)$, hence on the corresponding central extension (0.1), and the group of coinvariants $\pi_{G^{ad}(k)}$ is $K_2(k)$. The universal central extension hence gives rise to a central extension

(0.2) $K_2(k) \to G(k)^{\sim} \to G(k).$

The central extension (0.2) is canonically an extension by $K_2(k)$: the action on $G(k)$ of the group of automorphisms of G (and in particular of $G^{ad}(k)$) lifts to an action

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on $G(k)^{\sim}$, trivial on $K_2(k)$. We will call (0.2) *Matsumoto's central extension.* Notation: the *universal symbol* $\{\ ,\ \}$: $k^* \times k^* \to K_2(k)$ maps u, v to the image of $u \otimes v$ in $K_2(k)$.

Suppose now that k is a global field, i.e. a number field or the field of rational functions of a curve over a finite field. Let μ be the group of roots of unity in k. Matsumoto's central extensions, for k and its completions k_v, then give rise to a topological central extension of the adelic group $G(\mathbf{A})$ by μ, and to a splitting of this central extension over $G(k)$:

(0.3)
$$\begin{array}{ccccc} & & & & G(k) \\ & & & \swarrow & \downarrow \\ \mu & \longrightarrow & G(\mathbf{A})^{\sim} & \longrightarrow & G(\mathbf{A}). \end{array}$$

See C. Moore (1968).

In this paper, we consider reductive groups G over k, where reductive is meant to imply connected. We classify central extensions of $G(k)$ by $K_2(k)$ which, roughly speaking, are functorial in k. More precisely, we define objects called *central extensions of* G *by* $\mathbf{K}_2$, which give rise to a central extension of $G(k)$ by $K_2(k)$ and, for k a global field, to a diagram (0.3). We classify the central extensions of G by $\mathbf{K}_2$, see below, and we determine their functoriality in G and k. We hope that for k a global field this will prove useful in the study of "metaplectic" automorphic forms, that is, the harmonic analysis of functions on $G(\mathbf{A})^{\sim}/G(k)$.

Notation. — For X a scheme, the $K_n(X)$ are Quillen's higher K-theory groups of the exact category of vector bundles on X. For U open in X, $U \mapsto K_n(U)$ is a presheaf on X. We denote by $\mathbf{K}_n$ the associated sheaf.

An algebraic group G over a field k defines a sheaf $S \mapsto G(S) = \mathrm{Hom}_{\mathrm{Spec}(k)}(S, G)$ on the big Zariski site $\mathrm{Spec}(k)_{\mathrm{Zar}}$ (see 1.4) of $\mathrm{Spec}(k)$. A *central extension of* G *by* $\mathbf{K}_2$ is a central extension, on $\mathrm{Spec}(k)_{\mathrm{Zar}}$, of the sheaf of groups G by the sheaf of abelian groups $\mathbf{K}_2$. By SGA7 (tI, exp. VII, no. 1), it can be viewed as a $\mathbf{K}_2$-torsor P on G, provided with a multiplicative structure: an isomorphism m: $\mathrm{pr}_1^*P + \mathrm{pr}_2^*P \to \mu^*P$ of $\mathbf{K}_2$-torsors on $G \times G$, for μ: $G \times G \to G$ the group law, obeying an associativity condition on $G \times G \times G$.

The following alternate description, to be explained in §8, played for us a useful heuristic role. It is valid when k is infinite and G, as an algebraic variety over k, is connected and unirational. The face maps of the simplicial classifying space BG, that is, the simplicial scheme G^{Δ_n}/G, induces homomorphisms of fields of rational functions

$$k \rightrightarrows k(G) \equiv k(G \times G) \equiv k(G \times G \times G)...$$

Applying the functor K_2 and taking the usual alternating sum of face maps, one obtains

$$K_2k \xrightarrow{0} K_2k(G) \to K_2k(G \times G) \to K_2k(G \times G \times G) \to \ldots$$

This complex *incarnates* in degree 2 the category of central extensions of G by $\mathbf{K}_2$. This means that: (a) a 2-cocycle $c \in K_2k(G \times G)$ defines a central extension $E(c)$ of G by $\mathbf{K}_2$ and any isomorphism class of central extensions can be obtained in this way; (b) if c_0 and c_1 are two 2-cocycles, a 1-cochain b such that $c_1 - c_0 = db$ defines an isomorphism $E(b)\colon E(c_0) \to E(c_1)$, and this construction is a bijection

$$\{b | c_1 - c_0 = db\} \xrightarrow{\sim} \mathrm{Isom}(E(c_0), E(c_1)) \;;$$

(c) obvious compatibilities hold. The idea to go from the cocycle $c \in K_2k(G \times G)$ to the corresponding extension of $G(k)$ by $K_2(k)$ is as follows. One can write c as a finite sum $\sum n_i\{f_i, g_i\}$ with f_i, g_i in $k(G \times G)^*$. The f_i and g_i are all regular on some dense open subset U of $G \times G$ and

(**0.4**) $$c(x, y) = \sum n_i\{f_i(x, y), g_i(x, y)\}$$

is defined for $(x, y) \in U$. Because c is a cocycle, $c(x, y)$ obeys the cocycle condition $c(y, z) - c(xy, z) + c(x, yz) - c(x, y) = 0$ for (x, y, z) in some dense open subset of $G \times G \times G$. That such a generic cocycle suffices to define a group extension is similar to Weil's theorem that a birational group law determines an algebraic group (SGA3, t2, XVIII 3.7). We were also inspired by Mackey's theorem that for separable locally compact topological groups, borelian 2-cocycles define topological extensions (G. W. Mackey (1957)).

Example. — For G split simple simply-connected, Matsumoto's central extension carries a set theoretic section for which the cocycle describing the extension is expressed in terms of universal symbols. On some open subset U of $G \times G$, this cocycle is given by a formula (0.4). As this formula continues to define a cocycle after extension of scalars from k to $k(G \times G \times G)$, the element $\sum n_i\{f_i, g_i\} \in K_2k(G \times G)$ satisfies the cocycle condition in $K_2k(G \times G \times G)$: Matsumoto's central extension of $G(k)$ by $K_2(k)$ comes from an extension of G by $\mathbf{K}_2$.

We now sketch the classification of the central extensions of a reductive group G by $\mathbf{K}_2$. If G is simply-connected, the central extensions of G by $\mathbf{K}_2$ have no nontrivial automorphisms. In contrast, it can happen that $G(k)$ is not its own commutator subgroup and that the group of automorphisms $\mathrm{Hom}(G(k), K_2(k))$ of a central extension of $G(k)$ by $K_2(k)$ is not trivial. If G is in addition simple (over k), the group of isomorphism classes of central extensions of G by $\mathbf{K}_2$ is $\mathbf{Z}$. Corresponding to $1 \in \mathbf{Z}$, we have a canonical central extension of G by $\mathbf{K}_2$. It induces a canonical central extension of $G(k)$ by $K_2(k)$, well-defined up to unique isomorphism. Being canonical, it is acted upon by the group of automorphisms of G, and in particular by $G^{ad}(k)$. For G split, the central extension of $G(k)$ by $K_2(k)$ obtained is the one constructed by Matsumoto.

Fix a maximal torus T in G, and a Galois extension k' of k over which T splits. Let Y be the dual of the character group of T over k'. It is acted upon by $\mathrm{Gal}(k'/k)$ and by the Weyl group W. Rather than to say that central extensions of G by $\mathbf{K}_2$ are classified by integers, it is better to say that they are classified by the integer-valued quadratic forms Q on Y invariant by W and by $\mathrm{Gal}(k'/k)$. The classification then remains valid for G semi-simple simply-connected, and is functorial in G and k. For G simple, invariant quadratic forms correspond to integers by $Q \mapsto Q(\alpha^\vee)$, for $\alpha^\vee$ the (short) coroot corresponding to a long root α.

Let G be a reductive group over k. The group of automorphisms of any central extension of G by $\mathbf{K}_2$ is $\mathrm{Hom}(G, \mathbf{K}_2)$. The Hom group is in the category of sheaves on the big Zariski site of k. It is the subgroup $H^0(G, \mathbf{K}_2)^{\mathrm{prim}}$ of primitive elements of $H^0(G, \mathbf{K}_2)$. It is also the subgroup of elements $\mathbf{b}$ of $K_2k(G)$ for which $\mathrm{pr}_2^*\mathbf{b} - \mu^*\mathbf{b} + \mathrm{pr}_1^*\mathbf{b} = 0$ in $K_2k(G \times G)$. Example: if G is the quotient of a reductive group G_1 by a central μ_n, Kummer theory attaches to the μ_n-torsor G_1 over G a section f of $\mathscr{O}^*/\mathscr{O}^{*n}$ on G and for ζ a root of unity of k of order dividing n, $\mathbf{b} := \{\zeta, f\}$ belongs to $H^0(G, \mathbf{K}_2)^{\mathrm{prim}}$. Other example: $\mathrm{Hom}(\mathbf{G}_m, \mathbf{K}_2) = k^*$, with $a \in k^*$ giving rise to $x \mapsto \{a, x\}$.

Because of those automorphisms, the classification problem we consider, for G reductive, is not that of determining the set of isomorphism classes of central extensions by $\mathbf{K}_2$. It is that of determining the *category* of central extensions by $\mathbf{K}_2$. This turns out to be an easier problem. It means: to define an equivalence from the category of central extensions of G by $\mathbf{K}_2$ to a more down-to-earth category $\mathscr{C}$.

Suppose first that G is a torus T. Fix a Galois extension k' of k over which T splits and let Y be the dual of the character group of T over k'. The category $\mathscr{C}$ is here the category of pairs $(Q, \mathscr{E})$, where Q is a $\mathrm{Gal}(k'/k)$-invariant integer-valued quadratic form on Y and $\mathscr{E}$ a $\mathrm{Gal}(k'/k)$-equivariant central extension of Y by k'^*, for which (0.5) below holds. Let B be the bilinear form associated to Q: $B(y_1, y_2) := Q(y_1 + y_2) - Q(y_1) - Q(y_2)$. The condition is that the commutator of liftings $\widetilde{y}_1, \widetilde{y}_2$ to $\mathscr{E}$ of $y_1, y_2 \in Y$ should be given by

$$(\widetilde{y}_1, \widetilde{y}_2) = (-1)^{B(y_1, y_2)} \in k^*. \qquad (\mathbf{0.5})$$

In $\mathscr{C}$, there can be a morphism from $(Q', \mathscr{E}')$ to $(Q'', \mathscr{E}'')$ only if $Q' = Q''$. If $Q' = Q''$, morphisms are isomorphisms of equivariant central extensions from $\mathscr{E}'$ to $\mathscr{E}''$.

Let $T^\vee$ be the torus over k, split over k', whose character group is Y. It follows from the equivalence of the category of central extensions of T by $\mathbf{K}_2$ with $\mathscr{C}$ that

$$T^\vee(k) \xrightarrow{\sim} \mathrm{Hom}(T, \mathbf{K}_2).$$

Suppose that T is a maximal torus in a semi-simple simply-connected group G. A Weyl group and $\mathrm{Gal}(k'/k)$-invariant integer-valued quadratic form Q_0 on Y defines a central extension E_0 of G by $\mathbf{K}_2$. Restricting E_0 to T, we obtain a central extension E of T by $\mathbf{K}_2$, classified by a pair $(Q, \mathscr{E})$. One has $Q = Q_0$. In 11.7 we will describe $\mathscr{E}$ in terms of the root spaces decomposition of the Lie algebra of G over k', relative to the action of T.

The classification for general reductive groups is a mixture of the torus and simply-connected cases. Let G be a reductive group over k, with maximal torus T split over k'. Our main result, a particular case of Theorem 7.2, is:

Theorem. — The category of central extensions of G *by* $\mathbf{K}_2$ *is naturally equivalent to the category of triples* $(Q, \mathscr{E}, \varphi)$ *as follows:* Q *is a Weyl and Galois invariant integer-valued quadratic form on* Y*;* $\mathscr{E}$ *is a Galois equivariant central extension of* Y *by* k'^**, obeying (0.5). Let* $f : G_{sc} \to G$ *be the simply connected covering of the derived group of* G*,* $T_{sc} := f^{-1}(T)$ *and* $Y_{sc} \subset Y$ *be the dual character group of* T_{sc}*. The form* Q *induces* Q_{sc} *on* Y_{sc}*, from which we get a Galois-equivariant central extension* $\mathscr{E}_{sc}$ *of* Y_{sc} *by* k'^**, and* φ *is a Galois equivariant morphism from* $\mathscr{E}_{sc}$ *to* $\mathscr{E}$ *making the diagram*

$$\begin{array}{ccccc} k'^* & \longrightarrow & \mathscr{E}_{sc} & \longrightarrow & Y_{sc} \\ \| & & \varphi\downarrow & & \downarrow \\ k'^* & \longrightarrow & \mathscr{E} & \longrightarrow & Y \end{array}$$

commute.

Let H be the group of multiplicative type over k, split over k', with character group Y/Y_{sc}. It follows from the equivalence of the category of central extensions of G by $\mathbf{K}_2$ with $\mathscr{C}$ that

$$H(k) \xrightarrow{\sim} \mathrm{Hom}(G, \mathbf{K}_2).$$

For X connected and smooth over k, pointed by $x \in X(k)$, $\mathbf{K}_2$-torsors P over X, given with a trivialization of x^*P, obey Galois descent. This results from the Quillen resolution of $\mathbf{K}_2$ and from B. Kahn (1993) (corollary 1 to theorem 3.1).

Galois descent reduces the classification problem for central extensions by $\mathbf{K}_2$ to the split case. It matters here that the classification problem considered is that of describing a category, not just a set of isomorphism classes. The split case is handled using the Bruhat decomposition and known results on cohomology with coefficients in $\mathbf{K}_2$ for affine spaces or multiplicative group bundles (Quillen (1973), Sherman (1979)).

In 1978, the second named author constructed a canonical central extension of $G(k)$ by $K_2(k)$, for G absolutely simple and simply-connected, under the assumption that G is rational as an algebraic variety over k. This construction was exposed in a seminar at the IHES in 1977-1978. It used the results of Tate on the structure of $K_2k(T)$, and its consequence that $H^1(\mathrm{Gal}(k'/k), K_2k'(T)/K_2k') = 0$. In retrospect, one sees that this was a substitute for Galois descent.

Let n be an integer invertible in k. For $\bar{k}$ a separable closure of k, let us write simply $H^2(k, \mathbf{Z}/n(2))$ for $H^2(\mathrm{Gal}(\bar{k}/k), \mu_n(\bar{k})^{\otimes 2})$. By Tate (1976) one has a map

$$K_2(k)/nK_2(k) \to H^2(k, \mathbf{Z}/n(2)).$$

A central extension by $K_2(k)$ hence provides one by $H^2(k, \mathbf{Z}/n(2))$. For G simply-connected, Deligne (1996) attaches a central extension of $G(k)$ by $H^2(k, \mathbf{Z}/n(2))$ to each étale cohomology class in $H^4(BG \bmod Be, \mathbf{Z}/n(2))$. We will show that the central extension of $G(k)$ by $H^2(k, \mathbf{Z}/n(2))$ deduced from the canonical central extension by $K_2(k)$ is canonically isomorphic to that of Deligne (1996), applied to a natural generator of $H^4(BG \bmod Be, \mathbf{Z}/n(2))$.

In this introduction, we have emphasized the case of a ground field k. In the text, we consider a more general base S as well. This is useful to handle actions of an algebraic group H on an extension of G by $\mathbf{K}_2$. Cf. 1.7. For the ease of exposition, we often present the ground field case first, and explain later the changes required to work over S. Our reliance on the Quillen resolution of $\mathbf{K}_2$ usually forces us to assume that S is regular of finite type over a field. Using Bloch (1986), one could also handle the case where S is smooth over the spectrum of a discrete valuation ring.

We now review the successive sections of the paper. In § 1, we define central extensions of G by $\mathbf{K}_2$, we review from SGA7 their description as multiplicative torsors and we explain how to compute at the cocycle level. In § 2, we prove the Galois descent theorem for pointed $\mathbf{K}_2$-torsors. We also consider a relative version, for a base S which is regular of finite type over a field, with Galois descent generalized to descent for étale surjective maps.

In § 3, 4, 5 and 6, we consider split groups. We successively consider tori, semi-simple simply-connected groups, semi-simple groups and reductive groups. Our results partially duplicate those of § 3 and 4 of Esnault et al. (1998).

In § 7, we put together § 2 and 6 to classify the central extensions of not necessarily split reductive groups by $\mathbf{K}_2$. In § 8, we explain the formalism of generic cocycles. In § 9, we compare the central extensions constructed with those of Deligne (1996). In § 10, we explain how, over global fields, to construct diagram (0.3). The central extensions of Y by k^* alluded to above are made explicit in § 11. A number of examples are collected in § 12.

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0.N. Notation and terminology

0.N.1. We will systematically use the following notation:
T: an algebraic torus over a base scheme;
X: the character group $\mathscr{H}om(T, \mathbf{G}_m)$ of T;
Y: the dual $\mathscr{H}om(\mathbf{G}_m, T)$ of X.

If T is a split torus over a field k: T isomorphic to $\mathbf{G}_m^n$ for some n, one can think of X as simply being the free abelian group $\mathrm{Hom}(T, \mathbf{G}_m)$. For T a split torus group scheme over a base S, X is a locally constant sheaf of free abelian groups on S. For T not necessarily split, one should use the étale topology: for T over S, X is a locally constant sheaf of free abelian groups on the étale site of S. For S the spectrum of a field k, and k' a Galois extension of k on which T splits, one also writes X for the free abelian group $\mathrm{Hom}(T_{k'}, \mathbf{G}_m)$ with the natural action of $\mathrm{Gal}(k'/k)$, which carries the same information.

If a torus is denoted by T, with some decoration, for instance the subscript $_{\mathrm{sc}}$, its character and dual character groups will be X and Y, with the same decoration.

0.N.2. When considering a reductive group G, we will reserve the notation T for a maximal torus of G, and we denote by
N(T) : the normalizer of T;
W : the Weyl group N(T)/T.

Over a field, and for T split, one can think of W as simply being a finite group. Over S, or for T not necessarily split, it is a locally constant sheaf of groups on the étale site of S.

0.N.3. When considering a quadratic form Q, we will write B for the *associated bilinear form*

$$B(x, y) := Q(x + y) - Q(x) - Q(y).$$

The construction $Q \mapsto B$ is bijective, from integrer-valued quadratic forms on a free abelian group Y to even symmetric bilinear forms on Y.

0.N.4. Let

$$A \to E \to G$$

be a central extension of a group G by an abelian group A. For γ in G, we will denote by int[γ], or int_γ, the inner automorphism of E defined by any lifting $\tilde{\gamma}$ of γ to E:

$$\text{(1)} \qquad \mathrm{int}[\gamma]\colon x \longmapsto \tilde{\gamma} x \tilde{\gamma}^{-1}.$$

For γ and δ in G, we will denote by (γ, δ) the commutator in E of liftings $\tilde{\gamma}$ of γ and $\tilde{\delta}$ of δ:

$$\text{(2)} \qquad (\gamma, \delta) := \tilde{\gamma}\tilde{\delta}\tilde{\gamma}^{-1}\tilde{\delta}^{-1} = \mathrm{int}_\gamma(\tilde{\delta})\tilde{\delta}^{-1}.$$

Both (1) and (2) are independent of the choices of liftings.

If a group Δ acts on the central extension E, inducing the trivial action on A, for γ in G and δ in Δ, we will write

$$\text{(3)} \qquad (\delta, \gamma) := \delta(\tilde{\gamma})\tilde{\gamma}^{-1}$$

for $\tilde{\gamma}$ any lifting of γ. For the action (1) of G on E, we recover (2).

The same constructions make sense for a sheaf of central extensions, if liftings are replaced by local liftings.

0.N.5. The group law in K_n will be written additively, and the product map will be denoted by a dot. For f in the multiplicative group, $\{f\}$ is its image in K_1; one defines $\{f, g\} := \{f\}.\{g\}$ in K_2.

0.N.6. For K a cochain complex and n an integer, $\tau_{\leqslant n}K$ (resp. $\tau_{\geqslant n}K$) is the subcomplex (resp. quotient complex) of K defined by

$$\begin{array}{llllr} (\tau_{\leqslant n}K)^i & = 0 & \text{resp. } (\tau_{\geqslant n}K)^i & = K^i & \text{for } i > n \\ & = \text{cycles in } K^n & & = K^n/\text{coboundaries} & i = n \\ & = K^i & & = 0 & i < n. \end{array}$$

For $n \leqslant m$, $\tau_{[n,m]}K := \tau_{\geqslant n}\tau_{\leqslant m} K$ has as cohomology groups H^i those of K for $n \leqslant i \leqslant m$, 0 otherwise.

0.N.7. A *strict simplicial* object of a category $\mathscr{C}$ is a contravariant functor with values in $\mathscr{C}$ from the category of the finite ordered sets Δ_n $(n \geqslant 0)$ and injective increasing maps. Similarly for *strict cosimplicial*: face maps are given, but not degeneracies.

1. Central extensions, torsors and cocycles

1.1. If A is a sheaf of groups over some space, or site, S, an A-*torsor* (also called "principal homogeneous space") on S is a sheaf P on which A acts on the right and which locally on S is isomorphic to the following standard model: A, with the action of A on itself by right translations. If P_i is an A_i-torsor ($i = 1, 2$), $P_1 \times P_2$ is a $A_1 \times A_2$-torsor. A morphism $f : A \to B$ induces a functor f (also called "pushing by f") from A-torsors to B-torsors; one has a map of sheaves $f : P \to f(P)$ obeying $f(pa) = f(p)f(a)$ and this characterizes $f(P)$ up to unique isomorphism. Special case: for A commutative, and for the morphism $+\colon A \times A \to A$, one obtains the bifunctor "addition of A-torsors".

1.2. We recall from SGA7 (tI, exp. VII, no. 1) Grothendieck's description of central extensions in terms of torsors. We first consider the set-theoretic case. Let $A \to E \xrightarrow{p} G$ be a central extension of a group G by an abelian group A. The set E, with its projection p to G, and the action $e \mapsto ea$ of A on E, is then an A-torsor on G (viewed as a discrete topological space). The product map $E \times E \to E$ sits in a commutative diagram

$$\begin{array}{ccc} E \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

where μ is the product map of G. The top line can be reinterpreted as corresponding to a morphism of sets mapping to $G \times G$:

$$m_0\colon \mathrm{pr}_1^*E \times_{G\times G} \mathrm{pr}_2^*E \to \mu^*E.$$

This map is such that $m_0(e_1a_1, e_2a_2) = m_0(e_1e_2)a_1a_2$, hence defines an isomorphism of A-torsors over $G \times G$

$$m\colon \mathrm{pr}_1^*E + \mathrm{pr}_2^*E \to \mu^*E. \tag{1.2.1}$$

If $\mu_{123} = \mu\mu_{12} = \mu\mu_{23}\colon G \times G \times G \to G$ is the triple product map, the associativity of the multiplication of E is expressed by the commutativity of the following diagram of maps deduced from m between A-torsors on $G \times G \times G$:

$$\begin{array}{ccc} \mathrm{pr}_1^*E + \mathrm{pr}_2^*E + \mathrm{pr}_3^*E & \longrightarrow & \mathrm{pr}_{12}^*\mu^*E + \mathrm{pr}_3^*E \\ \downarrow & & \downarrow \\ \mathrm{pr}_1^*E + \mathrm{pr}_{23}^*\mu^*E & \longrightarrow & \mu_{123}^*E. \end{array} \tag{1.2.2}$$

From the central extension E we have deduced a *multiplicative* A-*torsor* on G, that is, an A-torsor E provided with a morphism (1.2.1) of A-torsors on $G\times G$ for which the diagram (1.2.2) is commutative. This construction is an equivalence from the category of central extensions of G by A to the category of multiplicative A-torsors on G.

1.3. As explained in SGA7 loc. cit., the natural framework for this construction is to work in any category of sheaves T, not just in the category of sets (= sheaves over the space with one point). Here, "sheaf" means sheaf over some site $\mathscr{S}$, not just the case of sheaves on a topological space: T is any topos (SGA 4 IV).

For X in T, one can "think of X as being a space", i.e. consider the topos T/X of pairs (F, f) with F in T and f a map from F to X. In the case of sheaves on a topological space S, this amounts to looking at a sheaf X on S as being a topological space mapped to S by a local homeomorphism. One has a morphism of topos $T/X \to T$.

Let G be a sheaf of groups and A be a sheaf of abelian groups. For $n = 1, 2, 3$, let us "think of G^n as being a space", and let us keep writing A for the pull-back of A to G^n. The same construction as in 1.2 then defines an equivalence from the category of central extensions of G by A to the category of multiplicative A-torsors on G, that is A-torsors on G provided with a multiplicative structure (1.2.1) on G^2 obeying (1.2.2) on G^3.

1.4. A big Zariski site S_{Zar} of a scheme S is a full subcategory $\mathscr{C}$ of the category of schemes over S, with the Zariski topology. If X is in $\mathscr{C}$, any open subscheme of X should be in $\mathscr{C}$ too, and $\mathscr{C}$ should be large enough to contain the schemes over S we are interested in. For us, the precise choice of S_{Zar} will be immaterial. To fix ideas, we take for S_{Zar} the category of schemes of finite type over S.

For A a sheaf on S_{Zar} and for X in S_{Zar}, define the sheaf A_X on the topological space X by $A_X(U) := A(U)$.

This construction identifies sheaves on S_{Zar} with systems (A_X) of sheaves on the topological spaces X, for X in S_{Zar}, contravariant in X, and with $A_U = (A_X$ restricted to U) for U open in X: a morphism $f \colon X \to Y$ in S_{Zar} defines $\varphi(f) \colon f^*A_Y \to A_X$, with a compatibility for composition of morphisms, and $\varphi(f)$ is an isomorphism if f is an open embedding.

The Zariski sheaves $\mathbf{K}_n$ form such a system. We denote by $\mathbf{K}_n$ the corresponding sheaf on S_{Zar}. For $n = 0$, it is the constant sheaf $\mathbf{Z}$. For $n = 1$, it is the sheaf $\mathscr{O}^*$.

We now unravel the description 1.3 of central extensions of sheaves in the case of sheaves over S_{Zar}, when G is representable.

For Y in S_{Zar}, the representable functor $h_Y \colon T \mapsto Y(T) := \mathrm{Hom}_S(T, Y)$ is a sheaf. To "think of h_Y as being a space", as in 1.3, simply means to consider the topos of sheaves over Y_{Zar}. If A is a sheaf of abelian groups on S_{Zar}, its "pull-back to h_Y, thought of as a space", is simply the sheaf on Y_{Zar} whose sections on $T \to Y$ is A(T). A further simplification is that the restriction from the big to the small Zariski site induces an equivalence of categories from A-torsors on S_{Zar} to A_S-torsors on S.

Let G be a group scheme of finite type over S. What matters is that the fiber powers G^n of G over S are in S_{Zar}. By 1.3, central extensions of h_G by A can be described as multiplicative A-torsors. In terms of the system of Zariski sheaves A_X, the description can be rephrased as follows. Notation: for $f: X \to Y$ in S_{Zar} and P a A_Y-torsor on Y, we write f^*P for the A_X-torsor on X obtained by pushing by $f^*A_Y \to A_X$ the f^*A_Y-torsor on X inverse image of P. With this notation, a multiplicative A-torsor is an A_G-torsor on G, provided with a morphism (1.2.1) of $A_{G\times G}$-torsors on $G \times G$ for which the associativity diagram (1.2.2) is a commutative diagram of $A_{G\times G\times G}$-torsors on $G \times G \times G$. A section over T of the corresponding central extension $\mathscr{E}$ of G by A is a pair (g, e): $g \in G(T)$: $T \to G$ and $e \in \Gamma(T, g^*E)$. Attaching to g the isomorphism class of the A_T-torsor e^*E on T, one obtains an exact sequence

$$(\mathbf{1.4.1}) \qquad 1 \to H^0(T, A_T) \to \mathscr{E}(T) \to G(T) \to H^1(T, A_T).$$

By abuse of language, we will often say "central extension of G by A" for "central extension of h_G by A".

Remark. — Suppose that S is the spectrum of a field k and write $A(k)$ for A(S). If $\mathscr{E}$ is the central extension of G by A corresponding to a multiplicative A-torsor E, (1.4.1) for T = S reduces to

$$1 \to A(k) \to \mathscr{E}(k) \to G(k) \to 1,$$

making $\mathscr{E}(k) := \mathscr{E}(S)$ a central extension of $G(k)$ by $A(k)$.

Our strategy to construct central extensions of $G(k)$ by $K_2(k)$ will be to apply this construction to central extensions of G by $\mathbf{K}_2$.

1.5. *Variant.* — Let X be a scheme in S_{Zar}, given with a section e. A *pointed* A-torsor on (X, e) is a A_X-torsor P on X, provided with a trivialization, on S, of the A_S-torsor e^*P.

If G is the trivial group scheme over S: G reduced to its neutral section, i.e. $G \xrightarrow{\sim} S$, a multiplicative structure on a G-torsor E is an isomorphism $E + E \xrightarrow{\sim} E$. It amounts to the data of a trivialization of E.

For a group scheme G, with neutral section e, it follows by restriction to e that a multiplicative torsor on G is automatically pointed. Multiplicative torsors could be redefined as pointed torsors on G, provided with a morphism (1.2.1) of pointed torsors on $G \times_S G$, obeying an associativity (1.2.2) on $G \times_S G \times_S G$.

1.6. Let E be a multiplicative A-torsor on a group scheme of finite type G over S. An automorphism σ of G transforms E into a new multiplicative A-torsor $\sigma(E)$ on G. If σ is inner, i.e. is $\mathrm{int}[g]$ for some g in G(S), then E and $\sigma(E)$ are isomorphic. This is clear if E is interpreted as a central extension $\mathscr{E}$: the construction (0.N.4) (1) lifts int[g] to an automorphism of $\mathscr{E}$ trivial on A.

Let G and H be group schemes of finite type over S, and let ρ be an action of H on G.

1.7. *Construction. — We will construct an equivalence of categories from the category of multiplicative* A*-torsors* E_0 *on the semi-direct product* $H \ltimes G$ *to the category of triples (a) (b) (c) as follows: (a) a multiplicative* A*-torsor* E *on* G*; (b) a multiplicative* A*-torsor* F *on* H*; (c) an action of* H *on* (G, E) *lifting the action of* H *on* G.

Here are equivalent definitions (1) (2) (3) of what an action (c) is.

(1) For any T of finite type over S, let (G_T, E_T) be deduced from (G, E) by base change to T. The group H(T) acts on G_T. Data (c) is a lifting of this action ρ to an action $\tilde{\rho}$ of H(T) on (G_T, E_T), functorial in T.

If H is smooth over S, we have a variant

(1′) Same as (1), except that T is restricted to be smooth over S.

The definition (2) is (1), in a universal case:

(2) Let $h_0 \in H(H) = \mathrm{Hom}(H, H)$ be the identity map of H. The group H(H) acts on G_H. Data (c) is a lifting of the automorphism $\rho(h_0)$ of G_H to an automorphism $\rho(h_0)^{\sim}$ of (G_H, E_H). If $\mu: H \times H \to H$ is the multiplication map, it is required that $\mu^*(\rho(h_0)^{\sim}) = \mathrm{pr}_1^*(\rho(h_0)^{\sim})\mathrm{pr}_2^*(\rho(h_0)^{\sim})$ over $H \times H$.

One goes from (1) or (1′) to (2) by taking $\rho(h_0)^{\sim} = \tilde{\rho}(h_0)$. For any $h \in H(T)$, h is the pull-back of $h_0 \in H(H)$ by $h: T \to H$ and one has $\tilde{\rho}(h) = h^*(\rho(h_0)^{\sim})$: the lifted action is determined by $\rho(h_0)^{\sim}$, and the condition on $H \times H$ in (2) is what is needed for $h \mapsto h^*(\rho(h_0)^{\sim})$ to be an action.

We wrote (2) to make clear the equivalence of (1) and (1′) when H is smooth over S. It is in order to construct actions in the sense of (1) or (1′) that we will systematically be considering multiplicative torsors not only on algebraic groups over a field, but also for group schemes.

(3) The multiplicative torsor E corresponds by 1.4 to a central extension $\mathscr{E}$ of the sheaf of groups h_G on S_{Zar} by the sheaf of abelian groups A. The sheaf of groups h_H acts on h_G. An action (c) is an action of h_H on $\mathscr{E}$, lifting the action of h_H on h_G and trivial on A.

To construct 1.7, we will use the interpretation 1.4 of multiplicative torsors as central extensions, and the interpretation (3) of data (c). Its equivalence with interpretation (1) is easily checked. To simplify notations, we write G, H, $H \ltimes G$ for h_G, h_H, $h_{H \ltimes G}$.

Let E_0 be a central extension of $H \ltimes G$ by A. Let E (resp. F) be the inverse image of G (resp. H) in E_0. It is a central extension of G (resp. H) by A. The central extension $F \subset E_0$ acts on $E \subset E_0$ by conjugation. The action is trivial on $A \subset F \subset E_0$ hence factors through an action of H on E, which lifts the action of H on G. This gives data (a) (b) (c), from which E_0 is recovered as the quotient of the semi-direct product of $F \ltimes E$, a central extension of $H \ltimes G$ by $A \times A$, by the antidiagonal copy of A in $A \times A$.

1.8. For any central extension $\mathscr{E}$ of G by A, if g and h in G commute, the lifted commutator (0.N.4) (2) is in A: on the subscheme $C \subset G \times_S G$ of commuting pairs of elements, the lifted commutator is a morphism of sheaves from C to A, i.e. is a section

(**1.8.1**) $$\mathrm{comm} \in H^0(C, A).$$

If G is commutative, then $C = G \times G$ and the section (1.8.1) of A over $G \times G$ is bimultiplicative.

More generally, if an algebraic group H acts on G, and if this action is lifted to an action on $\mathscr{E}$, trivial on A, we have a lifted commutator (0.N.4) (3) from $H \times G$ to $\mathscr{E}$. If $C \subset H \times G$ is the subscheme of pairs (h, g) for which h fixes g, this lifted commutator defines

(**1.8.2**) $$\mathrm{comm} \in H^0(C, A),$$

and if H acts trivially on G, the section (1.8.2) of A over $H \times G$ is bimultiplicative.

1.9. To go back and forth between cohomological computations and categorical interpretations, we will rely on the following construction of SGA4 XVIII 1.4. Let K be a complex and n an integer. Let $A \to B$ be the truncation $\tau_{[n-1,\, n]}K$ of K (0.N.6): one has $A := K^{n-1}/\text{coboundaries}$ and $B := n$-cocycles. We define the category $\mathscr{C}_n(K)$ as the category of n-cocycles, a morphism from b_0 to b_1 being $a \in A$ such that $b_1 - b_0 = da$. Composition of morphisms is given by addition. Addition of n-cocycles defines an addition law for objects, turning $\mathscr{C}_n(K)$ into a strictly commutative Picard category. The group of isomorphism classes of objects of $\mathscr{C}_n(K)$ is $H^n(K)$, and the automorphism group of any object is $H^{n-1}(K)$. A morphism of complexes $f : K \to L$ induces a functor $f : \mathscr{C}_n(K) \to \mathscr{C}_n(L)$, and if the morphism f induces isomorphisms on H^n and H^{n-1}, the functor f is an equivalence.

We will say that K *incarnates* a category $\mathscr{C}$, in degree n, if we are given an equivalence (usually: an equivalence of commutative Picard categories) from $\mathscr{C}_n(K)$ to $\mathscr{C}$. When no n is mentioned, $n = 1$ is assumed.

The heuristics for using this definition is as follows. Let $\mathscr{C}$ be a strictly commutative Picard category. Let $h^1(\mathscr{C})$ be the group of its isomorphism classes of objects and $h^0(\mathscr{C})$ be the group of automorphisms of any object. Suppose that $h^1(\mathscr{C})$ and $h^0(\mathscr{C})$ can be interpreted as cohomology groups H^n and H^{n-1}. Then, if K is any complex which naturally computes the cohomology in question, the complex K should incarnate $\mathscr{C}$ in degree n. The construction of the equivalence of $\mathscr{C}_n(\mathrm{K})$ with $\mathscr{C}$ will mimic the construction of the isomorphism of $\mathrm{H}^n(\mathrm{K})$ with $h^1(\mathscr{C})$.

Examples. — (i) Let $\mathscr{F}$ be a sheaf of abelian groups on a space S and let $\mathscr{F}^*$ be a resolution of $\mathscr{F}$, such that the complex $\Gamma(\mathrm{S}, \mathscr{F}^*)$ computes the cohomology of $\mathscr{F}$ in degree 0 and 1: the canonical maps $\mathrm{H}^i\Gamma(\mathrm{S}, \mathscr{F}^*) \to \mathrm{H}^i(\mathrm{S}, \mathscr{F})$ are isomorphisms for $i = 0$ and 1. Then, $\Gamma(\mathrm{S}, \mathscr{F}^*)$ incarnates the category of $\mathscr{F}$-torsors on S. The equivalence $\mathscr{C}(\Gamma(\mathrm{S}, \mathscr{F}^*)) \to (\mathscr{F}\text{-torsors})$ is as follows: a 1-cocycle c gives rise to the $\mathscr{F}$-torsor $\mathrm{P}(c)$ of local sections s of $\mathscr{F}^0$ with $ds = c$, and if $c_1 - c_0 = df$, f defines the isomorphism $s \mapsto s + f\colon \mathrm{P}(c_0) \to \mathrm{P}(c_1)$.

(ii) Suppose that a group G acts on $(\mathrm{S}, \mathscr{F})$ and that the resolution $\mathscr{F}^*$ is equivariant. The group G then acts on $\Gamma(\mathrm{S}, \mathscr{F}^*)$ and we can for each i form the standard complex $\mathrm{C}^*(\mathrm{G}, \Gamma(\mathrm{S}, \mathscr{F}^i))$ computing $\mathrm{H}^*(\mathrm{G}, \Gamma(\mathrm{S}, \mathscr{F}^i))$. These complexes organize into a double complex, whose component in bidegree (p, q) is the group $\mathrm{C}^q(\mathrm{G}, \Gamma(\mathrm{S}, \mathscr{F}^p))$ of q-cochains with values in $\Gamma(\mathrm{S}, \mathscr{F}^p)$; it is the group of maps $\mathrm{G}^q \to \Gamma(\mathrm{S}, \mathscr{F}^p)$. The associated simple complex incarnates the category of equivariant $\mathscr{F}$-torsors. If (c, r) is a 1-cocycle:

$$\begin{array}{lll} *_1 & & \\ r & *_2 & \\ f & c & *_3, \end{array}$$

c is a 1-cocycle in $\Gamma(\mathrm{S}, \mathscr{F}^*)$ (vanishing of $d(c, r)$ at $*_3$), $r(\gamma)$ gives an isomorphism between the corresponding torsor and its transform by γ (vanishing at $*_2$) and $r(\gamma)$ is a group action (vanishing at $*_1$). A 0-cochain f gives an equivariant morphism of torsors.

Standard operations on complexes can be given categorical meaning.

Examples. — (iii) Let $f\colon \mathrm{K} \to \mathrm{L}$ be a morphism of complexes. We view $f\colon \mathrm{K} \to \mathrm{L}$ as a double complex, with K (resp. L) in second degree 0 (resp. 1) and form the associated simple complex $\mathbf{s}(\mathrm{K} \to \mathrm{L})$. This shifted mapping cone construction gives rise to a long exact sequence of cohomology

$$\ldots \to \mathrm{H}^i(\mathbf{s}(\mathrm{K} \to \mathrm{L})) \to \mathrm{H}^i(\mathrm{K}) \to \mathrm{H}^i(\mathrm{L}) \to \ldots$$

Suppose that $\mathrm{K}^i = \mathrm{L}^i = 0$ for $i < 0$. Then, $\mathbf{s}(\mathrm{K} \to \mathrm{L})$ incarnates the category of objects of $\mathscr{C}(\mathrm{K})$ given with a trivialization (= isomorphism with the object 0) of its image in $\mathscr{C}(\mathrm{L})$. Indeed, a 1-cocycle is a 1-cocycle k^1 of K given with $\ell^0 \in \mathrm{L}^0$ such that $d\ell^0 = f(k^1)$.

(iv) We now view $f\colon \mathrm{K} \to \mathrm{L}$ as a double complex with K in second degree -1 and L in second degree 0. The long exact sequence of cohomology is

$$\ldots \to \mathrm{H}^i(\mathrm{K}) \to \mathrm{H}^i(\mathrm{L}) \to \mathrm{H}^i(\mathbf{s}(\mathrm{K} \to \mathrm{L})) \to \ldots$$

Suppose that $K^i = L^i = 0$ for $i < 0$, and that $H^i(K)$ injects into $H^i(L)$ for $i = 0$ and 2. By the long exact sequence, $H^1(L)$ maps onto $H^1(\mathbf{s}(K \to L))$: the category $\mathscr{C}(\mathbf{s}(K \to L))$ is equivalent to its subcategory with objects the 1-cocycles of L. This category has as objects the objects of $\mathscr{C}(L)$, a morphism from P to Q being an isomorphism class of pairs (A, a) with A an object of $\mathscr{C}(K)$ and a a morphism from $P + f(A)$ to Q. The assumption on H^0 ensures that pairs (A, a) have no automorphisms. If $H^1(K) = 0$, morphisms can more simply be described as classes modulo $H^0(K)$ of morphisms in $\mathscr{C}(L)$.

(v) Let $K \xrightarrow{f} L \xrightarrow{g} M$ be morphisms of complexes with zero composite. They give rise to functors $\mathscr{C}(K) \to \mathscr{C}(L) \to \mathscr{C}(M)$ and to an isomorphism of the composite functor with the constant functor with value the object 0. We view $K \to L \to M$ as a double complex, with K (resp. L, M) in second degree 0 (resp. 1, 2). We assume that the cohomology of K, L and M is zero in degree $i < 0$. The category $\mathscr{C}(\mathbf{s}(K \to L \to M))$ does not change (up to a natural equivalence) if K, L, M are replaced by their quasi-isomorphic truncation $\tau_{\geqslant 0}$ (0.N.6) and, using this remark, one checks that it is the category of objects P of $\mathscr{C}(K)$, given with a trivialization $0 \to f(P)$ in $\mathscr{C}(L)$ whose image $0 \to gf(P)$ in $\mathscr{C}(M)$ is the identity automorphism of the object zero of $\mathscr{C}(L)$.

In our applications, we will start from a cosimplicial system of complexes $K(p)$. We won't need degeneracies and so could start from a strict cosimplicial system (0.N.7). We assume that $H^iK(p) = 0$ for $i < 0$.

For each n one can consider $K(n) \to K(n+1) \to K(n+2)$, taking as morphisms of complexes the usual $\sum(-1)^i\partial_i$. The $\mathscr{C}(K(p))$ form a strict cosimplicial system of Picard categories. For $n = 0$ (resp. $n = 1$), $\mathscr{C}(\mathbf{s}(K(n) \to K(n+1) \to K(n+2)))$ has the following interpretation.

Case $n = 0$: Objects P of $\mathscr{C}(K(0))$, given with an isomorphism $f\colon \partial_0(P) \to \partial_1(P)$ in $\mathscr{C}(K(1))$ obeying a cocycle condition in $\mathscr{C}(K(2))$: for i_a $(a = 0, 1, 2)$ the three face maps from Δ_0 to Δ_2, commutativity of the diagram

$$\begin{array}{ccc} i_2(P) & \xrightarrow{\partial_1(f)} & i_0(P) \\ & {\scriptstyle\partial_0(f)}\searrow \quad \nearrow{\scriptstyle\partial_2(f)} & \\ & i_1(P) & \end{array}$$

Case $n = 1$: Objects P of $\mathscr{C}(K(1))$ given with a multiplicativity isomorphism $\partial_0(P) + \partial_2(P) \to \partial_1(P)$ in $\mathscr{C}(K(2))$ obeying an associativity condition in $\mathscr{C}(K(3))$.

(vi) The case $n = 1$ above will be met in the following context, explaining the terminology. Let $\mathscr{K}$ be a sheaf on the big Zariski site S_{Zar}, and G be a group scheme of finite type over S. In the inhomogeneous notation, the simplicial classifying space $BG = (G^{\Delta_n}/G)$ of G is

$$\ldots G^3 \mathrel{\substack{\to\\\to\\\to\\\to}} G^2 \mathrel{\substack{\to\\\to\\\to}} G \rightrightarrows e,$$

the three maps ∂_0, ∂_1, ∂_2 from G^2 to G being pr_2, μ and pr_1. Suppose that on each G^n we are given a resolution $\mathscr{K}^*$ of $\mathscr{K}_{G^n}$, contravariant in the face maps, and which computes $H^i(G^n, \mathscr{K}_{G^n})$ for $i = 0$ or 1. By the example (i), $\Gamma(G^n, \mathscr{K}^*)$ incarnates the category of $\mathscr{K}_{G^n}$-torsors on G^n. Applying example (v) to

$$\Gamma(G, \mathscr{K}^*) \to \Gamma(G^2, \mathscr{K}^*) \to \Gamma(G^3, \mathscr{K}^*),$$

one obtains a complex which incarnates the category of multiplicative torsors on G.

This complex can be replaced by the subcomplex of simplicial degree $\geqslant 1$ of $\mathbf{s}(\Gamma(G^*, \mathscr{K}^*))$: the complex

$$(\mathbf{1.9.1}) \qquad \mathbf{s}(\Gamma(G^n, \mathscr{K}^*) \quad \text{for} \quad n \geqslant 1)$$

incarnates multiplicative torsors in degree 2. Let us write e for the trivial group scheme G^0. For each n, $\Gamma(e, \mathscr{K}^*)$ maps to $\Gamma(G^n, \mathscr{K}^*)$, defining a cosimplicial map from $\Gamma((Be)_*, \mathscr{K}^*)$ to $\Gamma((BG)_*, \mathscr{K}^*)$. The composite

$$\mathbf{s}(\Gamma(G^n, \mathscr{K}^*) \text{ for } n \geqslant 1) \to \mathbf{s}\Gamma((BG)_*, \mathscr{K}^*) \to \mathbf{s}(\mathbf{s}\Gamma((Be)_*, \mathscr{K}^*) \to \mathbf{s}\Gamma((BG)_*, \mathscr{K})$$

is a quasi-isomorphism. It follows that

$$(\mathbf{1.9.2}) \qquad \mathbf{s}(\mathbf{s}(\Gamma((Be)_*, \mathscr{K}^*) \to \mathbf{s}(\Gamma((BG)_*, \mathscr{K}^*))))$$

which computes the relative cohomology of BG mod Be with values in $\mathscr{K}$, also incarnates the category of multiplicative $\mathscr{K}$-torsors in degree 2.

(vii) With notation as in example (vi), if $H^i(G^n, \mathscr{K}_{G^n}) = 0$ for $i > 0$, we can, on each G^n, take for $\mathscr{K}^*$ the complex reduced to $\mathscr{K}_{G^n}$ in degree 0. We find that the category of multiplicative torsors on G is incarnated by the complex

$$\Gamma(G^n, \mathscr{K}) \quad \text{for} \quad n \geqslant 1.$$

This is the familiar description of central extensions by 2-cocycles.

One could as well use the complex

$$\widetilde{H}^0(G^n, \mathscr{K}), \qquad d = \sum(-1)^i \partial_i,$$

where $\widetilde{H}^0$ is the direct summand $\Gamma(G^n, \mathscr{K})/\Gamma(e, \mathscr{K})$ of $\Gamma(G^n, \mathscr{K})$.

1.10. The constructions 1.9 can be sheafified. Let K be a complex of sheaves on S and n be an integer. As in 1.9, consider the truncation $\tau_{[n-1, n]}(K)$: $A \to B$ of K. Define $\mathscr{C}_n(K)$ to be the stack on S having as objects over U the A-torsors P over U provided with a trivialization of the B-torsor $d(P)$. It is a strictly commutative Picard stack. For S a point, i.e. for a complex of abelian groups, one recovers a category incarnated by K in degree n: to the cocycle $b \in B$ attach the trivial A-torsor A with the trivialization b of the trivial B-torsor $d(A)$. Indeed, over a point any A-torsor is isomorphic to the trivial one.

The analogue of 1.9, Example (i), is that for $f: X \to S$, $\mathscr{F}$ a sheaf on X, and $\mathscr{F}^*$ a resolution of $\mathscr{F}$ such that $f_*\mathscr{F}^*$ computes $R^i f_*\mathscr{F}$ in degree 0 and 1, $f_*\mathscr{F}^*$ incarnates the stack on S having as objects over U the $\mathscr{F}$-torsors on $f^{-1}(U)$.

2. Descent for K_2-torsors

The main result of this section is Theorem 2.2, which will allow us to reduce the construction of multiplicative $\mathbf{K}_2$-torsors on connected reductive groups to the case of split groups. From 2.6 on, we prove a relative variant of 2.2. This variant will not be needed for our main results, or rather for their restriction to the case of reductive groups over fields.

Our main tool is the following theorem of Colliot-Thélène and Suslin, see B. Kahn (1993) (corollary 1 to theorem 3.1):

Theorem **2.1.** — *Let* X *be a smooth geometrically irreducible variety over k. Assume that* X *has a k-rational point e. If k' is a finite Galois extension of k, with Galois group* Γ, *and* $X' := X \otimes_k k'$, *one has, denoting by* $k(X)$ *and* $k(X')$ *the fields of rational functions on* X *and* X′

$$K_2(k(X))/K_2(k) \xrightarrow{\sim} [K_2(k(X'))/K_2(k')]^{\Gamma} \tag{2.1.1}$$

and

$$H^1(\Gamma, K_2(k(X'))/K_2(k')) = 0. \tag{2.1.2}$$

In characteristic zero, (2.1.1) is Colliot-Thélène (1983) Th. 1. In any characteristic, it is in Suslin (1987), 3.6 and 5.8. The vanishing (2.1.2) can be deduced from Merkurjev-Suslin (1983) 14.1. The point of view of B. Kahn (1993) is that the Lichtenbaum complex $\Gamma(2)$ has good étale localization properties and that theorem 2.1 is a corollary of these. See also B. Kahn (1996). Under the assumptions of 2.1, one also knows that $K_2(k)$ injects as a direct summand in $K_2k(X)$.

The group Γ acts semi-linearly on $(X', e') = (X, e) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. A (Galois) *descent data* on a pointed $\mathbf{K}_2$-torsor P′ on (X', e') is an action of Γ on P′, covering the action of Γ on (X', e').

Theorem **2.2.** — *Under the assumptions of* 2.1, *the pull-back functor is an equivalence from*

(a) *the category of pointed* $\mathbf{K}_2$*-torsors on* (X, e), *to*
(b) *the category of pointed* $\mathbf{K}_2$*-torsors on* (X', e'), *provided with a descent data.*

Proof. — The category of $\mathbf{K}_n$-torsors on X is incarnated by the global sections of the Quillen resolution G_n^* of $\mathbf{K}_n$ by flabby sheaves on X. Let $X^{(i)}$ be the set of points of codimension i of X. One has

$$G_n^i = \bigoplus_{x \in X^{(i)}} i_{x*} K_{n-i}(k(x)).$$

For $n = 2$, the complex of global sections is

$$K_2 k(X) \to \bigoplus_{x \in X^{(1)}} k(x)^* \to \bigoplus_{x \in X^{(2)}} \mathbf{Z}.$$

We want to use the resolution G_n^* of $\mathbf{K}_n$ to construct a complex incarnating the category of pointed $\mathbf{K}_n$-torsors. The fact that the resolution G_n^*, while contravariant for smooth maps, is not contravariant for closed embeddings, creates a difficulty. We go around it as follows. The point e, viewed as a section of $f : X \to \mathrm{Spec}(k)$, defines an equivalence of the category of $\mathbf{K}_n$-torsors on X with the product of the categories of pointed torsors on (X, e) and the category of $K_n(k)$-torsors. This allows us to view the category of pointed torsors as a quotient category, rather than a subcategory, of the category of $\mathbf{K}_n$-torsors: it is naturally equivalent to the category of *relative torsors* having as objects the $\mathbf{K}_n$-torsors on X, a map from P to Q being a class mod $K_n(k)$ of maps of torsors from P to Q. The inverse of the forgetting functor from pointed torsors to relative torsors is the functor $P \mapsto P - f^* e^* P$.

By 1.9, Example (iv), the category of relative $\mathbf{K}_2$-torsors on X is incarnated by the complex

$$K_2 k(X)/K_2(k) \to \bigoplus_{x \in X^{(1)}} k(x)^* \to \bigoplus_{x \in X^{(2)}} \mathbf{Z}. \tag{2.2.1}$$

By a variant of 1.9, Example (ii), the category of relative $\mathbf{K}_2$-torsors on X', provided with a descent data, is incarnated by the simple complex associated to the double complex

$$C^*(\Gamma, K_2 k(X')/K_2(k')) \to C^*\Big(\Gamma, \bigoplus_{x \in X'^{(1)}} k(x)^*\Big) \to C^*\Big(\Gamma, \bigoplus_{x \in X'^{(2)}} \mathbf{Z}\Big) \tag{2.2.2}$$

These incarnations reduce 2.2 to the

Lemma **2.3.** — *The natural map of complexes* (2.2.1) $\to$ $\mathbf{s}$(2.2.2) *induces an isomorphism on* H^0 *and* H^1.

Proof. — More precisely, we will check that the cone on the natural map of complexes (2.2.1) $\to$ $\mathbf{s}$(2.2.2) has vanishing H^{-1}, H^0 and H^1. This cone is the associated simple complex of the double complex (2.2.2), coaugmented by (2.2.1) in second (= group cohomology) degree -1. It vanishes outside of $0 \leqslant p \leqslant 2$, $q \geqslant -1$.

The $p = 0$ column is $C^*(\Gamma, K_2 k(X')/K_2(k'))$, coaugmented by $K_2 k(X)/K_2(k)$ in degree -1. By 2.1, its cohomology vanishes in degree -1, 0 and 1.

The $p = 1$ column is the sum over x in $X^{(1)}$ of the $C^*(\Gamma, (k(x)\otimes k')^*)$, coaugmented by $k(x)^*$ in degree -1. As $k(x)^* \xrightarrow{\sim} (k(x)\otimes k')^{*\Gamma}$, its cohomology vanishes in degree -1 and 0.

The $p = 2$ column is the sum over x in $X^{(2)}$ of the $C^*(\Gamma, \mathbf{Z}^{I(x)})$, where $k(x)\otimes k'$ is a product of fields indexed by $I(x)$, coaugmented by $\mathbf{Z}$ in degree -1. As the diagonal map from $\mathbf{Z}$ to $\mathbf{Z}^{I(x)}$ is injective, its cohomology vanishes in degree -1.

The spectral sequence for the filtration by p gives the required vanishing.

If we apply 2.2 to the group of automorphisms of an object (resp. to the set of isomorphism classes of objects), we obtain

Corollary **2.4.** — *Under the assumptions of 2.1,*

(i) $H^0(X, \mathbf{K}_2)/K_2(k) \xrightarrow{\sim} (H^0(X', \mathbf{K}_2)/K_2(k'))^\Gamma$;

(ii) *one has an exact sequence*

$$0 \to H^1(\Gamma, H^0(X', \mathbf{K}_2)/K_2(k')) \to H^1(X, \mathbf{K}_2) \to H^1(X', \mathbf{K}_2)^\Gamma \to H^2(\Gamma, H^0(X', \mathbf{K}_2)/K_2(k')).$$

Remark **2.5.** — (i) If further $\mathbf{K}_2(k') \xrightarrow{\sim} H^0(X', \mathbf{K}_2)$, then $\mathbf{K}_2(k) \xrightarrow{\sim} H^0(X, \mathbf{K}_2)$ and $H^1(X, \mathbf{K}_2) \xrightarrow{\sim} H^1(X', \mathbf{K}_2)^\Gamma$. This results from 2.4.

(ii) Suppose that X is open and dense in a smooth geometrically irreducible variety X_1 over k and that X_1 has a k-rational point. Then 2.2 remains valid, with the same proof, if "pointed K_2-torsor" is replaced by "relative K_2-torsor".

2.6. Let $f\colon X \to S$ be a smooth morphism, with a section e. We assume that

(2.6.1.) S is regular and of finite type over some field k;

(2.6.2.) For each generic point η of S (there is one for each connected component of S), the generic fiber X_η/η is geometrically irreducible.

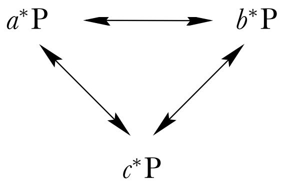
Assumption (2.6.1) is to have Quillen resolutions. Assumption (2.6.2) is to have 2.1 for X_η/η.

Let $u\colon S_0 \to S$ be an étale surjective map. We define S_n to be the $(n + 1)^{\text{st}}$ fiber power $(S_0/S)^{\Delta_n}$ of S_0 over S. Let (X_n, e_n) be the pull-back of (X, e) over S_n; $X_n := X\times_{S_0} S_n$ is also the $(n+1)^{\text{st}}$ fiber power $(X_0/X)^{\Delta_n}$ of X_0 over X. The X_n $(n \geqslant 0)$ form a simplicial scheme, augmented to $X_{-1} := X$.

A *descent data* for a $\mathbf{K}_2$-torsor P_0 on X_0 is an isomorphism between its two inverse images on X_1, obeying a compatibility condition on X_2. If Y maps to X, and if this map can be factorized through X_0, one can choose a factorization

$$Y \xrightarrow{a} X_0 \xrightarrow{u_X} X$$

and take the pull-back a^*P_0. If $Y \xrightarrow{b} X_0 \xrightarrow{u_X} X$ is another factorization, (a, b) is a map from Y to X_1 and the descent data gives an isomorphism between a^*P_0 and b^*P_0. The compatibility condition ensures that for three factorizations a, b and c, the diagram of isomorphisms



is commutative. More precisely, the compatibility condition is this commutativity, in the universal case where $Y = X_2$ and a, b, c are the three projections of X_2 to X_0.

This can be repeated for pointed torsors relative to the sections e_n of X_n/S_n.

Theorem **2.7.** — *Under the assumptions of 2.6, the pull-back functor is an equivalence from*

(a) *the category of pointed* $\mathbf{K}_2$*-torsors on* (X, e)*, to*

(b) *the category of pointed* $\mathbf{K}_2$*-torsors on* (X_0, e_0)*, provided with a descent data.*

Proof. — If S is the spectrum of a field k, 2.7 is little more than a rephrasing of 2.2. Indeed, if k' is a Galois extension of k, with Galois group Γ, the morphism $k' \otimes_k k' \to k'^{\Gamma}$: $x \otimes y \mapsto (x\sigma(y))_{\sigma \in \Gamma}$ is an isomorphism. More generally, if $S_0 = \mathrm{Spec}(k')$, S_n is a disjoint union of copies of S_0, indexed by Γ^n. Using those isomorphisms, one sees that descent data for $S_0 \to S$ are the same thing as Galois descent data. A general S_0 is the spectrum of an étale algebra A over k, product of finite separable extensions k_i of k. Let $\mathscr{C}(A/k)$ be the full subcategory of the category of finite separable extensions of k, consisting of the k'/k for which $k \to k'$ admits a factorization through A, i.e.through one of the k_i. Pointed torsors with descent data on (X_0, e_0) can be reinterpreted as the data for each k' in $\mathscr{C}(A/k)$ of a pointed torsor $P[k']$ on $(X', e') := (X, e) \otimes_k k'$, compatibly with base change k''/k'. Let K be a Galois extension of k in $\mathscr{C}(A/k)$. Restricting $P[\ \ ]$ from $\mathscr{C}(A/k)$ to $\mathscr{C}(K/k)$, one obtains a descent data for $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. By 2.2, it comes from P on (X, e), unique up to unique isomorphism. The isomorphism of $P[k']$ with $P_{k'}$ for k' in $\mathscr{C}(K/k)$ extends uniquely to all k' in $\mathscr{C}(A/k)$. If K' is a composed extension of k' and K, this results from the fully faithful part of 2.2, applied to the Galois extension K'/k'. We are here repeating Giraud (1964) 6.25. For S the spectrum of a field and under the assumptions of 2.5 (ii), the same argument proves 2.7 for relative $\mathbf{K}_2$-torsors.

We now prove 2.7. As in the proof of 2.2, the section e decomposes the category of $\mathbf{K}_2$-torsors on X into the product of the categories of pointed $\mathbf{K}_2$-torsors on (X, e) and the category of $\mathbf{K}_2$-torsors on S. If we view the category of pointed $\mathbf{K}_2$-torsors as a quotient category of the category of $\mathbf{K}_2$-torsors on X, it naturally identifies with the category of *relative torsors*, with objects the $\mathbf{K}_2$-torsors on X, a map $P \to Q$ being an isomorphism class of pairs (A, a), A a $\mathbf{K}_2$-torsor on S and a a morphism from $P + f^*A$ to Q.

Let us view the morphism of complexes

$$(\mathbf{2.7.1}) \qquad f^*\colon \Gamma(S, G_2^*) \to \Gamma(X, G_2^*)$$

as a double complex, with $\Gamma(X, G_2^*)$ in second degree 0 and $\Gamma(S, G_2^*)$ in second degree -1. By 1.9, Example (iv), we have:

Lemma **2.8.** — *The category of relative* $\mathbf{K}_2$*-torsors on* X *is incarnated by* $\mathbf{s}$(2.7.1).

2.9. The complexes $\mathbf{s}$(2.7.1) for the X_q/S_q

$$(\mathbf{2.9.1}) \qquad \mathbf{s}(\Gamma(S_q, G_2^*) \to \Gamma(X_q, G_2^*))$$

form a semi-simplicial system of complexes. Applying 1.9 Example (v) (case $n = 0$) to it, we see that the associated simple complex incarnates the category of relative (or, equivalently, pointed) torsors on X_0, given with a descent data. Those incarnations reduce 2.7 to

Lemma **2.10.** — *The natural map of complexes* $\mathbf{s}$(2.7.1) $\to$ $\mathbf{s}$(2.9.1) *induces an isomorphism on* H^0 *and* H^1.

More precisely, we will check that this map induces an isomorphism on H^0 and H^1 and a monomorphism on H^2, i.e. that its cone has vanishing H^i for $i \leqslant 1$. Let us denote by p the degree in the complex G_2^*. Filtering by p and using the corresponding long exact sequences of cohomology, one is reduced to check for $p = 0, 1, 2$ that the morphism of complexes

$$(\mathbf{2.10.1}) \qquad (\Gamma(S, G_2^p) \to \Gamma(X, G_2^p), \text{ in degree } -1 \text{ and } 0) \to \mathbf{s}(\Gamma(S_q, G_2^p) \to \Gamma(X_q, G_2^p))$$

induces an isomorphism in degree $i \leqslant 1 - p$ and a monomorphism in degree $2 - p$.

Case $p = 0$. We may and shall assume S connected. If η is the generic point of S, making the base change by $\eta \to S$ does not change (2.10.1). Indeed, the points of codimension 0 of the X_q and S_q are above η. We hence may and shall assume that S is the spectrum of a field k.

In $(2.10.1)_{p=0}$, because X/S has a section, $\Gamma(S, G_2^0) = K_2(k)$ injects into $\Gamma(X, G_2^0) = K_2(k(X))$ and similarly for X_q/S_q. This allows to replace $(2.10.1)_{p=0}$ by

$$(\Gamma(X, G_2^0)/\Gamma(S, G_2^0) \text{ in degree } 0) \to \mathbf{s}(\Gamma(X_q, G_2^0)/\Gamma(S_q, G_2^0)).$$

Both complexes are now in degree $\geqslant 0$, and injectivity on H^2 is trivial. It suffices to prove isomorphism on H^0 and H^1.

For S the spectrum of a field, we already know that 2.7 holds, i.e. that 2.10 holds. If one uses relative, rather than pointed, $\mathbf{K}_2$-torsors, this remains true if X is replaced by any non empty open subset U. Taking the inductive limit in U in $\mathbf{s}(2.7.1) \to \mathbf{s}(2.9.1)$, one obtains (2.10.1) and the case $p = 0$ follows.

Case $p = 1$ or 2. Taking the cone on (2.10.1), one obtains the simple complex associated to the coaugmented cosimplicial system of complexes

$$(\Gamma(S_q, G_2^p) \to \Gamma(X_q, G_2^p) \text{ in degree } -1 \text{ and } 0)_{q \geqslant -1}.$$

To show that it is acyclic in degree $\leqslant 1 - p$, it suffices to check that the coaugmented cosimplicial complexes $\Gamma(S_q, G_2^p)$ and $\Gamma(X_q, G_2^p)$ are acyclic in degree $\leqslant 2 - p$ (resp. $1 - p$). Both statements are instances of the following lemma, applied to $S_0 \to S$ or to $X_0 \to X$.

Lemma **2.11.** — *For* S *regular and* u: $S_0 \to S$ *étale and surjective, if* $p = 1$ *or* 2*, the coaugmented cosimplicial complex* $\Gamma((S_0/S)^{\Delta_q}, G_2^p)$ *is acyclic in degree* $\leqslant 2 - p$.

Proof. — Fix $p = 1$ or 2. The points of codimension p of $(S_0/S)^{\Delta_q}$ are just the points of $(S_0/S)^{\Delta_q}$ over a point of codimension p of S, and the coaugmented cosimplicial complex $\Gamma((S_0/S)^{\Delta_q}, G_2^p)$ is the sum over $s \in S^{(p)}$ of the following complexes: for $F = u^{-1}(s)$ and F^{Δ_q} a power over s, the complex

$$K_{2-p}(F^{\Delta_q}) \qquad (q \geqslant -1)$$

This is the étale cohomology coaugmented Čech complex, for the étale covering $F \to s$, and the sheaf $\mathscr{O}^*$ if $p = 1$, the sheaf $\mathbf{Z}$ if $p = 2$. For $p = 1$, Hilbert 90 gives acyclicity in degree $\leqslant 1$. For $p = 2$, the sheaf property gives acyclicity in degree $\leqslant 0$.

3. Split tori

Our aim in this section is to compute the category of multiplicative $\mathbf{K}_2$-torsors on a split torus. We also treat the case of split unipotent groups.

3.1. Let S be regular and of finite type over a field k. If p: $\mathbf{A}^1_S \to S$ is the affine line over S, Sherman (1979) proves that

$$(\mathbf{3.1.1}) \qquad p_* \mathbf{K}_j = \mathbf{K}_j \quad \text{and} \quad R^i p_* \mathbf{K}_j = 0 \quad \text{for} \quad i > 0 \qquad (\text{for } p\colon \mathbf{A}^1_S \to S).$$

If we remove the 0-section, we obtain the multiplicative group $p\colon \mathbf{G}_{mS} \to S$ over S. Sherman (1979) proves that

$$(\mathbf{3.1.2}) \qquad p_* \mathbf{K}_j = \mathbf{K}_j \oplus \mathbf{K}_{j-1} \quad \text{and} \quad R^i p_* \mathbf{K}_j = 0 \quad \text{for} \quad i > 0 \qquad (\text{for } p\colon \mathbf{G}_{mS} \to S).$$

The maps, from right to left, in (3.1.1) and (3.1.2) are as follows. The component $\mathbf{K}_j \to p_*\mathbf{K}_j$ expresses the contravariance of $\mathbf{K}_j$. For 3.1.2, the component $\mathbf{K}_{j-1} \to p_*\mathbf{K}_j$ comes from the contravariance and from the multiplicative structure on $\mathbf{K}_*$: the coordinate u on $\mathbf{G}_{m\mathrm{S}}$ is an invertible function, hence defines a class $\{u\}$ in $\mathrm{H}^0(\mathbf{G}_{m\mathrm{S}}, \mathbf{K}_1)$ and the map is $x \mapsto \{u\}.p^*(x)$.

Let U be a split unipotent group scheme over S: locally over S, U is an iterated extension of additive groups. If p: P $\to$ S is a U-torsor over S, an iterated application of (3.1.1) shows that the functor p^* is an equivalence from the category of $\mathbf{K}_2$-torsors on S to the category of $\mathbf{K}_2$-torsors on P. If G is a group scheme over S, extension of $\overline{\mathrm{G}}$ smooth over S by U, G is a U-torsor on $\overline{\mathrm{G}}$ and p: $\mathrm{G} \to \overline{\mathrm{G}} = \mathrm{G}/\mathrm{U}$ induces an equivalence from $\mathbf{K}_2$-torsors on $\overline{\mathrm{G}}$ to $\mathbf{K}_2$-torsors on G. The same applies to multiplicative $\mathbf{K}_2$-torsors. We conclude:

Proposition **3.2.** — *If* S *is regular of finite type over a field, and if the group scheme* G *over* S *is an extension of* $\overline{\mathrm{G}}$ *smooth over* S *by a split unipotent group scheme* U, *the pull back by* p: $\mathrm{G} \to \overline{\mathrm{G}} = \mathrm{G}/\mathrm{U}$ *is an equivalence from the category of central extensions of* $\overline{\mathrm{G}}$ *by* $\mathbf{K}_2$ *to the category of central extensions of* G *by* $\mathbf{K}_2$.

For $\overline{\mathrm{G}}$ the trivial group scheme e, 3.2 tells that a central extension of a split unipotent group scheme by $\mathbf{K}_2$ is trivial, and admits a unique trivialization.

3.3. Let p: T $\to$ S be a split torus over S and X, Y be as in (0.N.1). If we choose an isomorphism of Y with $\mathbf{Z}^n$, we obtain $\mathrm{T} \xrightarrow{\sim} \mathbf{G}^n_{m\mathrm{S}}$. Using induction on n we deduce from (3.1.2) that $\mathrm{R}^i p_*\mathbf{K}_j = 0$ for $i > 0$ and we obtain a description of $p_*\mathbf{K}_j$. Using the relation $\{u, u\} = \{u, -1\}$ in $\mathbf{K}_2$, the description of $p_*\mathbf{K}_j$ can be made intrinsic:

Lemma **3.3.1.** — *The sheaf of graded rings* $p_*\mathbf{K}_*$ *on* S *is generated by* $\mathbf{K}_*$ *and by* X *in degree* 1, *with the only relations that*

(i) $\mathrm{X} \to p_*\mathbf{K}_1$ *is additive;*
(ii) *for* x *in* X, $x.x = x.\{-1\}$.

3.4. If we filter $p_*\mathbf{K}_*$ by "how many factors in X", we obtain an increasing filtration V with

$$(\mathbf{3.4.1}) \qquad \mathrm{Gr}^{\mathrm{V}}_m p_*\mathbf{K}_* = \overset{m}{\wedge}\mathrm{X} \otimes \mathbf{K}_{*-m}.$$

The bottom piece $\mathrm{V}_0(p_*\mathbf{K}_*) = \mathbf{K}_*$ is a direct factor: the pullback by the 0-section e is a retraction to the embedding of $\mathbf{K}_*$ in $p_*\mathbf{K}_*$. The kernel of e^* is the *reduced* direct image $\widetilde{p}_*\mathbf{K}_*$.

The translation by $t \in \mathrm{T}(\mathrm{S})$ transforms $x \in \mathrm{X}$, viewed as a section of $p_*\mathbf{K}_1$, into $x - x(t)$, where $x(t) \in \mathrm{H}^0(\mathrm{S}, \mathbf{K}_1)$ is in V^0. It follows that a translation respects V and acts trivially on $\mathrm{Gr}^{\mathrm{V}} p_* \mathbf{K}_*$. If q: Q $\to$ S is a T-torsor on S, we still have $\mathrm{R}^i q_* \mathbf{K}_* = 0$ for $i > 0$. Moreover, we have a filtration V on $q_* \mathbf{K}_*$, for which

$$\mathrm{Gr}^{\mathrm{V}}_* p_* \mathbf{K}_* = \overset{*}{\wedge} \mathrm{X} \otimes p_* \mathbf{K}_{*-m}.$$

As those claims are local on S, it suffices to consider the case where Q is trivial. We leave this case to the reader.

Special cases: for $\mathbf{K}_1$, the filtration V on $p_* \mathbf{K}_1$ reduces to the exact sequence

$$1 \to \mathscr{O}^*_{\mathrm{S}} \to p_* \mathscr{O}^*_{\mathrm{T}} \to \mathrm{X} \to 1,$$

and the reduced direct image $\widetilde{p}_* \mathbf{K}_1$ is just X. For $\mathbf{K}_2$, $p_* \mathbf{K}_2$ has three graded pieces: $\mathbf{K}_2$, $\mathrm{X} \otimes \mathbf{K}_1$ and $\overset{2}{\wedge} \mathrm{X}$. The extension $\widetilde{p}_* \mathbf{K}_2$ of $\overset{2}{\wedge} \mathrm{X}$ by $\mathrm{X} \otimes \mathbf{K}_1$ can be described as follows.

Construction **3.5.** — *Under the assumptions of 3.1 and 3.3, the sheaf* $\widetilde{p}_* \mathbf{K}_2$ *is the sheaf of pairs* (A, q)*, with* A *an alternating form on* Y *and* q *a map of sheaves* $\mathrm{Y} \to \mathscr{O}^*_{\mathrm{S}}$ *such that*

$$(\mathbf{3.5.1}) \qquad q(y+z)/q(y)q(z) = (-1)^{\mathrm{A}(y,z)}.$$

In 3.5 and below, “form” means integer-valued form.

Using 3.3, one sees that $\widetilde{p}_{2*} \mathbf{K}_2$ is generated by $\mathrm{X} \otimes \mathrm{X}$ and by $\mathrm{X} \otimes \mathscr{O}^*_{\mathrm{S}}$, with the relations $x \otimes x = x \otimes \{-1\}$.

The following interpretations will be useful:

$\mathrm{X} \otimes \mathscr{O}^*_{\mathrm{S}}$: morphisms from Y to $\mathscr{O}^*_{\mathrm{S}}$: $x \otimes f \longmapsto$ morphism $f^{x(y)}$.

$\mathrm{X} \otimes \mathrm{X}$: the group of not necessarily symmetric bilinear forms $\mathrm{F}(y_1, y_2)$ on Y: $x_1 \otimes x_2 \longmapsto$ form $x_1(y_1)x_2(y_2)$.

$\overset{2}{\wedge} \mathrm{X}$: the group of alternating forms on Y: $x_1 \wedge x_2 \longmapsto$ (form $x_1(y_1)x_2(y_2) - x_2(y_1)x_1(y_2)$).

$\mathrm{Sym}^2(\mathrm{X})$: the group of quadratic forms $\mathrm{Q}(y)$ on Y: $x_1 x_2 \longmapsto$ form $x_1(y)x_2(y)$.

We will also meet $\Gamma^2 \mathrm{X}$, the component of degree 2 of the divided power algebra $\Gamma \mathrm{X}$. It is the target of the universal quadratic map with source X: γ_2: $\mathrm{X} \to \Gamma^2 \mathrm{X}$. It is also the symmetric part of $\mathrm{X} \otimes \mathrm{X}$, with $\gamma_2(x) = x \otimes x$.

Lemma **3.5.2.** — *The map* R: $\mathrm{X} \to (\mathrm{X} \otimes \mathrm{X}) \oplus (\mathrm{X} \otimes \mathscr{O}^*_{\mathrm{S}})$*:* $x \mapsto x \otimes x - x \otimes \{-1\}$ *is quadratic, i.e.* $\mathrm{R}(x_1 + x_2) - \mathrm{R}(x_1) - \mathrm{R}(x_2)$ *is biadditive and* $\mathrm{R}(nx) = n^2 \mathrm{R}(x)$.

Proof. — The summand $x \otimes x$ is clearly quadratic. For the summand $x \otimes \{-1\}$, one observes that it takes values in elements of order 2, and that $n^2 \equiv n (\mathrm{mod}\ 2)$.

By 3.5.2, the map $x \mapsto x \otimes x + x \otimes \{-1\}$ factors through $\Gamma^2(X)$ and $\widetilde{p}_* \mathbf{K}_2$ is the cokernel of

(**3.5.3**) $$\Gamma^2 X \to X \otimes X \oplus X \otimes \mathcal{O}_S^*\colon \gamma_2(x) \longmapsto x \otimes x - x \otimes \{-1\}.$$

To $C + F$ in $X \otimes X \oplus X \otimes \mathcal{O}_S^*$, we attach (A, q) defined as follows: with C and F interpreted as above,

$$\begin{aligned} A(y, z) &:= C(y, z) - C(z, y) \\ q(y) &:= F(y).(-1)^{C(y,y)}. \end{aligned}$$

Condition (3.5.1) holds for this (A, q). To check that this construction identifies the sheaf of pairs (A, q) with the cokernel of (3.5.3), one only needs to stare at the diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & \Gamma^2(X) & \longrightarrow & X \otimes X & \longrightarrow & \overset{2}{\wedge} X & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ & & \Gamma^2(X) & \longrightarrow & X \otimes X \oplus X \otimes \mathcal{O}_S^* & \longrightarrow & \{(A, q)\} & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & X \otimes \mathcal{O}_S^* & = & X \otimes \mathcal{O}_S^* & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

and conclude that the middle row is a short exact sequence.

3.6. A section s of $\mathbf{K}_2$ on a group scheme G over S is *multiplicative* if

$$\mu^*(s) = \mathrm{pr}_1^*(s) + \mathrm{pr}_2^*(s).$$

In the language of 1.4, this means that s is a morphism of sheaves of groups, from G to $\mathbf{K}_2$.

For group schemes G_1 and G_2, a section s of $\mathbf{K}_2$ on $G_1 \times_S G_2$ is *bimultiplicative* if it is multiplicative in G_1 as well as in G_2: for the group schemes $G_1 \times_S G_2$ over G_2 and $G_1 \times_S G_2$ over G_1.

Corollary **3.7.** — (i) *Under the assumptions of 3.1 and 3.3, the group of multiplicative sections of* $\mathbf{K}_2$ *over* T *is* $H^0(S, X \otimes \mathcal{O}_S^*)$.

(ii) *For split tori* T_1, T_2 *over* S, *a section of* $\mathbf{K}_2$ *over* $T_1 \times_S T_2$ *is bimultiplicative as soon as it has a trivial restriction to* $T_1 \times_S e$ *and* $e \times_S T_2$. *The group of bimultiplicative sections is* $H^0(S, X_1 \otimes X_2)$.

Proof. — The description 3.5 is functorial in T. It follows that multiplicative sections correspond to pairs (A, q) with A = 0. By (3.5.1), q is then an homomorphism from Y to $\mathscr{O}_S^*$, i.e. a section of $X \otimes \mathscr{O}_S^*$.

Sections with trivial restrictions to $T_1 \times_S e$ and $e \times_S T_2$ correspond to pairs (A, q) with trivial restrictions to Y_1 and Y_2: A reduces to a pairing between Y_1 and Y_2 and A determines q. The sheaf over S of such sections is identified to $X_1 \otimes X_2$ by $x_1 \otimes x_2 \mapsto \{\mathrm{pr}_1^*(x_1), \mathrm{pr}_2^*(x_2)\}$, and they are bimultiplicative.

3.8. *Remark* (cf. S. Bloch (1978)). — On the big Zariski site of S regular of finite type over a field k, 3.7 can be interpreted as

$$\textbf{(3.8.1)} \qquad \mathrm{Hom}(\mathbf{G}_m, \mathbf{K}_2) = \mathrm{H}^0(\mathrm{S}, \mathscr{O}_S^*)$$

$$\textbf{(3.8.2)} \qquad \mathrm{Hom}(\mathbf{G}_m \otimes_{\mathbf{Z}} \mathbf{G}_m, \mathbf{K}_2) = \mathbf{Z}.$$

On the big Zariski site of S, taken in the smooth sense (the category of T smooth over S), this can be repeated locally, giving

$$\textbf{(3.8.3)} \qquad \mathscr{H}om(\mathbf{G}_m, \mathbf{K}_2) = \mathbf{K}_1$$

$$\textbf{(3.8.4)} \qquad \mathscr{H}om(\mathbf{G}_m \otimes_{\mathbf{Z}} \mathbf{G}_m, \mathbf{K}_2) = \mathbf{K}_0.$$

3.9. If S is the spectrum of a field, it results from 1.9 Example (vii) and from the vanishing of $\mathrm{H}^i(\mathrm{T}, \mathbf{K}_n)$ for $i > 0$ that the category of central extension of T by $\mathbf{K}_n$ is incarnated in degree 2 by the complex $\widetilde{\mathrm{H}}^0(\mathrm{T}^p, \mathbf{K}_n)$. The cohomology of the complex $\mathrm{H}^0(\mathrm{T}^p, \mathbf{K}_n)$ is computed in Esnault et al. (1998) 4.6:

$$\textbf{(3.9.1)} \qquad \mathrm{H}^n(\mathrm{H}^0(\mathrm{T}^*, \mathbf{K}_m)) = \mathrm{Sym}^n(\mathrm{X}) \otimes \mathrm{K}_{m-n}(k).$$

For $m = 2$, one deduces from (3.9.1) that the complex $\widetilde{\mathrm{H}}^0(\mathrm{T}^p, \mathbf{K}_2)$ has cohomology only in degree 1 and 2, with

$$\textbf{(3.9.2)} \qquad \begin{aligned} \mathrm{H}^1\widetilde{\mathrm{H}}^0(\mathrm{T}^*, \mathbf{K}_2) &= \mathrm{X} \otimes k^* \\ \mathrm{H}^2\widetilde{\mathrm{H}}^0(\mathrm{T}^*, \mathbf{K}_2) &= \mathrm{Sym}^2\mathrm{X}. \end{aligned}$$

We will need to know how the isomorphisms (3.9.2) are obtained at the level of cocycles.

Let L be the cosimplicial group dual to the simplicial group BY. The component of degree n of L is the dual X^n of Y^n; it is the character group of the component T^n of BT. The coface maps are the

$$\partial_i \colon f(y_1, \ldots, y_n) \longmapsto f(y_1, \ldots, y_i + y_{i+1}, \ldots, y_{n+1}),$$

suitably modified for the extreme values 0 and $n+1$ of i. The codegeneracies are

$$s_i \colon f(y_1, \ldots, y_n) \longmapsto f(y_1, \ldots, y_{i-1}, 0, y_i, \ldots, y_{n-1}).$$

By 3.4, 3.5, the cosimplicial group $\widetilde{\mathrm{H}}^0(\mathrm{T}^n, \mathbf{K}_2)$ is an extension of $\overset{2}{\wedge}\mathrm{L}$ by $\mathrm{L} \otimes k^*$.

As a differential complex, L is homotopic to the subcomplex $\bigcap \mathrm{Ker}(s_i)$ of non-degenerate cochains. This subcomplex is reduced to X in degree 1. The cohomology of $L \otimes k^*$ is given by the Künneth formula: it is concentrated in degree 1, with $H^1(L \otimes k^*) = X \otimes k^*$. The subcomplex of non-degenerate cochains of $\overset{2}{\wedge} L$ is concentrated in degree 1 and 2. In degree 1, it is $\overset{2}{\wedge} X$. In degree 2, it is $\mathrm{Ker}(\overset{2}{\wedge}(X \oplus X) \to \overset{2}{\wedge} X \oplus \overset{2}{\wedge} X)$, identified with $X \otimes X$: with the identification of (3.5), a bilinear form $C \in X \otimes X$ on Y corresponds to the alternating form $C(y_1, z_2) - C(z_1, y_2)$ on $Y \oplus Y$. The differential maps A in $\overset{2}{\wedge} X$ to the alternating form $((y_1, y_2), (z_1, z_2)) \mapsto -(A(y_1, z_2) - A(z_1, y_2))$ on $Y \oplus Y$, identified with $-A$ in $X \otimes X$. The cohomology group $H^2 = \mathrm{coker}(\overset{2}{\wedge} X \to X \otimes X)$ is $\mathrm{Sym}^2(X)$, with $C \mapsto$ quadratic form $C(y, y)$ on Y.

The formula (3.9.2) now follows from the long exact sequence of cohomology. A bilinear form C on Y, identified with an element of $X \otimes X$, defines a 2-cocycle in the complex $\widetilde{H}^0(T^n, \mathbf{K}_2)$ by

(**3.9.3**) $$x_1 \otimes x_2 \longmapsto \{\mathrm{pr}_1^* x_1, \mathrm{pr}_2^* x_2\}$$

and the corresponding cohomology class is the quadratic form $C(y, y)$ on Y.

3.10. Let E be a multiplicative $\mathbf{K}_2$-torsor on T. By 3.9, for some bilinear form C on Y, E is isomorphic to the trivial torsor, with the multiplicative structure given by the cocycle image of C by (3.9.3). The group of isomorphism classes of multiplicative $\mathbf{K}_2$-torsors is $\mathrm{Sym}^2(X)$, and the class of E is given by the quadratic form $C(y, y)$ on Y.

Let us extend the scalars from k to $k[t, t^{-1}]$. By 1.4, we deduce from E a central extension (1.4.1), on $\mathrm{Spec}\, k[t, t^{-1}]$, of the sheaf of local sections of T by $\mathbf{K}_2$. As $H^1(\mathrm{Spec}\, k[t, t^{-1}], \mathbf{K}_2) = 0$ (3.1.2), taking global sections, we obtain a central extension

(**3.10.1**) $$H^0(\mathrm{Spec}(k[t, t^{-1}]), \mathbf{K}_2) \to \mathscr{E}_0 \to T(k[t, t^{-1}]).$$

We take the pull back of this central extension by

(**3.10.2**) $$Y \to T(k[t, t^{-1}]) = k[t, t^{-1}]^* \otimes Y : y \longmapsto t \otimes y,$$

and the push out by

(**3.10.3**) $$H^0(\mathrm{Spec}(k[t, t^{-1}], K_2) =_{(3.1.2)} K_2(k) \oplus k^* \to k^*,$$

obtaining a central extension

(**3.10.4**) $$k^* \to \mathscr{E} \to Y.$$

Proposition **3.11.** — *The construction 3.10 is an equivalence of categories from the category of multiplicative* $\mathbf{K}_2$*-torsors on* T *to the category of pairs* (Q, $\mathscr{E}$)*, where* Q *is a quadratic form on*

Y *and* $\mathscr{E}$ *is a central extension of* Y *by* k^* *for which the commutator map 0.N.4 (2):* $Y \times Y \to k^*$, *is given by*

(**3.11.1**) $$(y, z) = (-1)^{B(y, z)}.$$

In (3.11.1), B is the bilinear form associated to Q (0.N.3).

Proof. — We first check that (3.11.1) holds for the central extension (3.10.4). Let C be a (non-necessarily symmetric) bilinear form on Y such that $Q(y) = C(y, y)$. The multiplicative torsor E is then isomorphic to the trivial torsor, with multiplicative structure given by the cocycle image of C by (3.9.3). The extension (3.10.4) is then given by the cocycle

$$c(y, z) = C(y, z).(\text{projection to } k^* \text{ of } \{t, t\}).$$

Here, $\{t, t\}$ is the product in $\mathbf{K}_2$. It equals $\{t, -1\}$, with projection -1 in k^*, and $c(y, z) = (-1)^{C(y, z)}$ from which (3.11.1) follows. One has indeed

$$B(y, z) = C(y, z) + C(z, y) \equiv C(y, z) - C(z, y) \qquad (\text{mod } 2).$$

If $Q = 0$, a central extension for which (3.11.1) holds is commutative. As it is an extension of the free abelian group Y by k^*, it is trivial. It follows that the isomorphism class of $(Q, \mathscr{E})$ only depends on Q, and the construction 3.10 induces a bijection on isomorphism classes of objects.

It remains to see that it induces a bijection on automorphism groups. By 3.7 (i), the automorphism group of a multiplicative torsor E is $X \otimes k^* = \mathrm{Hom}(Y, k^*)$, with $x \otimes f$ acting as the addition of $\{x, f\}$. The automorphism group of a central extension of Y by k^* is $\mathrm{Hom}(Y, k^*)$ as well, and we leave it to the reader to check that the functoriality map is the identity.

Remark **3.12.** — In 3.10, instead of extending scalars from k to the ring $k[t, t^{-1}]$, we could have extended scalars to the field $k((t))$, obtaining from E instead of (3.10.1) a central extension

$$K_2 k((t)) \to \mathscr{E}_1 \to T(k((t)))$$

We still have a morphism

(**3.12.1**) $$Y \to T(k((t))) = k((t))^* \otimes Y : y \longmapsto t \otimes y$$

and the tame symbol defines

(**3.12.2**) $$K_2 k((t)) \to k^*.$$

Pushing by (3.12.2) and pulling back by (3.12.1), we obtain a central extension of Y by k^*, canonically isomorphic to the central extension (3.10.4).

Proposition **3.13.** — *Let* E *be a central extension of* T *by* $\mathbf{K}_2$*, that is a multiplicative* $\mathbf{K}_2$*-torsor on* T *(1.4). Let* Q *be the corresponding quadratic form on* Y*, and let* B *be the associated bilinear form. The corresponding commutator* comm *of (1.8.1), in* $H^0(T \times T, \mathbf{K}_2)$*, is bimultiplicative. It is the image of* B*, by the identification 3.7 (ii).*

Proof. — Choose C in $X \otimes X$ whose image in $\widetilde{H}^0(T \times T, \mathbf{K}_2)$ is a cocycle c defining E. One has $C(y, y) = Q(y)$. The commutator map (1.8.1) is $(t_1, t_2) \mapsto c(t_1, t_2) - c(t_2, t_1)$. The product $\mathbf{K}_1 \times \mathbf{K}_1 \to \mathbf{K}_2$ being anticommutative, the corresponding section of $\mathbf{K}_2$ over $T \times T$ is the image of $C + {}^tC$, where ${}^tC(y_1, y_2) = C(y_2, y_1)$. It remains to observe that $B = C + {}^tC$.

Corollary **3.14.** — *If* $B = \sum b_{ij} x^i \otimes x^j$*, the commutator for the central extension*

$$K_2(k) \to E(k) \to T(k)$$

is given by $(t_1, t_2) \mapsto \sum b_{ij} \{x^i(t_1), x^j(t_2)\}$.

3.15. We now return to the case of a split torus T over S regular of finite type over a field. Let p be the projection of T^n to S. Ambiguity on n will be avoided by writing $p_*(T^n, \mathbf{K}_2)$ for the direct image of $\mathbf{K}_2$ from T^n to S. Let $\widetilde{p}_*(T^n, \mathbf{K}_2)$ be the reduced direct image $p_*(T^n, \mathbf{K}_2)/\mathbf{K}_2$. Applying the variant 1.10 of 1.9 Example (vii), one deduces from the vanishing (3.1.2) that the complex of sheaves $\widetilde{p}_*(T^n, \mathbf{K}_2)$ on S incarnates in degree 2 the stack

$$U \longmapsto \text{ category of multiplicative } \mathbf{K}_2\text{-torsors on the split torus } T_U = p^{-1}(U) \text{ over } U.$$

As in 3.9, the local classification of $\mathbf{K}_2$-torsors on T is by quadratic forms on Y: locally on S, a multiplicative $\mathbf{K}_2$-torsor E on T defines a quadratic form on Y and E′, E″ are locally isomorphic if and only if the corresponding quadratic forms are equal. As in 3.10, a multiplicative $\mathbf{K}_2$-torsor on T gives rise to a central extension, on S, of the constant sheaf Y by $p_*\mathbf{K}_2$, for $p: \mathbf{G}_{m_S} \to S$, and from this we get a central extension of Y by $\mathcal{O}_S^*$. As in 3.11, one has

Theorem **3.16.** — *For* T *a split torus over* S *regular of finite type over a field* k*, the construction 3.15 induces an equivalence of commutative Picard categories from*

(i) *the category of multiplicative* $\mathbf{K}_2$*-torsors on* T*, to*

(ii) *the category of pairs* $(Q, \mathcal{E})$*:* Q *a quadratic form on the sheaf* Y *and* $\mathcal{E}$ *a central extension of* Y *by the sheaf* $\mathcal{O}_S^*$*, for which, as in 3.11,*

$$(y, z) = (-1)^{B(y, z)}.$$

The commutator (1.8.1) defined by a multiplicative $\mathbf{K}_2$*-torsor is given as in 3.13.*

4. Split semi-simple simply-connected groups

For G split semi-simple simply-connected over a field k, with split maximal torus T and Weyl group W, we will prove in this section results which, via the formalism of 1.9, amounts to $\widetilde{\mathrm{H}}^i(\mathrm{G}, \mathbf{K}_2) = 0$ for $i \neq 1$, $\widetilde{\mathrm{H}}^i(\mathrm{BG}, \mathbf{K}_2) = 0$ for $i \neq 2$, $\mathrm{H}^2(\mathrm{BG}, \mathbf{K}_2) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{G}, \mathbf{K}_2)$ and $\mathrm{H}^2(\mathrm{BG}, \mathbf{K}_2) \xrightarrow{\sim} \mathrm{H}^2(\mathrm{BT}, \mathbf{K}_2)^{\mathrm{W}}$, the group of W-invariant quadratic forms on the dual character group Y of T. We could have quoted those results from Esnault et al. (1996) (3.2, 4.7, 4.8). Our proofs are similar, but closer to the cocycle level and at places more elementary. In loc. cit. the computations use the projection $\mathrm{G} \to \mathrm{G}/\mathrm{T}$, the fact that G/T admits a stratification where the strata are affine spaces, and the fact that G can be viewed as a bundle over G/T, with fibers vector spaces minus coordinates hyperplanes. This leads to the spectral sequence loc. cit. (3.7) which presumably agrees with the spectral sequence of our complex (4.3.2), filtered by (4.3.5). At least, the E_1-terms and the abutment are the same. Our method is to use the Schubert cell decomposition of G/T (which proves the cellularity of G/T), and to use not the projection $\mathrm{G} \to \mathrm{G}/\mathrm{T}$, but rather the decomposition of G into the inverse images of the Schubert cells, i.e. the Bruhat decomposition of G. The reader may prefer to skip our proofs, obtain the relative version 4.8 from loc. cit. 3.22, look at 4.7 and 4.9 as categorical interpretations, via 1.9, and resume reading at 4.10.

4.1. Let G be a split reductive group over a field k. We fix a split maximal torus T of G. We will use the notation X, Y, W of (0.N.1) and (0.N.2), as well as the following:

- Φ : the set of roots (a subset of X). For α a root, we denote by
- $\alpha^\vee$: the corresponding coroot; $\alpha^\vee$ is in Y and $\langle \alpha, \alpha^\vee \rangle = 2$;
- U_α : the corresponding radicial subgroup; it is isomorphic to $\mathbf{G}_a$ and T acts on it by α;
- s_α : the corresponding reflection in W;
- S_α : the corresponding SL(2) or PGL(2) subgroup of G, generated by $\alpha^\vee(\mathbf{G}_m)$, U_α and $\mathrm{U}_{-\alpha}$.

The *walls* are the hyperplanes $\langle x, \alpha^\vee \rangle = 0$ of $\mathrm{X} \otimes \mathbf{R}$. For $w \in \mathrm{W}$, we will write $\dot{w}$ for a lifting of w in the normalizer N(T) of T.

We fix a Borel subgroup B containing T. We denote by

- Φ^+ : the corresponding set of positive roots: a root α is in Φ^+ if and only if U_α is in the unipotent radical U of B;
- I : the corresponding system of simple roots. A simple root $i \in \mathrm{I}$ will often be written α_i;
- $\ell(w)$: the length of $w \in \mathrm{W}$, relative to the system of generators s_i $(i \in \mathrm{I})$;
- Φ^- : the complement of Φ^+ in Φ. It is also $-\Phi^+$;
- C_0 : the fundamental chamber in $\mathrm{X} \otimes \mathbf{R}$, defined by $\langle x, \alpha^\vee \rangle \geqslant 0$ for α positive.

For $w \in W$, let $U_{\Phi^+ \cap w\Phi^-}$ be the group generated by the U_α with $\alpha \in \Phi^+ \cap w\Phi^-$. As a scheme, it is isomorphic to $\mathbf{A}^n$ for some n. Indeed, for any order on $\Phi^+ \cap w\Phi^-$, the product map is an isomorphism from $\prod_{\alpha \in \Phi^+ \cap w\Phi^-} U_\alpha$ to $U_{\Phi^+ \cap w\Phi^-}$.

The product map

(**4.1.1**) $$U_{\Phi^+ \cap w\Phi^-} \times wT \times U \to BwB$$

is an isomorphism. The map

(**4.1.2**) $$p\colon BwB \to wT\colon \qquad p = \mathrm{pr}_2 \circ (4.1.1)^{-1}$$

is the unique retraction of BwB to wT such that $p(u\dot{w}tv) = \dot{w}t$ for u and v in U, t in T and $\dot{w}$ any lifting of w in N(T). By 3.1,

(**4.1.3**) $$H^i(BwB, \mathbf{K}_j) = 0 \qquad \text{for } i > 0$$

and $p\colon BwB \to wT$ induces an isomorphism

(**4.1.4**) $$H^0(wT, \mathbf{K}_j) \xrightarrow{\sim} H^0(BwB, \mathbf{K}_j).$$

The length $\ell(w)$ of w is $|\Phi^+ \cap w\Phi^-|$: it is the number of walls separating the fundamental chamber C_0 from its transform $w(C_0)$. By (4.1.1), one has

(**4.1.5**) $$\dim BwB = \ell(w) + \dim B.$$

The Bruhat decomposition $G = \bigcup BwB$ is a stratification of G: the BwB are disjoint and the closure of a stratum BwB is the union of the strata Bw_1B with $w_1 \leqslant w$ in Bruhat's order. The Bw_1B of codimension one in the closure of BwB are obtained as follows: for some simple root i,

(**4.1.6**) $$\begin{aligned} w &= w's_iw'' \quad \text{with} \quad \ell(w) = \ell(w') + 1 + \ell(w''), \qquad \text{and} \\ w_1 &= w'w'' \quad \text{with} \quad \ell(w_1) = \ell(w') + \ell(w'') = \ell(w) - 1. \end{aligned}$$

If (4.1.6) holds, then

(**4.1.7**) $$BwB \cup Bw_1B = Bw'S_iw''B$$

is a smooth subscheme of G.

Fix a lifting $\dot{w}$ of w in N(T). Each character x of T gives rise to the function $x(\dot{w}^{-1}g)$ on wT. Pulling back by (4.1.2), we obtain an invertible function $x_{\dot{w}}$ on BwB. Let us extend x to a character of B by $x(tu) = x(t)$ for t in T and u in U. The invertible function $x_{\dot{w}}$ is characterized by the equivariance condition

(**4.1.8**) $$x_{\dot{w}}(b'gb'') = [w(x)](b')x_{\dot{w}}(g)x(b'')$$

and the normalization $x_{\dot w}(\dot w) = 1$. Changing the lifting $\dot w$ of w changes $x_{\dot w}$ by a multiplicative constant.

If w and w_1 are as in (4.1.6), Bw_1B is a smooth divisor in the closure of BwB and $x_{\dot w}$ has a valuation along Bw_1B. If the valuation is zero, $x_{\dot w}$ extends to an invertible function on $BwB \cup Bw_1B$, which satisfies the equivariance condition (4.1.8).

Lemma **4.2.** — (i) *With the notation above, the valuation of* $x_{\dot w}$ *along* Bw_1B *is given by*

$$v(x_{\dot w}) = \left\langle x,\, w''^{-1}(\alpha_i^\vee)\right\rangle . \tag{4.2.1}$$

(ii) *If* $v(x_{\dot w}) = 0$, *let* $x_{\dot w}|Bw_1B$ *be the restriction to* Bw_1B *of the extension of* $x_{\dot w}$ *to* $BwB \cup Bw_1B$. *Then,* $x_{\dot w}|Bw_1B$ *is a constant multiple of* $x_{\dot w_1}$.

(iii) *If further* $\dot w = \dot w' \dot s_i \dot w''$ *with* $\dot s_i$ *in* S_i *and* $\dot w_1 = \dot w' \dot w''$, *then*

$$x_{\dot w}|Bw_1B = x_{\dot w_1}. \tag{4.2.2}$$

For the proof (by reduction to the case of SL(2)) we refer the reader to Demazure (1974).

Writing $w_1 = w.(w''^{-1}s_i w'')$, one can rewrite (4.2.1) as follows: $w_1 = ws$, for s a reflection in W, and if α is the corresponding positive root, then

$$v(x_{\dot w}) = \langle x, \alpha^\vee\rangle . \tag{4.2.3}$$

4.3. The Bruhat decomposition induces a filtration F on any sheaf $\mathscr{F}$ on G, with $F^p\mathscr{F}$ the sheaf of sections with support in the union of the BwB of codimension $\geqslant p$. For the Quillen complex G_j^*, if $W^{(p)}$ is the set of $w \in W$ for which BwB is of codimension p, we get

$$\mathrm{Gr}_F^p\Gamma(G, G_j^*) = \bigoplus_{w\in W^{(p)}} \Gamma(BwB, G_{j-p}^*)[-p]. \tag{4.3.1}$$

The Quillen complexes in the second member of (4.3.1) are the Quillen complexes of the BwB. As $F^pG_j^*$ vanishes in degree $n < p$, $H^p\mathrm{Gr}_F^p\Gamma(G, G_j^*)$ maps to $\mathrm{Gr}_F^p\Gamma(G, G_j^p) = F^p\Gamma(G, G_j^p) \subset \Gamma(G, G_j^p)$; the $H^p\mathrm{Gr}_F^p\Gamma(G, G_j^*)$ form in this way a subcomplex of $\Gamma(G, G_j^*)$. By (4.1.3), $\Gamma(BwB, G_{j-p}^*)$ is a resolution of $H^0(BwB, \mathbf{K}_{j-p})$. It follows that the inclusion in $\Gamma(G, G_j^*)$ of the subcomplex

$$H^p\mathrm{Gr}_F^p\Gamma(G, G_j^*) = \bigoplus_{w\in W^{(p)}} H^0(BwB, \mathbf{K}_{j-p}) \tag{4.3.2}$$

is a quasi-isomorphism. If $\eta(w)$ is the generic point of BwB, $H^0(BwB, \mathbf{K}_{j-p})$ maps to $\Gamma(G, G_j^p)$ by the restriction map to $K_{j-p}(k(\eta(w)))$.

The only nonzero components of the differential of the subcomplex (4.3.2) of $\Gamma(G, G_j^*)$ are morphisms

$$(\mathbf{4.3.3}) \qquad d_w^{w_1}: H^0(BwB, \mathbf{K}_{j-p}) \to H^0(Bw_1B, \mathbf{K}_{j-p-1})$$

for w and w_1 as in (4.1.6). The local ring of $BwB \cup Bw_1B$ at $\eta(w_1)$ is a valuation ring $V(w_1, w)$ with special point $\eta(w_1)$ and generic point $\eta(w)$. The differential $d_w^{w_1}$ is induced by the corresponding residue map for K-groups:

$$(\mathbf{4.3.4}) \qquad \begin{array}{ccc} H^0(BwB, \mathbf{K}_{j-p}) & \xrightarrow{d_w^{w_1}} & H^0(Bw_1B, \mathbf{K}_{j-p-1}) \\ \downarrow & & \downarrow \\ K_{j-p}(k(\eta(w))) & \xrightarrow{\text{Res}} & K_{j-p-1}(k(\eta(w_1))). \end{array}$$

The commutative diagram (4.3.4) and the following facts suffice to compute $d_w^{w_1}$:

(i) the residue $K_1(k(\eta(w))) \to K_0(k(\eta(w_1)))$: $k(\eta(w))^* \to \mathbf{Z}$ is the valuation;

(ii) the residue map is right linear over $K^*(V(w_1, w))$.

Indeed, by (4.1.4), any element of $H^0(BwB, \mathbf{K}_j)$ is a sum of K-theory products $f_1 . \cdots . f_p . \varkappa$, where $\varkappa$ is in $K_{j-\ell}(k)$, the f_i are invertible functions of the form $x_{\dot{w}}$ on BwB and where, except for f_1, each f_i has valuation zero along Bw_1B, hence extends to $BwB \cup Bw_1B$. We transport to $H^0(BwB, K_{j-p})$, by the isomorphism (4.1.4), the increasing filtration V of $H^0(wT, K_{j-p})$ defined in 3.4. Using (i) (ii), one checks that $d_w^{w_1}$ maps V_n to V_{n-1}. We filter the complex (4.3.2) by the shifted filtration:

$$(\mathbf{4.3.5}) \qquad \text{in degree } p, \ (\text{V shifted})_n = V_{n-p}.$$

By (3.4.1),

$$(\mathbf{4.3.6}) \qquad \mathrm{Gr}_n^{\text{Vshifted}}((4.3.2)) = \bigoplus_{w \in W^{(p)}} \overset{n-p}{\wedge} X \otimes K_{j-n}(k).$$

For w and w_1 as in (4.1.6), the component $\mathrm{Gr}_n(d_w^{w_1})$ of the differential is given as follows: if $x_1 \wedge \ldots \wedge x_{j-p} \otimes \varkappa$ in $\overset{n-p}{\wedge} X \otimes K_{j-n}(k)$ is such that for $i \geqslant 2$ the $(x_i)_{\dot{w}}$ on BwB have valuation zero along Bw_1B, then it is mapped to $\langle x_1, w''^{-1}(\alpha_i^\vee)\rangle . x_2 \wedge \ldots \wedge x_{j-p} \otimes \varkappa$. In other words: one contracts with the linear form $\langle \ , w''^{-1}(\alpha_i^\vee)\rangle$ on X.

4.4. For $j = 2$, the $\mathrm{Gr}_n^{\text{Vshifted}}((4.3.2))$ are

$$(\mathbf{4.4.1}) \qquad \begin{aligned} n = 0 &: \quad K_2(k); \\ n = 1 &: \quad X \otimes k^* \to \bigoplus_{w \in W^{(1)}} k^*; \\ n = 2 &: \quad \overset{2}{\wedge} X \to \bigoplus_{w \in W^{(1)}} X \to \bigoplus_{w \in W^{(2)}} \mathbf{Z}. \end{aligned}$$

As $W^{(0)}$ is reduced to the longest element w_0 of W, we dispensed with indicating a sum over $W^{(0)}$. The set $W^{(1)}$ consists of the $w_0 s_i$, and $W^{(2)}$ consists of the $w_0 s_i s_j$ for $i \neq j$. On $W^{(0)} \cup W^{(1)} \cup W^{(2)}$, the Bruhat order is generated by the inequalities $w_0 s_i \leqslant w_0$ and $w_0 s_i s_j \leqslant w_0 s_i$, $w_0 s_j$ for $i \neq j$.

The maps $d_{w_0}^{w_0 s_i}$ and $d_{w_0 s_j}^{w_0 s_j s_i}$ are contraction with $\alpha_i^\vee$. The map $d_{w_0 s_j}^{w_0 s_i s_j}$ is contraction with $s_j(\alpha_i^\vee)$ (cf. (4.2.3)).

Let $Y_{sc} \subset Y$ be generated by the coroots. It is a free **Z**-module with basis the simple coroots $\alpha_i^\vee$. Let X_{sc} be the dual of Y_{sc}. We identify X_{sc} with $\bigoplus_{w \in W^{(1)}} \mathbf{Z}$ by $x \longmapsto (x(\alpha_i^\vee)$ at $w_0 s_i)$. With this identification, the complex $(4.4.1)_{n=1}$ becomes

$$X \otimes k^* \to X_{sc} \otimes k^*, \tag{4.4.2}$$

and can be identified with the restriction map from $\mathrm{Hom}(Y, k^*)$ to $\mathrm{Hom}(Y_{sc}, k^*)$. The complex $(4.4.1)_{n=2}$ becomes

$$\overset{2}{\wedge} X \to X_{sc} \otimes X \to \bigoplus_{W^{(2)}} \mathbf{Z}. \tag{4.4.3}$$

The first map is restriction, from alternating forms on Y to bilinear forms pairing $Y_{sc} \subset Y$ and Y. The set $W^{(2)}$ is the set of $w_0 s_i s_j$ for i and j distinct simple roots, and the only equalities among the $w_0 s_i s_j$ is that $w_0 s_i s_j = w_0 s_j s_i$ if the simple roots i and j are orthogonal. The $w_0 s_i s_j$ component of second map in (4.4.3) is

$$C \longmapsto C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_j^\vee, s_j(\alpha_i^\vee)). \tag{4.4.4}$$

The first term comes from $d_{w_0 s_i}^{w_0 s_i s_j}$, the second from $d_{w_0 s_j}^{w_0 s_i s_j}$. One can express the right side of (4.4.4) in terms of the quadratic form $Q(y) := C(y, y)$ on Y_{sc} and its associated bilinear form B:

$$\begin{aligned} C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_j^\vee, s_j(\alpha_i^\vee)) &= C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_j^\vee, \alpha_i^\vee - \alpha_j(\alpha_i^\vee)\alpha_j^\vee) \\ &= B(\alpha_i^\vee, \alpha_j^\vee) - \alpha_j(\alpha_i^\vee) Q(\alpha_j^\vee). \end{aligned} \tag{4.4.5}$$

Lemma **4.5.** — C *is in the kernel of the second map (4.4.3) if and only if the quadratic form* $Q(y) = C(y, y)$ *on* Y_{sc} *is* W*-invariant.*

Proof. — A quadratic form Q, with associated bilinear form B, is invariant by s_j if and only if

$$\begin{aligned} Q(s_j(y)) - Q(y) = Q(y - \alpha_j(y)\alpha_j^\vee) - Q(y) &= B(y, -\alpha_j(y)\alpha_j^\vee) + Q(-\alpha_j(y)\alpha_j^\vee) \\ &= -\alpha_j(y)(B(y, \alpha_j^\vee) - \alpha_j(y) Q(\alpha_j^\vee)) \end{aligned}$$

vanishes identically in y. As the linear form $\alpha_j(y)$ is not identically zero, this holds if and only if

$$B(y, \alpha_j^\vee) - \alpha_j(y) Q(\alpha_j^\vee) \tag{4.5.1}$$

vanishes identically in y. As (4.5.1) is linear in y and vanishes for $y = \alpha_j^\vee$, Q is invariant by s_j if and only if (4.4.5) vanishes for all $i \neq j$ and Q is W-invariant if and only if it is in the kernel of (4.4.3).

Until the end of this section, we assume that G is semi-simple simply-connected, i.e. that $Y = Y_{sc}$.

Proposition **4.6.** — *If* G *is simply-connected the complex* $(4.4.1)_{n=1}$ *is acyclic, and* $(4.4.1)_{n=2}$ *has cohomology only in degree* 1. *In the model (4.4.3), the morphism* $C \mapsto C(y, y)$ *identifies the* H^1 *of* $(4.4.1)_{n=2}$ *with the group of* W*-invariant quadratic forms on* Y.

As a consequence,

(**4.6.1**) $H^0(G, \mathbf{K}_2) = K_2(k)$, *hence* $\widetilde{H}^0(G, \mathbf{K}_2) = 0$;

(**4.6.2**) $H^1(G, \mathbf{K}_2) = \{$W*-invariant quadratic forms on* Y$\}$;

(**4.6.3**) $H^i(G, \mathbf{K}_2) = 0$ *for* $i \geqslant 2$.

Proof. — Acyclicity of $(4.4.1)_{n=1}$ is clear from the description (4.4.2). For $(4.4.1)_{n=2}$, we use the description (4.4.3), with $X_{sc} = X$. The cokernel of the map $\overset{2}{\wedge} X \to X \otimes X$ is the group $\mathrm{Sym}^2(X)$ of quadratic forms on $Y = Y_{sc}$. It hence follows from 4.5 that the H^1 of (4.4.3) is the group of W-invariant quadratic forms on Y. It remains to check the surjectivity of the map

(**4.6.4**) $$\mathrm{Sym}^2(X) \to \bigoplus_{W^{(2)}} \mathbf{Z}\colon Q \longmapsto (B(\alpha_i^\vee, \alpha_j^\vee) - \alpha_j(\alpha_i^\vee)Q(\alpha_j^\vee)) \text{ at } w_0 s_i s_j.$$

The surjectivity of (4.6.4) amounts to the vanishing of $H^2(G, \mathbf{K}_2)$, the Chow group $CH^2(G)$. In Esnault et al. (1998) 3.20, this is deduced from $K_0(G) = 0$, proved by Levine (1993) using the known topological K groups of the corresponding complex group. We give below an elementary verification. The result will not be needed in the rest of this article.

The Dynkin diagram of G is the graph I, with i and j joined by an edge if i and j are not orthogonal, plus additional information. To the decomposition of I into connected components corresponds a decomposition of (Y, W), and the invariant quadratic forms are the sums of invariant quadratic forms on the summands. On each summand, there is only one up to a factor. For each component I^λ, either

(a) They are two lengths of roots and I^λ is linear, with the short roots on one side and the long roots on the other. We order I^λ linearly, starting from a long root.

(b) All roots have the same length and I^λ is a tree. We order I^λ in such a way that each $i \in I^\lambda$, except the first, is joined to a unique previous root.

We will also use

(c) If simple roots α and β are not orthogonal and length $(\alpha) \geqslant$ length (β), then $\beta(\alpha^\vee) = -1$.

This can be checked case by case in rank 2, or one may observe that for any W-invariant positive inner product (,), the negative integer $\beta(\alpha^\vee) = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$ is < 2 in absolute value.

We identify $\bigoplus_{W^{(2)}} \mathbf{Z}$ with the systems of integers (d_{ij}) $(i, j \in \mathrm{I},\ i \neq j)$ for which $d_{ij} = d_{ji}$ whenever $i \perp j$, that is, $s_i s_j = s_j s_i$. We will show that for any such system (d_{ij}), and any family of integers d_λ, there is a unique Q in $\mathrm{Sym}^2(\mathrm{X})$ for which

$$\begin{cases} \mathrm{B}(\alpha_i^\vee, \alpha_j^\vee) - \alpha_j(\alpha_i^\vee)\mathrm{Q}(\alpha_j^\vee) = d_{ij} \\ \mathrm{Q}(\alpha_{1,\lambda}^\vee) = d^\lambda, \quad \text{for } \alpha_{1,\lambda} \text{ the first element of } \mathrm{I}^\lambda\text{:} \end{cases}$$

We take as coordinates for Q the integers $\mathrm{Q}(\alpha_i^\vee)$ and $\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee) = \mathrm{B}(\alpha_j^\vee, \alpha_i^\vee)$ $(i \neq j)$. If i is orthogonal to j, the first equation reduces to $\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee) = d_{ij}$. We now consider the remaining equations, as well as the remaining $\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee)$, and the $\mathrm{Q}(\alpha_i^\vee)$. The problem breaks into similar problems, one for each connected component I^λ. On I^λ, ordered as $\alpha_1, \ldots, \alpha_\ell$, the second equation tells us to start with

$$\mathrm{Q}(\alpha_1^\vee) = d^\lambda.$$

For each $j \geqslant 2$, α_i is not orthogonal to α_j for a unique $i < j$, and by (c) $\alpha_j(\alpha_i^\vee) = -1$. The equation

$$\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee) - \alpha_i(\alpha_j^\vee)\mathrm{Q}(\alpha_i^\vee) = d_{ji}$$

determines $\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee)$ from the previous $\mathrm{Q}(\alpha_i^\vee)$, while the equation

$$\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee) - \alpha_j(\alpha_i^\vee)\mathrm{Q}(\alpha_j^\vee) = d_{ij}$$

determines $\mathrm{Q}(\alpha_j^\vee)$ from $\mathrm{B}(\alpha_i^\vee, \alpha_j^\vee)$. This proves inductively existence and unicity of the solution: (4.6.4) is surjective, and there is a unique W-invariant quadratic form with prescribed $\mathrm{Q}(\mathrm{H}_1^\lambda)$.

Theorem **4.7.** — *For* G *split and simply-connected, pointed* $\mathbf{K}_2$*-torsors on* G *have no nontrivial automorphism. They admit a unique multiplicative structure. They are classified by* W-*invariant quadratic forms on* Y.

Proof. — The first statement amounts to (4.6.1). The last statement expresses (4.6.2).

A cartesian power G^n of G is again split and simply-connected, with maximal split torus T^n and Borel subgroup B^n. The Bruhat decompositions of G^n is the product of the Bruhat decomposition of the factors G and it follows that the classification 4.6 of (pointed) torsors by quadratic forms is functorial for the projections from G^n to G.

As a consequence, the morphism

$$(\mathbf{4.7.1}) \qquad (\mathrm{pr}_i^*)\colon \mathrm{H}^1(\mathrm{G}, \mathbf{K}_2)^n \to \mathrm{H}^1(\mathrm{G}^n, \mathbf{K}_2)$$

is an isomorphism. If inj_i is the embedding in G^n of the i^{th} factor G, the morphism

$$(\mathbf{4.7.2}) \qquad (\mathrm{inj}_i^*)\colon H^1(G^n, \mathbf{K}_2) \to H^1(G, \mathbf{K}_2)^n$$

is a retraction of 4.7.1, hence an inverse to (4.7.1): $\mathbf{K}_2$-torsors on G^n are determined by their restrictions to each factor G. If P is a pointed torsor on G, it follows that μ^*P is isomorphic to $\mathrm{pr}_1^*P + \mathrm{pr}_2^*P$. The unique isomorphism of pointed torsors

$$\mathrm{pr}_1^*P + \mathrm{pr}_2^*P \to \mu^*P$$

is a multiplicative structure: one has to check the equality of two isomorphisms of pointed $\mathbf{K}_2$-torsors on $G \times G \times G$, and such torsors have no nontrivial automorphisms.

4.8. For S regular of finite type over a field and G a split reductive group over S, with split maximal torus T and Borel subgroup B containing T, the arguments used in this section can be repeated, when sheafified over S. If a (resp. a_w) is the projection of G (resp. BwB) to S, the complex $a_*G_j^*$ is quasi-isomorphic to the subcomplex

$$(\mathbf{4.8.1}) \qquad \bigoplus_{w \in W^{(p)}} a_{w*}\mathbf{K}_{j-p}$$

which, for $j = 2$, can be filtered with associated graded Gr_n the following complexes of sheaves on S

$$(\mathbf{4.8.2}) \qquad \begin{aligned} n = 0 : &\quad \mathbf{K}_2 ; \\ n = 1 : &\quad X \otimes \mathcal{O}_S^* \to \bigoplus_{w \in W^{(1)}} \mathcal{O}_S^* ; \\ n = 2 : &\quad \overset{2}{\wedge} X \to \bigoplus_{w \in W^{(1)}} X \to \bigoplus_{w \in W^{(2)}} \mathbf{Z}. \end{aligned}$$

For G simply-connected the proof of 4.6 gives

(**4.8.3**) $a_*\mathbf{K}_2 = \mathbf{K}_2$, hence $\widetilde{a}_*\mathbf{K}_2 = 0$ for the reduced direct image;

(**4.8.4**) $R^1 a_*\mathbf{K}_2 =$ the locally constant sheaf of W-invariant forms on Y;

(**4.8.5**) $R^i a_*\mathbf{K}_2 = 0$ for $i \geqslant 2$.

As in 4.7, it follows that pointed $\mathbf{K}_2$-torsors on G have no nontrivial automorphisms, have a unique multiplicative structure, and are classified by (4.8.4).

If (T', B') is another choice of a split maximal torus and of a Borel subgroup containing it, locally on S, (T, B) and (T', B') are conjugate by some inner automorphism $\mathrm{int}(g)$ and the isomorphism $T \to T'$ induced by $\mathrm{int}(g)$ does not depend on g. The dual character groups Y and Y′ (they are rather locally constant sheaves on S) are hence canonically isomorphic.

Lemma **4.8.6.** — *The quadratic forms on* Y *and* Y′ *attached to a multiplicative* $\mathbf{K}_2$*-torsor on* G *correspond to each other by the isomorphism of* Y *with* Y′.

Proof. — The question is local on S, so that we may assume that (T, B) and (T′, B′) are conjugate by some int(g), $g \in$ G(S). An inner automorphism transforms a multiplicative torsor into an isomorphic one (1.6). The claim follows by transport of structures.

We now return to the case of a ground field k and to the notation 4.1.

Compatibility **4.9.** — *Assume that* G *is simply-connected. Let* E *be a multiplicative* $\mathbf{K}_2$*-torsor on* G*, and let* E_T *be its restriction to* T*. The quadratic form 3.11 describing the isomorphism class of the multiplicative* $\mathbf{K}_2$*-torsor* E_T *on* T *is the same as the quadratic form 4.7.*

Proof. — We will use the interpretation 1.4 of multiplicative $\mathbf{K}_2$-torsors as central extensions of G by $\mathbf{K}_2$, viewed as sheaves on a big Zariski site $\mathrm{Spec}(k)_{\mathrm{Zar}}$. As G is smooth over k, we may and shall take as big Zariski site the category of schemes smooth over k, with the Zariski topology. We denote by E_T the extension of T by $\mathbf{K}_2$ induced by the extension E of G by $\mathbf{K}_2$.

For S in $\mathrm{Spec}(k)_{\mathrm{Zar}}$, the product law of E(S) induces a product map

$$E(S) \times E_T(S) \to E(S)\colon (\tilde{g}, \tilde{t}) \longmapsto \tilde{g}\tilde{t}$$

and the right multiplication by $\tilde{t}$ is an automorphism of (G, E) with E viewed as a torsor over G. More precisely, the image t of $\tilde{t}$ in T(S) acts by a right translation r_t on $G_S = G \times S$, and the right multiplication $r_{\tilde{t}}$ by $\tilde{t}$ lifts this action of t on G_S to an automorphism of (G_S, E_S). By (4.8.3), any other lifting differs from $r_{\tilde{t}}$ by an element of $H^0(S, \mathbf{K}_2)$, hence is also of the form $r_{\tilde{t}'}$ for $\tilde{t}'$ in $E_T(S)$ projecting to t in T(S). This construction identifies $E_T(S)$ with the set of pairs (t, τ), with t in T(S) and τ a lifting of r_t. The lifting τ is an isomorphism $E \to r_t^* E$, and the product in E_T becomes

$$(t', \tau')(t'', \tau'') = (t't'', r_{t'}^*(\tau'') \circ \tau').$$

The complex (4.3.2) incarnates, for $j = 2$, the category of $\mathbf{K}_2$-torsors on G. Suppose that E is isomorphic to the $\mathbf{K}_2$-torsor defined by the cocycle (f_i), where f_i is an invertible function on Bw_0s_iB. It is the torsor whose sections on $V \subset G$ are the sections of $\mathbf{K}_2$ on $Bw_0B \cap V$, with residue f_i along $Bw_0s_iB \cap V$. Let $\overline{f}_i$ be the character of T such that $f_i(gt) = f_i(g)\overline{f}_i(t)$.

Fix S, and $t \in$ T(S). We will now tacitly work over S. The pullback torsor $r_t^*(E)$ is given by the cocycle $(r_t^*(f_i)) = (f_i(g)\overline{f}_i(t))$ in the complex (4.8.1). The $\mathbf{K}_2$-torsors E and $r_t^*(E)$ are both trivialized over Bw_0B. Suppose that σ is a section of $\mathbf{K}_2$ over Bw_0B with residue $\overline{f}_i(t)$ along Bw_0s_iB. Note that $\overline{f}_i(t)$ is a function on S, and that we continued to write $\overline{f}_i(t)$ for its pullback to Bw_0s_iB. Then, addition of σ defines an isomorphism of

$\mathbf{K}_2$-torsors $r : \mathrm{E} \to r_t^*(\mathrm{E})$. To construct σ, we start by choosing liftings $\dot{w}_0\dot{s}_i$ of $w_0 s_i$ in N(T). By abuse of notation, for x in X, let us write $x_{w_0 s_i}$ for $x_{\dot{w}_0\dot{s}_i}$. If x^i is the basis of X dual to the basis $\alpha_i^\vee$ of Y, with the notations of 4.1, we may take

$$\sigma = \sum \{x^i_{w_0 s_i} . \bar{f}_i(t)\}.$$

This σ depends on t, functorially in S, and $t \mapsto \sigma(t)$ defines a section $t \mapsto (t, \tau(t))$ of $\mathrm{E_T}$ over T. The central extension $\mathrm{E_T}$ by $\mathbf{K}_2$ is given by the cocycle $c(t', t'')$ such that $(t', \tau(t')(t'', \tau(t'') = (t't'', \tau(t't'') + c(t', t''))$. As

$$(t', \tau(t'))(t'', \tau(t'')) = (t't'', r_{t'}^*(\tau(t'') \circ \tau(t'))),$$

and as $r_{t'}^*(\tau(t'')) \circ \tau(t')$ is, on $\mathrm{B}w_0\mathrm{B}$, the addition of

$$r_{t'}^*\left(\sum\{x^i_{w_0 s_i}, \bar{f}_i(t'')\}\right) + \sum\{x^i_{w_0 s_i}, \bar{f}_i(t')\} = \sum\{x^i_{w_0 s_i}, \bar{f}_i(t't'')\} + \sum\{x^i(t'), \bar{f}_i(t'')\},$$

the cocycle $c(t', t'')$ is $\sum\{x^i(t'), \bar{f}_i(t'')\}$. It is defined (3.9.3) by the bilinear form $\mathrm{C} = \sum x^i \otimes \bar{f}_i$ in $\mathrm{X} \otimes \mathrm{X}$. As a bilinear form on Y, it is characterized by $\mathrm{C}(\alpha_i^\vee, y) = \bar{f}_i(y)$, and the quadratic form $\mathrm{C}(y, y)$ of 3.9 coincides with the quadratic form of 4.7.

4.10. Let $\mathrm{G_{ad}}$ be the adjoint group of G and $\mathrm{T_{ad}}$ be the maximal split torus of $\mathrm{G_{ad}}$ image of T. The group $\mathrm{G_{ad}}$ acts on G (adjoint action). Multiplicative $\mathbf{K}_2$-torsors on G have no nontrivial automorphism and their classification is discrete. It follows that if E is a multiplicative $\mathbf{K}_2$-torsor on G, the action of $\mathrm{G_{ad}}$ on G extends uniquely to an action of $\mathrm{G_{ad}}$ on (G, E). Let us make this argument explicit. If $\mathrm{G_S}$ is deduced from G by the base change $\mathrm{S} \to \mathrm{Spec}(k)$, any $g \in \mathrm{G_{ad}(S)}$ defines an automorphism of $\mathrm{G_S}$. We take $\mathrm{S} = \mathrm{G_{ad}}$ and for g the identity map of $\mathrm{G_{ad}}$. Let G(E) be the transform of E by g: a multiplicative $\mathbf{K}_2$-torsor on $\mathrm{G_{G_{ad}}}$. It is classified by a W-invariant quadratic form on the constant sheaf Y on $\mathrm{G_{ad}}$. As $\mathrm{G_{ad}}$ is connected, this form is constant and $g(\mathrm{E})$ is isomorphic to (the pullback to $\mathrm{G_{G_{ad}}}$ of) E. The isomorphism is unique by 4.8. It is the seeked-for action.

4.11. We pass to the language 1.4 of central extensions by $\mathbf{K}_2$. As $\mathrm{T_{ad}}$ fixes T, the commutator section (1.8.2) is a bimultiplicative section

(**4.11.1**) $\qquad$ $\mathrm{comm} \in \mathrm{H}^0(\mathrm{T_{ad}} \times \mathrm{T}, \mathbf{K}_2)$.

By 3.7 (ii), it corresponds to an element of $\mathrm{X_{ad}} \otimes \mathrm{X}$, or equivalently to a bilinear pairing between $\mathrm{Y_{ad}}$ and Y.

The pullback of (4.11.1) to $\mathrm{T} \times \mathrm{T}$ is induced by the commutator section (1.8.1). It can be computed using 4.9 and 3.14. This fixes (4.11.1) uniquely:

Proposition **4.12.** — *Suppose that* E *corresponds by 4.7 to the* W*-invariant quadratic form* Q *on* Y*. Then, the associated bilinear form* B *extends to a pairing* B_1 *between* Y_{ad} *and* Y *and, with the identifications of 3.7 (ii), the bimultiplicative section (4.11.1) of* $\mathbf{K}_2$ *corresponds to* B_1.

4.13. Fix E as in 4.10 to 4.12. Taking k-points, we deduce from E a central extension

(**4.13.1**) $$K_2(k) \to E(k) \to G(k)$$

as in (1.4). The action 4.10 of G_{ad} on E induces an action of $G_{ad}(k)$ on this central extension. By restriction to $T_{ad}(k)$ and $T(k)$, the corresponding "commutator" (0.N.4) (3): $G^{ad}(k) \times G(k) \to E(k)$ induces a bimultiplicative map

(**4.13.2**) $$T_{ad}(k) \times T(k) \to K_2(k).$$

We deduce from 4.12 that

Corollary **4.14.** — *The pairing (4.13.2):*

$$(Y_{ad} \otimes k^*) \times (Y \otimes k^*) \to K_2(k)$$

is given by $(y_1 \otimes a, y \otimes b) \mapsto B_1(y_1, y)\{a, b\}$.

Proposition **4.15.** — *Suppose that* G *is simple and that* Q *takes the value* 1 *on short coroots. Then the central extension (4.13.1) is Matsumoto's central extension (0.2).*

Proof. — The group $G(k)$ is its own commutator subgroup (except for small finite fields k, for which $K_2(k)$ is trivial anyway). It follows that a central extension E of $G(k)$ by an abelian group A has no nontrivial automorphism (trivial on $G(k)$ and A). Suppose that the action of $G_{ad}(k)$ on $G(k)$ extends to E, the action being trivial on A. As explained in Deligne (1996) 3.8, the central extension E is then determined up to unique isomorphism by the "commutator" pairing

$$T_{ad}(k) \times T(k) \to A.$$

We leave it to the reader to compare 4.14 with the "commutator" for Matsumoto's extension.

4.16. *Remark.* — We keep the notations of 4.1, with G simply-connected. The choice of a Borel subgroup B is not needed.

By 4.7, a W-invariant quadratic form Q on Y defines a canonical multiplicative $\mathbf{K}_2$-torsor E on G. By restriction to T, we obtain a multiplicative $\mathbf{K}_2$-torsor E_T on T well defined up to unique isomorphism, and whose isomorphism class is given by Q. It defines a central extension

$$k^* \to \mathscr{E} \to Y$$

with commutator map $(-1)^{B(y,z)}$. In 11.7, we will describe this central extension in terms of Q and of the root space decomposition of the Lie algebra of G.

5. Split semi-simple groups

We keep the notation of 4.1.

5.1. Let $(f_w)_{w\in W^{(1)}}$ be a 1-cochain in the complex (4.3.2) for $j = 2$. Each f_w is an invertible function on BwB. Write f_i for $f_{w_0 s_i}$ and, as in the proof of 4.9, let $\bar{f}_i$ be the character of T such that $f_i(gt) = f_i(g)\bar{f}_i(t)$. Whether (f_w) is a cocycle depends only on its image in the quotient $(4.4.1)_{n=2}$ of the complex (4.3.2). As in 4.4, we identify this image with a bilinear form $C \in X_{sc} \otimes X$ pairing Y_{sc} and Y. The form C is defined by $C(\alpha_i^\vee, y) = \langle \bar{f}_i, y\rangle$. By 4.5, (f_w) is a cocycle if and only if the quadratic form $C(y, y)$ on Y_{sc} is W-invariant.

Until the end of this section, we assume that G is semi-simple. This means that Y_{sc} is of finite index in Y: $Y_{sc} \otimes \mathbf{Q} \xrightarrow{\sim} Y \otimes \mathbf{Q}$. Dually, $X \otimes \mathbf{Q} \xrightarrow{\sim} X_{sc} \otimes \mathbf{Q}$ and X is of finite index in X_{sc}.

5.2. *Lemma.* — *Suppose that* (f_w) *is a cocycle and that* E *is the corresponding* $\mathbf{K}_2$*-torsor. The following conditions are then equivalent:*

(i) *the inverse image of* E *by* $pr_1 \colon G \times T \to G$ *is isomorphic to its inverse image by the multiplication map* μ*;*

(ii) *the inverse image of* E *by* $pr_2 \colon T \times G \to G$ *is isomorphic to its inverse image by the multiplication map* μ*;*

(iii) *the bilinear pairing* C *of* Y_{sc} *with* Y *extends to a bilinear pairing of* Y *with itself.*

If the $\mathbf{K}_2$-torsor E on G admits a multiplicative structure, the torsors μ^*E and $pr_1^*(E) + pr_2^*(E_{|T})$ on $G \times T$ are isomorphic. As, by 3.3, the restriction of E to T is a trivial torsor, 5.2(i) then holds. The main result of this section is a converse to be proven in 5.7.

Proof (i)$\Leftrightarrow$(iii). — The right action of T respects the Bruhat stratification. The difference $pr_1^*E - \mu^*E$ is given by the difference of the inverse image cocycles. We have $f_i(gt)/f_i(g) = \bar{f}_i(t)$ and this difference is the cocycle

$$\text{(5.2.1)} \qquad pr_2^*(\bar{f}_i) \qquad \text{on} \qquad (Bw_0 s_i B) \times T.$$

We have to show it is a coboundary if and only if (iii) holds.

The form of (5.2.1) makes it convenient to view $G \times T$ as a group scheme over T, and the complex (4.3.2) for $G \times T$ as the complex of global sections of the complex (4.8.1) of sheaves on T. This complex is filtered by the filtration V of 3.4, shifted as in (4.3.5). The associated graded is given by (4.8.2), where for basis S one takes T. The cocycle (5.2.1) is in V_1^{shifted}. In $\mathrm{Gr}_2^{\text{Vshifted}}$, $\overset{2}{\wedge} X$ injects into $\bigoplus_{w \in W^{(1)}} X = X_{sc} \otimes X$. It follows that (5.2.1) is a coboundary if and only if it is a coboundary in $\Gamma(T, V_1^{\text{shifted}}(4.8.1))$. The open stratum $(Bw_0B) \times T$ has over T a global section $\dot{w}_0$. It follows that the subcomplex $V_0^{\text{shifted}}(4.8.1)$ of (4.8.1), reduced to $\mathbf{K}_2$ in degree 0, is a direct summand, and the cocycle (5.2.1) is a coboundary if and only if it is one in $\Gamma(T, \mathrm{Gr}_1^{\text{Vshifted}}(4.8.1))$. The complex of sheaves $\mathrm{Gr}_1^{\text{Vshifted}}(4.8.1)$ is

(**5.2.2**) $$X \otimes \mathcal{O}_T^* \to \bigoplus_{w \in W^{(1)}} \mathcal{O}_T^* = X_{sc} \otimes \mathcal{O}_T^*.$$

Evaluation at the origin splits $\Gamma(T, (5.2.2))$ into two direct factors: $X \otimes k^* \to X_{sc} \otimes k^*$, and

(**5.2.3**) $$X \otimes X \to X_{sc} \otimes X.$$

Our cocycle (5.2.1) lives in the direct factor (5.2.3). It is the element C of $X_{sc} \otimes X$ and is a coboundary if and only if (iii) holds.

(ii)⇔(iii) We proceed in the same way. This time,

$$f_i(tg)/f_i(g) = \overline{f}_i((w_0 s_i)^{-1} t (w_0 s_i))$$

and we find that the relevant cocycle is a coboundary if and only if the pairing C_1 defined by

$$C_1(\alpha_i^\vee, y) = \left\langle \overline{f}_i, (w_0 s_i)^{-1}(y) \right\rangle$$

extends. As we are in the semi-simple case, a pairing between Y_{sc} and Y can be extended to $Y \times Y$ if and only if its restriction to $Y_{sc} \times Y_{sc}$ can. For the restrictions to $Y_{sc} \times Y_{sc}$, we have

$$C_1(\alpha_i^\vee, y) = \left\langle \overline{f}_i, s_i^{-1} w_0^{-1} y \right\rangle = C(\alpha_i^\vee, s_i^{-1} w_0^{-1} y).$$

By 5.1, the quadratic form $C(y, y)$ is W-invariant. This invariance is equivalent to the vanishing of (4.4.4) for $i \neq j$. As (4.4.4) vanishes identically for $i = j$, we have

(**5.2.4**) $$C_1(\alpha_i^\vee, y) = -C(w_0^{-1} y, \alpha_i^\vee).$$

One concludes by observing that the pairing $-C(w_0^{-1}y, z)$ extends to $Y \times Y$ if and only if $C(z, y)$ does.

Lemma **5.3.** — *If* G *is a product* $G_1 \times G_2$, *with* $T = T_1 \times T_2$, *the conditions of (i) (ii) (iii) 5.2 are equivalent to*

(iv) *the inverse image of* E *by* $pr_2 : T_1 \times G \times T_2 \to G$ *is isomorphic to its inverse image by the product map.*

Proof. — We have $Y = Y_1 \times Y_2$ and the condition (iv) translates, as in the proof of 5.2, into the condition that the pairing between Y_{sc} and Y defined by

$$\begin{cases} C_0(\alpha_i^\vee, y_1) = \left\langle \bar{f}_i, y_1 \right\rangle & \text{for } y_1 \in Y_1 \\ C_0(\alpha_i^\vee, y_2) = \left\langle \bar{f}_i, (w_0 s_i)^{-1} y_2 \right\rangle & \text{for } y \in Y_2 \end{cases}$$

extends to a pairing between Y and itself. We decompose C_0 into four pairings C_{0ij}, with $C_{0\,ij}$ between $Y_{i\,sc}$ and Y_j. The pairing C decomposes similarly, and we have

$$\begin{aligned} &C_{0ij} = C_{ij} \qquad \text{if } j = 1 \\ &C_{012}(y, z) = C_{12}(y, w_0^{-1} z) \\ &C_{022}(y, z) = -C_{22}(w_0^{-1} z, y). \end{aligned}$$

The second line comes from s_i invariance of $w_0 z$ for $\alpha_i^\vee$ in Y_1 and z in Y_2. The third is as in 5.2.4. It follows that each C_{0ij} extends if and only if C_{ij} does, proving 5.3.

Lemma **5.4.** — *For* $G = G_1 \times G_2$ *as in 5.3, the functor of restriction to* G_1 *and* G_2 *is an equivalence from*

(i) *the category of pointed* $\mathbf{K}_2$*-torsors on* G, *for which the equivalent conditions of 5.3 hold, to*

(ii) *the category of pairs* (E_1, E_2) *of pointed* $\mathbf{K}_2$*-torsors on* G_1 *and* G_2, *for which the equivalent conditions of 5.2 hold.*

Proof. — We first show that if 5.2 (iii) holds for a $\mathbf{K}_2$-torsor E on $G_1 \times G_2$, then E is isomorphic to a sum of pullbacks $pr_1^* E_1 + pr_2^* E_2$.

The decomposition $G = G_1 \times G_2$ induces decompositions $T = T_1 \times T_2$, $Y = Y_1 \times Y_2$, $W = W_1 \times W_2$, $B = B_1 \times B_2$, $I = I_1 \amalg I_2$. We have $w_0 = (w_{01}, w_{02})$ and we choose lifting $\dot{w}_{0i}$ of w_{0i} in $N(T_i) \subset G_i$.

Let (f_w) be a cocycle in the complex (4.3.2) defining E. We write f_i for $f_{w_0 s_i}$. For $i \in I_1$, f_i is an invertible function on

$$Bw_0 s_i B = B_1 w_{01} s_i B_1 \times B_2 w_{02} B_2 \subset G_1 \times G_2.$$

Let us restrict f_i to $B_1 w_{01} s_i B_1$ by $g \mapsto (g, \dot{w}_{02})$. This corresponds to restricting E to $G_1 \times \{\dot{w}_{02}\}$. Letting i run over I_1, we obtain a cocycle in the complex (4.3.2) for G_1, for which 5.2 (iii) holds. It corresponds to a $\mathbf{K}_2$-torsor E_1 on G_1. Let us do the same for $i \in I_2$, obtaining E_2 on G_2. The torsor $E - \mathrm{pr}_1^* E_1 - \mathrm{pr}_2^* E_2$ is described by the cocycle

$$f_i^0 = f_i(g_1, g_2)/f_i(g_1, \dot{w}_{02}) \qquad \text{for } i \in I_1$$
$$f_i^0 = f_i(g_1, g_2)/f_i(\dot{w}_{01}, g_2) \qquad \text{for } i \in I_2.$$

Let us show that the cocycle f^0 is a coboundary. We have

$$(\mathbf{5.4.1}) \qquad \begin{aligned} f_i^0(g_1, \dot{w}_{02}) &= 1 \qquad \text{for } i \in I_1, \\ f_i^0(\dot{w}_{01}, g_2) &= 1 \qquad \text{for } i \in I_2, \end{aligned}$$

and the equivalent conditions of 5.4 hold for f^0.

By (5.4.1), f_i^0 is determined by the $\bar{f}_i^0$, which are encoded in $C^0 \in X_{sc} \otimes X$. The conditions (5.4.1) translate into C^0 having trivial restrictions to $Y_{1sc} \times Y_1$ and $Y_{2sc} \times Y_2$. By 5.2 (iii), C^0 extends to $Y \times Y$: it is the sum of pairings C^0_{12} and C^0_{21} between Y_1 and Y_2 and Y_2 and Y_1, respectively. The W-invariance of $C^0(y, y)$, and the vanishing of $C^0(y, y)$ for y in Y_1 or Y_2, forces $C^0(y, y)$ to vanish: one has

$$C^0_{12}(y_1, y_2) = -C^0_{21}(y_2, y_1).$$

Let us write $C^0_{12} \in X_1 \otimes X_2$ as a sum $\sum x_1^\alpha \otimes x_2^\alpha$. The cocycle f^0 is then the coboundary of

$$\sum \{\mathrm{pr}_1^*\left(x^\alpha_{1\dot{w}_{01}}\right), \mathrm{pr}_2^*\left(x^\alpha_{2\dot{w}_{02}}\right)\}$$

on $B w_0 B$.

Restricting to G_1 or G_2, one sees that $\mathbf{K}_2$-torsors $\mathrm{pr}_1^* E_1 + \mathrm{pr}_2^* E_2$ and $\mathrm{pr}_1^* E_1' + \mathrm{pr}_2^* E_2'$ on $G_1 \times G_2$ are isomorphic if and only if E_1 is isomorphic to E_1' and E_2 to E_2'. It remains to check that

$$\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{pr}_1^* E_1 \times \mathrm{pr}_2^* E_2),$$

i.e. that

$$\widetilde{H}^0(G_1, \mathbf{K}_2) \oplus \widetilde{H}^0(G_2, \mathbf{K}_2) \xrightarrow{\sim} \widetilde{H}^0(G_1 \times G_2, \mathbf{K}_2).$$

In the semi-simple case, the first map $(4.4.1)_{n-2}$ is injective, and by the rewriting (4.4.2) of $(4.4.1)_{n=2}$, one has

$$(\mathbf{5.4.2}) \qquad \widetilde{H}^0(G) = \mathrm{Ker}(X \otimes k^* \to X_{sc} \otimes k^*).$$

The required decomposition follows, proving 5.4.

Proposition **5.5.** — *If* G *is split semi-simple, a pointed* $\mathbf{K}_2$*-torsor* E *on* G *has at most one multiplicative structure. It has one if and only if the equivalent conditions of 5.2 hold.*

Proof. — We have already shown, in the comment after 5.2, that if E has a multiplicative structure, 5.2 (i) holds.

Conversely, suppose that 5.2 (i) (ii) hold. Using that $t_1(gh)t_2 = (t_1g)(ht_2)$, one sees that 5.3(iv) holds for $\mu^*(E)$ on $G \times G$. Applying 5.4 one finds that there is a unique isomorphism

$$m\colon \mathrm{pr}_1^*E + \mathrm{pr}_2^*E \to \mu^*E$$

which extends the obvious isomorphisms on $G \times \{e\}$ and $\{e\} \times G$. It is the only possible multiplicative structure and it is a multiplicative structure, as the two isomorphisms (1.1.2) deduced from m on $G \times G \times G$:

$$\mathrm{pr}_1^*E + \mathrm{pr}_2^*E + \mathrm{pr}_3^*E \to \mu_{123}E$$

have the same restriction to $G \times \{e\} \times \{e\}$, $\{e\} \times G \times \{e\}$ and $\{e\} \times \{e\} \times G$ and, by (5.4), are characterized by those three restrictions.

5.6. The arguments leading to 5.5 extend to the case of G split semi-simple over a base S regular of finite type over a field. In 5.2, that (iii) is equivalent to (i) or (ii) holds locally on S, and similarly for 5.3. The Lemma 5.4 holds for torsors satisfying locally on S the equivalent conditions of 5.2, and from this we deduce as in 5.5 that

Proposition **5.7.** — *If* G *is split semi-simple over* S *regular of finite type over a field, a pointed* $\mathbf{K}_2$*-torsor on* G *has at most one multiplicative structure. It has one if and only if the locally equivalent conditions of 5.2 hold locally on* S.

6. Split reductive groups

6.1. Let S be regular of finite type over a field k and let G be a split reductive group scheme over S. We fix a split maximal torus T. Let G_{der} be the commutator subgroup of G, G_{sc} be its simply-connected covering and G_{ad} be the adjoint group:

$$G_{sc} \to G_{der} \to G \to G_{ad}.$$

The inverse images of T in G_{der} and G_{sc}, and its image in G_{ad}, are split maximal tori in the respective reductive groups. Notation: T_{der}, T_{sc}, T_{ad}. With the notation (0.N.1), (0.N.2), the morphisms $T_{sc} \to T_{der} \to T \to T_{ad}$ induce W-equivariant morphisms $Y_{sc} \hookrightarrow Y_{der} \hookrightarrow Y \to Y_{ad}$ and $X_{sc} \hookleftarrow X_{der} \leftarrow X \hookleftarrow X_{ad}$. The group Y_{sc} (resp. X_{ad}) is the subgroup of Y (resp. X) generated by the coroots (resp. roots). The group Y_{der} is $Y_{sc} \otimes \mathbf{Q} \cap Y$, and Y_{sc} and Y_{der} are subgroups of finite index in Y_{ad}.

If E is a multiplicative $\mathbf{K}_2$-torsor on G, its restriction to T is a multiplicative torsor E_T on T. By the equivalence of categories 3.16, E_T gives $(Q, \mathscr{E})$, with Q a quadratic form on Y and $\mathscr{E}$ a central extension, on S, of Y by $\mathscr{O}_S^*$, obeying (3.11.1).

Locally on S, any $w \in W$ can be lifted to G and, by 1.6, the automorphism w of T preserves the isomorphism class of E_T. It follows by transport of structure that the quadratic form Q is W-invariant.

The pullback E_{sc} of E to G_{sc} is a multiplicative torsor on G_{sc}. By 4.9, the corresponding quadratic form 4.7 on Y_{sc} is the restriction Q_{sc} of Q to Y_{sc}. The multiplicative torsor E_{sc} is determined up to unique isomorphism by Q_{sc}. Its restriction to T_{sc} gives rise to a central extension 3.15 of sheaves on S

$$\mathscr{O}_S^* \to \mathscr{E}_{sc} \to Y_{sc} \tag{6.1.1}$$

depending only on Q_{sc}. As E_{sc} restricted to T_{sc} is the pullback to T_{sc} of E_T, we have a commutative diagram

$$\begin{array}{ccccc} \mathscr{O}_S^* & \longrightarrow & \mathscr{E}_{sc} & \longrightarrow & Y_{sc} \\ \| & & \downarrow & & \downarrow \\ \mathscr{O}_S^* & \longrightarrow & \mathscr{E} & \longrightarrow & Y. \end{array} \tag{6.1.2}$$

Theorem **6.2.** — *Let* G *be a split reductive group scheme over* S *regular of finite type over a field. The construction 6.1 is an equivalence of categories from*

(i) *the category of multiplicative* $\mathbf{K}_2$*-torsors* E *on* G, *to*

(ii) *the category of triples* $(Q, \mathscr{E}, \varphi)$, *where*

(a) Q *is a* W*-invariant quadratic form on* Y. *By 4.16, the restriction of* Q *to* Y_{sc} *defines a central extension* $\mathscr{E}_{sc}$ *of* Y_{sc} *by* $\mathscr{O}_S^*$.

(b) $\mathscr{E}$ *is a central extension of* Y *by* $\mathscr{O}_S^*$, *obeying (3.11.1),*

(c) $\varphi\colon \mathscr{E}_{sc} \to \mathscr{E}$ *makes the diagram (6.12) commute.*

The morphisms in the category of triples $(Q, \mathscr{E}, \varphi)$ are as follows. There can be a morphism from $(Q', \mathscr{E}', \varphi')$ to $(Q'', \mathscr{E}'', \varphi'')$ only if $Q' = Q''$. If $Q' = Q''$, a morphism is an isomorphism f of central extensions

$$\begin{array}{ccccc} \mathscr{O}_S^* & \longrightarrow & \mathscr{E}' & \longrightarrow & Y_{sc} \\ \| & & \big\downarrow{\scriptstyle f} & & \| \\ \mathscr{O}_S^* & \longrightarrow & \mathscr{E}'' & \longrightarrow & Y_{sc} \end{array}$$

compatible with φ' and φ'': commutativity of the diagram

$$\begin{array}{ccc} \mathscr{E}_{sc} & = & \mathscr{E}_{sc} \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi''} \\ \mathscr{E}' & \xrightarrow{f} & \mathscr{E}''. \end{array}$$

6.3. *Proof of 6.2, when* G *is semi-simple.* — The group G is assumed to be semi-simple and for the ease of exposition, we begin by assuming that S is the spectrum of a field k.

We first show that any W-invariant **Z**-valued quadratic form Q on Y comes from some multiplicative $\mathbf{K}_2$-torsor on G. There exists a possibly nonsymmetric **Z**-valued bilinear form C on Y, such that $C(y, y) = Q(y)$. We have $C \in X \otimes X \subset X_{sc} \otimes X$ and there exists a 1-cochain (f_w) as in 5.1 giving rise to C. By 5.1, based on 4.5, this cochain is a cocycle. Let E be the corresponding $\mathbf{K}_2$-torsor, and choose a trivialization of E at $e \in G$. By 5.5, the pointed torsor E has a unique multiplicative structure. Applying 4.9 to the restriction of the multiplicative torsor E to G_{sc}, we obtain that the quadratic form 6.1 attached to E is Q.

As any W-invariant quadratic form on Y comes from some multiplicative $\mathbf{K}_2$-torsor, it suffices to prove 6.2 for the full subcategories of the categories described in (i) (ii) given by the condition $Q = 0$. Those categories can be incarnated as follows. For (i), if E is defined by a cocycle as in 5.1, giving rise to C in $X \otimes X$ by 5.5, the condition $Q = 0$ means that C is in the image of $\overset{2}{\wedge} X$. The category is incarnated by $(4.4.1)_{n=1}$, i.e. by the complex (4.4.2):

(**6.3.1**) $$X \otimes k^* \to X_{sc} \otimes k^*.$$

For (ii), we have to consider the category of pairs $(\mathscr{E}, \varphi)$, where $\mathscr{E}$ is a commutative central extension of Y by k^*, and φ is splitting of $\mathscr{E}$ over Y_{sc}. The extension $\mathscr{E}$ is commutative because $Q = 0$. This category is incarnated by (6.3.1) as well: an element $f \in X_{sc} \otimes k^* = \mathrm{Hom}(Y_{sc}, k^*)$ defines the following object $[f]$: the trivial extension $k^* \times Y$ of Y by k^*, with the splitting f on Y_{sc}. An element g of $X \otimes k^* = \mathrm{Hom}(Y, k^*)$ defines an isomorphism from $[f]$ to $[fg]$.

We leave it to the reader to check that the functor 6.2 is incarnated by the identity automorphism of (6.3.1).

Over a base S, one similarly finds that any Q can be locally obtained, and local incarnations of the subcategory of 6.2(i) (ii) where $Q = 0$ are given by

(**6.3.2**) $$X \otimes \mathscr{O}_S^* \to X_{sc} \otimes \mathscr{O}_S^*.$$

6.4. *Proof of 6.2 (general case).*

The quotient Y/Y_{der} is torsion free. It follows that Y_{der} is a direct factor in Y. We choose a decomposition $Y = Y_0 \times Y_{der}$. It gives us a product decomposition

$T = T_0 \times T_{der}$, and G is the semi-direct product

$$G = T_0 \ltimes G_{der}. \tag{6.4.1}$$

We will apply 1.7 to the semi-direct product decomposition (6.4.1) of G. For this, even if we cared only for the case when S is the spectrum of a field, we need the semi-simple case of 6.2 over a more general base, to describe actions of T_0 on a central extension of G_{der} by $\mathbf{K}_2$.

The inner action of G on itself factors through G^{ad}. The action of T_0 on G_{der} is hence the composite of q: $T_0 \subset T \to T_{ad}$, and of the action of $T_{ad} \subset G_{ad}$ on G_{der}. We continue to write q for the induced map

$$q: Y_0 \subset Y \to Y_{ad}. \tag{6.4.2}$$

The action of T_0 on G_{der} fixes T_{der}.

6.5. *Lemma.* — *Let* E_{der} *be a multiplicative* $\mathbf{K}_2$*-torsor on* G_{der}*, and let* Q_{der} *be the corresponding quadratic form on* Y_{der}.

(i) *If the action of* T_0 *on* G_{der} *can be lifted to* E_{der}*, the lifting is unique;*

(ii) *A lifting exists if and only if the quadratic form* Q_{der} *can be extended to a* W*-invariant quadratic form on* Y.

Proof. — Let E_{sc} be the pullback of E_{der} to G_{sc}. It is determined up to unique isomorphism by the restriction of the quadratic form Q_{der} to Y_{sc}. To E_{sc} corresponds a central extension

$$\mathcal{O}_S^* \xrightarrow{i} \mathcal{E}_{sc} \xrightarrow{p} Y_{sc} \tag{6.5.1}$$

on S. The sheaf of automorphisms of the central extension (6.5.1) is the torus $\mathrm{Hom}_S(Y_{sc}, \mathcal{O}_S^*)$ with character group Y_{sc}. To $f : Y_{sc} \to \mathcal{O}_S^*$ corresponds the automorphism $e \mapsto e.ifp(e)$ of $\mathcal{E}_{sc}$. The action of T_{ad} on G_{sc} lifts uniquely to E_{sc} and, as it fixes T_{sc}, it induces an action on the extension (6.5.1), that is, defines a morphism of tori from T_{ad} to $\mathrm{Hom}_S(Y_{sc}, \mathcal{O}_S^*)$. This morphism corresponds to a morphism of dual character sheaves, or equivalently to a pairing of Y_{ad} with Y_{sc}. One deduces from 4.12 that this pairing is $B(y_{ad}, y_{sc})$ for B the bilinear form associated to Q_{der}. The action of T_0 on (6.5.1), induced by that of T_{ad}, similarly corresponds to the pairing $B(q(y_0), y_{sc})$, with q given by (6.4.2).

The multiplicative torsor E_{der} gives rise to a central extension $\mathcal{E}_{der}$ of Y_{der} by $\mathcal{O}_S^*$, to which (6.5.1) maps:

$$\begin{array}{ccccc} \mathcal{O}_S^* & \longrightarrow & \mathcal{E}_{sc} & \longrightarrow & Y_{sc} \\ \| & & \downarrow & & \downarrow \\ \mathcal{O}_S^* & \longrightarrow & \mathcal{E}_{der} & \longrightarrow & Y_{der}. \end{array}$$

By 6.3, actions of T_0 on E_{der}, lifting the action of T_0 on G_{der}, are identified with actions of T_0 on the central extension $\mathscr{E}_{der}$, extending the known action of T_0 on $\mathscr{E}_{sc}$. Such actions correspond to extensions of the pairing $B(q(y_0), y_{sc})$ between Y_0 and Y_{sc} to a pairing between Y_0 and Y_{der}. As Y_{sc} is of finite index in Y_{der}, such an extension, if it exists, is unique. This proves (i), while (ii) is reduced to point (i) of the next lemma.

Lemma **6.6.** — *Let* Q_{der} *be a* W*-invariant quadratic form on* Y_{der}. *By 4.12, the restriction to* Y_{sc} *of the associated bilinear form extends to a pairing* B *of* Y_{ad} *with* Y_{sc}.

(i) *The form* Q_{der} *extends to a* W*-invariant quadratic form on* Y *if and only if* $B(q(y_0), y_{sc})$ *extends to a pairing between* Y_0 *and* Y_{der}.

(ii) *If* Q_{der} *extends to a* W*-invariant quadratic form on* Y, *the* W*-invariant extensions* Q *correspond one to one to quadratic forms on* Y_0, *by* $Q \mapsto Q|Y_0$.

Proof. — After we tensor with $\mathbf{Q}$, $Y \otimes \mathbf{Q}$ is the direct sum of $Y_{der} \otimes \mathbf{Q}$ and of $(Y \otimes \mathbf{Q})^W$. Invariant (rational) quadratic forms are the orthogonal direct sums of an invariant quadratic form on $Y_{der} \otimes \mathbf{Q}$ and an arbitrary quadratic form on $(Y \otimes \mathbf{Q})^W$.

The extension $Q_1 := Q_{der} \oplus 0$ of Q_{der}, expressed in terms of the decomposition $Y = Y_0 \oplus Y_{der}$, is

$$Q_1(y_0 + y_{der}) = B(q(y_0), y_{der}) + Q_{der}(y_{der}).$$

Any other W-invariant (rational) extension is the sum of Q_1 and of the pull back of a quadratic form on $Y \otimes Q/Y_{der} \otimes \mathbf{Q} \xrightarrow{\sim} Y_0 \otimes \mathbf{Q}$. It has the form

$$Q(y_0 + y_{der}) = Q_0(y_0) + B(q(y_0), y_{der}) + Q_{der}(y_{der}).$$

Integral extensions are given by the same formula: they exist if and only if $B(q(y_0), y_{der})$ is integral on $Y_0 \times Y_{der}$, and correspond one to one to Q_0.

6.7. We return to the proof of 6.2 for G, using the dictionary 1.4 between multiplicative torsors and central extensions. As explained in 1.7, the data of a central extension E of the semi-direct product $G = T_0 \ltimes G_{der}$ by $\mathbf{K}_2$ amounts to the data of

(a) its restriction E_{der} to G_{der};

(b) its restriction E_0 to T_0;

(c) an action of T_0 on the central extension E_{der} of G_{der} by $\mathbf{K}_2$, lifting the action of T_0 on G_{der}.

In (c), the action of t in T_0 is that of any lifting of t to E_0. One recovers E as a quotient of the semi-direct product $E_0 \ltimes E_{der}$.

We now translate (a) (b) (c), using 6.2 (semi-simple case), and 6.5. We get:

from (a): a W-invariant quadratic form Q_{der} on Y_{der}, a central extension $\mathscr{E}_{\mathrm{der}}$ of Y_{der} by $\mathscr{O}^*_S$, obeying (3.11.1), and a commutative diagram

$$\begin{array}{ccccc} \mathscr{O}^*_S & \longrightarrow & \mathscr{E}_{\mathrm{sc}} & \longrightarrow & Y_{\mathrm{sc}} \\ \| & & \downarrow & & \downarrow \\ \mathscr{O}^*_S & \longrightarrow & \mathscr{E}_{\mathrm{der}} & \longrightarrow & Y_{\mathrm{der}}, \end{array} \tag{6.7.1}$$

with $\mathscr{E}_{\mathrm{sc}}$ defined by the restriction Q_{sc} of Q_{der} to Y_{sc}.

from (c): that Q_{der} can be extended to a W-invariant form on Y. The action (c) then exists, and is unique.

from (b): a quadratic form Q_0 on Y_0, and a central extension

$$\mathscr{O}^*_S \to \mathscr{E}_0 \to Y_0 \tag{6.7.2}$$

obeying (3.11.1).

By 6.6 (ii), there is a unique W-invariant quadratic form Q on Y with restrictions Q_{der} and Q_0 to Y_{der} and Y_0. There is also a central extension

$$\mathscr{O}^*_S \to \mathscr{E} \to Y$$

obeying (3.11.1) and inducing respectively (6.7.2) and the second line of (6.7.1) on Y_0 and Y_{der}. It is unique up to unique isomorphism, and sits in a commutative diagram deduced from (6.7.1)

$$\begin{array}{ccccc} \mathscr{O}^*_S & \longrightarrow & \mathscr{E}_{\mathrm{sc}} & \longrightarrow & Y_{\mathrm{sc}} \\ \| & & \downarrow & & \downarrow \\ \mathscr{O}^*_S & \longrightarrow & \mathscr{E} & \longrightarrow & Y. \end{array} \tag{6.7.3}$$

The data (a) (b) (c) hence amounts to that of Q, W-invariant on Y, of a central extension $\mathscr{E}$ of Y by $\mathscr{O}^*_S$, obeying (3.13.1), and of a commutative diagram (6.7.3). One easily checks that the construction (6.1), applied to the corresponding central extension (or rather, multiplicative $\mathbf{K}_2$-torsor) give back Q, $\mathscr{E}$ and (6.7.3). This concludes the proof.

7. Reductive groups

7.1. Let S be regular of finite type over a field k and G be a reductive group scheme over S. Locally for the étale topology of S, G is split (SGA3 XXII 2.3): there exists a surjective étale map u: $S_0 \to S$ such that G_0 over S_0 deduced from G by base change is a split reductive group over S_0. By 2.7, the change of base from S to S_0 is an equivalence of categories from pointed $\mathbf{K}_2$-torsors on (G, e) to pointed $\mathbf{K}_2$-torsors on (G_0, e) provided with a descent data. Applying this to the G^n, and using that this equivalence of categories is compatible with pull-backs, one sees that giving a multiplicative $\mathbf{K}_2$-torsor on G amounts to giving one on G_0, and a descent data from S_0 to S.

Suppose T is a maximal torus of G. As in 6.1, one deduces from G semi-simple group schemes G_{sc}, G_{der} and G_{ad} over S, with maximal tori T_{sc}, T_{der} and T_{ad}. With the notations (0.N.1), (0.N.2), let Q be a W-invariant quadratic form on Y, and Q_{sc} be the induced quadratic form on Y_{sc}. Locally on S for the étale topology, G and T, hence G_{sc} and T_{sc}, are split, and by 4.7, amplified by 4.8, Q_{sc} defines a multiplicative $\mathbf{K}_2$-torsor E_{sc} on G_{sc}. Locally on S for the étale topology, more precisely after any étale change of base $S_0 \to S$ such that T splits on S_0, the multiplicative torsor E_{sc}, restricted to T_{sc}, defines by 3.16 a central extension of Zariski sheaves on S_0

$$(\mathbf{7.1.1}) \qquad 1 \to \mathcal{O}^* \to \mathcal{E}_{sc} \to Y_{sc} \to 1,$$

obeying (3.11.1). It is functorial in S_0, hence defines locally on S for the étale topology a central extension (7.1.1) of sheaves for the étale topology. To define a central extension of sheaves, it suffices to define it locally: we obtain a central extension (7.1.1) of étale sheaves on S obeying (3.11.1).

We can now consider triples $(Q, \mathcal{E}, \varphi)$ as in 6.2, with all sheaves for the étale topology. For triples $(Q, \mathcal{E}, \varphi)$, étale descent is trivially valid: to give a triple $(Q, \mathcal{E}, \varphi)$ on S amounts to giving one on S_0, and a descent data. By descent, one hence deduces from 6.2:

Theorem **7.2.** — *By étale descent, the equivalence of categories 6.2 induces an equivalence of categories from*

(i) *the category of multiplicative* $\mathbf{K}_2$*-torsors on* G, *to*

(ii) *the category of triples* $(Q, \mathcal{E}, \varphi)$ *as in 6.2, with all sheaves taken in the étale topology.*

7.3. *Special cases*

(i) If G is semi-simple simply-connected, the data of $(Q, \mathcal{E}, \varphi)$ reduces to that of Q: multiplicative $\mathbf{K}_2$-torsors on G have no nontrivial automorphisms, and are classified by W-invariant quadratic forms Q on Y.

(ii) For G reductive, the group $\mathrm{Hom}(G, \mathbf{K}_2)$ of automorphisms of a central extension E of G by $\mathbf{K}_2$ is the group of automorphisms for the corresponding $(\mathcal{E}, \varphi)$. For variable S, they form the étale sheaf $\mathcal{H}om(Y/Y_{sc}, \mathcal{O}^*)$.

(iii) Suppose S = Spec(k). A triple $(Q, \mathcal{E}, \varphi)$ determines a multiplicative $\mathbf{K}_2$-torsor. This torsor in turn defines a central extension $E(k)$ of $G(k)$ by $K_2(k)$. Automorphisms of $(Q, \mathcal{E}, \varphi)$ induce automorphisms of the central extension $E(k)$.

Applying descent to 3.5 and 3.7, one obtains

7.4. *Proposition.* — *Fix* S *as in 7.1.*

(i) *For* p: $T \to S$ *a torus over* S, $\widetilde{p}_*\mathbf{K}_2$ *is the sheaf of pairs* (A, q) *as in 3.5.*

(ii) *The group of multiplicative sections of* $\mathbf{K}_2$ *over* T *is* $H^0(S_{et}, X \otimes \mathcal{O}^*)$.

(iii) *For tori* T_1 *and* T_2 *over* S, *with sheaves of character groups* X_1 *and* X_2, *the group of bimultiplicative sections of* $\mathbf{K}_2$ *over* $T_1 \times_S T_2$ *is* $H^0(S_{et}, X_1 \otimes X_2)$.

7.5. *Special case.* — Suppose $S = \mathrm{Spec}(k)$ and let T_1 and T_2 be tori over S. For k_s a separable closure of k, the character groups of T_1 and T_2 are $\mathrm{Gal}(k_s/k)$-modules X_1 and X_2. Bimultiplicative sections of $\mathbf{K}_2$ are in bijection with the group of $\mathrm{Gal}(k_s/k)$-invariants in $X_1 \otimes X_2$. Such an invariant element B hence gives rise to a bimultiplicative map

$$(\ ,\ )_B : T_1(k) \times T_2(k) \to K_2(k). \tag{7.5.1}$$

This, completed by 7.6 below, answers the question 7.11 of Deligne (1996). Another, more natural, answer to this question has been communicated to us by B. Kahn. He systematically works in the étale topology, using the complex $\Gamma(2)$ of Lichtenbaum (1987).

7.6. Fix G over S as in 7.1, and a multiplicative $\mathbf{K}_2$-torsor on G. We view it as an extension E of G by $\mathbf{K}_2$, viewed as sheaves on a big Zariski site. If $C \subset G \times_S G$ is the subscheme of commuting elements, the commutator map $G \times G \to E$ induces a morphism from C to $\mathbf{K}_2$, i.e. an element

$$\mathrm{comm} \in H^0(C, \mathbf{K}_2). \tag{7.6.1}$$

For T a maximal torus in G, the restriction of comm to $T \times_S T$ is a bimultiplicative section of $\mathbf{K}_2$. To determine which, one may first localize for the étale topology. Applying 3.13, one sees that it corresponds by 7.5 to the bilinear form B associated to the quadratic form Q attached to the multiplicative torsor we started with.

In particular, for $S = \mathrm{Spec}(k)$, we have a central extension of $G(k)$ by $K_2(k)$ and on $T(k)$ the commutator map is (7.5.1).

Similarly, if G is semi-simple simply-connected, we have an action of G^{ad} on the multiplicative $\mathbf{K}_2$-torsor E corresponding to a W-invariant quadratic form Q. Let $\overline{C} \subset G^{ad} \times G$ be the subscheme of pairs (g, h) such that g fixes h. The scheme $C \subset G \times G$ is the inverse image of $\overline{C}$, and the section (7.6.1) comes from

$$\mathrm{comm} \in H^0(\overline{C}, \mathbf{K}_2). \tag{7.6.2}$$

As previously, its restriction to $T^{ad} \times T$ is bimultiplicative. It is given by the unique pairing B_1 between $Y^{ad} \supset Y$ and Y extending B.

For $S = \mathrm{Spec}(k)$, $G^{ad}(k)$ acts on the central extension of $G(k)$ by $K_2(k)$ defined by Q and the corresponding commutator map

$$T^{ad}(k) \times T(k) \to K_2(k)$$

is (7.5.1) for B_1.

8. Generic cocycles

8.1. Let k be a field and $\mathscr{K}$ be a sheaf on the big Zariski site of $\mathrm{Spec}(k)$. For X a reduced and irreducible scheme of finite type over k, with field of rational functions $k(X)$, we define $\mathscr{K}(k(X))$ to be the fiber of $\mathscr{K}$ at the generic point of X. It is the inductive limit over U of $\mathscr{K}(U)$, for U a nonempty open subscheme of X. For $\mathscr{K} = \mathbf{K}_2$, we get $K_2(k(X))$.

Let G be a connected algebraic group over k. Each component G^p of the simplicial classifying space $BG = (G^{\Delta_p}/G)$ is geometrically irreducible and the face maps ∂_i: $G^p \to G^{p-1}$ map the generic point of G^p to the generic point of G^{p-1}. The generic points of the components G^p of BG hence form a strict (0.N.7) simplicial system of spectrum of fields.

Applying the functor $\mathscr{K}$ to the $k(G^p)$, one obtains a strict cosimplicial abelian group. The *complex of generic cochains* is the corresponding complex

$$\mathscr{K}(k(G^p)), \quad d = \sum (-1)^i \partial_i^*.$$

8.2. *Construction. — Assume that the field k is infinite, and that the connected algebraic group* G *over k is unirational. Then the category of multiplicative* $\mathscr{K}$*-torsors on* G *is incarnated in degree 2 by the complex of generic cochains.*

The unirationality assumption holds if G is reductive (SGA3 XIV 6.7).

In cohomological degree one, the construction boils down to the following statement: the map "restriction to the generic point of G" is an isomorphism from the group of multiplicative sections of $\mathscr{K}$ over G to the group of elements of $\mathscr{K}(k(G))$ for which $s(g_1g_2) = s(g_1) + s(g_2)$ in $\mathscr{K}(k(G \times G))$.

As $d = 0$ on $\mathscr{K}(G^0)$, the complex of generic cochains and the subcomplex ($\mathscr{K}(G^p)$, for $p \geqslant 1$) incarnate in degree 2 the same Picard category.

Proof. — Let us view $\mathscr{K}$ as a functorial system of Zariski sheaves $\mathscr{K}_S$ on the schemes S in $\mathrm{Spec}(k)_{Zar}$ (1.4), and let us choose a functorial system of resolutions $\mathscr{K}_S^*$, with each $\mathscr{K}_S^q$ acyclic. Such systems of resolutions exist: one can use the canonical flabby resolutions of Godement (1958), or an injective resolution of the sheaf $\mathscr{K}$ on $\mathrm{Spec}(k)_{Zar}$.

As we saw in 1.9 Example (vi), the category of multiplicative $\mathscr{K}$-torsors on G is incarnated in degree 2 by the associated simple complex $\mathbf{s}(\Gamma(G^p, \mathscr{K}^*)$ for $p \geqslant 1)$. For each p, the fiber of $\mathscr{K}^*$ at the generic point of G^p is a resolution of $\mathscr{K}(k(G^p))$. It follows that the morphism of complexes

$$(\mathbf{8.2.1}) \qquad \mathscr{K}(k(G^p)) \text{ for } p \geqslant 1 \longrightarrow \mathbf{s}(\mathscr{K}^q(k(G^p)) \text{ for } p \geqslant 1)$$

is a quasi-isomorphism reducing 8.2 to

8.3. *Proposition. — Under the assumptions of 8.2, the morphism of restriction to the generic points*

$$(\mathbf{8.3.1}) \qquad \mathbf{s}(\Gamma(G^p, \mathscr{K}^q), \quad \textit{for} \quad p \geqslant 1) \to \mathbf{s}(\mathscr{K}^q(k(G^p)), \quad \textit{for} \quad p \geqslant 1)$$

is a quasi-isomorphism.

As G^0 is reduced to a point, 8.3 is equivalent to:

$$\mathbf{s}(\Gamma(G^p, \mathscr{K}^q)) \to \mathbf{s}(\Gamma(\mathscr{K}^q(k(G^p))))$$

is a quasi-isomorphism. This is what we will prove.

Some variants of 8.2 hold with more generality. To prepare for them, the next lemma will be proven in the following more general setting: S is a noetherian scheme, and G is a group scheme over S, flat of finite type over S and with connected fibers.

Fix n and, for $0 \leqslant i \leqslant n$, let U_i be an open subset of $G^i = G^{\Delta_i}/G$. For $i > n$, define $U_i := G^i$.

Lemma **8.4.** *— Assume that for* $0 \leqslant i \leqslant n$ *the fibers of* $U_i \to S$ *are not empty. Then, there exists a system of open subsets* $V_i \subset G^i$ $(i \geqslant 0)$ *such that*

(i) *the fibers of* $V_i \to S$ *are not empty;*
(ii) *for* $i \leqslant n$, $V_i \subset U_i$*;*
(iii) *each face map* ∂: $G^i \to G^{i-1}$ *maps* V_i *to* V_{i-1}*;*
(iv) *for* $i > 0$, ∂_0: $G^i \to G^{i-1}$ *maps* V_i *onto* V_{i-1}*;*
(v) *For* $i > n$, V_i *is the intersection of the* $\partial^{-1}(V_j)$ *for* ∂ *an iterated face map and* $j \leqslant n$.

Proof. — We will construct V_i as follows:

(a) define U'_i to be the intersection of the $\partial_0^j(U_{i+j})$ $(j \geqslant 0)$;

(b) define V_i as the intersection of all $\partial^{-1}(U'_j)$ for $0 \leqslant j \leqslant n$ and ∂ an iterated face map from G^i to G^j. The identity map is considered an iterated face map.

As $U_i = G^i$ for $i > n$, it suffices in (a) to take the intersection for $i + j \leqslant n$, and $U'_i = G^i$ for $i > n$. As G is flat over S, all face maps are open maps and the U'_i are open subsets. So are the V_i. That they satisfy (ii) (iii) (v) is clear. The construction of the V_i is compatible with passage to geometric fibers. To check (i) (iv), it hence suffices to prove (i), (iv) when S of the form Spec(k), for k an algebraically closed field. We assume S of this form.

We will use that a nonempty open subset of G^i is dense (irreducibility of G^i). As a finite intersection of open dense subsets is still open and dense, this gives us first that the U'_i are dense. Next that the V_i are. This proves (i). By construction, one has

$$(\mathbf{8.4.1}) \qquad @_0(U'_i) \supset U'_{i-1} \qquad \text{for} \qquad i > 0.$$

We have $G^i = G^{\Delta_i}/G$. Let $\widetilde{U}'_i$ and $\widetilde{V}_i$ be the inverse image of U'_i and V_i in G^{Δ_i}. We have to check that $@_0$ maps $\widetilde{V}_i(k)$ onto $\widetilde{V}_{i-1}(k)$, i.e. that for $(g_1, \dots, g_i)$ in $\widetilde{V}_{i-1}(k)$ there exist g_0 such that $(g_0, g_1, \dots, g_i)$ is in $\widetilde{V}_i(k)$. The conditions on g_0 are that for any subsequence $j_0, \dots, j_\ell$ of $0, \dots, i$, $(g_{j_0}, \dots, g_{j_\ell}) \in \widetilde{U}'_\ell(k)$. If $j_0 \neq 0$, this results from $(g_1, \dots, g_i)$ being in $\widetilde{V}_{i-1}(k)$. It remains to find g_0 such that for each subsequence $j_1, \dots, j_\ell$ of $1, \dots, i$, $(g_0, g_{j_1}, \dots, g_{j_\ell})$ is in $\widetilde{U}'_\ell(k)$. If · = 0, this holds for any g_0 by (i). For · > 0, $(g_{j_1}, \dots, g_{j_\ell})$ is in $\widetilde{V}_{\ell-1}(k)$ by (iii), and as $\widetilde{V}_{\ell-1}(k) \subset \widetilde{U}_{\ell-1}(k) \subset @_0\widetilde{U}_\ell()$ ((ii) and (8.4.1)), it holds for g_0 in a nonempty open subset. The intersection of these nonempty open subsets is nonempty, proving (iv).

Variant. Suppose S is the spectrum of a field k. If G is not necessarily connected, 8.4 should be modified as follows. The U_i should be assumed dense, not just nonempty. In (i), the V_i should be Zariski dense too. In (iv), the fibers of $@_0 \colon V_i \to V_{i-1}$ should be dense in the fibers of $@_0 \colon G^i \to G^{i-1}$.

In the proof, each time one has to take an image by $@_0$ (or an iterated image, by some $@_0^\ell$), it should be replaced by the following modified image: modified $@_0(U)$ is the set of x such that the trace of U is dense in $@_0^{-1}(x)$. In our setting, the modified image of an open set is open. Indeed, modified images are compatible with a finite extension of scalars k'/k: we may and shall assume that the geometric connected components G^α of G are defined over k. Our maps $@_0$ are isomorphic to projections $p \colon G \times X \to X$, and

$$\text{modified } p(U) = \bigcap p(U \cap (G^\alpha \times X)).$$

Lemma **8.5.** — *Under the assumptions of 8.2, suppose that open subsets* $V_i \subset G^i$ *obey (i) (iii) (iv) of 8.4 (for* S = Spec(k)*). Let* $\widetilde{V}_i$ *be the inverse image of* V_i *in* G^{Δ_i}*. Then, for any scheme* T *over* k*, any* $i > 0$*, and any finite family of morphisms* $(g_1^\alpha, \dots, g_k^\alpha)$ *from* T *to* $\widetilde{V}_{i-1}$*, locally on* T *(for the Zariski topology) there exist* $g_0 \colon T \to G$ *such that all* $(g_0, g_1^\alpha, \dots, g_i^\alpha)$ *are with values in* $\widetilde{V}_i$*.*

Proof. — For each α, $(g_0, g_1^\alpha, \ldots, g_i^\alpha)$ is in $\widetilde{V}_i$ if and only if g_0, viewed as a section of $pr_2\colon G \times S \to S$, takes value in the open set U^α, inverse image by $(g, s) \mapsto (g, g_1^\alpha, \ldots, g_i^\alpha)\colon S \to G^{\Delta_i}$ of $\widetilde{V}_i \subset G^{\Delta_i}$:

$$\begin{array}{ccccc} U^\alpha & \hookrightarrow & G \times S & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{V}_i & \hookrightarrow & G^{\Delta_i} & \longrightarrow & G^{\Delta_{i-1}} \end{array}$$

By assumption each U^α maps onto S. As G is irreducible, so does their intersection U.

By the unirationality assumption on G, there exist an open subset W of an affine space $\mathbf{A}^N$ and a dominant morphism f from W to G. Let $W' \subset W \times S$ be the inverse image of U in $W \times S$. It is an open subset of $\mathbf{A}^N_S/S$, fiber by fiber not empty. The residue fields of S being infinite, W'/S has local sections. Their images by f are the required g_0.

Lemma **8.6.** — *Under the same assumptions as in 8.5, the restriction map*

(**8.6.1**) $$\mathbf{s}\,\Gamma(G^p, \mathscr{K}^q) \to \mathbf{s}\,\Gamma(V_p, \mathscr{K}^q)$$

is a quasi-isomorphism.

Proof. — We will show that both sides of (8.6.1) have as cohomology groups the same Ext groups in the big Zariski site of Spec(k). We identify each scheme of finite type X over k with the corresponding representable sheaf, and we let $\mathbf{Z}[X]$ be the sheaf of abelian groups freely generated by X. If G acts on X, $\mathbf{Z}[X]$ is a sheaf of modules over the sheaf of rings $\mathbf{Z}[G]$.

The $\mathbf{Z}[\widetilde{V}_i]$ form a complex of $\mathbf{Z}[G]$-modules, augmented to $\mathbf{Z}$. The augmentation ε is $\mathbf{Z}[\widetilde{V}_0] = \mathbf{Z}[G] \to \mathbf{Z}\colon [g] \mapsto 1$. The complex $\mathbf{Z}[\widetilde{V}_i]$ is a resolution of $\mathbf{Z}$. Indeed, a section c of $\mathbf{Z}[\widetilde{V}_{i-1}]$ over S is a finite linear combination $\Sigma n_\alpha(g_1^\alpha, \ldots, g_i^\alpha)$ and, by 8.5, there exists locally on S a g_0 such that the $(g_0, g_1^\alpha, \ldots, g_i^\alpha)$ are in $\widetilde{V}_i$. If c is a cocycle (resp., for $i = 0$, if c is in the kernel of ε), it is the coboundary of $\Sigma n_\alpha(g_0, g_1^\alpha, \ldots, g_i^\alpha)$.

For each i, the subscheme $g_0 = 1$ of $\widetilde{V}_i$ maps isomorphically to V_i, and the action of G gives an equivariant isomorphism

(**8.6.2**) $$G \times V_i \xrightarrow{\sim} \widetilde{V}_i.$$

The isomorphism (8.6.2) induces an isomorphism

(**8.6.3**) $$\mathbf{Z}[\widetilde{V}_i] = \mathbf{Z}[G] \otimes_{\mathbf{Z}} \mathbf{Z}[V_i].$$

For any sheaf of $\mathbf{Z}[G]$-modules $\mathscr{F}$, one hence has

(**8.6.4**) $$\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}[\widetilde{V}_i], \mathscr{F}) = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[V_i], \mathscr{F}) = \Gamma(V_i, \mathscr{F}).$$

The functor "forgetting the $\mathbf{Z}[G]$-module structure" has an exact left adjoint: the extension of scalars functor from sheaves of $\mathbf{Z}$-modules to sheaves of $\mathbf{Z}[G]$-modules. Hence it transforms injectives into injectives, and, deriving (8.6.4) in $\mathscr{F}$, one finds that

$$\text{(8.6.5)} \qquad \mathrm{Ext}^*_{\mathbf{Z}[G]}(\mathbf{Z}[\widetilde{V}_i], \mathscr{F}) = H^*(V_i, \mathscr{F}).$$

If $\mathscr{F}^*$ is a resolution of $\mathscr{F}$ by sheaves of $\mathbf{Z}[G]$-modules such that $H^p(V_i, \mathscr{F}^q) = 0$ for $p > 0$, it follows that the cohomology groups of the complex

$$\mathbf{s}\,\Gamma(V_p, \mathscr{F}^q)$$

are the $\mathrm{Ext}^i_{\mathbf{Z}[G]}(\mathbf{Z}, \mathscr{F})$. This applies to $\mathscr{K}$, and its resolution $\mathscr{K}^*$, with the trivial action of G:

$$\text{(8.6.6)} \qquad H^i\mathbf{s}\,\Gamma(V_p, \mathscr{K}^q) = \mathrm{Ext}^i_{\mathbf{Z}[G]}(\mathbf{Z}, \mathscr{K}).$$

the same applies to the left side of (8.6.1) which corresponds to taking $V_p = G^p$, and 8.6 follows.

8.7 *Proof of 8.3.* — Let $\mathscr{V}$ be the set of systems of open subsets $V_i \subset G^i$ obeying 8.4 (i) (iii), (iv) as well as (v) for some n. We order it by inclusion: $V'_* \leqslant V''_*$ if $V'_i \supset V''_i$ for all i. By 8.4, this order is filtering, and each V_i can be arbitrarily small. The morphism (8.3.1) is hence the inductive limit of the quasi-isomorphisms (8.6.1) and, as such, is a quasi-isomorphism.

Corollary **8.8.** — *Under the assumptions of 8.2, the complexes*

$$\text{(8.8.1)} \qquad \mathscr{K}(k(G^p)) \qquad (p \geqslant 1)$$

as well as

$$\text{(8.8.2)} \qquad \mathrm{coker}(\mathscr{K}(k) \to \mathscr{K}(k(G^p)))$$

are quasi-isomorphic to the complex (1.9.2) computing the cohomology of $H^*(BG \bmod Be, \mathscr{K})$.

Proof. — The first quasi-isomorphism follows from 8.2 applied to G and to the trivial group.

For each p, the map $\mathscr{K}(k) \to \mathscr{K}(k(G^p))$ is injective: indeed, $\mathscr{K}(k(G^p))$ is the inductive limit of the $\mathscr{K}(U)$, for $U \subset G^p$ open and nonempty, and each map $\mathscr{K}(k) \to \mathscr{K}(U)$ is injective, because by unirationality of G, U has a rational point. It follows that the complex (8.8.2) is quasi-isomorphic to (8.8.1).

8.9. *Remark.* — The proof SGA3 XIV 6.7 of the unirationality of reductive groups holds in a relative situation, for G over S, provided that the torus G/G^{der} is isotrivial. Isotriviality holds, for instance, if S is normal. It follows that 8.6 holds for G reductive over S, and S normal with infinite residue fields.

8.10. *Remark.* — Let G be a smooth connected linear algebraic group over an infinite field k. Assume that G is unirational. Let k' be a finite Galois extension of k. We have seen that extension of scalars to k' induces an equivalence of categories from

(a) the category of multiplicative $\mathbf{K}_2$-torsors on G, to

(b) the category of multiplicative $\mathbf{K}_2$-torsors on $G' = G \otimes_k k'$, provided with a (Galois) descent data.

The proof in 7.1 was based on 2.2. Using 8.2, it is possible to base it instead directly on 2.1. Indeed, the categories (a) (b) are incarnated in degree 2, respectively by the complex $K_2 k(G^p)/K_2(k)$ and by the complex deduced from the complex $K_2 k(G'^p)/K_2(k')$ by applying the derived functor of invariants by $\mathrm{Gal}(k'/k)$. By 2.1, they are quasi-isomorphic in degree $\leqslant 2$.

8.11. *Remark.* — Suppose that $\mathscr{K}_1$ and $\mathscr{K}_2$ are sheaves on the big Zariski site of Spec(k) and that $u\colon \mathscr{K}_1 \to \mathscr{K}_2$ induces isomorphisms $\mathscr{K}_1(k(X)) \to \mathscr{K}_2(k(X))$ for any scheme X reduced irreducible and of finite type over k. For instance, one can take for $\mathscr{K}_1$ the Zariski sheaf $\mathbf{K}_2^{M}$ associated to the presheaf of Milnor K-groups $\mathscr{O}^*(U) \otimes \mathscr{O}^*(U)/\langle u \otimes v | u + v = 1\rangle$, and for $\mathscr{K}_2$ the sheaf $\mathbf{K}_2$. Under the assumptions of 8.2, it then follows from 8.2 that u induces an equivalence from the category of multiplicative $\mathscr{K}_1$-torsors on G to that of multiplicative $\mathscr{K}_2$-torsors.

8.12. Let G be an algebraic group over a field k, with $G(k)$ Zariski dense in G. The methods used to prove 8.2 apply as well to the study of central extensions of $G(k)$ by an abelian group A.

The category of central extensions of $G(k)$ by A is incarnated in degree 2 in the standard complex $C^*(G(k), A)$ computing the cohomology $H(BG(k), A)$ of the $G(k)$-module A, with the trivial action of $G(k)$. The group $C^n(G(k), A)$ is the group $\mathscr{F}(G(k)^n, A)$ of A-valued functions on $G(k)^n$.

Let $C^n_{\mathrm{gen}}(G(k), A)$ (generic cocycles) be the inductive limit of the groups $\mathscr{F}(U(k), A)$ of functions from $U(k)$ to A, for U an open Zariski dense subset of G^n.

Proposition **8.13.** — *Under the assumptions and with the notations of 8.12, the morphism of complexes*

$$C^*(G(k), A) \to C^*_{\mathrm{gen}}(G(k), A) \qquad (\mathbf{8.13.1})$$

is a quasi-isomorphism.

Proof. — For $V_i \subset G^i$ a system of open dense subsets of the G^i, obeying 8.4 (i) (iii) (iv) modified as in 8.4 variant, the morphism of complexes

(**8.13.2**) $$C^*(G(k), A) \to \mathscr{F}(V_*(k), A)$$

is a quasi-isomorphism. This is proved as in 8.6, identifying the cohomology groups of both sides with the $\mathrm{Ext}^i_{\mathbf{Z}[G(k)]}(\mathbf{Z}, A)$. Indeed, as in 8.6, if $\widetilde{V}_i$ is the inverse image of V_i in G^{Δ_i}, $\mathbf{Z}[\widetilde{V}_i(k)]$ is a resolution of $\mathbf{Z}$ by free $\mathbf{Z}[G]$-modules, because for any finite family $(g_1^\alpha, \ldots, g_i^\alpha)$ of elements of $\widetilde{V}_{i-1}(k)$, there exists $g_0 \in G(k)$ such that the $(g_0, g_1^\alpha, \ldots, g_i^\alpha)$ are in $\widetilde{V}_i(k)$, and

$$\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}[\widetilde{V}_i(k)], A) = \mathscr{F}(V_i(k), A).$$

Taking an inductive limit as in 8.7, one obtains (8.13.1) as an inductive limit of quasi-isomorphisms (8.13.2).

8.14. Under the assumptions of 8.12, it follows that the category of central extensions of $G(k)$ by A is incarnated in degree 2 by the complex $C^*_{\mathrm{gen}}(G(k), A)$ of generic cochains. In particular, a generic 2-cochain c defines a central extension $E(c)$, unique up to unique isomorphisms. More precisely, c is the image of a cocycle $\widetilde{c}$ in $C^*(G(k), A)$, defining $E(\widetilde{c})$ and if $\widetilde{c}'$, $\widetilde{c}''$ are two such liftings of c, there exists a unique 1-cochain d, vanishing on some $U(k)$, with U open and Zariski dense in G, with coboundary $\widetilde{c}'' - \widetilde{c}'$. It defines an isomorphism from $E(\widetilde{c}')$ to $E(\widetilde{c}'')$.

9. Comparison with Galois cohomology

9.1. Let k be a field and n an integer prime to the characteristic. We write $\mathbf{Z}/n(1)$ for the group of n^{th} roots of unity in a separable closure $\bar{k}$ of k, $\mathbf{Z}/n(d)$ for its d^{th} tensor power and $H^p(k, \ )$ for a Galois cohomology group $H^p(\mathrm{Gal}(\bar{k}/k), \ )$. Let cl be the Kummer coboundary

$$\mathrm{cl}\colon k^* \to H^1(k, \mathbf{Z}/n(1)).$$

By Tate (1970), if $u + v = 1$, one has $\mathrm{cl}(u) \cup \mathrm{cl}(v) = 0$ in $H^2(k, \mathbf{Z}/n(2))$. As $K_2(k) = k^* \otimes k^* / \langle u \otimes v | u + v = 1\rangle$, the morphism $u, v \mapsto \mathrm{cl}(u) \cup \mathrm{cl}(v)$ induces a morphism

(**9.1.1**) $$K_2(k) \to H^2(k, \mathbf{Z}/n(2)).$$

More generally, if n is invertible on a scheme S, $\mathbf{Z}/n(1)$ is the étale sheaf of n^{th} root of 1 and $\mathbf{Z}/n(d)$ its d^{th}-tensor power. Let ε be the morphism from the étale to the Zariski site of S, and let

$$\mathscr{H}^p(\mathbf{Z}/n(d)) := R^p \varepsilon_* \mathbf{Z}/n(d)$$

be the Zariski sheaf associated to the presheaf $U \mapsto H^p_{et}(U, \mathbf{Z}/n(d))$ (étale cohomology). We will mainly use the $\mathscr{H}^p(\mathbf{Z}/n(2))$ and denote them simply $\mathscr{H}^p$.

For U affine, Soulé (1979) has defined Chern class maps $c_2\colon K_2(U) \to H^2_{et}(U, \mathbf{Z}/n(2))$. They define a morphism of sheaves

$$(\mathbf{9.1.2}) \qquad \mathbf{K}_2 \to \mathscr{H}^2.$$

By loc. cit. II 3, for S spectrum of a field k, (9.1.2) reduces to (9.1.1) possibly up to a sign s. If needed, we change the sign of (9.1.2) so that it agrees with (9.1.1). The sheaf $\mathscr{H}^2$ is contravariant in S. By 1.7, it defines a sheaf on the big Zariski site S_{Zar} of S, and the morphisms (9.1.2) define a morphism of sheaves on S_{Zar}.

9.2. Let G be a simply-connected absolutely simple algebraic group over a field k. Let T be a maximal torus, Y be its cocharacter group (an étale sheaf over Spec(k), i.e. a $\mathrm{Gal}(\bar{k}/k)$-module) and Q be the minimal Weyl group invariant positive quadratic form on Y: one has $Q(\alpha^\vee) = 1$ for α a long root. Applying 7.3(i), we obtain a multiplicative $\mathbf{K}_2$-torsor E on G, hence a central extension of $G(k)$ by $K_2(k)$. Pushing it by (9.1.1), one obtains a central extension

$$(\mathbf{9.2.1}) \qquad H^2(k, \mathbf{Z}/n(2)) \to E_n(k) \to G(k).$$

Construction **9.3.** — *We construct an isomorphism of the central extension (9.2.1) with the opposite of the canonical central extension of Deligne (1996) 3.6.*

For simplicity, we assume k infinite. If k is finite, both central extensions are by the trivial group.

Lemma **9.4.** — *Both central extensions are deduced, as in 1.4, from a multiplicative* $\mathscr{H}^2$-*torsor on* G.

Proof. — For the central extension (9.2.1), the required multiplicative $\mathscr{H}^2$-torsor is deduced from E by pushing by (9.1.2). To describe the other central extension, we have to review the construction of Deligne (1996).

The quadratic form Q defines an étale cohomology class c_Q in $H^4(BG \bmod Be, \mathbf{Z}/n(2))$. This H^4 is canonically isomorphic to $\mathbf{Z}/n$, with c_Q as generator (loc. cit. §1). If $\mathscr{B}^*$ is a functorial acyclic resolution of $\mathbf{Z}/n(2)$, for instance an injective resolution in the category of sheaves of $\mathbf{Z}/n$-modules on the big étale site of Spec(k), c_Q is given by a 4-cocycle in $\mathbf{s}(\Gamma(G^p, \mathscr{B}^q)_{p\geqslant 1})$ with components $\gamma_i \in \Gamma(G^{4-i}, \mathscr{B}^i)$ $(0 \leqslant i \leqslant 3)$.

The component γ_3 defines a class $[\gamma_3]$ in $H^3_{et}(G, \mathbf{Z}/n(2))$. This class is primitive. By loc. cit. 1.13, it vanishes on some open subset U of G. Translating U, one sees that $[\gamma_3]$ vanishes locally on G for the Zariski topology.

If $[\gamma_3]$ vanishes on $U \subset G$, there exists $b \in H^0(U, \mathscr{B}^2)$ such that $db = \gamma_3$. The possible b's form a torsor under the group Z^2 of cocycles, and, pushing by $Z^2 \to H^2_{et}(U, \mathbf{Z}/n(2))$, one obtains a $H^2_{et}(U, \mathbf{Z}/n(2))$-torsor. In this way, γ_3 gives rise to a $\mathscr{H}^2$-torsor on G. The component γ_2 provides a multiplicative structure on this $\mathscr{H}^2$-torsor. The resulting multiplicative $\mathscr{H}^2$-torsor is the one promised by 9.4.

Here is another description of its isomorphism class, viewed as a class in $H^2(BG \bmod Be, \mathscr{H}^2)$. The projection ε from étale to Zariski sites gives rise to a spectral sequence

$$E_2^{pq} = H^p_{Zar}(BG \bmod Be, \mathscr{H}^q) \Rightarrow H^{p+q}_{et}(BG \bmod Be, \mathbf{Z}/n(2)). \qquad (\mathbf{9.4.1})$$

Lemma **9.5.** — *In the spectral sequence (9.4.1),*

(i) $E_2^{0,q} = 0$

(ii) *the morphism* $H^4 \to E_2^{1,3}$ *vanishes*

(iii) *the resulting morphism* $H^4 \to E_2^{2,2} = H^2(BG \bmod Be, \mathscr{H}^2)$ *carries* c_Q *to the class of the multiplicative* $\mathscr{H}^2$*-torsor on* G *we constructed.*

Proof. — Let $\mathscr{B}^*$ be as in 9.4. Following Grothendieck (1957) 2.4, the spectral sequence (9.4.1) is obtained by viewing $\mathbf{s}(\Gamma_{et}(G^p, \mathscr{B}^q)$ for $p \geqslant 1, q \geqslant 0)$ as the hypercohomology of $BG \bmod Be$ with coefficients in $\varepsilon_* \mathscr{B}^*$, and using the canonical filtration τ of $\varepsilon_* \mathscr{B}^*$. Functorially in G^p, one should take a filtered resolution $(\mathscr{B}^*_1, F)$ of $(\varepsilon_* \mathscr{B}^*, \tau)$, so that on each G^p $(p \geqslant 1)$, $Gr^F_n(\mathscr{B}^*_1)$ is a resolution of $\mathscr{H}^n$ with acyclic components. The morphism

$$\mathbf{s}(\Gamma_{et}(G^p, \mathscr{B}^q)_{p\geqslant 1}) = \mathbf{s}(\Gamma_{Zar}(G^p, \varepsilon_* \mathscr{B}^q))_{p \geqslant 1} \to \mathbf{s}\Gamma_{Zar}(G^p, \mathscr{B}^q_1)_{p \geqslant 1}$$

is a quasi-isomorphism, and one uses on the target the filtration F: $E_2^{pq} = H^{p+q} Gr^F_q = H^p \mathbf{s}\Gamma_{zar}(G^*,$ resolution of $\mathscr{H}^q)_{* \geqslant 1}$.

The term E_2^{0q} vanishes, being the H^0 of a complex which starts in degree 1. If a class in H^4 is given by a cocycle (γ_i), as in 9.4, the term $E_2^{1,3}$ is the primitive part of $H^0(G, \mathscr{H}^3)$, and the map $H^4 \to E_2^{1,3}$ is $c \mapsto$ class in $\mathscr{H}^3$ of γ_3. It vanishes (see 9.4) and we leave it to the reader to check that the resulting map $H^4 \to E_2^{2,2}$ attaches to c the class of the multiplicative $\mathscr{H}^2$-torsor constructed as in 9.4.

Lemma **9.6.** — (i) *In the spectral sequence (9.4.1), the morphism*

$$H^4 \to E_2^{2,2} = H^2(BG \bmod Be, \mathscr{H}^2)$$

(defined thanks to 9.5 (i) (ii)) is an isomorphism.

(ii) *One has* $E_2^{1,2} = H^1(BG \bmod Be, \mathscr{H}^2) = 0$: *multiplicative* $\mathscr{H}^2$*-torsors on* G *have no nontrivial automorphisms.*

Proof. — We first prove the vanishing of E_2^{pq} for $q = 0$ or 1. By the spectral sequence

$$'E_1^{r,s} = H^s(G^r \bmod e, \mathscr{H}^q) \Rightarrow H^{r+s}(BG \bmod Be, \mathscr{H}^q),$$

it suffices to show that for G simply-connected, one has

$$H^*(G \bmod e, \mathscr{H}^q) = 0 \qquad \text{for } q = 0, 1 \ :$$

applied to the G^r, which are simply-connected, this gives $'E_1 = 0$.

For $q = 0$, the Zariski sheaf $\mathscr{H}^0$ is the constant sheaf with value $H^0(k, \mathbf{Z}/n(2))$. Its higher cohomology group vanishes, as well as all its reduced cohomology groups $H^*(G \bmod e, \mathscr{H}^0)$.

To handle the case $q = 1$, we use the spectral sequence

$$''E_2^{u,v} = H^u(G \bmod e, \mathscr{H}^v) \Rightarrow H_{et}^{u+v}(G \bmod e, \mathbf{Z}/n(2))$$

and what we know of its abutment to obtain $''E_2^{*1} = 0$. The case $q = 0$ gives us $''E_2^{*,0} = 0$. By Bloch-Ogus (1974), $''E_2^{u,v} = 0$ for $u > v$. The $''E_2^{u,v}$ for $u, v \geqslant 0$ hence look as follows:

$$\text{(9.6.1)} \qquad \begin{matrix} ''E_2^{02} & ''E_2^{12} & ''E_2^{22} & 0 & \cdots \\ ''E_2^{01} & ''E_2^{11} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{matrix}$$

The abutment $H_{et}^n(G \bmod e, \mathbf{Z}/n(2))$ vanishes for $n \leqslant 2$. Indeed, in the geometric case of an algebraically closed field k, this follows from similar results for complex groups (cf. Deligne (1996) 1.5, 1.7), and the general case follows from the geometric case by the Hochschild-Serre spectral sequence

$$H^p(k, H^q(G_{\bar{k}} \bmod e, \quad)) \to H^{p+q}(G \bmod e, \quad).$$

a diagram chase in (9.6.1) shows that the vanishing of the abutment for $n \leqslant 2$ implies the vanishing of $''E_2^{uv}$ for $u + v \leqslant 2$. In particular, $''E_2^{0,1} = ''E_2^{1,1} = 0$, as required.

By 9.5 (i), we know now that the E_2^{pq} of (9.4.1) look as follows:

$$\text{(9.6.2)} \qquad \begin{matrix} 0 & & & & & \\ 0 & E_2^{13} & & & & \\ 0 & E_2^{12} & E_2^{22} & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \end{matrix}$$

As the abutment vanishes for $n \leqslant 3$, (Deligne (1996) 1.10), (i) and (ii) of 9.6 follow from 9.5 (ii) by a diagram chase in (9.6.2).

9.7. *Proof of 9.3.* — We will construct the isomorphism 9.3 as the image of an isomorphism between the multiplicative $\mathscr{H}^2$-torsors of 9.4. By 9.6 (ii), such an isomorphism, if it exists, is unique: we are left with showing that the two multiplicative $\mathscr{H}^2$-torsors of 9.4 are isomorphic. The group $H^2(BG \bmod Be, \mathscr{H}^2)$ of isomorphism classes coincides with $H^4_{et}(BG \bmod Be, \mathbf{Z}/n(2))$ (9.6 (i)). This group is $\mathbf{Z}/n$, independently of k. To prove the isomorphism of the two multiplicative $\mathscr{H}^2$-torsors of 9.4, it therefore suffices to do so after a field extension k'/k: we may and shall assume that G is split.

The two $\mathscr{H}^2$-torsors to be compared have the same functoriality in G. As in loc. cit. 3.7, we can hence reduce the general case to that of $G = SL(2)$, and reduce the case of SL(2) to that of SL(3).

The multiplicative $\mathscr{H}^2$-torsors on G can be detected as follows: extend scalars from k to $k(U, V)$; consider the resulting central extension

$$1 \to H^2(\mathrm{Gal}(\overline{k(U, V)}/k(U, V)), \mathbf{Z}/n(2)) \to E \to G(k(U, V)) \to 1;$$

fix coroots y_1, y_2 in Y with $B(y_1, y_2) = 1$, where B is the bilinear form attached to Q; take the commutator

$$(y_1(U), y_2(V)) \in H^2(\mathrm{Gal}(k(U, V)^-/k(U, V)), \mathbf{Z}/n(2)).$$

It will be a power of the Galois symbol (U, V). This Galois symbol is of order n, as one sees by taking its residue along $U = 0$, and which power of (U, V) we get tells us which $\mathscr{H}^2$-torsor we started from. To prove 9.2, it then suffices to compare the commutator formulae of 3.14 and of loc. cit. 3.5.

9.8. Let T_1 and T_2 be two tori over a field k. We let Y_1 and Y_2 be their groups of cocharacters and view them as Galois-modules. To a bilinear form $B: Y_1 \otimes Y_2 \to \mathbf{Z}$, supposed to be Galois invariant, we have attached a bimultiplicative map (7.5)

(**9.8.1**) $$T_1(k) \times T_2(k) \to K_2(k).$$

Fix an integer n prime to the characteristic of k. Composing (9.8.1) with the map (9.1.1): $K_2(k) \to H^2(k, \mathbf{Z}/n(2))$, we obtain

(**9.8.2**) $$T_1(k) \times T_2(k) \to H^2(k, \mathbf{Z}/n(2)).$$

The short exact sequences of sheaves for the étale topology,

$$0 \to Y_1 \otimes \mathbf{Z}/n(1) \to T_i \xrightarrow{t^n} T_i \to 1,$$

give rise to coboundaries

$$T_i(k) \to H^1(k, Y_i \otimes \mathbf{Z}/n(1)).$$

Taking a cup product, and applying B, we obtain

(**9.8.3**) $$\mathrm{T}_1(k) \times \mathrm{T}_2(k) \to \mathrm{H}^2(k, \mathrm{Y}_1 \otimes \mathrm{Y}_2 \otimes \mathbf{Z}/n(2)) \to \mathrm{H}^2(k, \mathbf{Z}/n(2)).$$

Proposition **9.9.** — *The maps (9.8.2) and (9.8.3) are equal.*

Both (9.8.2) and (9.8.3) come from a bimultiplicative section of $\mathscr{H}^2$ on $\mathrm{T}_1 \times \mathrm{T}_2$. For (9.8.2): the map (9.8.1) comes from a bimultiplicative section of $\mathbf{K}_2$, to which one applies (9.1.2): $\mathbf{K}_2 \to \mathscr{H}^2$. For (9.8.3): the same construction continues to make sense, using étale cohomology, over any base, and one applies it in the universal case: over the base $\mathrm{T}_1 \times \mathrm{T}_2$, for the universal $\mathrm{T}_1 \times \mathrm{T}_2$-points pr_1 and pr_2 of T_1 and T_2.

If T_1 and T_2 are split, it is clear from the definitions that those two bimultiplicative sections of $\mathscr{H}^2$ on $\mathrm{T}_1 \times \mathrm{T}_2$ are equal. We reduce to the split case by using the

Lemma **9.10.** — *Let* $\mathrm{H}^0(\mathrm{T}_1 \times \mathrm{T}_2 \bmod \mathrm{T}_1, \mathrm{T}_2, \mathscr{H}^2)$ *be the group of sections of* $\mathscr{H}^2$ *with trivial restrictions to* T_1 *and* T_2. *If we extend scalars from k to a separable closure* k' *of k, this relative* H^0 *injects into the similar relative* H^0 *on* k'.

Proof. — The inclusion and projections $\mathrm{T}_i \hookrightarrow \mathrm{T}_1 \times \mathrm{T}_2 \to \mathrm{T}_i$ give commuting idempotent endomorphisms $\mathrm{pr}_i^* \, \mathrm{inj}_i^*$ of $\mathrm{H}^0(\mathrm{T}_1 \times \mathrm{T}_2, \mathscr{H}^2)$, and the relative H^0 we consider is one of the four corresponding direct summands.

The projection from the étale site to the Zariski site gives a spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathrm{T}_1 \times \mathrm{T}_2 \bmod \mathrm{T}_1, \mathrm{T}_2, \mathscr{H}^q) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{et}}(\mathrm{T}_1 \times \mathrm{T}_2 \bmod \mathrm{T}_1, \mathrm{T}_2, \mathbf{Z}/n(2)).$$

By Bloch-Ogus (1974), $\mathrm{H}^p(\mathrm{T}_1 \times \mathrm{T}_2, \mathscr{H}^q)$ vanishes for $p > q$. So does the direct summand E_2^{pq} and the E_2^{pq} for $p, q \geqslant 0$ look as follows:

$$\begin{matrix} * & * & * & 0 \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \end{matrix}$$

At the abutment of the spectral sequence, $\mathrm{H}^i = 0$ for $i < 2$, while H^2 is the group of Galois invariant bilinear forms from $\mathrm{Y}_1 \times \mathrm{Y}_2$ to $\mathbf{Z}/n$ (Deligne (1996) 3.1). From the spectral sequence, we get a short exact sequence

(**9.10.1**) $$0 \to \mathrm{E}_2^{1,1} \to \mathrm{H}^2 \to \mathrm{E}_2^{0,2} \to 0.$$

Let us compare (9.10.1) and the similar sequence obtained after extension of scalars from k to k':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{E}_2^{1,1} & \longrightarrow & \mathrm{H}^2 & \longrightarrow & \mathrm{E}_2^{0,2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}'\mathrm{E}_2^{1,1} & \longrightarrow & {}'\mathrm{H}^2 & \xrightarrow{①} & {}'\mathrm{E}_2^{0,2} & \longrightarrow & 0. \end{array}$$

The middle vertical arrow is injective. The map ① is injective as well: over k', choosing bases $x_1, \ldots, x_N$ and $y_1, \ldots, y_M$ of the characters group of T_1 and T_2, one can write any element of $'H^2$ uniquely in the form $\sum n_{ij} \mathrm{cl}(x_i) \cup \mathrm{cl}(y_j)$ with n_{ij} in $\mathbf{Z}/n$. If $K = k'(x_1, \ldots, x_N, y_1, \ldots, y_M)$ is the field of rational functions on $T_1 \times T_2$, the element is uniquely determined by its image in $H^2(K, \mathbf{Z}/n(2))$, as one sees by taking an iterated residue along $x_i = 0$, and then along $y_j = 0$.

Those two injectivities force $'E_2^{1,1} = 0$, then $E_2^{1,1} = 0$ and finally the required injectivity of $E_2^{0,2} \to {'E_2}^{0,2}$.

10. Local and global fields

10.1. Let F be a local field: F a finite extension of $\mathbf{R}$, $\mathbf{Q}_p$ or $\mathbf{F}_p((T))$ for some p. The group $K_2(F)$ is the target of the universal symbol $\{\ ,\ \}$, where a symbol with values in an abelian group A is a bimultiplicative map $F^* \times F^* \to A$, with $(x, y) = 0$ for $x + y = 1$. If one considers only locally constant symbols, there is also a universal one. Its target $K_2^{\mathrm{cont}}(F)$ is trivial if $F \simeq \mathbf{C}$, and is the group μ_F of roots of unity of F otherwise, with $\{\ ,\ \}$ = Hilbert symbol. See Moore (1968), Theorem 3.1.

Lemma **10.2.** — *Let* X *be a scheme of finite type over* F *and let* s *be a section of* $\mathbf{K}_2$ *over* X. *Evaluating* s *at each* $x \in X(F)$, *one obtains a map* $s_1 \colon X(F) \to K_2(F) \to K_2^{\mathrm{cont}}(F)$. *The map* s_1 *is locally constant.*

Proof. — Let n be the order of μ_F. One has

$$H^2(F, \mathbf{Z}/n(2)) = \mu_F,$$

the universal locally constant symbol being the Galois symbol. Let s' be the image of s in $H^0(X, \mathscr{H}^2)$ by (9.1.2). The map s_1 is obtained by evaluating s' at $s \in X(F)$ and 10.2 reduces to Deligne (1996) 2.10.

By 10.2, a $\mathbf{K}_2$-torsor on X gives rise to a $K_2^{\mathrm{cont}}(F)$-torsor on the topological space X(F). Applying this to multiplicative torsors, one obtains the

Construction **10.3.** — *Let* E *be a multiplicative* $\mathbf{K}_2$*-torsor on the algebraic group* G *over* F. *The resulting central extension*

$$(\mathbf{10.3.1}) \qquad K_2^{\mathrm{cont}}(F) \to \widetilde{G}(F) \to G(F)$$

deduced from 1.4 by pushing by $K_2(F) \to K_2^{\mathrm{cont}}(F)$ *is a topological central extension of the locally compact group* G *by the finite (and discrete) group* $K_2^{\mathrm{cont}}(F)$.

For G absolutely simple and simply-connected, the central extensions (10.3.1) coincide with the central extensions constructed in Deligne (1996). See §9 and loc.

cit. 5.4. For E a generator of the group of central extensions of G by $\mathbf{K}_2$, Prasad-Rapinchuk (1996) 8.4 proves that if F is nonarchimedian and G isotropic, the central extension (10.3.1) is a universal topological central extension of G(F).

10.4. Let now F be a global field. For each place v of F, let F_v be the completion of F at v. Let μ_F, or simply μ, be the group of roots of unity of F. For each place v, if v is not a complex place, let μ_v be the group of roots of unity in F_v, and let $[\mu_v: \mu]$ be the index of μ in μ_v. For v a complex place, define μ_v to be trivial. The reciprocity law for the Hilbert symbols says that for x, y in F^*, $((x,y)_v)$ is in the kernel of the map

(**10.4.1**) $$(\zeta_v) \longmapsto \prod_v \zeta_v^{[\mu_v:\mu]}$$

from $\oplus\mu_v$ to μ. By Moore (1968), Theorem 7.4 one has more precisely an exact sequence

(**10.4.2**) $$K_2(F) \to \bigoplus_v K_2^{\mathrm{cont}}(F_v) \xrightarrow{10.4.1} \mu \to 0.$$

From this, we will deduce that for G a linear algebraic group over F and E a multiplicative $\mathbf{K}_2$-torsor over G, one has a topological central extension of the adelic group $G(\mathbf{A})$ by μ, canonically split over G(F):

(**10.4.3**) $$\begin{array}{ccccc} & & & & G(F) \\ & & & \swarrow & \downarrow \\ \mu & \longrightarrow & G(\mathbf{A})^{\sim} & \longrightarrow & G(\mathbf{A}). \end{array}$$

A little care is required because torsors on G are for the Zariski topology, and that for $U \subset G$ an affine open subset, $U(\mathbf{A})$ is not in general open in $G(\mathbf{A})$. We will not work directly with $G(\mathbf{A})$, but rather obtain (10.4.3) as an inductive limit of similar diagrams, relative to larger and larger finite sets of places of F.

10.5. If F is a number field, let $\mathcal{O}_F$ be its ring of integers. For S_1 a finite set of closed points of $X := \mathrm{Spec}(\mathcal{O}_F)$, identified with places of F, let $S := S_1 \cup \{\text{infinite places}\}$. If F is a function field, let X be the projective nonsingular curve of which it is the field of rational functions. We let $S = S_1$ be a finite set of closed points, identified with places. In both cases, we note $|X|$ the set of closed points of X.

Fix S_1 large enough so that G is the generic fiber of a group scheme G_{X-S_1} over $X - S_1$ and that the $\mathbf{K}_2$-torsor E comes from a multiplicative $\mathbf{K}_2$-torsor E_{X-S_1} on G_{X-S_1}. Fix G_{X-S_1} and E_{X-S_1}. Two choices become isomorphic after restriction to some $X - S_1'$, and two isomorphisms between two choices become equal after restriction to some $X - S_1''$. The following lemma is analogous to Deligne (1996) 6.2.

Lemma **10.6.** — *Except for* F *a function field and* S_1 *empty, one has*

$$H^1(X - S_1, \mathbf{K}_2) = 0.$$

As K_2 of a finite field vanishes, we have an exact sequence of sheaves over X

$$0 \to \mathbf{K}_2 \to \text{constant sheaf } K_2(F) \to \bigoplus_{x \in |X|} (i_x)_* k_x^* \to 0$$

with i_x the inclusion of x in X and k_x the residue field. From this, we get an exact sequence

$$K_2(F) \to \bigoplus_{x \in X - S_1} k_x^* \to H^1(X - S_1, \mathbf{K}_2) \to 0.$$

By (10.4.2), the image of $K_2(F)$ in the sum of the k_x^* $(x \in X - S_1)$ coincide with the image of $\mathrm{Ker}(\oplus \mu_v \to \mu)$, where the sum runs over all noncomplex places. We have to show that, except for F a function field and S_1 empty, the natural map

(**10.6.1**) $$\mathrm{Ker}(\oplus \mu_v \to \mu) \to \bigoplus_{x \in X - S_1} k_x^*$$

is onto. If F is a function field and if $v \in S_1$, this results from the surjectivity of $\zeta \mapsto \zeta^{[\mu_v : \mu]}: k_v^* = \mu_v \to \mu$. If F is a number field, for any prime ℓ, let v be a place of characteristic ℓ. The map $\zeta \mapsto \zeta^{[\mu_v : \mu]}$ induces a surjection from $\mathrm{Ker}(\mu_v \to k_v^*)$ to the ℓ-primary component of μ, and it follows that (10.6.1) is onto on the ℓ-primary components. This holding for each ℓ, 10.6 follows.

10.7. If S_1 is large enough so that the conclusion of 10.6 holds, we have a central extension

(**10.7.1**) $$H^0(X - S_1, \mathbf{K}_2) \to E_1 \to G(X - S_1).$$

For v a place of F, it maps to the local central extension

(**10.7.2**) $$\mu_v \to \widetilde{G}(F_v) \to G(F_v).$$

For $v \in |X|$, the map factors through a central extension

$$K_2(\mathcal{O}_v) \to G(\mathcal{O}_v)^{\sim} \to G(\mathcal{O}_v).$$

If μ_v is of order prime to the residue characteristic, the exact sequence

$$K_2(\mathcal{O}_v) \to K_2(F_v) \to k_v^*$$

shows that $K_2(\mathcal{O}_v)$ maps trivially to μ_v. We obtain a trivialization of $\widetilde{G}(F_v)$ over $G(\mathcal{O}_v)$:

$$\begin{array}{ccccc} K_2(\mathcal{O}_v) & \longrightarrow & G(\mathcal{O}_v)^{\sim} & \longrightarrow & G(\mathcal{O}_v) \\ \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow \\ \mu_v & \longrightarrow & \widetilde{G}(F_v) & \longrightarrow & G(F_v). \end{array}$$

and the map (10.7.1) $\to$ (10.7.2) maps E_1 to a lifting of $G(\mathcal{O}_v)$ in $\widetilde{G}(F_v)$. Taking a product, we obtain

$$(\mathbf{10.7.3}) \quad \begin{array}{ccccc} H^0(X-S_1, \mathbf{K}_2) & \longrightarrow & E_1 & \longrightarrow & G(X-S_1) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{v\in S} \mu_v & \longrightarrow & \prod_{v\in S} \widetilde{G}(F_v) \times \prod_{v\in X-S_1} G(\mathcal{O}_v) & \longrightarrow & \prod_{v\in S} G(F_v) \times \prod_{v\in X-S_1} G(\mathcal{O}_v) \end{array}$$

provided that for all v in $X - S_1$, μ_v is of order prime to the residual characteristic. This holds for S_1 large enough. The first vertical map, composed with the reciprocity map $\Pi\zeta_v^{[\mu_v:\mu]}$ with values in μ, vanishes. From 10.7.3 we hence get

$$(\mathbf{10.7.4}) \quad \begin{array}{ccccc} & & & \swarrow & G(X-S_1) \\ & & & & \downarrow \\ \mu & \longrightarrow & \Big[\prod_{v\in S} G(F_v)\Big]^{\sim} \times \prod_{v\in X-S_1} G(\mathcal{O}_v) & \longrightarrow & \prod_{v\in S} G(F_v) \times \prod_{v\in X-S_1} G(\mathcal{O}_v). \end{array}$$

The diagram 10.4.3 is the inductive limit of the diagrams 10.7.4.

The geometric analogue of those constructions will be considered in 12.8 to 12.15.

11. Associated central extensions of dual characters groups

If G is a split simply connected group over a field k, with split maximal torus T, a Weyl group invariant integer-valued quadratic form Q on the dual character group Y of T determines a central extension of Y by k^*. This central extension is an ingredient in the classification 6.2 of central extensions of split reductive groups by $\mathbf{K}_2$. Our aim in this section is to describe it in terms of the root spaces decomposition of Lie(G). The classification 6.2 depends on the choice of a split maximal torus in the split reductive group. We will explain also how the classifications for different maximal torus are related.

To simplify the exposition, we will work in this section over a field k.

11.1. We first consider the case when G is isomorphic to SL(2). Choose a split maximal torus T. Let U^+, U^- be the root subgroups normalized by T. We write α_+

and α_- for the corresponding roots. In the constructions which follow, U^+ and U^- play symmetric roles: the constructions are invariant by the group of automorphisms of (G, T), i.e. by the normalizer of T_{ad} in the adjoint group G_{ad}.

Define $U^{\pm *} := U^{\pm} - \{e\}$ and $N^1 := N(T) - T$. We will use the trijection of Tits (1966) 1.1 between U^{+*}, U^{-*} and N^1. Trijection means: transitive system of isomorphisms. For the standard SL(2) and its torus of diagonal matrices, the triples in trijection are the

$$(\mathbf{11.1.1})\qquad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}.$$

The elements e^+, e^- and n are in trijection if and only if $n = e^+e^-e^+$. One then has

$$(\mathbf{11.1.2})\qquad n = e^+e^-e^+ = e^-e^+e^-.$$

If e^+, e^- and n are in trijection, so are their inverses (take the inverses in (11.1.2)) and the conjugation by n exchanges e^+ and e^- (apply conjugation by n to (11.1.2) and observe that conjugation by n fixes n and exchanges U^+ and U^-). The square of any $n \in N^1$ is the central element $\alpha_+^\vee(-1)$, hence $n^{-1} = \alpha_+^\vee(-1)n$.

Let E be the extension of G by $\mathbf{K}_2$ defined by a quadratic form Q on Y. By 3.2, it is trivial on the unipotent subgroup U^+ (resp. U^-). We note $u \mapsto \widetilde{u}$ the unique trivializing section. We define a section of E over N^1 by $\widetilde{n} = \widetilde{e^+}\widetilde{e^-}\widetilde{e^+}$, for n decomposed as in (11.1.2). Applying int$[n]$, which fixes $\widetilde{n}$, one obtains that $\widetilde{n} = \widetilde{e^-}\widetilde{e^+}\widetilde{e^-}$. Taking the inverse, one obtains $\widetilde{n}^{-1} = (n^{-1})^{\sim}$.

Fix n in N^1. The right translation by n induces an isomorphism of schemes from T to N^1, and $h \mapsto s[n](h) := (hn)^{\sim}\widetilde{n}^{-1}$ is a section $s[n]$ of E over T, trivial at e. One has

$$(\mathbf{11.1.3})\qquad \widetilde{n}^{-2} = (n^{-1})^{\sim}\widetilde{n}^{-1} = (\alpha_+^\vee(-1)n)^{\sim}\widetilde{n}^{-1} = s[n](\alpha_+^\vee(-1))$$

Steinberg's cocycle $c \in H^0(T \times T, \mathbf{K}_2)$ is the 2-cocycle defined by

$$(\mathbf{11.1.4})\qquad s[n](h_1)s[n](h_2) = s[n](h_1h_2).c(h_1, h_2).$$

Let us identify T with $\mathbf{G}_m$ and Y with $\mathbf{Z}$, using α_+. The cocycle $c(h_1, h_2)$ is trivial for $h_1 = e$ or $h_2 = e$. By 3.7 (ii), it is of the form $c(h_1, h_2) = q.\{h_1, h_2\}$ for some integer q. Applying 3.13 and 4.9, one obtains that the quadratic form Q is qn^2:

$$(\mathbf{11.1.5})\qquad c(\alpha_+^\vee(a), \alpha_+^\vee(b)) = Q(\alpha_+^\vee)\{a, b\}.$$

The adjoint group G_{ad} acts on G, with T_{ad} respecting T. One has $Y_{ad} = \frac{1}{2}Y$. The group U^+ is naturally a vector space, with

(**11.1.6**) $$\left(\frac{1}{2}\alpha^\vee\right)(a)[e^+] = ae^+.$$

The action of G^{ad} on G extends uniquely to E (4.10). If $(\frac{1}{2}\alpha_+^\vee)(a)$ transforms n into n', it will transform the section $s[n]$ of E over T into $s[n']$. By 4.12,

(**11.1.7**) $$s[n'](t) = s[n](t).[Q(\alpha_+^\vee)\{a, t\}].$$

The central extension $\mathscr{E}$ of Y by k^* attached to Q is obtained as follows. One extends the scalars from k to $k((\tau))$, one obtains from $E|_T$ a central extension of $T(k((\tau)))$ by $K_2k((\tau))$, one pulls it back by $Y \to T(k((\tau)))$: $y \mapsto y(\tau)$ and pushes it out by the residue map Res: $K_2k((\tau)) \to k^*$. We normalized the residue map to be the tame symbol

(**11.1.8**) $$\mathrm{Res}\{f, g\} = (-1)^{v(f)v(g)}.(g^{v(f)}f^{-v(g)})(0).$$

We choose to call the variable τ, not t as in 3.10 or 3.12, to avoid confusion with elements of T.

For e^+, e^- and n in Tits trijection, we define $[e^+]$ to be the image in $\mathscr{E}$ of $s[n](\alpha_+^\vee(\tau))$. Applying this definition to e^-, e^+, n, which are also in trijection, we get $[e^-] := s[n](\alpha_-^\vee(\tau))$. The element $[e^+]$ (resp. $[e^-]$) of $\mathscr{E}$ is a lifting to $\mathscr{E}$ of the generator $\alpha_+^\vee$ (resp. $\alpha_-^\vee$) of Y. By (11.1.6), (11.1.7) and (11.1.8), one has

(**11.1.9**) $$[ae^+] = a^{-Q(\alpha_+^\vee)}[e^+]$$

in $\mathscr{E}$. By (11.1.4), (11.1.5) and (11.1.8), one has

(**11.1.10**) $$[e^+][e^-] = (-1)^{Q(\alpha_+^\vee)}.$$

As U^+ is a vector space, it can be identified with its Lie algebra $\mathfrak{u}^+$. By (11.1.9), the inverse image of $\alpha_+^\vee$ in $\mathscr{E}$ can be identified with the complement of 0 in the N^{th} tensor power of the line u^+, for $N = -Q(H^+)$, by $(e^+)^{\otimes N} \mapsto [e^+]$. The central extension $\mathscr{E}$ of Y by k^* is characterized up to unique isomorphism as being a central extension of Y by k^*, provided with maps [ ]: $U^{+*} \to \{\text{liftings of } \alpha_+^\vee\}$ and $U^{-*} \to \{\text{liftings of } \alpha_-^\vee\}$, obeying (11.1.9), (11.1.10).

11.2. Suppose that G is a split semi-simple simply-connected algebraic group, with split maximal torus T. We will use the notation of 4.1. For each root α, we denote by T_α the maximal torus $\alpha^\vee(\mathbf{G}_m)$ of the SL(2)-subgroup S_α, and define $N_\alpha^1 := N(T_\alpha) - T_\alpha$

in S_α. Let E be the central extension of G by $\mathbf{K}_2$ defined by a quadratic form Q on Y, and $\mathscr{E}$ be the corresponding central extension of Y by k^*. Let us apply 11.1 to S_α. We obtain a canonical section $n \mapsto \widetilde{n}$ of E over N^1_α; the choice of n_α in N^1_α defines a section $s[n_\alpha]$ of E over T_α; $U^*_\alpha := U_\alpha - \{e\}$ maps to the inverse image of $\alpha^\vee$ in $\mathscr{E}$, by a map [ ], and for e_α in U^*_α, one has

(**11.2.1**) $$[ae_\alpha] = a^{-Q(\alpha^\vee)}[e_\alpha].$$

Proposition **11.3.** — *With the notations of 11.2, for n_α in N^1_α, for t in* T, *and for $\widetilde{t}$ a lifting of t in* E, *one has*

(**11.3.1**) $$\mathrm{int}[n_\alpha](\widetilde{t}) = \widetilde{t}.s[n_\alpha](\alpha^\vee(\alpha(t)^{-1})).$$

Proof. — One has

$$\mathrm{int}[n_\alpha](\widetilde{t}) = \widetilde{n}_\alpha\widetilde{t}\widetilde{n}_\alpha^{-1} = \widetilde{t}.((\widetilde{t}^{-1}\widetilde{n}_\alpha\widetilde{t})\widetilde{n}_\alpha^{-1}) = \widetilde{t}(\mathrm{int}[t^{-1}](\widetilde{n}_\alpha).\widetilde{n}_\alpha^{-1}).$$

As $\mathrm{int}[t]$ respects N^1_α and the canonical section of E over N^1_α, the second factor is of the form $s[n_\alpha](u)$, for some u in T_α. The conjugation by n_α acts on T as the Weyl group reflection s_α, and $s_\alpha(t) = t.\alpha^\vee(\alpha(t)^{-1})$. This gives $u = \alpha^\vee(\alpha(t)^{-1})$ and (11.3.1).

We will not need the next proposition, which could be used to give a complete description of the restriction of E to the normalizer N(T) of T. We choose a simple root system I. For simple roots α and β, let $m_{\alpha\beta}$ be the order of $s_\alpha s_\beta$. We write $\mathrm{prod}(m; x, y)$ for a product $xyxy...$ with m factors.

Proposition **11.4.** — *Let α and β be distinct simple roots. For n_α in N^1_α and n_β in N^1_β, one has in* E *the braid group relation*

(**11.4.1**) $$\mathrm{prod}(m_{\alpha\beta}; \widetilde{n}_\alpha, \widetilde{n}_\beta) = \mathrm{prod}(m_{\alpha\beta}; \widetilde{n}_\beta, \widetilde{n}_\alpha).$$

Proof. — The proof is inspired by Tits (1966) 2.4. Define β' to be β when $m_{\alpha\beta}$ is even, α when $m_{\alpha\beta}$ is odd, and let q and q' be, respectively, the left and right sides of (11.4.1). One has

$$q'q^{-1} = \widetilde{n}_\beta q \widetilde{n}_{\beta'}^{-1} q^{-1}.$$

By loc. cit, q and q' have the same image in G, i.e. $n_\beta = q n_{\beta'}^{-1} q^{-1}$. As the system of sections $\widetilde{n}$ of E over the N^1_α is respected by passage to the inverse and by conjugation by N(T), one has $\widetilde{n}_\beta = q\widetilde{n}_{\beta'}^{-1}q^{-1}$ and $q = q'$.

Lemma **11.5.** — *With the notation of 11.2 and 4.1,* Y *admits the following presentation. Generators: the coroots. Relations:*

(**11.5.1**) $$s_\alpha(\beta)^\vee = \beta^\vee - \alpha(\beta^\vee)\alpha^\vee.$$

Proof. — The coroots form a root system Φ^*, the *dual* of Φ. By Bourbaki Lie VI §1 no. 6 corollary 2 to proposition 9, applied to Φ^*, all relations among the coroots follow from those of the form $\alpha^\vee + \beta^\vee = 0$ or $\alpha^\vee + \beta^\vee + \gamma^\vee = 0$. The first, i.e. the relations $(-\alpha)^\vee = -\alpha^\vee$, are the relations (11.5.1) for $\beta = \alpha$. If $\alpha^\vee + \beta^\vee + \gamma^\vee = 0$, then $\alpha^\vee$, $\beta^\vee$ and $\gamma^\vee$ belong to a subroot system of type A_2, B_2 or G_2, and we leave it to the reader to check that in the roots systems A_2, B_2, and G_2, any relation among coroots is, up to permutation of $\alpha^\vee$, $\beta^\vee$ and $\gamma^\vee$, of the form $\alpha^\vee + \beta^\vee + (-s_\alpha(\beta))^\vee = 0$, with $\alpha(\beta^\vee) = -1$.

11.6. With notation as in 11.2, the central extension $\mathscr{E}$ of Y by k^* is deduced from an extension of $T(k((\tau)))$ by $K_2(k((\tau)))$.

In S_α, fix e_α, $e_{-\alpha}$ and n_α in Tits trijection. For any root β and for e_β in U_β^*, one has by 11.3

$$\begin{aligned}[][\mathrm{int}(n_\alpha)(e_\beta)] &= \mathrm{int}(n_\alpha)([e_\beta]) \\ &= [e_\beta].(\text{projection in } \mathscr{E} \text{ of } s[n_\alpha](\alpha^\vee(\tau^{-\alpha(\beta^\vee)}))).\end{aligned}$$

Applying (11.1.4), (11.1.5), one obtains

$$s[n_\alpha](\alpha^\vee(\tau^N)) = s[n_\alpha](\alpha^\vee(\tau))^N.\{\tau, \tau\}^{\varepsilon(N)Q(\alpha^\vee)},$$

with $\varepsilon(N) = N(N+1)/2$. As $\{\tau, \tau\} = \{\tau, -1\}$, with residue -1 in k^*, we get

(**11.6.1**) $$[\mathrm{int}(n_\alpha)(e_\beta)] = [e_\beta][e_\alpha]^{-\alpha(\beta^\vee)}.(-1)^{\varepsilon(-\alpha(\beta^\vee)).Q(\alpha^\vee)}.$$

For $\alpha = \beta$ and $e_\alpha = e_\beta$, this gives back (11.1.10).

Proposition **11.7.** — *Under the assumption of 11.2, the central extension* $\mathscr{E}$ *of* Y *by* k^* *defined by* Q *(4.16) is characterized up to unique isomorphism as being a central extension of* Y *by* k^**, provided for each root* α *with a map* [ ] *from* U_α^* *to the inverse image of* $\alpha^\vee$*, obeying (3.11.1) (11.2.1) and (11.6.1).*

The unicity claim results from 11.5.

This description extends by Galois descent to the non split case.

11.8. Let G be a split reductive group, with split maximal torus T. We will use the notations of 6.1. Let E be a central extension of G by $\mathbf{K}_2$, and E_{sc} its pullback to G_{sc}. We write q for the projection from G_{sc} to G, and from E_{sc} to E.

Proposition **11.9.** — *With the notations of 11.8, for* n_α *in* $N^1_\alpha \subset G_{sc}$ *and for* t *in* T, *lifted to* $\tilde{t}$ *in* E, *one has*

$$\text{int}[q(n_\alpha)](\tilde{t}) = \tilde{t}\, q(s[n_\alpha](\alpha^\vee(\alpha(t)^{-1}))). \quad (\mathbf{11.9.1})$$

Proof. — For g in G, with image g_{ad} in G_{ad}, and h in G_{sc}, the commutator (g, qh) in E is the image by q of the commutator (g^{ad}, h) of 0.N.4 (3). Indeed, the difference between the two commutators is given by a section of $\mathbf{K}_2$ on $G \times G_{sc}$, vanishing for $h = e$, and one applies (4.8.3) to a: $G \times G_{sc} \to G$.

One can then repeat the proof of 11.3.

11.10. The group G acts on E by inner automorphisms. The induced action of the normalizer N(T) of T respects T, hence acts on the central extension $\mathcal{E}$ of Y by k^* attached to E. Here is how to describe the action of N(T) on $\mathcal{E}$ in terms of the triple $(Q, \mathcal{E}, \varphi)$ attached to E, and of the description 11.7 of the central extension $\mathcal{E}_{sc}$ of Y_{sc} by k^*.

Proposition **11.11.** — *With the notation of 11.10, let* B *be the bilinear form associated to* Q.

(i) *For* y *in* Y, *the linear form* $B(y, \ )$ *on* Y *can be identified with a character of* T. *For* t *in* T, *and* $\tilde{y}$ *in* E *lifting* y, *the inner automorphism* $\text{int}[t]$ *acts on* $\mathcal{E}$ *by*

$$\text{int}[t](\tilde{y}) = \tilde{y}.B(y, t)^{-1}. \quad (\mathbf{11.11.1})$$

(ii) *For* e_α, $e_{-\alpha}$ *and* n_α *in Tits trijection,*

$$\text{int}[n_\alpha](\tilde{y}) = \tilde{y}.\varphi([e_\alpha])^{-\alpha(y)}.(-1)^{\varepsilon(-\alpha(y))Q(\alpha^\vee)}. \quad (\mathbf{11.11.2})$$

Proof. — The formula (11.11.1) follows from the commutator formula 3.14 for $(t, y(\tau))$. The formula (11.11.2) follows from (11.9.1) applied to any lifting $\tilde{t}$ of $y(\tau)$. Cf. the proof of (11.6.1).

11.12. As N(T) is generated by T and the images of the N^1_α, 11.11 determines the action of N(T) on $\mathcal{E}$. If T′ is another split maximal torus of G, 6.2 gives an equivalence from the category of central extensions of G by $\mathbf{K}_2$ with the category of triples $(Q, \mathcal{E}, \varphi)$ for T, or for T′. The resulting equivalence

$$\text{triples } (Q, \mathcal{E}, \varphi) \text{ for } T \to \text{triples } (Q', \mathcal{E}', \varphi') \text{ for } T'$$

can be constructed directly as follows. Let P be the set of g in $G(k)$ conjugating T to T′. It is a $N(T)(k)$-torsor. Any g in P transforms Q into Q′. The group $N(T)(k)$ acts on

$\mathscr{E}$ by 11.11, and $\mathscr{E}'$ is deduced from $\mathscr{E}$ by twisting by P. The same rule applies to $\mathscr{E}_{sc}$, and φ' is obtained by twisting as well.

This can be repeated over a base S regular of finite type over a field, and localized for the étale topology. It shows how the description 7.2 of central extensions of G by $\mathbf{K}_2$ by triples (Q, $\mathscr{E}$, φ) depends of the chosen maximal torus T.

12. Examples

12.1 SL(2): Let G be a semi-simple simply-connected algebraic group over a field k, and let E be a multiplicative $\mathbf{K}_2$-torsor on G. Arguing as in the proof of 4.9, one finds the following description for the central extension E(k) of G(k) by $\mathbf{K}_2(k)$: it is the group of automorphisms of the pair (space G, $\mathbf{K}_2$-torsor E on G), acting on G as a left translation.

Take G = SL(2). Let U be the unipotent subgroup of matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. The quotient SL(2)/U is identified with the punctured affine plane $A^2 - \{0\}$ by $g \mapsto g.(1, 0)$. The projection p: SL(2) $\to$ SL(2)/U = $A^2 - \{0\}$ is a U-torsor. The pull-back functor p^* is hence an equivalence of categories from $\mathbf{K}_2$-torsors on $A^2 - \{0\}$ to $\mathbf{K}_2$-torsors on SL(2). Applying 4.7, we conclude:

Proposition **12.2.** *Central extensions* E *of* SL(2) *by* $\mathbf{K}_2$ *correspond one to one to isomorphism classes of* $\mathbf{K}_2$*-torsors on* $A^2 - \{0\}$. *For the central extension* E *corresponding to* P, E(k) *is the group of g in* SL(2, k) *given with a lifting of their action on* $A^2 - \{0\}$ *to an action on* ($A^2 - \{0\}$, P).

The punctured affine plane $A^2 - \{0\}$ is covered by the two open sets $x \neq 0$ and $y \neq 0$. The group $H^1(A^2 - \{0\}, \mathbf{K}_2)$ of isomorphism classes of $\mathbf{K}_2$-torsors on $A^2 - \{0\}$ is $\mathbf{Z}$, generated by the Čech cocycle $\{x, y\}$ on the intersection $x \neq 0, y \neq 0$.

12.3. Let G be a reductive group over k, with maximal torus T, and ρ: G $\to$ SL(V) be a unimodular representation of G. Let E_0 be the central extension of SL(V) by $\mathbf{K}_2$ corresponding to the quadratic form Q_0 taking the value 1 on each coroot. By pullback to G, it gives a central extension E_V of G by $\mathbf{K}_2$. To compute the quadratic form Q on Y defined by E_V, we may first extend the scalars to a splitting field for T: we may and shall assume that T is split.

In characteristic zero, one can identify Y with a subgroup of the Lie algebra $\mathfrak{t}$ of T, by attaching to y: $\mathbf{G}_m \to$ T the element $dy(1)$ of $\mathfrak{t}$. The quadratic form Q_0 is then induced by the quadratic form $\frac{1}{2}\mathrm{Tr}(x^2)$ on the Lie algebra $\mathfrak{sl}$(V), and the quadratic form Q is hence induced by the quadratic form $\frac{1}{2}\mathrm{Tr}(\rho(x)^2)$ on the Lie algebra of G.

In any characteristic, similar arguments show that

$$Q(y) = \frac{1}{2} \sum \mu(y)^2,$$

where the sum runs over the weights μ of V, counted with their multiplicity.

Suppose G split simple and simply connected, fix a simple root system, let α be the largest root and let $\alpha^\vee = \sum n_i \alpha_i^\vee$ be the decomposition of the coroot $\alpha^\vee$ into simple coroots. When V is the adjoint representation, $Q(\alpha^\vee)$ is twice the dual Coxeter number $h^\vee = 1 + \sum n_i$ of G. One can case by case check that for any V, each n_i divides $Q(\alpha^\vee)$ and that for V nontrivial of the smallest possible dimension, $Q(\alpha^\vee)$ is the l.c.m. of the n_i.

12.4 PGL(n): Let T be the image in PGL(n) of the torus T_1 of diagonal matrices in GL(n). The cocharacter group Y of T is the quotient of $Y_1 = \mathbf{Z}^n$ by the diagonal $\mathbf{Z}$. The Weyl group invariant integer-valued quadratic forms Q on Y are the Weyl group invariant integer-valued quadratic forms on $\mathbf{Z}^n$, which vanish on the diagonal $\mathbf{Z}$. For n even, the minimal positive Q is

$$Q_1(y) := n \sum y_i^2 - \left(\sum y_i\right)^2.$$

For n odd, it is $1/2\, Q_1$. One has $Q_1(\alpha^\vee) = 2n$. The dual Coxeter number is n, and Q_1 is obtained as in 12.3, for the adjoint representation.

12.5 GL(n)**.** Let T be the split maximal torus of diagonal matrices in GL(n). On $Y = \mathbf{Z}^n$, take the quadratic form

$$Q(y) = \frac{1}{2}\left(\sum y_i^2 - \left(\sum y_i\right)^2\right).$$

The construction of the §7 of Deligne (1996) agrees with those of §11. Applying loc. cit. 7.10, one finds that for G an inner form of GL(n), to give a central extension of G by $\mathbf{K}_2$ corresponding to the form Q amounts to giving a representation V of G, which is a form of the defining representation of GL(n). Such a representation exists only if G is isomorphic to GL(n).

12.6. Over $\mathbf{R}$, any torus is a product of tori T of the following three types: $\mathbf{G}_m$, U^1 and $\prod_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$, with groups of points $\mathbf{R}^*$, the circle group and $\mathbf{C}^*$. The corresponding Y are $\mathbf{Z}$ with the trivial action of Gal($\mathbf{C}/\mathbf{R}$), $\mathbf{Z}$ with the sign action: $F_\infty(n) = -n$ and $\mathbf{Z} \oplus \mathbf{Z}$, with F_∞ interchanging $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For E a central extension of T by $\mathbf{K}_2$, we will determine the resulting topological central extensions of the groups of real points by $\mu_{\mathbf{R}} = \{\pm 1\}$ (10.3).

In the split case: $T = \mathbf{G}_m$, isomorphism classes of central extensions by $\mathbf{K}_2$ correspond to quadratic forms Q on $\mathbf{Z}$. For $Q(n) = qn^2$, the central extension is given by the cocycle $(x, y)^q$, with $(x, y) = -1$ if $x, y < 0$ and $(x, y) = 1$ otherwise. It is trivial for q even, and for q odd, liftings of $-1 \in \mathbf{R}^*$ have order 4.

For $T = U^1$ or $\prod_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$, $T(\mathbf{R})$ is connected and central extensions by $\mu_{\mathbf{R}}$ have no nontrivial automorphism. The group $\mathrm{Hom}(\pi_1, \mu_{\mathbf{R}})$ of their isomorphism classes is $\mathbf{Z}/2$, and the question is to determine which central extensions E of T by $\mathbf{K}_2$ give the trivial (resp. nontrivial) central extension $E(\mathbf{R})$ of $T(\mathbf{R})$ by $\mu_{\mathbf{R}}$. The inclusion of U^1 in $\mathbf{C}^*$ induces an isomorphism on π_1, and this reduces the case of $\prod_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$ to that of U^1.

Central extensions of T by $\mathbf{K}_2$ are classified by pairs $(Q, \mathscr{E})$ (7.2) and, for $T = U^1$, the group of isomorphism classes of pairs $(Q, \mathscr{E})$ is $\mathbf{Z} \times \mathbf{Z}/2$. The first factor gives Q: $q \mapsto$ the form qn^2. The second factor gives $\mathscr{E}$: the central extension $\mathscr{E}$ is commutative, and equivariant extensions of $\mathbf{Z}$, with the sign action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$, by $\mathbf{C}^*$ are classified by

$$H^1(\mathbf{R}, \mathbf{C}^* \text{ with } F_\infty \text{ acting by } z \longmapsto \bar{z}^{-1}) = \mathbf{Z}/2.$$

If u is a lifting in $\mathscr{E}$ of 1, $\bar{u}$ is a lifting of -1, $u\bar{u}$ is in $\mathbf{R}^* \subset \mathbf{C}^*$ and the extension $\mathscr{E}$ is trivial if and only if $u\bar{u} > 0$. We have to compute the homomorphism

$$(\mathbf{12.6.1}) \qquad \mathbf{Z} \times \mathbf{Z}/2 \to \mathrm{Hom}(\pi_1(U_1), \mu_R) = \mathbf{Z}/2 : (Q, \mathscr{E}) \mapsto \text{ extension } E(\mathbf{R}).$$

Let us embed U^1 in SL(2) (resp. SU(2)), and let E_0 be the central extension of SL(2) (resp. SU(2)) by $\mathbf{K}_2$ corresponding to the quadratic form Q_0 taking the value 1 on coroots. For the induced central extension E of U^1, $q = 1$.

Case of SL(2): the action of $SL(2, \mathbf{R})$ on $\mathbf{R}^2 - \{0\}$ lifts to an action of $E_0(\mathbf{R})$ on the double covering of $\mathbf{R}^2 - \{0\}$ (cf. 12.1), and $E(\mathbf{R})$ is the nontrivial double covering of U^1. If e_0 is an eigenvector of U^1 acting on $\mathbf{R}^2 \otimes \mathbf{C}$, e_0 and $\bar{e}_0$ form a basis of $\mathbf{R}^2 \otimes \mathbf{C}$. Take (e^+, e^-, n) to be the triple (11.1.1), in this basis. One has $e^- = -\bar{e}^+$, where $-\bar{e}^+$ refers to the vector space structure (11.1.6). For $u = [e^+]$, it follows from 11.1.9 and 11.1.10 that $u\bar{u} = 1$: the extension $\mathscr{E}$ is trivial.

Case of SU(2): As the topological group SU(2) is simply-connected, the central extension $E_0(\mathbf{R})$ is trivial, and so is the central extension $E(\mathbf{R})$ of U^1. It is not isomorphic to the central extension of U^1 obtained by embedding U^1 into $SL(2, \mathbf{R})$. The same must hold for the extension $\mathscr{E}$, which gives rise to it: the extension $\mathscr{E}$ must be nontrivial.

From those two examples, it follows that

Proposition **12.7.** — *The morphism (12.6.1) is*

$$(q, n) \longmapsto q + n.$$

12.8. Let K be a local field, by which we mean: the field of fractions of a complete discrete valuation ring. Let V be the valuation ring, and k be the residue field. Let E be a central extension by $\mathbf{K}_2$ of an algebraic group G over K. Pushing the central extension E(K) by the residue map $K_2(K) \to k^*$, we obtain a central extension of G(K) by k^*, which we denote by $\widetilde{G}(k)$:

$$k^* \to \widetilde{G}(k) \to G(K). \tag{12.8.1}$$

Let K′ be an unramified Galois extension of K, with valuation ring V′ and residue field k'. The field extension k'/k is Galois, with $\mathrm{Gal}(K'/K) \xrightarrow{\sim} \mathrm{Gal}(k'/k)$. Let G′, E′ be deduced from G, E by extension of scalars and write $\widetilde{G}(k')$ for the resulting central extension (12.8.1). The determination of $\widetilde{G}(k)$ is reduced to that of $\widetilde{G}(k')$ by the

Proposition **12.9.** — *One has* $\widetilde{G}(k) \xrightarrow{\sim} \widetilde{G}'(k')^{\mathrm{Gal}(K'/K)}$.

Proof. — As k and G(K) are the groups of invariants of Gal(K′/K) acting on k' and G(K′), respectively, this follows from the commutativity of the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \widetilde{G}(k) & \longrightarrow & G(K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & k'^* & \longrightarrow & \widetilde{G}(k') & \longrightarrow & G(K') & \longrightarrow & 1. \end{array}$$

12.10. Suppose that $K = k((t))$, and that G is absolutely simple and simply-connected, and splits over some unramified extension $k'((t))$ of K. In this situation, Garland (1980) constructed a central extension of G(K) by k^*. It agrees with the central extension 12.8.1, for E defined by the quadratic form Q taking the value 1 on small coroots. Indeed, both obey 12.9, and for G split, Garland's extension is derived from Matsumoto's central extension by pushing by $K_2(K) \to k^*$.

Assume now that G is the general fiber of a group scheme G_V smooth over V, with special fiber G_s.

12.11. *Construction.* — *From the central extension* E *of* G *by* $\mathbf{K}_2$ *we will construct a central extension* $\widetilde{G}_s$ *of* G_s *by* $\mathbf{G}_m$ *such that the central extension* $G_V(V)^\sim$ *restriction of (12.8.1) to* $G_V(V)$ *is the pull back of* $\widetilde{G}_s(k)$*: commutativity of the diagram*

$$\begin{array}{ccccc} k^* & \longrightarrow & G_V(V)^\sim & \longrightarrow & G_V(V) \\ \| & & \downarrow & & \downarrow \\ k^* & \longrightarrow & \widetilde{G}_s(k) & \longrightarrow & G_s(k). \end{array} \tag{12.11.1}$$

In this construction, we will treat central extensions as multiplicative torsors.

Lemma **12.12.** — *Let* X_V *be a smooth scheme over* V, *with general and special fibers* $j\colon X_\eta \hookrightarrow X_V$ *and* $i\colon X_s \hookrightarrow X_V$. *Let* E *be a* $\mathbf{K}_2$*-torsor on* X_η, *for which*

(∗) *each point of* X_s *has a neighborhood* U *in* X *such that* E *admits a section over* $U_\eta = U \cap X_\eta$.

To E, *we will attach a* $\mathbf{G}_m$*-torsor* $\overline{E}$ *over* X_s, *by a construction compatible with a pull back* $Y_V \to X_V$, *with the addition of torsors, and which for* $X_V = V$ *is simply pushing a* $K_2(K)$*-torsor by* Res: $K_2(K) \to K_1(k) = k^*$.

Proof. — The assumption (∗) amounts to saying that j_*E is a $j_*\mathbf{K}_2$-torsor. The residue map in K-theory: $K_2(U \cap X_\eta) \to K_1(U \cap X_s)$ for $U \subset X$, induce a morphism of sheaves $j_*\mathbf{K}_2 \to i_*\mathbf{K}_1 = i_*\mathcal{O}^*$. Pushing $j_*\mathbf{K}_2$ by this morphism, we obtain a $i_*\mathcal{O}^*$-torsor on X or, what amounts to the same, a $\mathcal{O}^*$-torsor on X_s, i.e. a $\mathbf{G}_m$-torsor over X_s. We leave the compatibilities to the reader.

Proof of 12.11. If condition (∗) of 10.12 holds for E, the multiplicative structure of E induces one on $\overline{E}$, which is the required multiplicative $\mathbf{G}_m$-torsor. Its formation is compatible with an étale extension V′/V and, by Galois descent, it suffices to show that (∗) holds after such an "extension of the residue field" in V.

The $\mathbf{K}_2$-torsor E is trivial over some Zariski dense open subset U of G. Define $U_V := G_V - (G-U)^-$. The closure $(G-U)^-$ is flat, of relative dimension strictly smaller than that of G, and it follows that the special fiber U_s of U_V is dense in G_s. The property (∗) holds at each point of U_s: it holds for U_V. By the multiplicativity of E. It also holds at the points of any translate of U_s by an element g of $G_s(k)$. Indeed, V being henselian, g can be lifted to $\tilde{g}$ in $G_V(V)$ and (∗) holds for $\tilde{g}U_V$. After a finite extension of k, such translates cover G_s.

Questions **12.13.** — (i) Suppose that G is reductive, and that E is given as in 7.2, for T a maximally split maximal torus of G. Suppose that G_V is given as in Bruhat Tits (1984) 4.6. It would be interesting to compute the central extension $\widetilde{G_s}$ in that case, especially for $G_V(V)$ a maximal bounded subgroup of G(K), given by a vertex of the building of G.

(ii) Suppose that $V = k[[t]]$ and that G is a split reductive group. How to relate the point of view where central extensions of G(K) by k^* are viewed as infinite dimensional groups of k, corresponding to affine Cartan matrices, with our point of view of central extensions by $\mathbf{K}_2$, classified by 6.2?

(iii) For $V = k[[t]]$, not all natural central extensions by k^* are captured by 12.8. For example, if G is the multiplicative group, $G(K) = K^*$ acts projectively on the semi-infinite exterior algebra of $k((t))$, relative to $k[[t]]$, giving rise to a central extension E of K^* by k^*. For x in K^*, the line

$$[xk[[t]] : k[[t]]] = \det(xk[[t]]/\Lambda)\det(k[[t]]/\Lambda)^{-1}$$

is independent of Λ, supposed to contain $t^N k[[t]]$ for N large enough and to be contained in $k[[t]]$ as well as in $xk[[t]]$. A lifting of x to E is the choice of a generator of $[xk[[t]]: k[[t]]]$. The commutator 0.N.4(2) defined by E is, up to a sign, the tame symbol. See E. Arbarello, C. De Concini and V. Kac, The infinite wedge representation and the reciprocity law for algebraic curves, in: Theta functions, *Proc. Symp. Pure Math*, **49** 1 AMS (1987), p. 171-190. The *square* of this commutator is the commutator for the central extension (12.8.1) defined by the tame symbol used as a cocycle. We expect that "natural" central extensions of G(K) by k^* are attached to data as follows: a Weyl group and Galois group invariant integer-valued symmetric bilinear form B on Y, even on Y_{sc}; a central extension $\mathscr{E}$ fo Y by $\mathbf{G}_m$, for which the commutator is

$$(y_1, y_2) = (-1)^{B(y_1, y_2) + \varepsilon(y_1)\varepsilon(y_2)}$$

for $\varepsilon(y) \equiv B(y, y)$ mod 2; φ: $\mathscr{E}_{sc} \to \mathscr{E}$ as in 6.2.

Remarks **12.14.** — (i) 12.11 holds for any discrete valuation ring V, complete or not: if $\hat{V}$ is the completion of V, and if $G_{\hat{V}}$ is deduced from G_V by base change, the central extension $\widetilde{G_s}$ attached by 12.11 to $G_{\hat{V}}$ gives rise to a diagram 12.11.1 for G_V.

(ii) The property (∗) of 12.12 holds, as shown by the proof of 12.11, as soon as V is henselian and $G_s(k)$ Zariski dense in G_s.

(iii) Suppose that V is henselian and essentially of finite type over a field. For j (resp. i) the inclusion of G (resp. G_s) in G_V, Quillen resolution gives a short exact sequence of sheaves on G_V

$$0 \to \mathbf{K}_2 \to j_*\mathbf{K}_2 \to i_*\mathbf{K}_1 \to 0,$$

and a trivialization of the central extension $\widetilde{G_s}$ of 12.11 defines an extension of E, as a multiplicative $\mathbf{K}_2$-torsor, over G_V.

12.15. Let X be a projective and smooth curve over a field k. Let K be its field of rational functions. For x a closed point of X, let V_x be the completion of the local ring of X at x, and K_x be its field of fractions, the completion of K at x.

For G a linear algebraic group over K, we write G(**A**) for the restricted product of the $G(K_x)$. It is also the group of **A**-points of G, for $\mathbf{A} = \coprod K_x$.

Let E be a central extension of G by $\mathbf{K}_2$. For some Zariski dense open subset U of X, G extends as an affine group scheme of finite type G_U over U, and E as a multiplicative $\mathbf{K}_2$-torsor over G_U. By extension of scalars to K_x, E defines central extensions (12.8.1) of $G(K_x)$ by $k(x)^*$, split over $G(V_x)$ for x in U. For $U_1 \subset U$ Zariski dense, we get commutative diagrams

$$\begin{array}{ccccccc}
H^0(U_1, \mathbf{K}_2) & \longrightarrow & G(U_1)^{\sim} & \longrightarrow & G(U_1) & \longrightarrow & H^1(U_1, \mathbf{K}_2) \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_{x \notin U_1} k(x)^* & \longrightarrow & \prod_{x \in U_1} G(V_x) \times \prod_{x \notin U_1} G(K_x)^{\sim} & \longrightarrow & \prod_{x \in U_1} G(V_x) \times \prod_{x \notin U_1} G(K_x). & &
\end{array}$$

with inductive limit

(12.15.1)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_2(K) & \longrightarrow & G(K)^{\sim} & \longrightarrow & G(K) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \oplus k(x)^* & \longrightarrow & G(\mathbf{A})^{\sim} & \longrightarrow & G(\mathbf{A}) & \longrightarrow & 1.
\end{array}$$

By the reciprocity law for tame symbols, the first vertical arrow has its image in the kernel of the product of the norm maps N: $\oplus k(x)^* \to k^*$. Pushing by N, we hence get from (12.15.1) a central extension of $G(\mathbf{A})$ by k^*, canonically split over G(K):

(12.15.2)
$$\begin{array}{ccccc}
 & & & & G(K) \\
 & & & \swarrow & \downarrow \\
k^* & \longrightarrow & G(\mathbf{A})^{\sim} & \longrightarrow & G(\mathbf{A}),
\end{array}$$

the geometric analogue of 10.4.3. For G simply-connected, it is also an analogue, in algebraic geometry, of the analytic constructions of Segal (1988) and Brylinski-McLaughlin (1994).

As in 12.13(iii), we don't obtain all natural diagrams (12.15.2) by this method.

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