Central extensions of smooth 2–groups and a finite-dimensional string 2–group

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We provide a model of the String group as a central extension of finite-dimensional 2–groups in the bicategory of Lie groupoids, left-principal bibundles, and bibundle maps. This bicategory is a geometric incarnation of the bicategory of smooth stacks and generalizes the more naive 2–category of Lie groupoids, smooth functors and smooth natural transformations. In particular this notion of smooth 2–group subsumes the notion of Lie 2–group introduced by Baez and Lauda [5]. More precisely we classify a large family of these central extensions in terms of the topological group cohomology introduced by Segal [56], and our String 2–group is a special case of such extensions. There is a nerve construction which can be applied to these 2–groups to obtain a simplicial manifold, allowing comparison with the model of Henriques [23]. The geometric realization is an A_∞–space, and in the case of our model, has the correct homotopy type of $\mathrm{String}(n)$. Unlike all previous models [58; 60; 33; 23; 7] our construction takes place entirely within the framework of finite-dimensional manifolds and Lie groupoids. Moreover within this context our model is characterized by a strong uniqueness result. It is a canonical central extension of $\mathrm{Spin}(n)$.

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1 Introduction

The String group is a group (or A_∞–space) which is a 3–connected cover of $\mathrm{Spin}(n)$. It has connections to string theory, the generalized cohomology theory *topological modular forms* (*tmf*), and to the geometry and topology of loop space. Many of these relationships can be explored homotopy theoretically, but a geometric model of the String group would help provide a better understanding of these subjects and their interconnections. Over the past decade there have been several attempts to provide geometric models of the String group; see Stolz [58], Stolz and Teichner [60], Jurco [33], Henriques [23] and Baez, Stevenson, Crans and Schreiber [7]. The most recent of these use the language of higher categories, and consequently string differential geometry also provides a test case for the emerging field of higher categorical differential geometry; see Waldorf [64] and Sati, Schreiber and Stasheff [51; 52].

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Nevertheless, progress towards the hard differential geometry questions, such as a geometric understanding of the connection to elliptic cohomology or the Höhn–Stolz Conjecture [58], remains slow. Perhaps one reason is that all previous models of the string group, including the higher categorical ones, are fundamentally infinite-dimensional. In a certain sense, which will be made more precise below, it is impossible to find a finite-dimensional model of String(n) as a *group*. However, there remains the possibility that String(n) can be modeled as a finite-dimensional, but higher categorical object, namely as a finite-dimensional 2–group. This idea is not new, and models for the string group as a Lie 2–group have been given by Henriques [23] and Baez et al [7]. However, these models are also infinite-dimensional.

In this paper we consider 2–groups in the bicategory of finite-dimensional Lie groupoids, left principal bibundles and bibundle maps. This bicategory, which is equivalent to the bicategory of smooth stacks, is an enhancement of the usual bicategory of Lie groupoids, smooth functors and smooth natural transformations. We call such 2–groups *smooth 2–groups*. We classify a large family of central extensions of smooth 2–groups in terms of easily computed cohomological data. Our model of the string group comes from such a finite-dimensional central extension. We begin this paper with a more detailed look at the string group and the ideas needed for constructing our model. The main ingredients are, of course, the above mentioned bicategory and also a certain notion of topological group cohomology introduced by Graeme Segal in the late 60s.

What is the String group?

The String group is best understood in relation to the Whitehead tower of the orthogonal group $O(n)$. The Whitehead tower of a space X consists of a sequence of spaces $X\langle n+1\rangle \to X\langle n\rangle \to \cdots \to X$, which generalize the notion of universal cover. A (homotopy theorist's) universal cover of a connected space X is a space $X\langle 2\rangle$ with a map to X, which induces an isomorphism on all homotopy groups except π_1, and such that $\pi_1(X\langle 2\rangle) = 0$. For more highly connected spaces, there is an obvious generalization, and the Whitehead tower assembles these together. The maps $X\langle n+1\rangle \to X\langle n\rangle \to \cdots \to X$ induce isomorphisms $\pi_i(X\langle n\rangle) \cong \pi_i(X)$ for $i \geq n$, and each space satisfies $\pi_i(X\langle n\rangle) = 0$ for $i < n$.

For large n, the orthogonal group $O(n)$ has the following homotopy groups:

i	0	1	2	3	4	5	6	7
$\pi_i(O(n))$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The first few spaces in the Whitehead tower of $O(n)$ are the familiar Lie groups SO(n) and Spin(n). These are close cousins to the String group, String(n). The maps in the

Whitehead tower are realized by Lie group homomorphisms

$$\mathrm{SO}(n) \hookrightarrow O(n)$$
$$\mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n).$$

This raises the question: can we realize the remaining spaces in the Whitehead tower of $O(n)$ as Lie groups? or as topological groups? The next in the sequence would be $O(n)\langle 4\rangle = \cdots = O(n)\langle 7\rangle$, a space which now goes under the name $\mathrm{String}(n)$. It is the 3–connected[1] cover of $\mathrm{Spin}(n)$. Since π_1 and π_3 of this space are zero, it cannot be realized by a finite-dimensional Lie group[2]. Moreover, since this hypothetical group is characterized homotopy-theoretically, it is not surprising that there are many models for this group.

The easiest candidates arise from the machinery of homotopy theory. If we relax our assumption that $\mathrm{String}(n)$ be a topological *group* and allow it to be an A_∞–space[3] then there is an obvious model. First we look at the classifying space $BO(n)$. We can mimic our discussion above and construct the Whitehead tower of $BO(n)$. The homotopy groups of $BO(n)$ are the same as those of $O(n)$, but shifted:

i	0	1	2	3	4	5	6	7	8
$\pi_i(BO)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

It is well known that the pointed loop space of a classifying space satisfies $\Omega(BG) \simeq G$ for topological groups G. It then follows that the space $\Omega(BO(n)\langle 8\rangle)$ is an A_∞–space with the right homotopy type. With more care, the homomorphism to $\mathrm{Spin}(n)$ can also be constructed[4].

If one insists on getting an actual *group*, then more sophisticated but similar homotopy theoretic techniques succeed. One replaces the space $BO(n)\langle 8\rangle$ with its singular simplicial set and applies Kan's simplicial loop group (see for example Goerss and Jardine [21, Chapter 5.5]). This produces a simplicial group, which models $\Omega(BO(n)\langle 8\rangle)$. Taking the geometric realization gives an honest topological group with the correct homotopy type. Needless to say, this construction is not very geometric.

[1] For $n \geq 7$ $\mathrm{String}(n)$ is 6–connected.

[2] This often cited fact follows from two classical results: A theorem of Malcev which states that any connected Lie group deformation retracts onto a compact subgroup [41; 42] (see also Iwasawa [31, Theorem 6]), and the classification of finite-dimensional compact simply connected Lie groups, which may be found in many standard text books on Lie groups.

[3] Since $\mathrm{String}(n)$ is connected, having an A_∞–space structure is equivalent to having the homotopy type of a loop space.

[4] For Lie groups G, the map $\Omega BG \to G$ may be constructed as the holonomy map of the universal connection on EG over BG. Thus the composite $\Omega(B\mathrm{Spin}(n)\langle 8\rangle) \to \Omega B\mathrm{Spin}(n) \to \mathrm{Spin}(n)$ is one way to construct the desired map.

In both of these approaches the homomorphism $\mathrm{String}(n) \to \mathrm{Spin}(n)$ realizes the String group as a fiber bundle whose fiber is an Eilenberg–Mac Lane space $K(\mathbb{Z},2)$. This is a general feature of all approaches. Suppose that we are given a model of $\mathrm{String}(n)$ as a topological group equipped with a continuous homomorphism to $\mathrm{Spin}(n)$, realizing it as the 3–connected cover. Let K be the kernel of this map and suppose that this forms a fiber bundle

$$K \to \mathrm{String}(n) \to \mathrm{Spin}(n).$$

By the long exact sequence of homotopy groups associated to this bundle, we have $K \simeq K(\mathbb{Z},2)$.

The primary method of building models of the String group is consequently finding group extensions where the kernel is topologically an Eilenberg–Mac Lane $K(\mathbb{Z},2)$–space. The first geometric models, constructed by Stolz and Teichner [58; 60], were of this kind. Any CW–complex with the homotopy type of a $K(\mathbb{Z},2)$ must have cells of arbitrarily high dimension, and is thus infinite-dimensional[5]. Although the groups K used in these models were not CW–complexes, they too were infinite-dimensional and hence resulted in infinite-dimensional models of the String group.

While $K(\mathbb{Z},2)$ is infinite-dimensional, it still has a well known finite-dimensional description, but at the cost of working higher categorically (or equivalently through the language of S^1–gerbes; see Giraud [20], Murray [45] and Behrend and Xu [10]). This suggests that there might be a finite-dimensional model of $\mathrm{String}(n)$, but as a higher categorical object. This idea is not new and goes back to the work of Baez and Lauda [5], Henriques [23] and Baez et al [7]. The latter were able to construct a Lie 2–group modeling $\mathrm{String}(n)$ in a precise sense, but their model is also infinite-dimensional.

Baez and Lauda [5] considered (weak) group objects in the bicategory LieGpd of Lie groupoids[6], smooth functors and smooth natural transformations. These objects are now commonly called Lie 2–groups, and the finite-dimensional incarnation of $K(\mathbb{Z},2)$ in this context is the Lie 2–group we call $[\mathrm{pt}/S^1]$. The Lie group $\mathrm{Spin}(n)$ also provides a basic example of a Lie 2–group. We will elaborate on this in due course.

Baez and Lauda [5] considered certain "extensions" $[\mathrm{pt}/S^1] \to E \to \mathrm{Spin}(n)$, and under certain restrictive assumptions (which can be removed), they proved that such central extensions are in bijection with smooth group cohomology $H^3_{\mathrm{grp}}(G;A)$. Herein lies the problem. Since the work of Hu [30; 29], van Est [17; 18] and Mostow [25] we have that for all compact 1–connected simple Lie groups G, $H^3_{\mathrm{grp}}(G;S^1)=0$. Thus in LieGpd,

[5] In fact an easy Serre spectral sequence argument shows that $\mathrm{String}(n)$ itself has cohomology in arbitrarily high degrees and hence has no finite-dimensional CW–model.

[6] A *Lie groupoid* is the common name for a groupoid object internal to the category of smooth manifolds, in which the source and target maps are surjective submersions.

the only such central extension is the trivial one. This is why Baez et al [7] were led to infinite-dimensional groups. Essentially, they replace $G = \mathrm{Spin}(n)$ with an infinite-dimensional 2–group for which the above central extension exists. This Lie 2–group is not equivalent to $\mathrm{Spin}(n)$, but nevertheless its geometric realization is homotopy equivalent to $\mathrm{Spin}(n)$, and the resulting central extension does model $\mathrm{String}(n)$. The model of Henriques [23] uses different techniques but produces essentially the same object as [7], but cast in the language of simplicial spaces.

In this paper we work entirely within the context of finite-dimensional manifolds and Lie groupoids, never passing into the infinite-dimensional setting. As a result our model is fundamentally finite dimensional. The cost is that we must consider groups not in LieGpd, but in the bicategory Bibun of Lie groupoids, left-principal bibundles and bibundle maps. This bicategory is a natural generalization of LieGpd, in which the 1–morphisms have a simple geometric description. Hence, this notion of 2–group, which we call *smooth 2–group*, subsumes the notion previously introduced by Baez and Lauda. The bicategory Bibun has other familiar guises. It is equivalent to the bicategory of smooth stacks and also to the (derived) localization of LieGpd with respect to the *local equivalences*; see Pronk [47] and Lerman [37]. This later has a description in terms of "smooth anafunctors" by Bartels [9].

The structure and results of this paper

The bicategory Bibun, sadly, does not appear to be widely known, and so we provide a brief review of some key results about this bicategory that we will use. We then review the notion of weak group object (and also weak abelian group object) in a general bicategory. These are commonly called *2–groups*. More importantly we make precise the notion of extension and central extension of 2–groups, particularly in the context of the bicategory Bibun. This generalizes those central extensions

$$1 \to A \to E \to G \to 1$$

of topological groups in which the A–action on E realizes it as an A–principal bundle over G. We also show that the geometric realization of a group object in Bibun is naturally a (group-like) A_∞–space.

To make the notion of central extension precise in the context of Bibun, we must consider certain pullbacks. However, just like the category of smooth manifolds, Bibun does not admit all pullbacks. Nevertheless if two maps of smooth manifolds are transverse, then the fiber product exists. We extend this notion to Bibun, introducing *transversality for bibundles* in a way which generalizes the usual notion of transversality for smooth maps. We prove that for transverse bibundles the fiber product indeed exists. To our knowledge this is the first time such a result has appeared in the literature. We also introduce a notion of *surjective submersion* for bibundles, generalizing the usual notion. This permits us to make precise the central extensions of 2–groups we wish to consider.

Given an abelian 2–group $\mathbb{A}$ and a 2–group $\mathbb{G}$ (both in Bibun) there is a bicategory of central extensions of $\mathbb{G}$ by $\mathbb{A}$, $\mathrm{Ext}(\mathbb{G};\mathbb{A})$. This bicategory is contravariantly functorial in $\mathbb{G}$ and covariantly functorial in $\mathbb{A}$, and so the Baer sum equips $\mathrm{Ext}(\mathbb{G};\mathbb{A})$ with the structure of a symmetric monoidal bicategory (see Gordon, Power and Street [22], Kapranov and Voevodsky [35; 34], Baez and Neuchl [6], Day and Street [16] and especially the author's thesis [54, Chapter 3]). In this paper we prove the following theorem (Theorem 99):

Theorem 1 *Let G be a Lie group and A an abelian Lie group, viewed as a trivial G–module. Then we have an (unnatural) equivalence of symmetric monoidal bicategories*

$$\mathrm{Ext}(G;[\mathrm{pt}/A]) \simeq H^3_{\mathrm{SM}}(G;A) \times H^2_{\mathrm{SM}}(G;A)[1] \times H^1_{\mathrm{SM}}(G;A)[2],$$

where $H^i_{\mathrm{SM}}(G;A)$ denotes the smooth version Segal–Mitchison topological group cohomology [56]. Moreover, isomorphism classes of central extensions

$$1 \longrightarrow \begin{array}{c} A \\ \downdownarrows \\ \mathrm{pt} \end{array} \longrightarrow \begin{array}{c} \Gamma_1 \\ \downdownarrows \\ \Gamma_0 \end{array} \longrightarrow \begin{array}{c} G \\ \downdownarrows \\ G \end{array} \longrightarrow 1$$

are in natural bijection with $H^3_{\mathrm{SM}}(G;A)$.

In the above theorem an abelian group M is regarded as a symmetric monoidal bicategory in three ways. It can be viewed as a symmetric monoidal bicategory M with only identity 1–morphisms and 2–morphisms. It can be viewed as a symmetric monoidal bicategory $M[1]$ with one object, M many 1–morphisms and only identity 2–morphisms. Finally, it may be viewed as $M[2]$, a symmetric monoidal bicategory with one object, one 1–morphism, and M many 2–morphisms. Specializing to the case relevant to the String group we obtain the following Theorem:

Theorem 2 *If $n \geq 5$, $A = S^1$ and $G = \mathrm{Spin}(n)$, we have*

$$H^i_{\mathrm{SM}}(\mathrm{Spin}(n);S^1) \cong H^{i+1}(B\mathrm{Spin}(n);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 3, \\ 0 & i = 1,2. \end{cases}$$

Thus for each class $[\lambda] \in H^3_{\mathrm{SM}}(\mathrm{Spin}(n);S^1) \cong \mathbb{Z}$ the bicategory of central extensions with that class is contractible[7]*, hence such extensions are coherently unique. Moreover, the central extension corresponding to a generator of $H^3_{\mathrm{SM}}(\mathrm{Spin}(n);S^1)$ gives a finite-dimensional model for $\mathrm{String}(n)$.*

[7] equivalent to the terminal bicategory

The uniqueness in the above theorem is the strongest possible given the category number of the problem. It has the following interpretation. Given a class $[\lambda] \in H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1)$, there exists a central extension realizing that class. Any two such extensions are isomorphic, and moreover any two 1–morphisms realizing such an isomorphism are isomorphic by a unique 2–isomorphism.

Importance of the String group

The importance of the String group was first noticed in physics. It is well known that in order to define the 1–dimensional supersymmetric sigma model with target space a manifold X, one needs X to be a spin manifold. A similar problem for the 2–dimensional supersymmetric sigma model was studied by Killingback [36] and later by Witten [65]. They realized that the 2–dimensional supersymmetric sigma model in a space X requires a "spin structure on the free loop space LX". Witten's investigations eventually lead him to what is now called the Witten genus, which associates to an oriented manifold a formal power series whose coefficients are given by certain combinations of characteristic numbers. For string manifolds, this is the q–expansion of an integral modular form.

One way to understand spin structures on a manifold X is homotopy-theoretically. The stable normal bundle induces a classifying map $X \to BO$, and a homotopy-theoretic spin structure is a lift of this map to $B\mathrm{Spin}$. Classical obstruction theory arguments show such a lift exists only if the first and second Stiefel–Whitney classes vanish. If both w_1 and w_2 vanish, then there is a new characteristic class $p_1/2$, such that $2 \cdot (p_1/2) = p_1$ is the first Pontryagin class. A further lift to $BO\langle 8\rangle$ exists if and only if $p_1/2$ vanishes. Such a lift is the homotopy theoretic version of a string structure. A "spin structure on loop space" exists if the transgression of $p_1/2$ vanishes, and it satisfies a further locality property if $p_1/2$ itself vanishes; see Stolz and Teichner [61].

Standard techniques allow one to construct for each of the spaces $BO\langle n\rangle$ a corresponding bordism theory of $BO\langle n\rangle$–manifolds. These bordism theories gives rise to generalized cohomology theories, or more precisely E_∞–ring spectra, $MO\langle n\rangle$. The Witten genus is an $BO\langle 8\rangle$–bordism invariant, and thus gives rise to a map $MO\langle 8\rangle(\mathrm{pt}) \to MF$, where MF is the ring of integral modular forms.

The Witten genus has a refinement as a map of cohomology theories (see Ando, Hopkins and Strickland [2], Ando, Hopkins and Rezk [1] and Henriques [24]):

$$MO\langle 8\rangle \to \mathit{tmf}.$$

Here *tmf* is the theory constructed by Hopkins and Miller of *topological modular forms* [26; 27]. There is a map of graded rings $\mathit{tmf}^*(\mathrm{pt}) \to MF$, which factors the Witten genus. This map is rationally an isomorphism, but is not surjective or injective, integrally. The ring $tmf^*(\mathrm{pt})$ contains a significant amount of torsion. The refinement of the Witten genus is similar to the refinement of the $\widehat{A}$–genus, which can also be viewed as a map of cohomology theories,

$$M\mathrm{Spin} \to KO.$$

Here KO is real K–theory. These refinements have the following consequences. If $E \to X$ is a family of string manifolds parametrized by X, then there is a family Witten genus which lives in $tmf^*(X)$. Similarly a family of spin manifolds has a family version of the $\widehat{A}$–genus, which lives in $KO^*(X)$. While there are homotopy theoretic descriptions of both of these based on the Thom isomorphisms for string and spin vector bundles, respectively, the $\widehat{A}$–genus also has an analytic/geometric interpretation derived from the concrete geometric model of $\mathrm{Spin}(m)$.

Given a manifold with a geometric spin structure, we can form the associated bundle of spinors and the corresponding Clifford-linear Dirac operator. If we have a family of spin manifolds parametrized by a space X, we get a corresponding family of Fredholm operators. This represents the class in $KO^*(X)$. The Witten genus has no corresponding geometric definition[8], and nor does the cohomology theory tmf. A suitable geometric model for the String group will lead to a better *geometric* understanding of string structures and might provide insight into these problems.

Finally, we should mention an as yet unresolved conjecture due independently to Höhn and Stolz relating string structures and Riemannian geometry. Stolz conjectures in [58] that a $4k$–dimensional string manifold which admits a positive Ricci curvature metric necessarily has vanishing Witten genus. Some progress has been made towards this (and related conjectures [59]) in the dissertation of Redden [49], but a clear answer remains out of reach. A better geometric understanding of string structures would doubtless shed light on this problem as well.

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[8] Note however that Witten's original argument is based on the construction of a Spin–structure on the free loop space. His heuristic derivation was to take the S^1–equivariant index of the "Dirac operator on loop space".

2 Lie groupoids and smooth stacks

2.1 Lie groupoids

Definition 3 A *Lie groupoid* is a groupoid object, $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$, in the category of (finite-dimensional) smooth manifolds in which the source and target maps

$$s,t\colon \Gamma_1 \to \Gamma_0$$

are surjective submersions. (In particular the iterated fiber products $\Gamma_1 \times_{\Gamma_0} \Gamma_1$ and $\Gamma_1 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Gamma_1$ exist as smooth manifolds). *Functors* and *natural transformations* are defined as functors and natural transformations internal to the category of smooth manifolds.

Together Lie groupoids, functors and natural transformations form a 2–category, LieGpd. There are many examples of Lie groupoids. The most common are special cases of the following two kinds:

Example 4 (G–Spaces) Let G be a Lie group acting smoothly (say, on the right) on a manifold X. Then we can form the *action groupoid* $\Gamma = [X/G]$. The objects are $\Gamma_0 = X$ and morphisms are $\Gamma_1 = X \times G$. The target map is projection, and the source is the action map. Composition

$$m\colon (X \times G) \times_X (X \times G) = X \times G \times G \to X \times G$$

is given by multiplication in G. The identity map is $x \mapsto (x,e)$ and the inverse map is $(x,g) \mapsto (xg, g^{-1})$.

When the group is trivial, this allows any smooth manifold X to be viewed as a Lie groupoid with $X_0 = X_1 = X$ and all maps identity maps. When the manifold $X = \mathrm{pt}$ is trivial, this allows any Lie group G to be viewed as a Lie groupoid with $G_0 = \mathrm{pt}$ and $G_1 = G$. In this case the composition is the group multiplication, with the usual identities and inverses.

Example 5 (Čech groupoids) If $Y \to X$ is a submersion, then we can form the *Čech groupoid* X_Y. We have objects $(X_Y)_0 = Y$ and morphisms $(X_Y)_1 = Y^{[2]} := Y \times_X Y$. The source and target maps are the canonical projections, the identities come from the diagonal. Inversion comes from the flip map and composition comes from forgetting the middle factor. We will only be interested in the case where Y is a *surjective* submersion, and in particular when $Y = U \to X$ is an ordinary cover. The special case $Y = M \to \mathrm{pt} = X$ yields a Lie groupoid known as the *pair groupoid* EX.

There are also many examples of functors and natural transformations:

Example 6 (Smooth maps) Let X, Y be manifolds viewed as Lie groupoids. Smooth functors from X to Y are the same as smooth maps $X \to Y$. Given two such functors f, g there are no natural transformations unless we have equality $f = g$. In that case there is just the identity natural transformation. This gives a (fully faithful) inclusion functor $\mathsf{Man} \to \mathsf{LieGpd}$.

Example 7 (Lie homomorphisms) Let G and H be Lie groups viewed as Lie groupoids. The functors from G to H are precisely the Lie group homomorphisms. A natural transformation between f and g is the same as an element of H, conjugating f into g.

Example 8 Let X be a manifold and let EX be the corresponding pair groupoid. There is a unique functor to the one-point groupoid pt. A choice of point $x_0 \in X$, determines a functor $x_0\colon \mathrm{pt} \to EX$. The composition $\mathrm{pt} \to EX \to \mathrm{pt}$ is the identity functor. The other composition $EX \to \mathrm{pt} \to EX$ sends every object to x_0 and every morphism to $\iota(x_0)$. This is naturally isomorphic to the identity functor via the natural transformation

$$\eta\colon x \mapsto (x, x_0).$$

Thus EX and pt are equivalent as Lie groupoids.

Let $U \to X$ be a cover, and let X_U be the resulting Čech groupoid. Recall that the Čech groupoid can be thought of as the pair groupoid, but in the category of spaces over X. Again there is a canonical functor $X_U \to X$, and X serves the same role as the point, but in the category of spaces over X. Thinking in this line, one is tempted to guess that $X_U \to X$ is an equivalence. However, usually this is false. The canonical functor $X_U \to X$ is an equivalence if and only if the cover admits a global section $s\colon X \to U$. More precisely, we have the following lemma, whose proof is a straightforward calculation left to the reader.

Lemma 9 *Let $Y \to X$ and $Z \to X$ be spaces over X. Then the corresponding Čech groupoids are equivalent if and only if there exist maps over X, $f\colon Y \to Z$ and $g\colon Z \to Y$. In that case the equivalence is given by the canonically induced functors and the natural transformations are given by*

$$Y \to Y^{[2]}, \quad y \mapsto (y, gf(y))$$
$$Z \to Z^{[2]}, \quad z \mapsto (z, fg(z)).$$

In particular, X_U is equivalent to X if and only if the cover $U \to X$ admits a global section.

The last example highlights one of the well-known deficiencies of the 2–category of Lie groupoids. The functor $X_U \to X$ is both fully faithful and essentially surjective (in fact, actually surjective), but it fails to be an equivalence.

2.2 Bibundles and smooth stacks

In the following let Man_X denote the category of *manifolds over* X, that is the category whose objects are manifolds Y equipped with a smooth map $Y \to X$, and whose morphisms are smooth maps $Y \to Y'$ making the following triangle commute.

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Definition 10 Let $G = (G_1 \rightrightarrows G_0)$ and $H = (H_1 \rightrightarrows H_0)$ be Lie groupoids. A *(left principal) bibundle* from H to G is a smooth manifold P together with

(1) a map $\tau\colon P \to G_0$, and a surjective submersion $\sigma\colon P \to H_0$,

(2) action maps in $\mathsf{Man}_{G_0 \times H_0}$

$$G_1 \times^{s,\tau}_{G_0} P \to P$$
$$P \times^{\sigma,t}_{H_0} H_1 \to P$$

which we denote on elements as $(g, p) \mapsto g \cdot p$ and $(p, h) \mapsto p \cdot h$,

such that

(i) $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in G_1 \times^{s,t}_{G_0} G_1 \times^{s,\tau}_{G_0} P$,

(ii) $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \times^{\sigma,t}_{H_0} H_1 \times^{s,t}_{H_0} H_1$,

(iii) $p \cdot \iota_H(\sigma(p)) = p$ and $\iota_G(\tau(p)) \cdot p = p$ for all $p \in P$,

(iv) $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in G_1 \times^{s,\tau}_{G_0} P \times^{\sigma,t}_{H_0} H_1$,

(v) The map

$$G_1 \times^{s,\tau}_{G_0} P \to P \times^{\sigma,\sigma}_{H_0} P$$
$$(g, p) \mapsto (g \cdot p, p)$$

is an isomorphism. (The G–action is simply transitive.)

Bibundles combine several widely used notions into a single useful concept, as these examples illustrate.

Example 11 (Smooth maps) Let X and Y be smooth manifolds, viewed as Lie groupoids. Let P be a (left principal) bibundle from X to Y. Then $\sigma\colon P \to X$ is an isomorphism. Thus P is "the same" as a smooth map $\tau\colon X \to Y$.

Example 12 (Lie homomorphisms) Let G and H be Lie groups, thought of as Lie groupoids as in Example 4. Let P be a bibundle from H to G. Then P is (noncanonically) isomorphic to G with its left G–action. After identifying P with G, the right action of H on P is equivalent to a Lie group homomorphism $H \to G$. Thus Lie homomorphisms are "the same" as bibundles. (More precisely, as we will see shortly, conjugacy classes of Lie homomorphisms correspond to isomorphism classes of bibundles).

This next example shows where bibundles derive their name.

Example 13 (Principal bundles) A (left principal) bibundle from a manifold X to a Lie group G, viewed as Lie groupoids as in Example 4, is the same as a (left) principal G–bundle over X.

Example 14 Generalizing this last example, let Y be a manifold with a (left) action of a Lie group G. Denote the associated action Lie groupoid by $[Y/G]$. Let X be a manifold. A (left principal) bibundle from X to $[Y/G]$ consists of a (left) principal G–bundle P over X together with a G–equivariant map to Y. In particular, we may take the action of G on $Y = \mathrm{Aut}(G)$ to be by left multiplication by the conjugation automorphism, ie for $h \in \mathrm{Aut}(G)$, $g \cdot h = c_g \circ h$, where $c_g(g') = gg'g^{-1}$ is the conjugation automorphism. A (left principal) bibundle from X to $[\mathrm{Aut}(G)/G]$ is a "G–bibundle" in the sense of Aschieri, Cantini and Jurčo [3].

Example 15 (Identities) Let G be a Lie groupoid. There is a G-G bibundle given by $P = G_1$ with $\tau = t, \sigma = s$ and the obvious action maps. This is called the *identity bibundle* for reasons which will become obvious later.

Example 16 If $f\colon X \to G_0$ is a map, then we can form the pullback bibundle. $f^*G_1 = X \times^{f,s}_{G_0} G_1 \to X$. The induced action of G_1 on f^*G_1 makes this a bibundle from the trivial groupoid X (with only identity morphisms) to the groupoid G.

Example 17 Let $f\colon H \to G$ be a functor of Lie groupoids. Then we form the bibundle $\langle f \rangle$ as follows. As a space we have $\langle f \rangle = f_0^* G_1$, which we've already seen is a bibundle from H_0 to G. We need only supply the action of H_1. This is given by applying $f_1\colon H_1 \to G_1$ and using right action of G_1 on $f_0^*G_1$. Thus any functor gives rise to a bibundle. The association $f \mapsto \langle f \rangle$ is known as *bundlization*.

Example 18 If $f\colon U \to G_0$ is a submersion, then we may form the pullback groupoid f^*G. The objects consist of U, the morphisms consist of $(U \times U) \times_{G_0 \times G_0} G_1$ with source and target the natural projections. Composition is defined in the obvious way, as a confluence of the composition the pair groupoid of U and of the composition of G. There is a functor from f^*G to G which on object is the original map $U \to G_0$ and on morphisms is the projection $f^*G \to G_1$. In particular, there is a canonical bibundle from f^*G to G given by the bundlization of this functor.

Remark 19 Right principal bibundles can be defined in a similar manner. The only difference being that now τ, instead of σ, is required to be a surjective submersion and the action of H is simply transitive, ie

$$P \times_{H_0}^{\sigma,t} H_1 \cong P \times_{G_0}^{\tau,\tau} P.$$

In particular any left-principal bibundle P from H to G gives rise to a right-principal bibundle $\overline{P}$ from G to H, given by swapping the maps σ and τ, and precomposing the action maps with the inverse maps.

Definition 20 A *bibundle map* is a map $P \to P'$ over $H_0 \times G_0$ which commutes with the G– and H–actions, ie the following diagrams commute.

$$\begin{array}{ccc} G_1 \times_{G_0}^{s,\tau} P & \longrightarrow & P \\ \downarrow & & \downarrow \\ G_1 \times_{G_0}^{s,\tau} P' & \longrightarrow & P' \end{array} \qquad \begin{array}{ccc} P \times_{H_0}^{\sigma,t} H_1 & \longrightarrow & P \\ \downarrow & & \downarrow \\ P' \times_{H_0}^{\sigma,t} H_1 & \longrightarrow & P' \end{array}$$

Thus for each pair of groupoids we have a category $\mathsf{Bibun}(H,G)$ of bibundles from H to G. If $f,g\colon H \to G$ are two smooth functors between Lie groupoids, then the bibundle maps from $\langle f \rangle$ to $\langle g \rangle$ are in natural correspondence with the smooth natural transformations from f to g. In this sense the category $\mathsf{LieGpd}(H,G)$ is a subcategory of $\mathsf{Bibun}(H,G)$.

Example 21 A left principal bibundle from a Lie group H to a Lie group G always arises as $\langle f \rangle$ for some functor $f\colon H \to G$.

Example 22 A left principal bibundle whose target is a space is also always of the form $\langle f \rangle$ for some functor $f\colon X \to Y$. Hence if X and Y are spaces this is the same as a map of spaces. If X is an action groupoid, then this is just a G–invariant map.

Proposition 23 (Lerman [37]) *A bibundle P from H to G admits a section of $\sigma\colon P \to H_0$ if and only if $P \cong \langle f \rangle$ for some smooth functor f.*

Bibundles can be composed, and this gives us a bicategory Bibun. If P is a bibundle from H to G and Q is a bibundle from K to H, then we define the bibundle $P \circ Q$ as the coequalizer

$$P \times_{H_0}^{\sigma,t} H_1 \times_{H_0}^{s,\tau} Q \rightrightarrows P \times_{H_0}^{\sigma,\tau} Q \to P \circ Q.$$

Since σ is a surjective submersion, these pullbacks are manifolds and since our action on Q is simply transitive this coequalizer exists as a smooth manifold. In fact it is a bibundle from K to G. See [37] for details. Equivalence in this bicategory is sometimes referred to as *Morita equivalence*. They are characterized as those bibundles which are simultaneously left principal and right principal. The identity bibundle of Example 15 above serves as the identity 1–morphism. If the submersion in Example 18 is surjective, then the pullback groupoid is easily seen to be Morita equivalent to the original groupoid via the constructed bibundle.

Example 24 Let G and H be Lie groupoids and $P\colon H \to G$ a left-principal bibundle. If P is also a right-principal bibundle, then we may form a new left principal bibundle $P^{-1}\colon G \to H$. P^{-1} is the space P with τ and σ switched, and with a right (resp. left) action of G (resp. H) induced by the composition of the inversion map and the original action on P. In this case we have that $P \circ P^{-1}$ and $P^{-1} \circ P$ are isomorphic to identity bibundles. Is this case P and P^{-1} are Morita equivalences, and this characterizes Morita equivalences.

Example 25 As a special case of the above, suppose that G is a Lie group with a free and transitive action on the manifold X. Suppose further that the quotient space Y is a manifold, with smooth quotient map, $q\colon X \to Y$. If the quotient map admits local sections (so that X is a fiber bundle over Y), then we have a bibundle

$$\begin{array}{ccccc} Y & & X & & X \times G \\ \downarrow\downarrow & \overset{q}{\swarrow} & & \searrow & \downarrow\downarrow \\ Y & & & & X \end{array}$$

with the obvious induced actions. This bibundle, which is the bundlization $\langle q \rangle$ of the induced quotient functor, is simultaneously a left- and right-principal bibundle and hence Y and $[X/G]$ are equivalent in Bibun. Conversely, if Y and $[X/G]$ are equivalent in Bibun, then the quotient map $q\colon X \to Y$ necessarily admits local sections.

Theorem 26 (Pronk [47]) *There are canonical equivalences of bicategories between* Bibun, $\mathsf{Stack}_{\mathrm{pre}}$ *and* $\mathsf{LieGpd}[W^{-1}]$*, where* $\mathsf{Stack}_{\mathrm{pre}}$ *is the* 2*–category of (presentable) smooth stacks (in the surjective submersion topology) and* $\mathsf{LieGpd}[W^{-1}]$ *is the (derived) localization of the* 2*–category of Lie groupoids, functors, and natural transformations with respect to the* essential equivalences.

Remark 27 There is a forgetful 2–functor $\mathsf{LieGpd} \to \mathsf{Gpd}$ which forgets the topology of the Lie groupoid. This functor sends essential equivalences to equivalences and hence extends in an essentially unique way to a 2–functor $\mathsf{Bibun} \to \mathsf{Gpd}$. This 2–functor is product preserving.

2.3 Transversality for stacks

Definition 28 Let X, Y, Z be Lie groupoids and let $G\colon X \to Y$ and $F\colon Z \to Y$ be two left-principal bibundles. F and G are *transverse* (written $F \pitchfork G$) if the maps $F \to Y_0$ and $G \to Y_0$ are transverse.

This extends the usual notion of transversality for maps of spaces.

Lemma 29 *Let X, Y, Z be Lie groupoids and let $G\colon X \to Y$ and $F\colon Z \to Y$ be left-principal bibundles. If $F \pitchfork G$ then each of the four pairs of maps*

(1) $t \circ p_1\colon Y_1 \times_{Y_0}^{s,\tau} F \to Y_0$ *and* $G \to Y_0$,

(2) $F \to Y_0$ *and* $s \circ p_1\colon Y_1 \times_{Y_0}^{t,\tau} G \to Y_0$,

(3) $s \circ p_1\colon Y_1 \times_{Y_0}^{t,\tau} F \to Y_0$ *and* $G \to Y_0$,

(4) $F \to Y_0$ *and* $t \circ p_1\colon Y_1 \times_{Y_0}^{s,\tau} G \to Y_0$

are transverse.

Proof By symmetry, it is enough to consider only the first two pairs of maps. Moreover, the transversality of the first pair is easily seen to be equivalent to the second pair, thus it is enough to prove that the first pair of maps are transverse. The map $t \circ p_1\colon Y_1 \times_{Y_0}^{s,\tau} F \to Y_0$ factors through the action map

$$Y_1 \times_{Y_0}^{s,\tau} F \to F$$

which is surjective and surjective on tangent spaces. Therefore the images agree $d(t \circ p_1)(T(Y_1 \times_{Y_0}^{s,\tau} F)) = d\tau(TF)$, and the result follows. □

Recall that given a left principal bibundle G from the Lie groupoid X to the Lie groupoid Y, we may form a right principal bibundle $\overline{G}$ from Y to X by flipping the structure maps τ and σ and by using the inverse maps to switch left and right actions.

Lemma 30 *If $F \pitchfork G$, then the following coequalizer is a smooth manifold:*

$$\overline{G} \times_{Y_0}^{\tau,t} Y_1 \times_{Y_0}^{s,\tau} F \rightrightarrows \overline{G} \times_{Y_0} F \to \overline{G} \circ F.$$

Proof This is a local question. For each point $x \in X_0$ there exists an open neighborhood $U \subset X_0$ and a map $g_0\colon U \to Y_0$, so that over U we have $G|_U \cong Y_1 \times_{Y_0}^{s,g_0} U$. Similarly, for each point in Z_0, there exists a open neighborhood $f_0\colon V \subset Z_0$, a map $V \to Y_0$ so that $F|_V \cong Y_1 \times_{Y_0}^{s,f_0} U$. The transversality conditions ensure that f_0 and g_0 are also transverse. Locally the above equalizer is isomorphic to

$$U \times_{Y_0}^{g_0,t} Y_1 \times_{Y_0}^{s,f_0} V$$

which is again a manifold by our transversality assumptions. □

A similar calculation shows that $X_1 \times_{X_0} (\overline{G} \circ F) \times_{Z_0} Z_1$ is a smooth manifold. The primary reason for introducing the notion of transversality between maps of spaces is that it is a condition which ensures that pullbacks exist as smooth manifolds. The notion of transversality introduced here generalizes this property to the bicategory Bibun.

Proposition 31 *Let X, Z, Y be Lie groupoids and let $G\colon X \to Y$ and $F\colon Z \to Y$ be two left-principal bibundles. If F and G are transverse then the pullback exists in* Bibun.

In the above proposition, *pullback* is meant as a weak categorical limit (also known as *bilimit*) of the obvious diagram. See Street [62; 63] for details concerning such limits. In this case, such a pullback consists of a Lie groupoid W, equipped with bibundles $P_1\colon W \to X$ and $P_2\colon W \to Y$, together with an isomorphism of bibundles $G \circ P_1 \cong F \circ P_2\colon W \to Y$, which is universal for such Lie groupoids.

Proof of Proposition 31 We explicitly construct a pullback. The underlying Lie groupoid is given as follows:

- objects $\overline{G} \circ F$,
- morphisms $X_1 \times_{X_0}^{s,\sigma} (\overline{G} \circ F) \times_{Z_0}^{\sigma,s} Z_1$,

with source map given by p_2 onto the middle factor and target given by the action. Composition is given by the formula

$$(\alpha, [g, f], \beta) \circ (\alpha', [g', f'], \beta') = (\alpha \circ \alpha', [g, f], \beta \circ \beta').$$

The identities and inverses are given by the obvious maps. Call this Lie groupoid Γ. This Lie groupoid comes equipped with two smooth functors, which we regard as bibundles. The first is a functor $p_1\colon \Gamma \to X$ and is given on objects by the natural projection $(\overline{G} \circ F) \to X_0$. On morphisms it is also the projection

$$X_1 \times_{X_0}^{s,\sigma} (\overline{G} \circ F) \times_{Z_0}^{\sigma,s} Z_1 \to X_1.$$

One can check that this indeed defines a functor. The functor $p_2\colon \Gamma \to Z$ is defined similarly. The bundlization of the first functor is a bibundle whose total space is $X_1 \times_{X_0} (\overline{G} \circ F)$.

Composing the first map with the bibundle G we have

$$\begin{aligned} G \circ (X_1 \times_{X_0} (\overline{G} \circ F)) &\cong G \times_{X_0} (\overline{G} \circ F) \\ &\cong (G \times_{X_0} \overline{G}) \circ F \\ &\cong (\overline{G} \times_{Y_0} Y_1) \circ F \\ &\cong \overline{G} \times_{Y_0} F, \end{aligned}$$

where the later isomorphism follows from the simple transitivity of the Y action on G. A similar calculation shows that composing the second map with F gives a canonically isomorphic bibundle.

To prove that Γ is the pullback, we must now check the universal property. In particular given a Lie groupoid W and bibundles $f\colon W \to Z$ and $g\colon W \to X$, together with an isomorphism of bibundles $\phi\colon G \circ g \to F \circ f$, we must construct a bibundle $P\colon W \to \Gamma$ and isomorphisms $g \cong P_1 \circ P$, $f \cong P_2 \circ P$. The total space of P is given by $P = g \times_{W_0} f$, with its canonical map to W_0, and diagonal action. We must construct the projection to $\overline{G} \circ F$.

The isomorphism $\phi\colon G \circ g \to F \circ f$ is essential for this map. ϕ induces a map

$$G \times_{X_0} g \times_{W_0} f \to (G \circ g) \times_{W_0} f \to (F \circ f) \times_{W_0} f \cong F \circ (Z_1 \times_{Z_0} f) \cong F \times_{Z_0} f.$$

Let $(a, b) \in g \times_{W_0} f$. Consider the image of a in X_0 under the projection $g \to X_0$. Choose a lift $\tilde{a} \in G \to X_0$, which always exists since $G \to X_0$ is a surjective map. The above map says that given $\tilde{a}, a, b$, we get an element in $\tilde{b} \in F$.

We define the image of $(a, b) \in g \times_{W_0} f$ in $\overline{G} \circ F$ to be the equivalence class $[\tilde{a}, \tilde{b}]$. The only ambiguity in this construction is the choice of the lift $\tilde{a}$. Since the action of Y is simply transitive on G, the choices of $\tilde{a}$ differ precisely by the action of Y. Since ϕ is equivariant with respect to the Y–action, it follows that we have a well defined element in $\overline{G} \circ F$. Moreover since the lift $\tilde{a}$ is given by a section of $G \to X_0$, which can locally be chosen to be smooth, the resulting projection map is smooth.

The left action on $g \times_{W_0} f$ is given by the usual action map via

$$[X_1 \times_{X_0} (\overline{G} \circ F) \times_{Z_0} Z_1] \times_{\overline{G} \circ F} [g \times_{W_0} f] \cong (X_1 \times_{X_0} g) \times_{W_0} (Z_1 \times_{Z_0} f) \to g \times_{W_0} f.$$

One can check that there are canonical isomorphisms $g \cong P_1 \circ P$ as desired $f \cong P_2 \circ P$, and consequently that Γ satisfies the universal property of a pullback. □

Example 32 If X, Y, Z are manifolds, then transversality is transversality in the usual sense and the pullback is the usual pullback. More generally if X and Z are manifolds and $Y = [W/G]$ is a quotient Lie groupoid, then locally in X a bibundle to Y is given by a G–equivariant map $f\colon X \times G \to W$, or equivalently by a map $X \to W$. (This is only the *local* picture. Globally these maps f are glued together by the action of G on W. There is usually no global map.) If $y \in W$ is a point which is in the image of the corresponding (local) maps $f\colon X \to W$ and $g\colon Z \to W$, then transversality at y is equivalent to the identity:

$$df(T_x X) + dg(T_z Z) + T_y \mathcal{O}_G(y) = T_y W,$$

where $\mathcal{O}_G(y)$ is the G–orbit through the point y.

Definition 33 A morphism $F \in \mathsf{Bibun}(X, Y)$ is called *representable* if for all manifolds M and all maps $G \in \mathsf{Bibun}(M, Y)$, the pullback exists and is equivalent to a manifold.

Example 34 Let X, Y, Z be Lie groups thought of as Lie groupoids with one object. Then F and G are equivalent to group homomorphisms and are always transverse. The pullback is the action groupoid of $X \times Z$ acting on the space Y by $(x, z) \cdot y = G(x) y F(z)^{-1}$.

Example 35 An important special case of the previous example is when $X = \mathrm{pt}$ corresponds to the trivial group, and the homomorphism $Z \to Y$ corresponds to a closed embedding of Lie groups. In this case the action is free and the quotient is a manifold. Thus the groupoid of Z acting on Y is equivalent to the quotient space. In this case the map $Z \to Y$ is representable.

Example 36 Let $X = (X_1 \rightrightarrows X_0)$ be a groupoid. We may view X_0 as a Lie groupoid with only identity morphisms. Then there is the canonical bibundle $X_0 \to X$, which is the bundlization of the inclusion functor $X_0 \subset X$. If M is any manifold with a bibundle $F \in \mathsf{Bibun}(M, X)$, then the pullback is canonically isomorphic to the total space of F, viewed as a manifold. In particular, the pullback of X_0 with itself over X is the space X_1, thought of as a Lie groupoid with only identity morphisms.

Definition 37 Let $F \in \mathsf{Bibun}(X, Y)$. F is a *covering bibundle* if it is representable and the map $\tau\colon F \to Y_0$ is a surjective submersion.

Remark 38 A bibundle $F \in \mathsf{Bibun}(X, Y)$ such that the map $\tau\colon F \to Y_0$ is a surjective submersion is transverse to every bibundle $G \in \mathsf{Bibun}(Z, Y)$.

Example 39 For any groupoid $X = (X_1 \rightrightarrows X_0)$, the canonical bibundle from X_0 to X is a covering bibundle.

3 2–Groups in stacks

3.1 2–Groups

Groups are pervasive in all subjects of mathematics and are an important and well studied subject. 2–Groups are a categorification of the notion of group, and have been playing an increasingly important role in many areas of mathematics and even physics. Recall the following slightly nonstandard but equivalent definition of a group. Let $(G, 1, \cdot)$ be a monoid. We say G is a *group* if the map

$$\begin{aligned} G \times G &\to G \times G \\ (x, y) &\mapsto (x, x \cdot y) \end{aligned}$$

is a bijection. The inverse of this map allows one to find an element g^{-1} for each element g such that $gg^{-1} = g^{-1}g = 1$. Categorifying this definition yields the most succinct definition of 2–group of which I am aware.

Definition 40 A monoidal category $(M, \otimes, 1, a, \ell, r)$ is a 2*–group* if the functor

$$(p_1, \otimes)\colon M \times M \to M \times M$$

is an equivalence of categories, where p_1 is projection onto the first factor. The 2–category of 2–groups is the full sub-bicategory of the bicategory of monoidal categories whose objects consist of the 2–groups.

There are many equivalent descriptions of 2–groups which have arisen in various branches of mathematics. While the precise history of 2–groups is too intricate and convoluted to be done justice in this article, a few key highlights are in order. One of the earliest appearances of 2–groups arose in topology, without the aid of (higher) category theory. Since a 2–group is automatically a groupoid, its simplicial nerve will be a Kan simplicial set. Hence the geometric realization of a 2–group is automatically a homotopy 1–type (ie $\pi_i = 0$ at all base points for all $i > 1$). The geometric realization of a monoidal category is well known to be an A_∞–space, and for 2–groups it is group-like. Thus it may be de-looped once to obtain a pointed connected homotopy 2–type $B|M|$. The (pointed) mapping spaces between pointed connected homotopy 2–types are automatically homotopy 1–types, and so by replacing the mapping space with its fundamental groupoid we obtain a bicategory which captures essentially all the homotopical information of homotopy 2–types. This bicategory is equivalent to the bicategory of 2–groups, and so the study of pointed connected 2–types (going back to the work of Whitehead and Mac Lane in the 1940s and 1950s) can be regarded as one of the earliest studies of 2–groups.

It is well known that small monoidal categories can be strictified, that is replaced with equivalent monoidal categories where associativity and unit identities are satisfied on the nose. Doing this to a 2–group yields a so-called "categorical group", ie a (strict) group object in categories. A construction, known in the 1960s, shows that such categorical groups are essentially the same thing as *crossed modules*, a concept introduced by J H C Whitehead in 1946 and later used by Whitehead and Mac Lane to classify pointed connected homotopy 2–types.

Finally, another method of studying 2–groups is via *skeletalization* (introduced for 2–groups in [5]) in which the 2–group is replaced by an equivalent 2–group which is skeletal[9]. This yields a particularly simple description of each 2–group in terms of invariants: two ordinary groups π_1, π_2, and certain other data known collectively as the k-invariant. This classification is in direct correspondence with the usual classification of connected pointed 2–types in terms of Postnikov data.

3.2 2–Groups in general bicategories

Monoid and group objects can be defined in any category with finite products, and a similar statement holds true for 2–groups. Following Baez and Lauda [5] we introduce 2–group objects in arbitrary bicategories with finite products. Such a "2–group" consists of an object, G, together with a multiplication 1–morphism $m\colon G \times G \to G$, a unit 1–morphism $e\colon 1 \to G$, and several coherence 2–isomorphisms. Additionally it must satisfy a property which ensures that a coherent inverse map may be chosen. This is essentially a mild generalization of the definition of "coherent 2–group objects" defined in [5], modified to make sense in an arbitrary bicategory.

Definition 41 [5] Let C be a bicategory with finite products. A 2–*group* in C consists of an object G together with 1–morphisms $e\colon 1 \to G$, $m\colon G \times G \to G$, and invertible 2–morphisms

$$a\colon m \circ (m \times \mathrm{id}) \to m \circ (\mathrm{id} \times m)$$
$$\ell\colon m \circ (e \times \mathrm{id}) \to \mathrm{id}$$
$$r\colon m \circ (\mathrm{id} \times e) \to \mathrm{id}$$

such that
$$(p_1, m)\colon G \times G \to G \times G$$

is an equivalence in C and the diagrams in Figures 1 and 2 commute.

The names of these diagrams have been chosen so as to correspond to the names of diagrams when C is a strict 2–category. Thus the "pentagon" identity is no longer

[9] A category C is skeletal if for all objects $x, y \in C$, the property $x \cong y$ implies $x = y$.

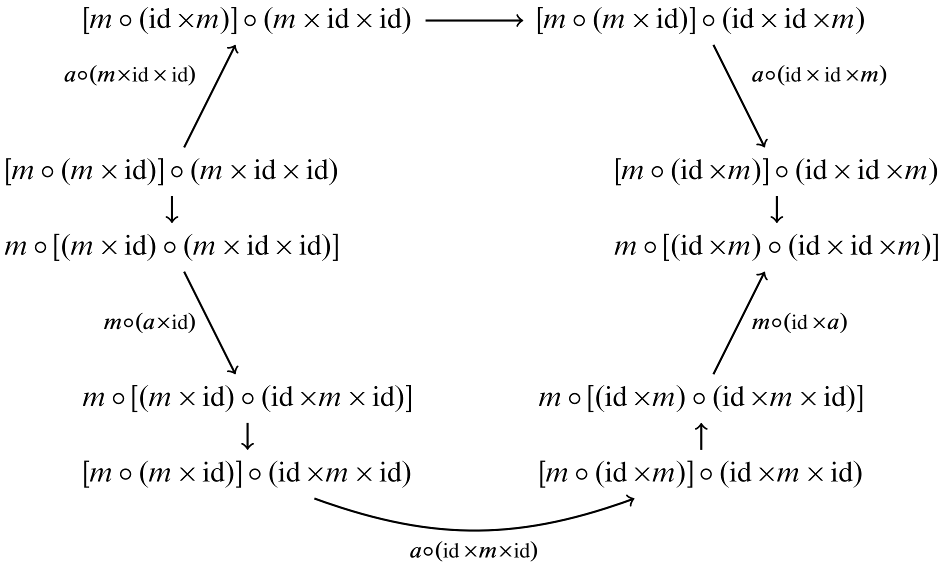


Figure 1: The "pentagon" identity

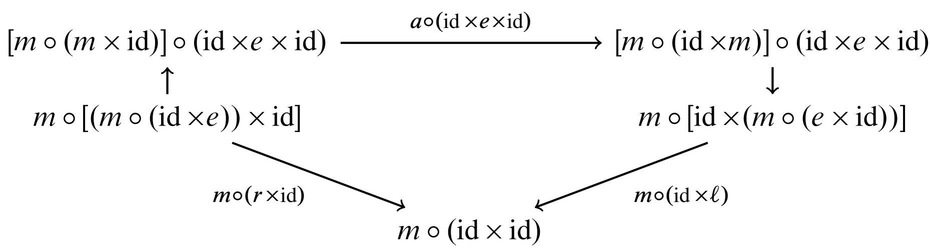


Figure 2: The "triangle" identity

pentagonal in shape. The unlabeled arrows are the canonical isomorphisms given from associativity and products in C.

Just as the notion of 2–group presented in [5] extends to an arbitrary bicategory C, so too do the notions of homomorphism and 2–homomorphism. Homomorphisms and 2–homomorphisms compose making a bicategory of 2–groups in C. A direct calculation shows that all 2–homomorphisms are invertible[10].

[10] Thus the category of 2–groups is an example of an $(\infty, 1)$–category.

Definition 42 Let G and G' be 2–groups in C. A *homomorphism* of 2–groups $G \to G'$ consists of

- a 1–morphism $F\colon G \to G'$,
- 2–isomorphisms $F_2\colon m' \circ (F \times F) \to F \circ m$ and $F_0\colon e' \to F \circ e$

such that the three diagrams in Figures 3, 4 and 5 commute. In these diagrams the unlabeled arrows are the canonical isomorphisms given from associativity and products in C.

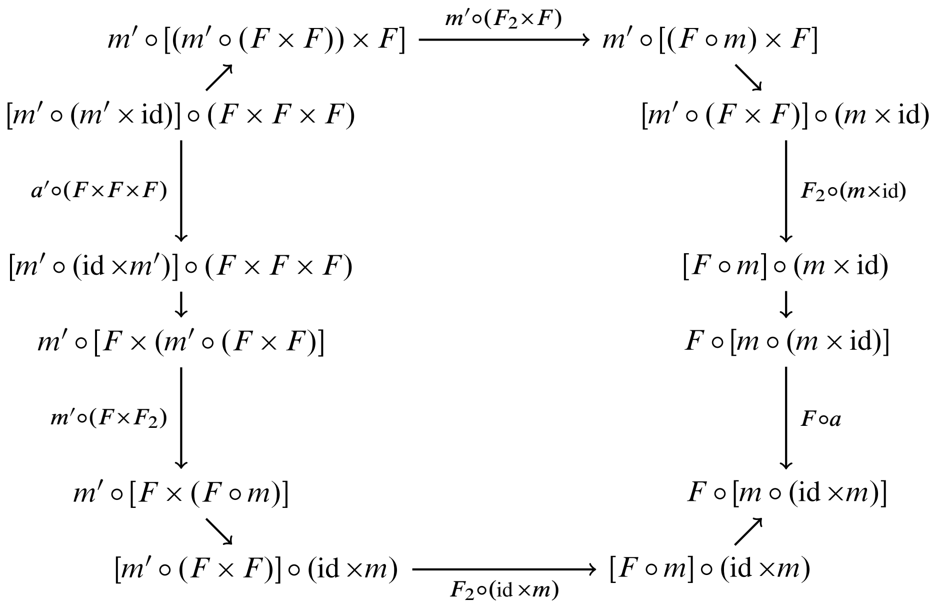


Figure 3: Axiom 1 for 2–group homomorphisms

Definition 43 Given two homomorphisms $F, K\colon G \to G'$ between 2–groups in C, a 2–homomorphism $\theta\colon F \Rightarrow K$ is a 2–morphism such that the diagrams in Figure 6 commute.

Definition 44 Let C be a bicategory with finite products and G be a 2–group in C. Let X be an object in C. A *(left) G–action* on X consists of

- a 1–morphism $f\colon G \times X \to X$,
- invertible 2–morphisms

$$a_f\colon f \circ (m \times \mathrm{id}) \to f \circ (\mathrm{id} \times f)$$
$$\ell_f\colon f \circ (e \times \mathrm{id}) \to \mathrm{id}$$

such that the diagrams in Figures 7 and 8 commute.

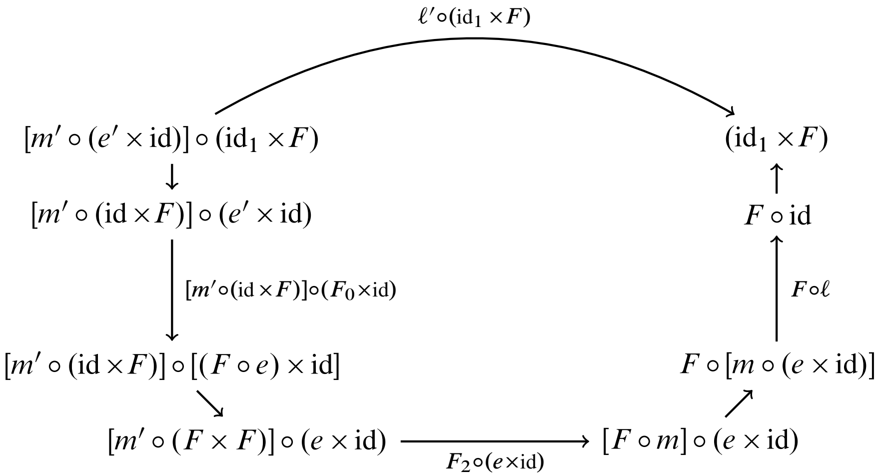


Figure 4: Axiom 2 for 2–group homomorphisms

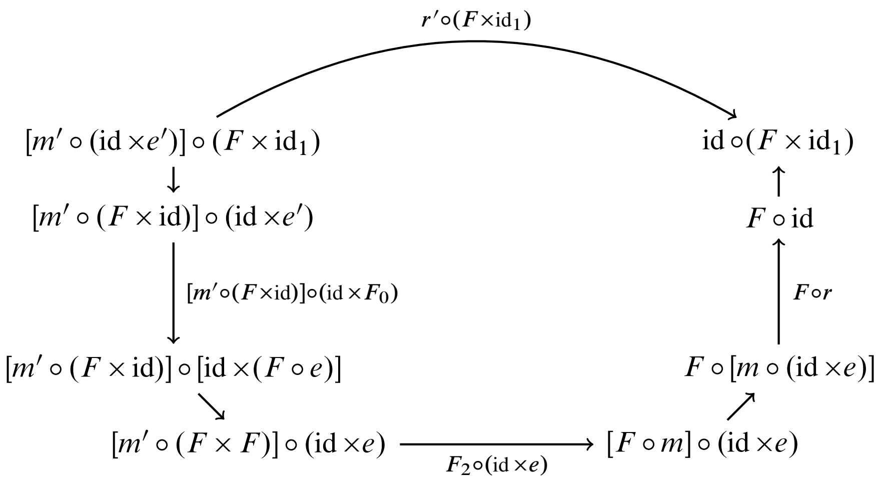


Figure 5: Axiom 3 for 2–group homomorphisms

Definition 45 Let C be a bicategory with finite products, G a 2–group in C and X and Y two objects in C equipped with G–actions. A *G–equivariant 1–morphism* from X to Y consists of

- a 1–morphism $g\colon X \to Y$ in C,
- invertible 2–isomorphisms $\phi\colon f_Y \circ (1 \times g) \to g \circ f_X$ in C

such that the diagrams in Figure 9 and Figure 10 commute.

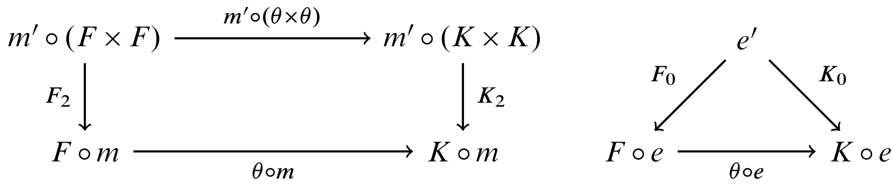


Figure 6: Axioms of 2–homomorphisms

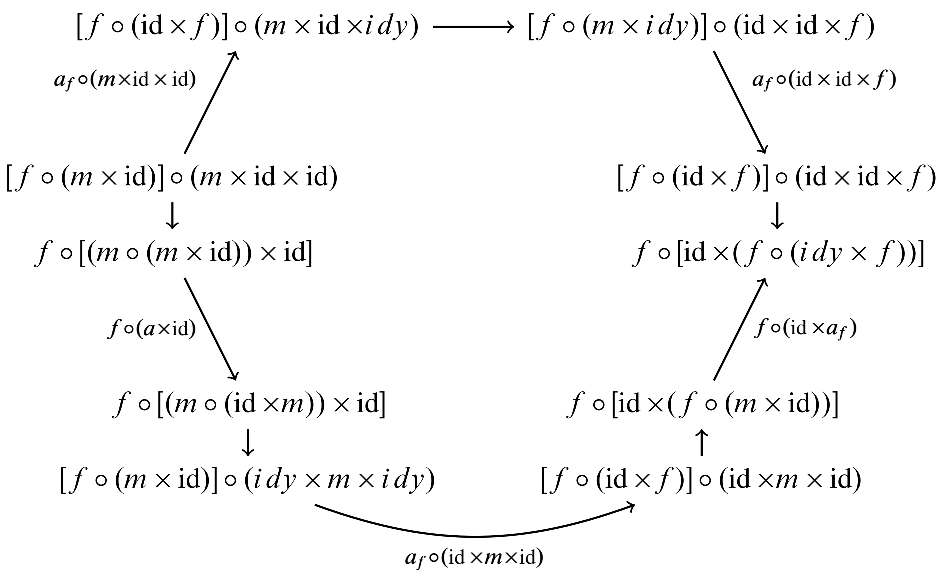


Figure 7: The "pentagon" identity for 2–group actions

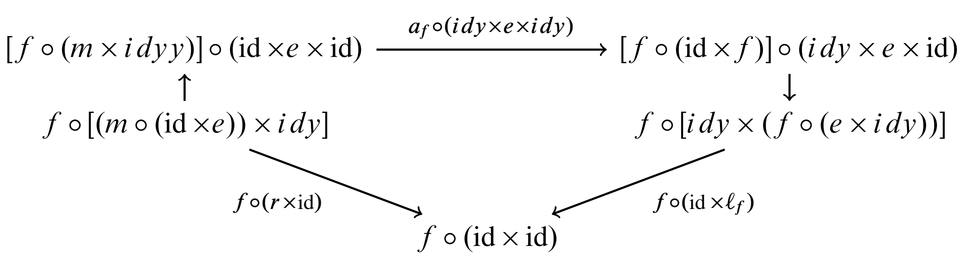


Figure 8: The "triangle" identity for 2–group actions

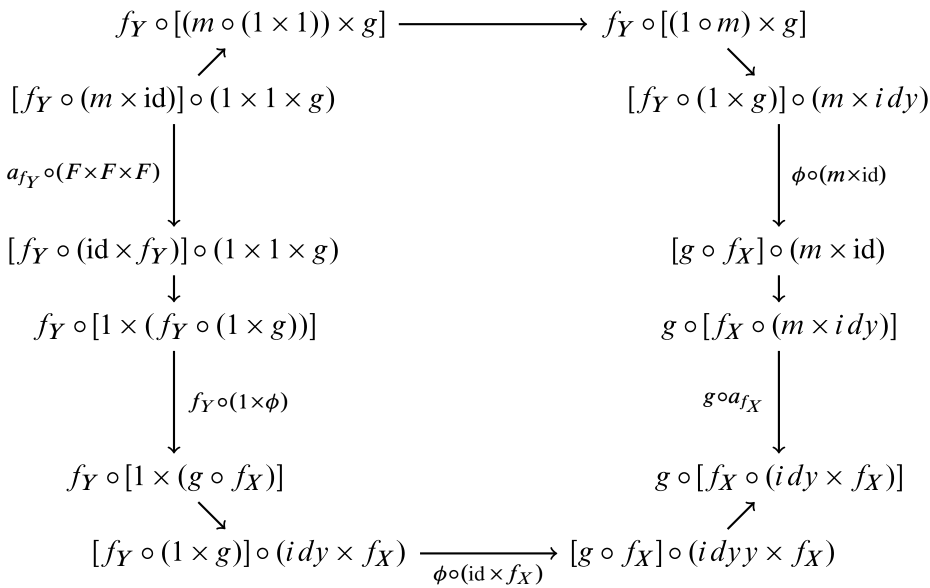


Figure 9: Axiom 1 of equivariant 1–morphism

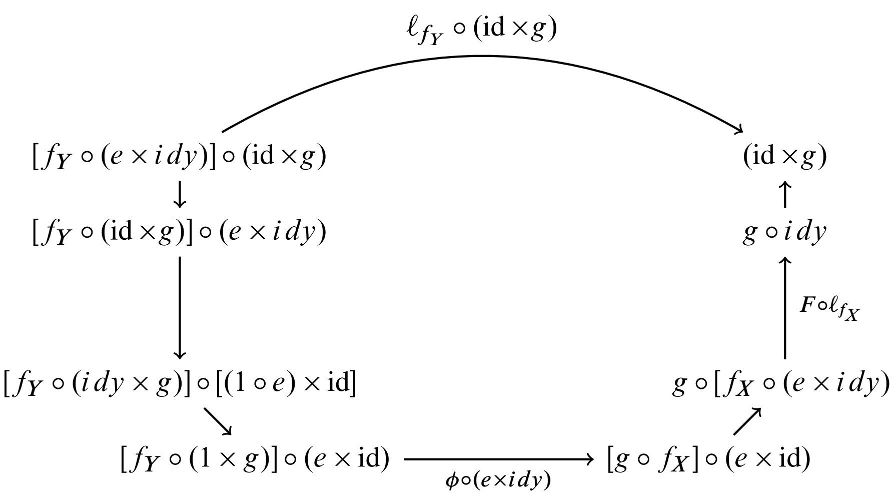


Figure 10: Axiom 2 of equivariant 1–morphism

Definition 46 Given two G–equivariant 1–morphisms $g, g'\colon X \to Y$, an *equivariant 2–morphism* $\rho\colon g \to g'$ is a 2–isomorphism such that the following square commutes:

$$\begin{array}{ccc} f_Y \circ (1 \times g) & \xrightarrow{f_Y \circ (1\times\rho)} & f_Y \circ (1 \times g') \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ g \circ f_X & \xrightarrow[\rho\circ f_X]{} & g' \circ f_X \end{array}$$

3.3 Abelian groups in bicategories

The description in terms of monoidal categories makes it clear that there are two related notions of 2–group which generalize the notion of abelian group: braided 2–groups and symmetric 2–groups. In this work we will only be interested in the later, most highly commutative structure. A braided monoidal category is a monoidal category G, equipped with natural isomorphisms $\beta_{x,y}\colon x \otimes y \to y \otimes x$, which satisfy the requirement that two different hexagonal diagrams commute. A symmetric monoidal category further satisfies the condition: $\beta_{x,y}\beta_{y,x} = id y$.

Following the discussion in Joyal and Street [32], if the above symmetry equation is satisfied, then the hexagonal diagrams become redundant: only one is necessary, the other is a consequence. Thus if one were interested only in defining symmetric monoidal categories and not braided monoidal categories, one could omit one of the hexagonal diagrams from the definition. This is the approach we take here.

Definition 47 Let C be a bicategory with finite products. An *abelian 2–group* in C consists of a 2–group (G, e, m, I, a, ℓ, r) in C, together with a 2–isomorphism $\beta\colon m \to m \circ \tau$, where $\tau\colon G \times G \to G \times G$ is the "flip" 1–morphism in C, such that the diagrams in Figure 11 and Figure 12 commute (in these figures the unlabeled arrows are canonical 2–morphisms from C).

$$\begin{array}{ccc} m \circ \tau & \xrightarrow{\beta\circ\tau} & (m \circ \tau) \circ \tau \\ \uparrow{\scriptstyle \beta} & & \downarrow \\ m & \xrightarrow{1} & m \end{array}$$

Figure 11: Axiom 1 of abelian 2–group

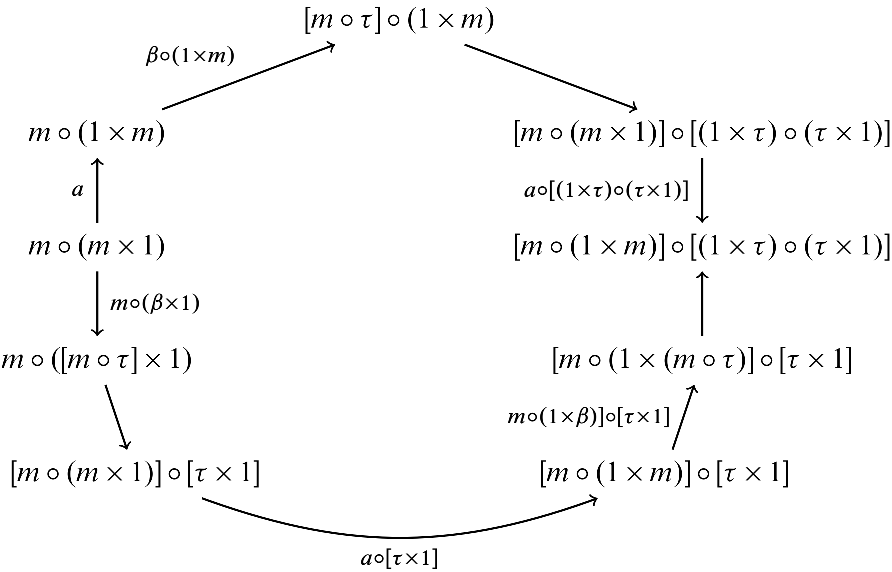
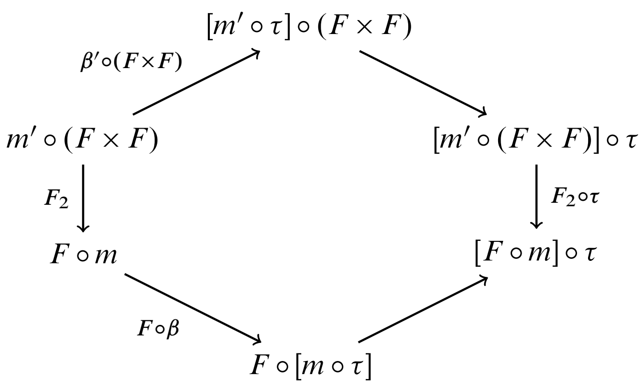


Figure 12: Axiom 2 of abelian 2–group

Definition 48 Let G and G' be abelian 2–groups in C. A *homomorphism* of abelian 2–groups consists of a homomorphism (F, F_2, F_0): $G \to G'$ of underlying groups such that the following diagram commutes:



A 2*–homomorphism* between homomorphisms of abelian 2–groups in C consists of a 2–homomorphism of underlying homomorphisms of groups in C.

Remark 49 2–Groups and abelian 2–groups in bicategories are defined diagrammatically. Thus if h: $\mathsf{C} \to \mathsf{C}'$ is a product preserving 2–functor and G is an (abelian) 2–group object in C, then $h(G)$ is canonically an (abelian) 2–group object in C'. Similarly, h sends G–objects in C to $h(G)$–objects in C'.

3.4 2–Groups as a localization

The bicategory of 2–groups (in Cat) admits a succinct description as a localization of the bicategory of monoidal groupoids (and hence also as a localization of monoidal categories). This description will play a small technical role in this paper and so we offer a brief summary of this approach and a proof of its equivalence to the one already introduced (Definition 40). Along the way we will encounter several equivalent descriptions of 2–groups.

Definition 50 A monoidal category M *admits functorial inverses* if there exists a functor $i\colon M \to M$ and a natural isomorphism $x \otimes i(x) \cong 1$. A *choice of functorial inverses* shall refer to a specific choice of functor i and corresponding natural isomorphism.

Lemma 51 *If M is a monoidal category which admits functorial inverses, then for any choice of functorial inverses i there is a natural isomorphism $i^2(x) \cong x$.*

Proof $i^2(x) \cong 1 \otimes i^2(x) \cong (x \otimes i(x)) \otimes i^2(x) \cong x \otimes (i(x) \otimes i^2(x)) \cong x \otimes 1 \cong x$. □

Lemma 52 *If M is a monoidal category which admits functorial inverses, then the underlying category of M is a groupoid.*

Proof Let $f\colon x \to y$ be a morphism in M. Its inverse $f^{-1}\colon y \to x$ is given by the composition

$$f^{-1}\colon y \cong i^2(y) \cong (x \otimes i(x)) \otimes i^2(y)$$
$$\xrightarrow{(1\otimes i(f))\otimes 1} (x \otimes i(y)) \otimes i^2(y) \cong x \otimes (i(y) \otimes i^2(y)) \cong x$$

of natural morphisms. □

Lemma 53 *If M is a monoidal category admitting functorial inverses and $f\colon x \to x'$ is a morphism in M, then for all objects $y, z \in M$ the following maps are bijections:*

$$(-) \otimes f\colon C(y,z) \to C(y \otimes x, z \otimes x'),$$
$$f \otimes (-)\colon C(y,z) \to C(x \otimes y, x' \otimes z).$$

Proof The inverse to the first bijection is obtained by choosing a functorial inverse i, applying the functor $(-)\otimes i(f)$, and using the natural isomorphisms $y \cong (y\otimes x)\otimes i(x)$ and $z \cong (z \otimes x') \otimes i(x')$. The inverse to the second is obtained similarly, making use of the isomorphism $i^2(x) \cong x$. □

The bicategory of monoidal categories admits all small weak colimits. This can be seen, for example, by observing that the bicategory of monoidal categories is equivalent to the bicategory of algebras for a 2–monad on the bicategory of categories; see Blackwell, Kelly and Power [11, Section 6]. Similarly the bicategory $\mathsf{Gpd}^{\otimes}$ of monoidal groupoids (ie of those monoidal categories whose underlying categories are groupoids) is equivalent to the bicategory of algebras for the same 2–monad restricted to the bicategory of groupoids. All the 2–morphisms of $\mathsf{Gpd}^{\otimes}$ are invertible, hence it fits into the formalism of $(\infty, 1)$–categories as considered by Lurie [40; 39], and we may therefore bring to bear the sophisticated machinery developed in those sources in our study of 2–groups. In particular $\mathsf{Gpd}^{\otimes}$ is *presentable* in the sense of [40, Definition 5.5.0.1]. We will see the relevance of this shortly.

Any monoid may be regarded as a monoidal groupoid in which all morphisms are identity morphism and where the monoidal structure is given by multiplication in the monoid. The natural numbers $\mathbb{N}$, viewed as a monoidal category in this way, are free in the sense that the we have a natural equivalence of categories $\hom(\mathbb{N}, C) \simeq C$ for any monoidal category C, where $\hom(A, B)$ denotes the category of monoidal functors from A to B.

Definition 54 Define the monoidal category $\mathbb{F}$ as the weak pushout in the following diagram of monoidal categories.

(55)
$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\Delta} & \mathbb{N} \times \mathbb{N} \\ \downarrow & \Downarrow \quad \lrcorner & \downarrow \\ 0 & \longrightarrow & \mathbb{F} \end{array}$$

The inclusion $\mathbb{N} \times 0 \to \mathbb{N} \times \mathbb{N}$ induces a map of monoidal categories $s\colon \mathbb{N} \to \mathbb{F}$.

Definition 56 A monoidal category M is *s–local* if the induced functor

$$s^*\colon M^{\mathbb{F}} = \hom(\mathbb{F}, M) \to \hom(\mathbb{N}, M) \simeq M$$

is an equivalence of categories.

Theorem 57 *For a monoidal category M the following conditions are equivalent:*

(1) *M is a 2–group (ie $(p_1, \otimes)\colon M \times M \to M \times M$ is an equivalence).*

(2) *M is s–local.*

(3) *M admits functorial inverses.*

Proof Let M be a monoidal category. Applying $\hom(-, M)$ to the commutative diagram in Equation (55) we obtain the first of the following pair of weak pullback squares of categories:

$$\begin{array}{ccc} M \simeq M^{\mathbb{N}} & \xleftarrow{\otimes} & M \times M \simeq M^{\mathbb{N}\times\mathbb{N}} \\ \uparrow & \Downarrow \quad \lrcorner & \uparrow \\ 0 & \longleftarrow & M^{\mathbb{F}} \end{array} \qquad \begin{array}{ccc} M & \xleftarrow{p_2} & M \times M \\ \uparrow & \Downarrow \quad \lrcorner & \uparrow{\scriptstyle i_1} \\ 0 & \longleftarrow & M \end{array}$$

If M is a 2–group, then these pullback squares are equivalent via the equivalence $(p_1, \otimes)\colon M \times M \to M \times M$. Thus the natural map $s^*\colon M^{\mathbb{F}} \to M$ is an equivalence, and so 2–groups are s–local.

By construction, a functor $\mathbb{F} \to M$ is equivalent to a pair of objects $x, \overline{x} \in M$ together with an equivalence $\alpha\colon x \otimes \overline{x} \cong 1$. The functor $s^*\colon M^{\mathbb{F}} \to M$ sends the triple $(x, \overline{x}, \alpha)$ to the object x. If M is s–local then we have an inverse equivalence $M \to M^{\mathbb{F}}$, and hence a *functorial* choice of inverse $\overline{x}$ for every object $x \in M$. In other words, M admits functorial inverses.

Finally, suppose that M admits functorial inverses. We wish to show that M is a 2–group, ie that the natural functor $(p_1, \otimes)\colon M \times M \to M \times M$ is an equivalence of categories. Given a monoidal category C, the collection of isomorphism classes of objects, $\pi_0 C$, is a monoid. For a category which admits functorial inverses, M, the monoid $\pi_0 M$ is a group, and hence $(p_1, \otimes)$ is a bijection on isomorphism classes of objects. It remains to show that $(p_1, \otimes)$ is fully faithful, ie that for all objects $x, y, x', y' \in M$, the natural map

$$M(x, y) \times M(x', y') \to M(x, y) \times M(x \otimes x', y \otimes y')$$

is a bijection. This in turn is equivalent to the statement that for each $f\colon x \to y$, the map $f \otimes (-)\colon M(x', y') \to M(x \otimes x', y \otimes y')$ is a bijection, which is part of the statement of Lemma 53. □

Remark 58 Examining the last part of the above proof and the proof of Lemma 53, one observes that if the underlying category of M is a groupoid, then $(p_1, \otimes)$ is an equivalence (and hence M is a 2–group) precisely if $\pi_0 M$ is a group. Thus our definition agrees with the notion of "weak 2–group" given in [5, Definition 2]. This characterization allows one to deduce that $\mathbb{F}$ itself is a 2–group: $\mathbb{F}$ is a monoidal groupoid, being a colimit of such, and moreover $\pi_0 \mathbb{F} \cong \mathbb{Z}$ is a group (this last follows from the definition of $\mathbb{F}$ and from the fact that π_0 sends colimits of monoidal groupoids to colimits of monoids). Finally, using the skeletal classification of 2–groups

[5, Section 8.3] and the universal property that $M^{\mathbb{F}} \simeq M$ for all 2–groups M, one may deduce the monoidal equivalence $\mathbb{F} \simeq \mathbb{Z}$. Alternatively, one may simply compute the pushout defining $\mathbb{F}$ and deduce this equivalence. We will not make use of this in what follows.

Corollary 59 *The bicategory of 2–groups is cocomplete, the inclusion of 2–groups into monoidal groupoids admits a weak left adjoint, and this adjunction is 2–monadic.*

Proof By the above theorem, 2–groups are precisely the s–local objects of $\mathsf{Gpd}^{\otimes}$. Since this later is a presentable $(\infty, 1)$–category, the first two claims are direct statements from [40, Proposition 5.5.4.15], from which it also follows that 2–groups form a *strongly reflective sub-bicategory* of $\mathsf{Gpd}^{\otimes}$ [40, page 482].

The final statement follows from general principles as the localizing adjunction from any presentable $(\infty, 1)$–category to a strongly reflective sub–$(\infty, 1)$–category is monadic. This is classical for ordinary categories and an identical argument applies to the higher categorical setting, as follows. Let $i\colon 2\,\mathsf{Grp} \to \mathsf{Gpd}^{\otimes}$ be the inclusion functor, L its left adjoint and $T = iL$ the corresponding 2–monad. Then for every 2–group X, the adjunction induces the structure of a T–algebra on iX. Moreover, since i is fully faithful, any T–algebra structure on iX is equivalent to this canonical one. Thus it is sufficient to show that if Y is an arbitrary T–algebra, then Y is in fact a 2–group. This follows since for any T–algebra Y the structure morphism $h\colon TY \to Y$ is an equivalence, with inverse $\eta_Y\colon Y \to TY$. □

Corollary 60 *The forgetful functor from 2–groups to groupoids admits a weak left adjoint and the resulting adjunction is 2–monadic.*

Proof The composition of 2–monadic adjunctions remains 2–monadic, and so the statement follows from the previous corollary and from the fact that the forgetful functor from monoidal groupoids to groupoids is part of a 2–monadic adjunction [11, Section 6]. □

The results of this section can be applied to 2–groups in a bicategory C much more general then $\mathsf{C} = \mathsf{Cat}$. Let S be an essentially small category[11]. Let C be a localization (ie reflexive sub-bicategory) of the functor bicategory $\mathrm{Fun}(\mathsf{S}^{\mathrm{op}}, \mathsf{Gpd})$. For each object $U \in \mathsf{S}$, let $L_U\colon \mathsf{Gpd} \to \mathsf{C}$ be the left-adjoint to evaluation at U. Let $\mathsf{C}^{\otimes}$ denote the bicategory of monoidal objects in C.[12] The functor L_U induces a functor $L_U\colon \mathsf{Gpd}^{\otimes} \to \mathsf{C}^{\otimes}$.

[11] The following construction works when S is an essentially small bicategory, but we will only need the case where S is an ordinary category.

[12] These are defined identically to 2–groups in C, except for the requirement that $(p_1, m)\colon M \times M \to M \times M$ is an equivalence.

Let $s_U \colon L_U(\mathbb{N}) \to L_U(\mathbb{Z})$ denote the canonical map of objects in $\mathsf{C}^{\otimes}$. An object of $\mathsf{C}^{\otimes}$ is *S–local* if it is local with respect to all s_U.[13]

Theorem 61 *Let C be a localization of $\mathrm{Fun}(\mathsf{S}^{\mathrm{op}}, \mathsf{Gpd})$, as above, and let $M \in \mathsf{C}^{\otimes}$. Then the following statements are equivalent:*

(1) *M is a 2–group in C.*

(2) *$M(U)$ is a 2–group (in Gpd) for every object $U \in \mathsf{S}$.*

(3) *$M(U)$ admits functorial inverses for every object in $U \in \mathsf{S}$.*

(4) *$M(U)$ is s–local for every object in $U \in \mathsf{S}$.*

(5) *M is an S–local object of $\mathsf{C}^{\otimes}$.*

Moreover the adjunction $F \colon \mathsf{C} \leftrightarrows 2\,\mathsf{Grp}(\mathsf{C}) : U$ (induced by the forgetful functor from $2\,\mathsf{Grp}(\mathsf{C})$ to C) is monadic.

Proof The equivalences (1) ⇔ (2) and (4) ⇔ (5) follow from the bicategorical Yoneda lemma, and the fact that C is a full sub-bicategory of $\mathrm{Fun}(\mathsf{S}^{\mathrm{op}}, \mathsf{Gpd})$. The equivalences (2) ⇔ (3) ⇔ (4) follow from Theorem 57. The proof of Corollaries 59 and 60 carry over immediately to show the final statement. □

Example 62 The category $\mathsf{C} = \mathsf{Stack}$ of all stacks (not necessarily presentable) on the site $\mathsf{S} = \mathsf{Man}$ of smooth manifolds with the surjective submersion topology is a localization of $\mathrm{Fun}(\mathsf{Man}^{\mathrm{op}}, \mathsf{Gpd})$. Hence the adjunction $F \colon \mathsf{Stack} \leftrightarrows 2\,\mathsf{Grp}(\mathsf{Stack}) : U$ between stacks and 2–groups in stacks is monadic.

3.5 Smooth 2–groups and gerbes

We now specialize to the case $\mathsf{C} = \mathsf{Bibun}$, the bicategory of Lie groupoids, bibundles, and bibundle morphisms. We will also refer to the objects of Bibun as *smooth* stacks.

Definition 63 A *smooth 2–group* (resp. *smooth abelian 2–group*) is a 2–group object (resp. abelian 2–group object) in Bibun. Let G be a smooth 2–group. Then a smooth G–stack X is a G–object in Bibun. Similarly, if X is a smooth stack, a *smooth 2–group over X* is a group object of Bibun/X. Let G be a smooth 2–group over X, then a *smooth G–stack over X* is a G–object in Bibun/X.

[13] Since S is essentially small, the S–local objects of C are a further reflexive sub-bicategory of $\mathrm{Fun}(\mathsf{S}^{\mathrm{op}}, \mathsf{Gpd})$.

Remark 64 If X is a discrete space, then any surjective submersion $P \to X$ admits a *global* section. Hence a bibundle whose source is a discrete groupoid is equivalent to one arising from a functor (see Proposition 23). In particular we have that the composite 2–functor $\mathsf{Gpd} \hookrightarrow \mathsf{LieGpd} \to \mathsf{Bibun}$ is fully faithful. Thus any smooth 2–group whose underlying Lie groupoid is discrete arises from a discrete 2–group and we may regard the theory of discrete 2–groups as a special case of smooth 2–groups.

Example 65 (Lie groups) Let G be a Lie group, viewed as a Lie groupoid with only identity morphisms. Then G is a smooth 2–group with monoidal structure coming from the multiplication in G.

Example 66 (Abelian Lie groups) Let A be an abelian Lie group. Let $[\mathrm{pt}/A]$ denote the Lie groupoid with a single object and with automorphism of this object equal to A. Since A is a abelian, addition is a group homomorphism $A \times A \to A$. Thus addition equips $[\mathrm{pt}/A]$ with a monoidal structure. Moreover, there is a (trivial) braiding making this into a smooth abelian 2–group.

Example 67 (Crossed modules) A crossed module of Lie groups, $\beta\colon H \to G$, is well known to be equivalent to a group object in the category of Lie groups. Thus a crossed module gives rise to a Lie 2–group (and hence a smooth 2–group) in which the associator and unitor structures are trivial. The translation from a crossed module to a Lie 2–group is as follows; see [5]. The objects consist of the manifold G. The morphisms consist of the manifold $G \times H$. The source map is projection onto the G–factor. The target map is given by

$$t(g,h) = g \cdot \beta(h).$$

Composition is given by $(g_0, h_0) \circ (g\beta(h_0), h_1) = (g, h_0 h_1)$. Viewing the morphisms as the group $G \rtimes H$, both the source and target maps are group homomorphisms. Group multiplication in G and $G \rtimes H$ equip this Lie groupoid with a strict monoidal structure, with strict inverses. The underlying stack of this smooth 2–group is the quotient stack $[G/H]$.

A particularly important example of such a crossed module is $H \to \mathrm{Aut}(H)$, sending an element to the conjugation automorphism. The corresponding smooth 2–group $[\mathrm{Aut}(H)/H]$ plays a key role in the theory of nonabelian bundle gerbes described in [3], to which we will turn shortly.

Example 68 (Smooth cocycles) Let G be a Lie group and A an abelian Lie group equipped with an action of G. Let $a \in Z^3(G, A)$ be a smooth normalized group cocycle. Following [5] we may form the following smooth 2–group $\Gamma = (G, A, a)$.

The objects of Γ consist of the manifold G, the morphisms are the space $G \times A$. There are no morphisms from an element g to g', unless $g = g'$. In this case the morphisms are identified with A, which is the fiber of the projection map $G \times A \to A$. The monoidal structure is given by group multiplication at the level of objects and by the multiplication in the group $G \rtimes A$ at the level of morphisms. This is a strictly associative multiplication. Nevertheless, we equip Γ with the nontrivial associator determined by a. This determines the unitor and inversion structures up to natural isomorphism.

Given a object $X \in \mathsf{Bibun}$ we can consider the product 2–functor

$$(-) \times X\colon \mathsf{Bibun} \to \mathsf{Bibun}/X.$$

This is product preserving, and sends any smooth 2–group to a smooth 2–group over X. If G is a smooth 2–group, then by a smooth G–stack over X we will mean a smooth $G \times X$–stack over X. If $U \to X$ is any surjective submersion, then the pullback functor

$$\mathsf{Bibun}/X \to \mathsf{Bibun}/U$$

is product preserving as well, hence sends smooth G–stacks over X to smooth G–stacks over U.

Example 69 Let A, B, and C be smooth 2–group, let $f\colon A \to B$ and $g\colon B \to C$ be homomorphisms. Then f gives B the structure of a smooth A–stack. Moreover, if $\phi\colon gf \to 0$ is a 2–homomorphism, then the action of A on B induced by f may be canonically augmented via ϕ to an action over C. Hence in this case B is an A–object over C.

Definition 70 (Principal bundles) Let G be a smooth 2–group. A smooth G–stack Y over X is a *G–principal bundle* if it is locally trivial as a G–stack, ie there exists a covering bibundle $f\colon U \to X$ such that f^*Y is G–equivariantly equivalent to $U \times G$ as a smooth G–stack over U.

Example 71 (Ordinary principal bundles) Let G be a Lie group, thought of as a smooth 2–group with only identity morphisms. Let X be a manifold thought of as a Lie groupoid with only identity morphisms. Then a G–principal bundle over X is the same as a G–principal bundle over X in the usual sense.

Example 72 (Abelian cocycle data) Let A be an abelian Lie group, X a manifold. Let $Y \to X$ be a surjective submersion, and fix a Čech 2–cocycle $\lambda\colon Y^{[3]} = Y \times_X Y \times_X Y \to A$. We construct a $[\mathrm{pt}/A]$–principal bundle E^λ over X as follows. The objects of E^λ consist of the manifold Y. The morphisms of E^λ consist of the manifold $Y^{[2]} \times A$. Composition is given by the formula

$$E_1^\lambda \times_Y E_1^\lambda \to E_1^\lambda$$
$$(y_0, y_1, a) \times (y_1, y_2, b) \mapsto (y_0, y_2, a + b + \lambda(y_0, y_1, y_2)).$$

This is readily seen to be a Lie groupoid equipped with a map to X and an [pt/A]–action over X. The pullback of E^λ along $Y \to X$ is a trivial [pt/A]–space over Y, so that E^λ is indeed a principal bundle.

Example 73 (Abelian bundle gerbes) More generally, an *A–bundle gerbe over X* in the sense of Murray [45] is an [pt/A]–principal bundle over X. Let A be an abelian Lie group, X a manifold. An A–bundle gerbe over X consists of $Y \to X$ a surjective submersion, $L \to Y^{[2]}$ an A–principal bundle, together with an isomorphism of A–principal bundles over $Y^{[3]}$,

$$\lambda\colon d_2^* L \otimes_A d_0^* L \to d_1^* L,$$

such that the induced map $d\lambda\colon Y^{[4]} \to A$ is trivial. From this we construct an [pt/A]–principal bundle E^λ over X as follows. The objects of E^λ are the manifold Y. The morphisms are the elements of the manifold L, with source and target maps induced from the maps

$$L \to Y^{[2]} \rightrightarrows Y.$$

Composition is induced from the map λ, and is associative because $d\lambda = 0$. There exists a covering $c\colon U \to Y$ such that the induced A–bundle $c^* L \to U^{[2]}$ is trivial, and pulling E^λ back to U consequently yields a trivial [pt/A]–stack over U.

The last example can be given a more conceptual description. Just as maps from a space to a topological group again form a group, maps from a manifold to a smooth 2–group form a 2–group. A map from a manifold Z to the 2–group [pt/A] consists precisely of an A–bundle L over Z. The (2-)group structure is given by tensoring A–bundles. Thus the above abelian bundle gerbe data consists of a map of stacks $L\colon Y^{[2]} \to$ [pt/A], together with an isomorphism of bibundles $\lambda\colon dL \to 0$ from $Y^{[3]}$ to [pt/A], such that $d\lambda$ is the canonical isomorphism $d^2 L \cong 0$ of bibundles from $Y^{[4]}$ to [pt/A].

Example 74 (Classical nonabelian bundle gerbes) The classical data of a nonabelian bundle gerbe as described in [3] consists of a nonabelian Lie group H, a manifold X, a surjective submersion Y, a bibundle $\mathcal{E}$ from $Y^{[2]}$ to the smooth crossed module 2–group [Aut(H)/H] from Example 67, and an isomorphism of bibundles $\lambda\colon d\mathcal{E} \to 1$

from $Y^{[3]}$ to $[\mathrm{Aut}(H)/H]$, such that $d\lambda$ is the canonical isomorphism $d^2\mathcal{E} \cong 0$ of bibundles from $Y^{[4]}$ to $[\mathrm{Aut}(H)/A]$.

This can be made into an $[\mathrm{Aut}(H)/H]$–principal bundle over X in a manner analogous to the above constructions. The objects of this Lie groupoid are the manifold $Y \times \mathrm{Aut}(H)$, and the morphisms are the manifold $\mathcal{E} \times \mathrm{Aut}(H)$. Composition is defined using the above structures and there is an induced action of $[\mathrm{Aut}(H)/H]$. By choosing a covering of Y appropriately (so that the pullback of $\mathcal{E}$ can be trivialized) we see that this yields a principal bundle over X.

One of the advantages of using the bicategory Bibun to define the notion of gerbe is that it automatically produces the correct notion of *equivalence* of gerbe over X. To see this, consider a covering $U \to X$ and the corresponding Čech groupoid $X_U = (U^{[2]} \rightrightarrows U)$. There is a canonical functor $X_U \to X$ given by projection. This functor is almost never an equivalence in LieGpd; see Example 8. However, the bundlization is *always* an invertible bibundle. It is an equivalence in Bibun. For this reason the stable equivalences which need to be formally inverted in the approaches given in [45; 3], correspond to honest equivalences of G–stacks over X. This approach is similar to the ones presented in [9; 8].

In the last section of this paper we will provide a model of the String group as a smooth 2–group which is not of the form of the Examples 65, 66, 67 or 68. Nevertheless, the above material allows us to discuss principal $\mathrm{String}(n)$–bundles over a given manifold X. The notion of string structure introduced in [64] yields such a principal $\mathrm{String}(n)$–bundle for the model of $\mathrm{String}(n)$ constructed in the final section of this paper.

3.6 Extensions of 2–groups

Definition 75 An *extension* of a smooth 2–group G by a smooth 2–group A consists of a smooth 2–group E, a homomorphisms $f\colon A \to E$, a homomorphism $g\colon E \to G$, and a 2–homomorphism $\phi\colon gf \to 0$, such that E is an A–principal bundle over G.

Lemma 76 *For any extension of smooth 2–groups as above, the following diagram is a (homotopy) pullback in the bicategory of smooth 2–groups.*

(77)
$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & E \\ \downarrow & \Downarrow\phi & \downarrow g \\ 0 & \longrightarrow & G \end{array}$$

Proof The forgetful functor from smooth 2–groups to stacks reflects equivalences and is a (weak) right adjoint, hence preserves (homotopy) pullbacks. Since A is the pullback in stacks, it follows that A is also the pullback in smooth 2–groups. □

There is an obvious generalization to extensions of smooth abelian 2–groups. In that context, the above square is also a pushout square. In the nonabelian setting this fails, a fact which was graciously pointed out to us by an anonymous reviewer. Nevertheless, the above square does satisfy a closely related universal property, which we now formulate.

Definition 78 A *kernel square* of smooth 2–groups consists of a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \Downarrow\phi & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

ie homomorphisms $f\colon X \to Y$, $g\colon Y \to Z$, and a 2–homomorphism $\phi\colon gf \cong 0$, in which X is the pullback.

The 1–morphisms and 2–morphisms of kernel squares are the obvious ones for diagrams in bicategories. Given an extension of smooth 2–groups $A \stackrel{f}{\to} E \stackrel{g}{\to} G$, we may consider the sub-bicategory $\mathsf{KS}(f)$ of kernel squares

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow & \Downarrow\phi & \downarrow h \\ 0 & \longrightarrow & Z \end{array}$$

and those 1–morphisms and 2–morphism which restrict to the identity of $f\colon A \to E$.

Proposition 79 *Given an extension $A \to E \to G$ of smooth 2–groups as above, the kernel square $A \to E \to G$ is the initial object of $\mathsf{KS}(f)$.*

Before proving the above proposition, we must first introduce a technical lemma.

Lemma 80 *If $A \to E \to G$ is an extension of smooth 2–groups, then the induced augmented simplicial object in smooth 2–groups*

$$G \leftarrow E \leftleftarrows E \times_G E \Lleftarrow E \times_G E \times_G E \cdots$$

is a (homotopy) colimit diagram.

Proof of Proposition 79, assuming Lemma 80 Let $E^\bullet$ denote the simplicial 2–group in stacks,

$$E \leftleftarrows E \times_G E \Lleftarrow E \times_G E \times_G E \cdots .$$

Given a kernel square $A \to E \to X$, there exists a 2–homomorphism from $E^\bullet$ to the constant simplicial object X, which agrees with the given homomorphism $E \to X$ on the zero$^{\text{th}}$ objects. Moreover the category of such 2–homomorphisms of simplicial objects is contractible. This can be seen, for example, by identifying the A–stack $E \times_G \cdots \times_G E$ with the A–stack $E \times A \times \cdots \times A$.

Thus, by Lemma 80, the homomorphism $E \to X$ factors uniquely through the homomorphism $E \to G$. Because both $A \to E \to G$ and $A \to E \to X$ are pullback squares, this extends to an essentially unique morphism of kernel squares. □

It remains to prove Lemma 80. Up to this point we have been primarily concerned with 2–groups in the bicategory of presentable stacks in the surjective submersion topology on smooth manifolds. It is surely possible to provide a direct proof of Lemma 80 entirely within this bicategory. However, for the sake of brevity, we will now also contemplate the bicategory Stack of *all stacks* for the surjective submersion topology on manifolds. In other words, Stack is the bicategory of all fibered categories over Man (fibered in groupoids) which satisfy stack descent with respect to the surjective submersion topology.

Unlike Bibun this larger bicategory is complete and cocomplete (in the higher categorical sense), and there exists a monadic adjunction

$$F\colon \mathsf{Stack} \rightleftarrows 2\,\mathsf{Grp}(\mathsf{Stack}) : U$$

between stacks and the bicategory of 2–groups in stacks (see Example 62). Thus we may now apply the higher categorical Barr–Beck Theorem [39, Theorem 3.4.5], which has the following corollary.

Corollary 81 *Let $F\colon \mathsf{C} \rightleftarrows \mathsf{D} : U$ be a monadic adjunction of bicategories (for example $\mathsf{C} = \mathsf{Stack}$ and $\mathsf{D} = 2\,\mathsf{Grp}(\mathsf{Stack})$). Then for any object $X \in \mathsf{D}$, the colimit of the simplicial object $FU^\bullet(X)$ exists in D and agrees with the object X.*

Proof This follows immediately from [39, Theorem 3.4.5], as the augmented simplicial object $X \leftarrow FU^\bullet(X)$ is U–split. □

Proof of Lemma 80 Consider the augmented simplicial object

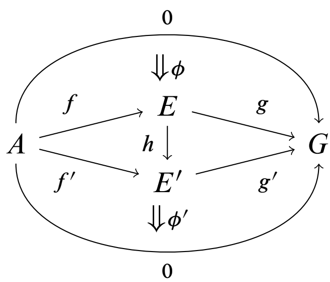
$$G \leftarrow E \leftleftarrows E \times_G E \Lleftarrow E \times_G E \times_G E \cdots$$

which we write $G \leftarrow E^\bullet$. Because E is an A–principal bundle over G, the diagram $U(G) \leftarrow U(E^\bullet)$ is a colimit diagram in Stack (and hence in presentable stacks). Since F is a left adjoint, it preserves colimits. Hence the diagram $FU(G) \leftarrow FU(E^\bullet)$ is a colimit diagram in 2–groups in stacks.

We may now consider the bisimplicial diagram of 2–groups in stacks $\{FU^p(E^{[q]})\}$, where $E^{[q]} = E \times_G \cdots \times_G E$, q–times. We may compute the colimit of this diagram in two ways, each consisting of two steps. We may first take the colimit in the p–direction, after which we obtain the simplicial diagram $E^\bullet$. Hence the colimit of this bisimplicial 2–group is precisely the colimit of $E^\bullet$.

On the other hand, we may instead take the colimit first in the q–direction. For each fixed p, this is the colimit of $FU^p(E^\bullet)$, which we have already observed is $FU^p(G)$. Thus the colimit of this bisimplicial 2–group is also the colimit of the simplicial diagram $FU^\bullet(G)$. By Corollary 81, this is precisely G. □

Given a smooth abelian 2–group A and a smooth 2–group G the central extensions of G by A form a bicategory $\mathrm{Ext}(G; A)$. A 1–morphism of extensions $(E, f, g, \phi) \to (E', f', g', \phi')$, consists of an equivalence of 2–groups $h\colon E \to E'$, together with 2–isomorphisms $\alpha\colon hf \cong f'$ and $\beta\colon g'h \cong g$, such that the following pasting diagram is the identity:



The 2–morphisms in $\mathrm{Ext}(G; [\mathrm{pt}/A])$ are given by 2–isomorphisms $\psi\colon h \to h'$, such that $\alpha = \alpha' \circ (\psi * id y_f)$ and $\beta = \beta' \circ (if_g * \psi)$. By construction the bicategory $\mathrm{Ext}(G; A)$ is contravariantly 2–functorial in G, covariantly 2–functorial in A, and commutes with products. Thus the usual Baer sum operation equips $\mathrm{Ext}(\mathbb{G}; \mathbb{A})$ with the structure of a symmetric monoidal bicategory (see Gordon, Power and Street [22], Kapranov and Voevodsky [35; 34], Baez and Nuechl [6], Day and Street [16] and the author's thesis [54, Chapter 3]).

Since the 2–category of discrete 2–groups embeds into the bicategory of smooth 2–groups (see Remark 64) this gives a notion of extension of discrete 2–groups. Just as the theory of discrete groups is more elementary then the theory of topological groups, so too the theory of discrete 2–groups is easier then the theory of topological (or smooth) 2–groups. Nevertheless, we can learn many things by comparing these two settings. For example, extensions of groups G by abelian groups A are categorized according to the induced action of G on A. In the topological setting, such actions are more problematic because such an action should be required to be a *continuous* homomorphism $G \to \mathrm{Aut}(A)$, where this latter group is the group of continuous automorphisms of A. This is defined using the internal hom for topological spaces.

While this doesn't pose a significant problem for finite-dimensional Lie groups, there are further issues to contend with in the case of smooth stacks. Noohi [46] and Carchedi [14] have independently gone to some pains to understand conditions under which internal homs exist in the setting of topological stacks.[14] The smooth situation is likely more difficult.

These difficulties disappear in the discrete setting, where internal homs for the 2–category of groupoids are well known to exist. Even though the general theory of smooth actions of groups presents problems, it is still possible to define *central* extensions merely by comparing with the discrete case. An extension of topological groups $A \to E \to G$ is a central extension precisely when it is a central extension of discrete groups, after forgetting the topology. We will employ the same strategy to define central extensions of smooth 2–groups.

Lemma 82 *Given an extension of discrete 2–groups $A \to B \to C$ with A an abelian 2–group, there exists a homomorphism of 2–groups $C \to \mathrm{Aut}(A)$, unique up to unique 2–homomorphism, where $\mathrm{Aut}(A)$ is the automorphism 2–group of A.*

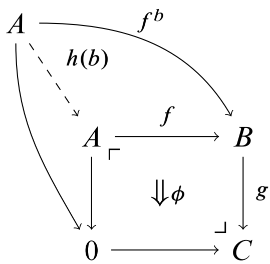
Proof Let $f\colon A \to B$, $g\colon B \to C$, and $\phi\colon gf \to 0$ be the homomorphisms and 2–homomorphism of the extension of 2–groups. Choose a functorial assignment $b \mapsto \overline{b}$ of weak inverses for the elements of B together with functorial isomorphisms $b \otimes \overline{b} \cong 1$. Such an assignment is unique up to unique isomorphism.

For each object $b \in B$ we may form an automorphism of B given by conjugation. Specifically we consider the functor defined on objects $x \in B$ by $x \mapsto b \otimes [x \otimes \overline{b}]$. This can be made a self homomorphism of B by using the structure maps of B. It is compatible with composition and induces a homomorphism of 2–groups $B \to \mathrm{Aut}(B)$.

Precomposing with f yields, for each b, a new homomorphism $f^b\colon A \to B$ together with a 2–homomorphism $\phi^b\colon gf^b \to 0$. On objects we have $f^b(a) = b \otimes [f(a) \otimes \overline{b}]$, where $\overline{b}$ is the functorial weak inverse of b. The structure morphisms of f^b and ϕ^b are

[14] The internal hom always exists as a fibered category and automatically satisfies stack descent. The main problem is in proving that such fibered categories are *presentable* by smooth or topological groupoids.

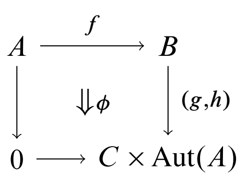
canonically induced by those of f, ϕ, g, and the 2–group structure of B. A morphism $b \to b'$ in B induces a natural isomorphism $f^b \to f^{b'}$. Thus by the universal property of the pullback, for each object of b we have a homomorphism as in the following diagram.



Morphisms $b \to b'$ in B induce 2–homomorphisms $h(b) \to h(b')$. The assignment $h\colon b \mapsto f^b$, $h\colon (b \to b') \mapsto (f^b \to f^{b'})$ is not strictly canonical, but depends upon a contractible category of choices.

The assignment $b \mapsto h(b)$ is compatible with the multiplication in B in the sense that $h(b \otimes b') \cong h(b) \circ h(b')$. These isomorphisms may be chosen to be functorial and yield a homomorphism of 2–groups $h\colon B \to \mathrm{Aut}(A)$. Again this choice is unique up to unique isomorphism. Precomposition yields a homomorphism $A \to \mathrm{Aut}(A)$. However, if A is abelian, then the braiding allows us to canonically trivialize this composite.

We now use the universal property of the extension $A \to B \to C$ to factor h by an essentially unique homomorphism $B \to C \to \mathrm{Aut}(A)$. The trivialization of $A \to \mathrm{Aut}(A)$ permits us to form the following square:



This square is readily checked to be a kernel square. Since $A \to B \to C$ is the initial such kernel square, there exists an essentially unique morphism of kernel squares from $A \to B \to C$ to the above kernel square. In particular this consists of a homomorphism $C \to C \times \mathrm{Aut}(A)$, and projecting to the second factor yields the desired homomorphism $C \to \mathrm{Aut}(A)$. □

Given a smooth 2–group, we obtain a discrete 2–group by applying the forgetful 2–functor $U\colon \mathsf{Bibun} \to \mathsf{Gpd}$, which forgets the topology. Thus to every extension of smooth 2–group we get a corresponding extension of discrete 2–groups.

Definition 83 An extension of discrete 2–groups $A \to B \to C$, with A abelian, is *central* if the induced homomorphism $C \to \mathrm{Aut}(A)$ is isomorphic to the trivial homomorphism. An extension of smooth 2–groups is *central* if the corresponding extension of discrete 2–groups is central.

By the work of [5], every discrete 2–group is equivalent to a *skeletal* 2–group[15] and these are classified by the following invariants:

(1) a group $\pi_0\Gamma$ = isomorphism classes of objects,

(2) an abelian group $\pi_1\Gamma = \mathrm{Hom}_\Gamma(1,1)$,

(3) an action ρ of $\pi_0\Gamma$ on $\pi_1\Gamma$, (induced by conjugating an automorphism of 1 by an object in G),

(4) the k-invariant $[a] \in H^3(\pi_0\Gamma; \pi_1\Gamma, \rho)$, which is determined by the associator of Γ.

(See [5] for details.) Part of this classification is the construction of a 2–group from a this data. This construction will play a role in what follows, so we review it. Consider the data (G, A, ρ, α) where G is a group, A is an abelian group, ρ is an action of G on A and $\alpha \in Z^3_{\mathrm{grp}}(G; A, \rho)$ is a group cocycle. Then we may form the following skeletal 2–group $\Gamma(G, A, \rho, \alpha)$: The objects of Γ are the elements G, the morphisms of Γ are the product space $G \times A$, with both source and target maps the projection to G. The automorphisms of each object are identified with the fiber A, as a group. The monoidal structure is given by the group multiplication in G on objects and the group multiplication of $A \rtimes_\rho G$ on morphisms. It is strictly associative, nevertheless we equip it with a nontrivial associator determined by α. The associator is given by $a_{g_0,g_1,g_2} = \alpha(g_0, g_1, g_2) \in A$, using the identification of A with the automorphisms of $g_0g_1g_2$. That α is a cocycle ensures that the pentagon identity is satisfied. The left and right unitors are uniquely determined by α and the requirement that the triangle identity hold.

There exist canonical homomorphisms of 2–groups $f\colon [\mathrm{pt}/A] \to \Gamma(G, A, \rho, \alpha)$ and $g\colon \Gamma(G, A, \rho, \alpha) \to G$ given by the obvious inclusion and projection. The composition gf is equal to the zero map $[\mathrm{pt}/A] \to G$, which in this case has no nonidentity automorphisms.

Lemma 84 *Consider an extension of discrete 2–groups of the form*

$$\begin{array}{ccccc} A & & E_1 & & G \\ \downdownarrows & \xrightarrow{f} & \downdownarrows & \xrightarrow{g} & \downdownarrows \\ \mathrm{pt} & & E_0 & & G \end{array}$$

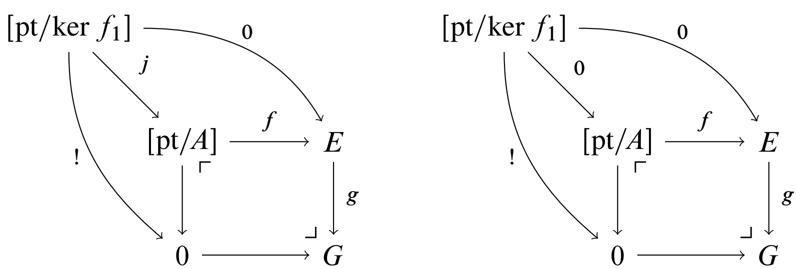
[15] A discrete 2–group Γ is skeletal if the for all objects $x, x' \in \Gamma$ the condition $x \cong x'$ implies $x = x'$.

(the 2–homomorphism ϕ: $gf \to 0$ is unique if it exists, hence is determined by f and g). Then this extension is equivalent to an extension such that $E = \Gamma(G, A, \rho, \alpha)$, with f and g the canonical inclusion an projection homomorphisms. Moreover, all such inclusion-projection sequences are extensions, and such an extension is central if and only if the action ρ is trivial.

Proof By the work of [5] we know that E is equivalent to *some* skeletal 2–group $E \simeq \Gamma(H, B, \rho, \alpha)$, so it suffices to consider that case. The homomorphisms f and g are determined by their component homomorphisms f_1: $A \to B$ and g_0: $H \to G$.

Consider the 2–group [pt/B]. This has a canonical inclusion functor ℓ: [pt/B] $\to E$ whose composition with g is zero. Thus by the universal property of the pullback, there exists a (unique) homomorphism [pt/B] $\to$ [pt/A] which factors this inclusion. In particular f_1 must be a split surjection.

Now consider the kernel $\ker f_1$, with its inclusion j: $\ker f_1 \to A$. This yields a homomorphism of 2–groups j: [pt/$\ker f_1$] $\to$ [pt/A], such that the following two diagrams commute (strictly):



By the universal property of the pullback this implies that $j \cong 0$ and hence f_1: $A \to B$ is an isomorphism.

Dually, consider the group H viewed as a 2–group with only identity morphisms. The canonical projection $E = \Gamma(H, B, \rho, \alpha) \to H$ is a homomorphism such that the composite

$$[\mathrm{pt}/A] \to E = \Gamma(H, B, \rho, \alpha) \to H$$

is isomorphic to the zero homomorphism (if such an isomorphism exists it is unique). Thus there exists an essentially unique morphism [pt/A] $\to K$, where $K = \ker(E \to H)$. Conversely, since the map $E \to G$ factors as $E \to H \to G$, we obtain a unique map $K \to$ [pt/A]. These are easily checked to be inverses so that $K =$ [pt/A], and hence [pt/A] $\to E \to H$ is a kernel square.

Thus by the universal property of the extension $[\mathrm{pt}/A] \to E \to G$, there exists a (unique) group homomorphism $k\colon G \to H$ making the following diagram commute.

$$\begin{array}{ccc} H & & \\ {\scriptstyle g_0}\downarrow & \searrow^{\mathrm{id}} & \\ G & \underset{k}{\dashrightarrow} & H \end{array}$$

In particular the kernel of $g_0\colon H \to G$ is zero.

We may compose the map $E \to H$ with the homomorphism $g_0\colon H \to G$, and thereby obtain a map of kernel squares from $[\mathrm{pt}/A] \to E \to G$ to itself which restricts to the map $g_0 \circ k\colon G \to G$. Since this kernel square is initial, this composite must be the identity on G. Thus g_0 is an isomorphism.

A similar argument shows that the inclusion-projection sequence $[\mathrm{pt}/A] \to E \to G$ is always an extension. The automorphism 2–group of the 2–group $[\mathrm{pt}/A]$ is equivalent to the group $\mathrm{Aut}(A)$ viewed as a 2–group with only identity morphisms. Following the previous construction, we see that the induced map $G \to \mathrm{Aut}([\mathrm{pt}/A]) = \mathrm{Aut}(A)$ is precisely the action ρ, and thus the extension is central precisely when ρ is trivial. □

The above notion of central extension of smooth 2–group is more general then the notion introduced by Wockel [66]. In particular it is invariant under equivalence of smooth 2–group, and includes the following examples not covered by Wockel's treatment.

Example 85 Let $\Gamma = (G, A, a)$ be the 2–group from Example 68. There exists a canonical central homomorphism (inclusion) $i\colon [\mathrm{pt}/A] \to \Gamma$ and a homomorphism (projection) $\pi\colon \Gamma \to G$. The composite is equal to the zero homomorphism $[\mathrm{pt}/A] \to G$. This has a unique automorphism ϕ, and with this choice the triple (i, π, ϕ) is a central extension. Γ is a trivial principal bundle over G in the sense that it is equivalent to $G \times [\mathrm{pt}/A]$ as a $[\mathrm{pt}/A]$–stack over G.

Example 86 Let A be an abelian Lie group. There is a unique homomorphism from A to the abelian 2–group $[\mathrm{pt}/A]$. This homomorphism factors as the composite

$$A \xrightarrow{f} 0 \xrightarrow{g} [\mathrm{pt}/A]$$

The automorphisms of $0 = gf\colon [\mathrm{pt}/A] \to A$ are in canonical bijection with $\phi \in \hom(A, A)$. The triple (f, g, ϕ) is a central extension precisely when ϕ is an automorphism.

3.7 Smooth 2–groups and A_∞–spaces

Given a smooth 2–group Γ we may obtain a space by taking the geometric realization $|\Gamma|$; see Segal [55]. In the trivial case, when $\Gamma = G$, the resulting space is $|G| \cong G$ and hence is a topological group. This is too much to expect in general, and indeed the geometric realization functor, viewed as an assignment in the category of spaces, is not precisely functorial with respect to bibundles. It does however lead to a functor $\mathsf{Bibun} \to \text{h–}\mathsf{Top}$ with values in the *homotopy category* of spaces, as we shall see.

In particular any smooth 2–group gives rise to a group-like H-space, and this assignment is functorial. In this section we will show that this can be improved upon to give an infinitely coherent multiplication (A_∞–structure) on the geometric realization of any smooth 2–group. Everything in this section holds equally well in the topological setting, provided the source and target maps admit local sections.

Given a Lie groupoid $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ we may construct an associated groupoid. The target $t\colon \Gamma_1 \to \Gamma_0$ is a surjective submersion so we may construct the corresponding Čech groupoid $E^{(t)}\Gamma = (\Gamma_1 \times^{t,t}_{\Gamma_0} \Gamma_1 \rightrightarrows \Gamma_1)$; see Example 5. As always with Čech groupoids, there is a functor to the space Γ_0, viewed as a Lie groupoid. In this case that functor is given by the target map $t\colon E^{(t)}\Gamma \to \Gamma_0$. This map is an equivalence of Lie groupoids: the identity map $\iota\colon \Gamma_0 \to \Gamma_1$ induces a functor the other direction which is an inverse to t. This equivalence can be taken to be over the Lie groupoid Γ_0.

Moreover, there is a functor $\sigma\colon E^{(t)}\Gamma \to \Gamma$ which on objects is $s\colon \Gamma_1 \to \Gamma_0$, and on morphisms is given by $(f,g) \mapsto g \circ f^{-1}$. Passing to the nerve, we see on each level that we have a space $E^{(t)}\Gamma_n$, which consists of n–tuples of morphisms of Γ, all with the same target. There is an action by Γ in the sense that postcomposition gives a map

$$\Gamma_1 \times^{s,t}_{\Gamma_0} E^{(t)}\Gamma_n \to E^{(t)}\Gamma_n$$

This action map is over $\Gamma_0 \times \Gamma_n$, and $E^{(t)}\Gamma_n$ becomes a bibundle from the space Γ_n to Γ.

Geometric realization of these simplicial spaces is stable under fiber products by May [43, Corollary 11.6] (see also Lewis [38] and Rezk [50]) and thus upon geometric realization we find that $|E^{(t)}\Gamma|$ is a (left principal) bibundle from the classifying space $|\Gamma|$ to the groupoid Γ, now viewed in the topological category rather then the smooth category[16]. The bibundle $|E^{(t)}\Gamma|$ is the analog of the classifying bundle of a

[16]It is not clear from our description that $|E^{(t)}\Gamma|$ will be locally trivial over $|\Gamma|$, ie admit local sections. Indeed this fails for general topological groupoids. However, in the case that the spaces involved are locally contractible, local triviality follows from an argument identical to the proof of [56, Proposition A.1]. Since local triviality is not used in our argument we will omit these details.

group: when $\Gamma = (G \rightrightarrows \mathrm{pt})$ is a Lie group, $|E^{(t)}G| = |EG|$ is exactly the classifying bundle in the usual sense.

Moreover, the equivalence between $E^{(t)}\Gamma$ and Γ_0 induces a homotopy equivalence between $|E^{(t)}\Gamma|$ and Γ_0 in the category of spaces over Γ_0, ie $|E^{(t)}\Gamma|$ is homotopy equivalent to the terminal object of Top_{Γ_0}. This is the appropriate analog of *contractible* in the relative category Top_{Γ_0}. When $\Gamma = G$ is a group, then $\Gamma_0 = \mathrm{pt}$ and hence EG is contractible in the usual sense. Starting with the source map $s\colon \Gamma_1 \to \Gamma_0$ yields an analogous story, but the outcome is a *right* principal bibundle $|E^{(s)}\Gamma|$ from Γ to $|\Gamma|$. Again $|E^{(s)}\Gamma| \simeq \Gamma_0$ as spaces over Γ_0. The inversion isomorphism allows us to canonically identify $|E^{(t)}\Gamma| \cong |E^{(s)}\Gamma|$, thus we obtain an isomorphism of spaces $|E^{(t)}\Gamma| \times_{|\Gamma|} |E^{(s)}\Gamma| \cong \Gamma_1 \times_{\Gamma_0} |E^{(t)}\Gamma|$. Projection gives rise to a map

$$|E^{(t)}\Gamma| \times_{|\Gamma|} |E^{(s)}\Gamma| \to \Gamma_1 \tag{87}$$

which commutes with both the left and right Γ–actions.

Lemma 88 *Let P be a (left principal) bibundle from the space X to the Lie groupoid Γ. Then the composition $|E^{(s)}\Gamma| \circ P$ with the* right *principal bundle $|E^{(s)}\Gamma|$ is a space over X homotopy equivalent to X over X. In particular the space of sections is a contractible space.*

Proof This is a local statement and so it is enough to consider the case when we have a map $f\colon X \to \Gamma_0$ and $P \cong \Gamma_1 \times^{s,f}_{\Gamma_0} X$. The composition with $|E^{(s)}\Gamma|$ then becomes $|E^{(s)}\Gamma| \circ P \cong |E^{(s)}\Gamma| \times_{\Gamma_0} X$. But since $|E^{(s)}\Gamma| \simeq \Gamma_0$ over Γ_0 the space $P \circ |E^{(s)}\Gamma|$ is homotopy equivalent to X over X. □

Corollary 89 *A bibundle P from the Lie groupoid G to the Lie groupoid Γ gives rise to a contractible family of morphisms from $|G|$ to $|\Gamma|$ (described in the proof below). This association is compatible with composition, and therefore yields a functor $\mathsf{Bibun} \to h\text{–}\mathsf{Top}$.*

Proof Consider the following chain of bibundles:

$$\begin{array}{ccccccccccc} & |E^{(s)}\Gamma| & & \Gamma_1 & & P & & G_1 & & |E^{(t)}G| & \\ \swarrow & & \searrow & \downdownarrows & \swarrow & & \searrow & \downdownarrows & \swarrow & & \searrow \\ |\Gamma| & & & \Gamma_0 & & & & G_0 & & & |G| \end{array}$$

Note that P and $|E^{(t)}G|$ are left-principal bibundles, while $|E^{(s)}\Gamma|$ is a right-principal bibundle. Composing these three yields a space $K = |E^{(s)}\Gamma| \circ P \circ |E^{(t)}G|$, with two maps, one to $|\Gamma|$ and one to $|G|$. Moreover, by the previous lemma, K is homotopy equivalent to $|G|$ over $|G|$. Hence the space $\mathcal{S}$ of sections of K over $|G|$ is contractible. Composing a section $|G| \to K$ with the projection $K \to |\Gamma|$ induce the desired family $\mathcal{S} \to \mathrm{Maps}(|G|, |\Gamma|)$.

For a composable pair of bibundles $P\colon G \to \Gamma$, and $Q\colon H \to G$, a pair of sections in $\mathcal{S}_P$ and $\mathcal{S}_Q$ gives rise to a section over $|H|$ of the space

$$|E^{(s)}\Gamma| \circ P \circ |E^{(t)G}| \times_{|G|} |E^{(s)}G| \circ Q \circ |E^{(t)H}|.$$

From this, we get an section over $|H|$ of $|E^{(s)}\Gamma| \circ P \circ Q \circ |E^{(t)H}|$, and hence a map $\mathcal{S}_P \times \mathcal{S}_Q \to \mathcal{S}_{P\circ Q}$, by composing with the map of spaces

$$|E^{(t)}G| \times_{|G|} |E^{(s)}G| \to G_1$$

described in Equation (87). The compatibility of this map with composition follows from the biequivariance of the map in Equation (87). □

Remark 90 A more sophisticated approach is to consider the bicategory Bibun as an $(\infty, 1)$–category. Then the above corollaries and proposition may be summarized by saying that there is an ∞–functor from Bibun to the $(\infty, 1)$–category of topological spaces.

Corollary 91 *The geometric realizations of Morita equivalent Lie groupoids are homotopy equivalent.*

Proof In the setting of the previous proof, a bibundle P between Lie groupoids gives rise to a space K with maps to $|G|$ and $|\Gamma|$. In the case that P is a Morita equivalence, both these maps are homotopy equivalences. □

Definition 92 A *topological operad* consists of a collection of spaces S_n for each $n \geq 0$, together with composition maps

$$S_n \times S_{i_1} \times \cdots S_{i_n} \to S_{i_1+\cdots+i_n}$$

which are associative in the obvious way. An *algebra* for a topological operad S is a space X together with actions maps

$$S_n \times \underbrace{X \times \cdots X}_{n \text{ times}} \to X$$

which again are associative and compatible with the maps from S, in the obvious way.

Definition 93 (May [43]) An A_∞*–operad* is any topological operad with contractible spaces. An A_∞*–space* is a space X which is an algebra for an A_∞–operad.

Theorem 94 *The geometric realization of a smooth* 2*–group in* Bibun *is naturally an* A_∞*–space.*

Proof We must construct an A_∞–operad and an action of this operad on $|\Gamma|$. Consider the composition of bibundles, $|E^{(s)}\Gamma| \circ m \circ (|E^{(t)}\Gamma| \times |E^{(t)}\Gamma|)$. This is a space with a map to $|\Gamma|$ coming from $|E^{(s)}\Gamma|$ and a map to $|\Gamma|\times|\Gamma|$ coming from $|E^{(t)}\Gamma|\times|E^{(t)}\Gamma|$. Thus if we choose a section over $|\Gamma| \times |\Gamma|$ we get a map of spaces. Let S_2 denote the space of sections over $|\Gamma| \times |\Gamma|$. Putting these maps together gives us a map

$$S_2 \times |\Gamma| \times |\Gamma| \to |\Gamma|$$

which is continuous. Recall, however, that the space S_2 is contractible. Thus the contractible space S_2 parametrizes several multiplications for the space $|\Gamma|$. We mimic this and define contractible spaces of sections S_n for all n. For $n \geq 2$ S_n is the space of sections (over $|\Gamma|^n$) of

$$|E^{(s)}\Gamma| \circ \underbrace{m \circ (m \times 1) \circ (m \times 1 \times 1) \circ \cdots}_{n \text{ times}} \circ \underbrace{(|E^{(t)}\Gamma| \times |E^{(t)}\Gamma| \times \cdots \times |E^{(t)}\Gamma|)}_{n \text{ times}}.$$

This is again a contractible space. We set $S_0 = S_1 = \mathrm{pt}$. Since we started with a Lie 2–group we have a specified isomorphism of bibundles $m \circ (m \times 1) \cong m \circ (1 \times m)$, given by the associator. Mac Lane's coherence theorem ensures us that this extends to a canonical isomorphism between any two possible bracketings. For example the composition

$$m \circ (m \times 1) \circ (m \times m \times m) \circ (1 \times 1 \times m \times 1 \times m \times 1)$$

is canonically isomorphic to the composition

$$\underbrace{m \circ (m \times 1) \circ (m \times 1 \times 1) \circ \cdots}_{7 \text{ times}}.$$

We turn the collection of spaces S_n into an A_∞–operad as follows. A point in the space, $S_{i_1} \times \cdots S_{i_n}$ is a section of the corresponding product of bibundles (over $|\Gamma|^{(i_1+\cdots+i_n)}$). These bundles project to $|\Gamma|^n$ and so we get a map

$$|\Gamma|^{(i_1+\cdots+i_n)} \to |\Gamma|^n.$$

A point in S_n then gives us a section (over $|\Gamma|^n$) of its corresponding bundle. When we compose these bundles, we get a corresponding composition of sections. This is a section of a certain bundle over $|\Gamma|^{(i_1+\cdots+i_n)}$, similar in construction to S_n, but with a different bracketing. The canonical identification from the associator allows us to identity this with a point of $S_{i_1+\cdots+i_n}$. Hence we have assembled maps

$$S_n \times S_{i_1} \times \cdots S_{i_n} \to S_{i_1+\cdots+i_n}.$$

It can readily be checked that this is an operad (the compositions involving S_1 and S_0 are similar, where S_1 corresponds to sections of the identity bibundle $|\Gamma| \to |\Gamma|$ and S_0 to sections of the unit bibundle $\iota\colon 1 \to |\Gamma|$). Moreover since the spaces are contractible, this is an A_∞–operad.

We have also seen how $|\Gamma|$ is naturally an algebra for this operad. S_n is a space of sections of a bundle over $|\Gamma|^n$ and this bundle has a map to $|\Gamma|$, hence there is an induced action map

$$S_n \times |\Gamma|^n \to |\Gamma|$$

which makes $|\Gamma|$ into an A_∞–space. □

Remark 95 With more work one sees that a homomorphism of smooth 2–groups yields a morphism of A_∞–spaces.

4 A finite-dimensional string 2–group

In this section we prove a theorem which interprets Segal–Mitchison topological group cohomology in terms of certain central extensions of smooth 2–groups. The model of the String group presented in this paper is a special case of such an extension.

4.1 Segal's topological group cohomology

Segal [56; 57] introduced a version of cohomology for (locally contractible) topological groups, which mimics the derived functor definition of ordinary group cohomology. A few years later, Quillen introduced the notion of *exact category* in his work on algebraic K–theory [48]. Roughly speaking, an exact category is an additive category equipped with a distinguished class of *short exact sequences*. Such a category is not required to be an abelian category, and there are many examples, among them the category of topological groups considered by Segal. It is now realized that essentially all the constructions and machinery of homological algebra carry over to the setting of exact categories; see Buehler [13] for a fairly comprehensive introduction and overview. In particular resolutions and derived functors can often be defined in this setting and Segal's cohomology is an example.

Segal's cohomology was rediscovered by Brylinski [12] in the smooth setting. This group cohomology solves many of the defects of the naive "group cohomology with continuous/smooth cochains", and certain cocycle representatives will serve as our basic input in constructing the String(n) 2–group. Let us summarize some of the special features of this cohomology theory. Proofs of these facts can be found in Segal [56; 57].

If G is a topological group and A is a topological G–module[17] then we can form the *Segal–Mitchison group cohomology* $H^n_{\mathrm{SM}}(G;A)$.

(1) In low dimensions, $q = 0, 1, 2$, $H^q_{\mathrm{SM}}(G;A)$ may be interpreted in the usual manner.
   (a) $H^0_{\mathrm{SM}}(G;A) = A^G$, the G–invariant subgroup,
   (b) $H^1_{\mathrm{SM}}(G;A)$ is the group of continuous crossed homomorphisms $G \to A$, modulo the principal crossed homomorphisms, and
   (c) $H^2_{\mathrm{SM}}(G;A)$ is the group of isomorphism classes of group extensions
$$A \to E \to G$$
   inducing the action of G on A, where $E \to G$ is topologically a locally trivial fibration, ie a fiber bundle.

(2) If A is contractible, then the Segal–Mitchison cohomology coincides with the continuous group cohomology.

(3) If A is discrete, then the Segal–Mitchison cohomology is isomorphic to the twisted cohomology of the space BG with coefficients in A.[18] In particular if the G action is trivial we have $H^n_{\mathrm{SM}}(G;A) \cong H^n(BG;A)$, the ordinary cohomology of the space BG with coefficients in A.

(4) A sequence of topological G–modules $A' \to A \to A''$ is a *short exact sequence* if it is a short exact sequence of underlying abelian groups and the action of A' on A realizes A as an A'–principal bundle over A''. If $A' \to A \to A''$ is such a short exact sequence then there is a long exact sequence of cohomology groups
$$0 \to H^0_{\mathrm{SM}}(G,A') \to H^0_{\mathrm{SM}}(G,A) \to H^0_{\mathrm{SM}}(G,A'') \to$$
$$\to H^1_{\mathrm{SM}}(G,A') \to H^1_{\mathrm{SM}}(G,A) \to \cdots.$$

(5) If G is a topological group and A is a topological G–module, then A determines a simplicial sheaf $\mathcal{O}_A$ on the simplicial space $BG_\bullet$. When the action of G on A is trivial, then $\mathcal{O}^n_A$ is simply the sheaf of continuous functions with values in A. In general we have $H^q_{\mathrm{SM}}(G;A) \cong H^q(BG_\bullet;\mathcal{O}_A)$, where $BG_\bullet$ is the simplicial nerve of G, $\mathcal{O}_A$ is the simplicial sheaf corresponding to A, and this latter group denotes the hypercohomology (see Friedlander [19] for details about simplicial hypercohomology).

[17] As mentioned in the introduction, and action of a topological group G on a topological abelian group A is an action in the usual sense such that the map $G \times A \to A$ is continuous, where $G \times A$ is given the compactly generated topology.

[18] If A is discrete then the action of G factors as $G \to \pi_0 G \to \mathrm{Aut}(A)$ and since $\pi_1 BG \cong \pi_0 G$, we have a canonical locally constant sheaf over BG. This can also be obtained as the sheaf associated to the fiber bundle $EG \times_G A \to BG$. This sheaf is used to define the twisted cohomology of BG.

The category of (locally contractible) topological abelian groups becomes an exact category with the short exact sequences introduced above. Segal's cohomology is then defined to be the derived functor of the invariant subgroup functor

$$\Gamma^G\colon A \mapsto A^G.$$

Segal [56] proves that this functor is derivable by demonstrating a class of objects adapted to this functor (his so-called "soft" modules). He also proves this cohomology has the above properties.

In the finite-dimensional smooth setting, there is an analogous exact structure. More precisely fix a Lie group G and consider the category of abelian Lie groups equipped with smooth actions of G. A sequence of such G–modules $A' \to A \to A''$ will be called a *short exact sequence* if it is a short exact sequence of underlying abelian groups and the action of A' on A realizes A as an A'–principal bundle over A''. Unfortunately this category will not contain enough adapted objects in order to derive the invariant subgroup functor.

This can be overcome by embedding abelian Lie groups into a lager category of "smooth" abelian group objects. For example abelian group objects in one of the "convenient categories of smooth spaces" discussed by Baez and Hoffnung [4] provide such an enhancement. Alternatively one could use the sheaf cohomology of the resulting simplicial sheaf on $BG_\bullet$; see Friedlander [19] for the relevant definitions. Both of these approaches result in the same cohomology theory which Brylinski [12] shows may be computed as the cohomology of the total complex of a certain double complex, which we now describe.

Definition 96 A *simplicial cover* (or just *cover*) of a simplicial manifold $X_\bullet$ is a simplicial manifold $U_\bullet$ and a map $U_\bullet \to X_\bullet$, such that each $U_n \to X_n$ is a surjective submersion. A cover is *good* if each of the spaces

$$U_n^{[p]} = \underbrace{U_n \times_{X_n} \cdots \times_{X_n} U_n}_{p \text{ times}}$$

is the union of paracompact contractible spaces, where $p, n \geq 0$.

Consider the simplicial manifold $BG_\bullet$. Brylinski [12] provides an inductive construction of a good simplicial cover of $BG_\bullet$[19]. For compact G, using techniques developed by Meinrenken [44] we may construct a canonical such cover. For our

[19]For general simplicial spaces it is not possible to construct *good* simplicial covers. In that case one must instead use good hypercovers. Brylinski's construction allows us to avoid this subtlety entirely.

purposes, however, any good simplicial cover will do and so we will not dwell on this aspect. For $A \in \mathsf{Top\,Ab}_G$, we get an induced double complex, where

$$C^{p,q} = C^\infty(U_q^{[p+1]}, A),$$

and the differentials are induced by the two simplicial directions. The cohomology of the total complex computes the smooth version of Segal's group cohomology. Let us fix some notation. Let G be a Lie group and A an abelian Lie group with a G–action.

- $H^k_{\mathrm{SM}}(G;A)$ is the smooth version of Segal's cohomology which we take to be the total cohomology of the double complex $C^{p,q} = C^\infty(U_q^{[p+1]}, A)$ computed from a good simplicial cover of $BG_\bullet$.
- $H^k_{\mathrm{smooth}}(G;A)$ denotes the cohomology of G computed with smooth group cocycles.
- We will primarily be interested in the case where the action of G on A is trivial. In this case $\check{H}^k(G;A)$ is the Čech cohomology of the space G with coefficients in the sheaf of smooth functions with values in A.

Corollary 97 *If G is a compact Lie group and $A = S^1$ then we have the isomorphism of smooth Segal–Mitchison cohomology*

$$H^i_{\mathrm{SM}}(G;S^1) \cong H^{i+1}_{\mathrm{SM}}(G;\mathbb{Z}) \cong H^{i+1}(BG)$$

for all $i \geq 1$, where $H^k(BG)$ is integral cohomology of the space BG, and moreover in low degrees we have an exact sequence

$$0 \to H^0_{\mathrm{SM}}(G;\mathbb{Z}) \to H^0_{\mathrm{SM}}(G;\mathbb{R}) \to H^0_{\mathrm{SM}}(G;S^1) \to H^1_{\mathrm{SM}}(G;\mathbb{Z}) \to 0$$

where S^1, $\mathbb{R}$ and $\mathbb{Z}$ are considered G–modules with trivial action.

Proof The short exact sequence of Lie groups $\mathbb{Z} \to \mathbb{R} \to S^1$ induces a long exact sequence in Segal cohomology. However since $\mathbb{R}$ is contractible, Segal cohomology agrees with cohomology computed with smooth cochains. Since G is compact, these vanish in degrees larger then zero. □

4.2 Classifying extensions of smooth 2–groups

The goal of this section is to prove Theorem 1, which classifies the bicategory of central extensions of certain smooth 2–groups. We begin with some elementary results on symmetric monoidal bicategories. Results on general symmetric monoidal bicategories may be found in [22; 35; 34; 6; 16; 54, Chapter 3].

The simplest kinds of symmetric monoidal bicategories arise from 3–term cochain complexes of abelian groups. Let

$$C^3 \overset{d}{\leftarrow} C^2 \overset{d}{\leftarrow} C^1$$

be a 3–term cochain complex of abelian groups. We may form a strict bicategory D as follows. The objects D_0 consist of the elements of the group C^3. The 1–morphisms consist of the product $D_1 = C^3 \times C^2$. The source is the projection to C^3, the target map is given by $t(c_3, c_2) = c_3 + d(c_2)$, and the strict horizontal composition is given by $(c_3, c_2) \circ (c_3', c_2') = (c_3, c_2 + c_2')$, when $c_3' = c_3 + d(c_2)$. Similarly, the 2–morphisms consist of $C^3 \times C^2 \times C^1$ with source map the projection, target map given by $t(c_3, c_2, c_1) = (c_3, c_2 + d(c_1))$, and vertical composition of composable elements given by addition of the C^1 terms. The horizontal composition of composable 2–morphisms is given by $(c_3, c_2, c_1) * (c_3', c_2', c_1') = (c_3, c_2 + c_2', c_1 + c_1')$, which is again a strict operation.

The bicategory D comes equipped with the structure of a strict symmetric monoidal bicategory. The monoidal structure is induced by the abelian group multiplication in C^i, $i = 1, 2, 3$, and the braiding is trivial. In this way we obtain a number of examples of elementary symmetric monoidal bicategories. For example, an abelian group M may be regarded as a cochain complex concentrated in a single degree. There are three possibilities for 3–term cochain complexes arising in this manner, and hence we obtain three symmetric monoidal bicategories, M, $M[1]$ and $M[2]$. The notation $M[2]$ denotes the symmetric monoidal 2–category with one object, one 1–morphism, and M many 2–morphisms, whose compositions are induced from multiplication in M. Similarly $M[1]$ denotes the symmetric monoidal bicategory with one object, M many 1–morphisms, and only identity 2–morphisms, and M without decoration denotes the symmetric monoidal bicategory with objects M and only identity 1–morphisms and 2–morphisms. The following lemma is presumably well known to experts.

Lemma 98 *Let (C^i, d) be a 3–term cochain complex and let D be the resulting symmetric monoidal bicategory. Let $H^* = H^*(C^i, d)$ be the cohomology groups of (C^i, d). Then there is an equivalence*

$$D \simeq H^3 \times H^2[1] \times H^1[2]$$

of symmetric monoidal bicategories. In general this equivalence is unnatural, but nonetheless there is a natural isomorphism $\pi_0(D) \cong H^3$.

Proof This can be proven in several ways. A global approach is to analyze the k-invariants[20] of Picard symmetric monoidal bicategories as was done for Picard symmetric monoidal categories in [28, Appendix B.2] and for certain braided monoidal categories in [32]. Since D is both *strict* and *symmetric monoidal*, these k-invariants vanish. Thus D splits up to unnatural equivalence as the product of its "homotopy groups". Notice, however that there is natural map of cochain complexes from (C^i, d) to the complex with a single nonzero group H^3 in the top term. This is an isomorphism on third cohomology groups and induces the natural isomorphism $\pi_0(D) \cong H^3$.

Alternatively, one may simply choose a skeleton of D, as in [54, Lemma 3.4.5 - 3.4.7]. A direct calculation, following the proofs of these lemmas, shows that D splits as in the statement of Lemma 98. Producing such a splitting usually requires choices. □

Let G be a Lie group and A an abelian Lie group. Let $Z_{\mathrm{SM}}(G;A)$ denote the 3–term chain complex

$$Z^3_{\mathrm{SM}}(G;A) \overset{d}{\leftarrow} C^2_{\mathrm{SM}}(G;A) \overset{d}{\leftarrow} C^1_{\mathrm{SM}}(G;A)$$

given by the smooth Segal–Mitchison cohomology of G with values in the trivial G–module A. By abuse of notation, let $Z_{\mathrm{SM}}(G;A)$ also denote the corresponding symmetric monoidal bicategory.

Theorem 99 *Let G and A be as above. There is a natural equivalence of symmetric monoidal bicategories $Z_{\mathrm{SM}}(G;A) \overset{\simeq}{\to} \mathrm{Ext}(G;[\mathrm{pt}/A])$. Thus, we have a (generally unnatural) equivalence*

$$\mathrm{Ext}(G;[\mathrm{pt}/A]) \simeq H^3_{\mathrm{SM}}(G;A) \times H^2_{\mathrm{SM}}(G;A)[1] \times H^1_{\mathrm{SM}}(G;A)[2],$$

where $H^i_{\mathrm{SM}}(G;A)$ denotes the smooth version Segal–Mitchison topological group cohomology [56] . In particular isomorphism classes of central extensions

$$1 \longrightarrow \begin{matrix} A \\ \downdownarrows \\ \mathrm{pt} \end{matrix} \longrightarrow \begin{matrix} \Gamma_1 \\ \downdownarrows \\ \Gamma_0 \end{matrix} \longrightarrow \begin{matrix} G \\ \downdownarrows \\ G \end{matrix} \longrightarrow 1$$

are in natural bijection with $H^3_{\mathrm{SM}}(G;A)$.

Proof The bicategory $Z_{\mathrm{SM}}(G;A)$ is covariantly functorial in G, contravariantly functorial in A, and preserves products. Moreover, just as for $\mathrm{Ext}(G;[\mathrm{pt}/A])$, the symmetric monoidal structure is induced from the Baer sum. Thus it suffices to

[20]The 1– and 2–morphisms in the bicategory D arising from a 3–term chain complex are invertible and hence D is a 2–groupoid. Moreover, $\pi_0 D$ is a group and so D is a 3–group. Thus its k-invariants are well understood and coincide with the classical k-invariants of a stable homotopy 2–type.

produce a natural equivalence of bicategories $Z_{\mathrm{SM}}(G;A) \to \mathrm{Ext}(G;[\mathrm{pt}/A])$. It will automatically be an equivalence of *symmetric monoidal* bicategories. See also [54, Theorem 3.4.10]. The remaining statements in the theorem follow from this equivalence and Lemma 98.

Before getting into the details, which are somewhat computational, let us explain the philosophy behind why this theorem is true. This result is the offspring of two well established ideas. On the one-hand, following [5], there is a direct relationship between smooth functors between Lie groupoids and between certain Lie groupoid cocycles. This link extends to the level of smooth natural transformations, as well. If the multiplication in a smooth 2–group was given by a smooth functor, then we would be able to translate the axioms it must satisfy into certain concrete statements about cocycle data and be able to classify central extensions in terms of this data. See [5, Theorem 55] for an example of a result along these lines.

On the other hand the multiplication in a smooth 2–group is a bibundle and these come from functors precisely when there exists a global section of the bibundle over the source object space [37] (see also Proposition 23). However, every bibundle admits sections *locally* in the sense that for every bibundle P from G to H, there exists a cover $f\colon U \to G_0$, such that the composition of P with the canonical bibundle from f^*G to G admits a global section; see Example 18.

So while the multiplication bibundle in a 2–group, which is a bibundle m from $G \times G$ to G, may not admit global sections, we may choose a cover $f\colon U_2 \to G_0 \times G_0$ such that the pullback of m to $f^*(G \times G)$ *does* admit global sections. Hence this induced bibundle comes from a functor, and may be described by appropriate classical cocycle data. The associator will have a similar description via cocycle data on the pullback of $G \times G \times G$ to an appropriately chosen cover of $G_0 \times G_0 \times G_0$. In this way we may extract from a smooth 2–group precisely the cocycle data of a smooth Segal–Mitchison cocycle. Conversely, given such cocycle data we may push it forward to bibundle data via the equivalences between, say, $G \times G$ and its pullback along $U_2 \to G_0 \times G_0$.

We now proceed to prove Theorem 99. Let us now note that there is a slight ambiguity in the definition of the cochain complex $Z_{\mathrm{SM}}(G;A)$. In defining the cochain complex computing Segal–Mitchison cohomology, we were free to use any good simplicial covering. The resulting cohomology is independent of this choice. However the cochain complex itself clearly depends upon this choice. However, in the course of proving Theorem 99, we will show that these choices are irrelevant. More precisely, we will first fix a simplicial cover U and construct a functor $Z_{SM,U}(G;A) \to \mathrm{Ext}(G;[\mathrm{pt}/A])$ from the chain complex bicategory defined relative to this fixed chosen cover. If the cover is good, this functor is an equivalence of bicategories. Refining a simplicial cover $U' \to U$

induces a (strict) functor $Z_{SM,U'}(G;A) \to Z_{SM,U}(G;A)$ which is also an equivalence of bicategories. Our construction is compatible with refinement and since any two covers have a common refinement the choice of cover is irrelevant. Equivalently, we may consider $Z_{\mathrm{SM}}(G;A)$ to consist of the directed colimit over all simplicial covers. These considerations produce an equivalence $Z_{\mathrm{SM}}(G;A) \to \mathrm{Ext}(G;[\mathrm{pt}/A])$.

First we fix a good simplicial cover and construct a canonical central extension from a given cocycle representative $\lambda \in Z^3_{\mathrm{SM}}(G;A)$. A 3–cocycle has three nontrivial parts, which are smooth maps:

$$\lambda_3\colon U_3^{[1]} \to A$$
$$\lambda_2\colon U_2^{[2]} \to A$$
$$\lambda_1\colon U_1^{[3]} \to A$$

We will see that these three data give rise to the three most important structures on a smooth 2–group. λ_1 will give rise to an A–gerbe over G, which will be the underlying Lie groupoid of E^λ. λ_2 will give rise to the multiplication bibundle for E^λ, and λ_3 will give rise to its associator. These three maps $(\lambda_1, \lambda_2, \lambda_3)$ form a cocycle in the double complex $C^{pq} = C^\infty(U_q^{[p+1]}; A)$, which computes the (smooth version of) Segal's group cohomology, thus they satisfy the following relations:

$$\delta_h \lambda_1 = 0$$
$$\delta_v \lambda_1 = \delta_h \lambda_2$$
$$\delta_v \lambda_2 = \delta_h \lambda_3$$
$$\delta_v \lambda_3 = 0.$$

The first of these states that λ_1 is a Čech cocycle in $\check{C}^2_{U_1}(G;A)$.

In Example 72 we constructed a $[\mathrm{pt}/A]$–principal bundle (also known as an A–gerbe) given precisely such a Čech cocycle. This principal bundle will be the underlying Lie groupoid of our smooth 2–group E^λ. Recall that the objects of E^λ consist the manifold U_1 and the morphisms consist of the manifold $U_1^{[2]} \times A$, with composition being given by the formula

$$E_1^\lambda \times_{U_1} E_1^\lambda \to E_1^\lambda$$
$$(u_0, u_1, a) \times (u_1, u_2, b) \mapsto (u_0, u_2, a + b + \lambda_1(u_0, u_1, u_2)).$$

There are several associated objects we may build out of the cocycle λ. The function $(d_0^*\lambda_1, d_2^*\lambda_1)$ from $U_2^{[3]}$ to $A \times A$ defines a Čech cocycle in $\check{C}^2_{U_2}(G \times G; A \times A)$ and hence gives rise to a $[\mathrm{pt}/A \times A]$–principal bundle F^λ over $G \times G$. Here d_i is the simplicial map in the simplicial manifold $U_\bullet$. There is a functor $(d_0, d_2)\colon F^\lambda \to E^\lambda \times E^\lambda$, which is given on morphisms by the map

$$\begin{aligned} U_2^{[2]} \times A^2 &\to U_1^{[2]} \times A \times U_1^{[2]} \times A \\ (v_0, v_1, a, b) &\mapsto (d_0(v_0), d_0(v_1), a) \times (d_2(v_1), d_2(v_1), b). \end{aligned}$$

This realizes F^λ as a pullback of the groupoid $E^\lambda \times E^\lambda$ and so becomes an equivalence upon bundlization.

The function $(d_0^* d_0^* \lambda_1, d_0^* d_2^* \lambda_1, d_2^* d_2^* \lambda_1)$ defines a Čech cocycle in $\check{C}^2_{U_3}(G^3; A^3)$ and hence a $[\mathrm{pt}/A^3]$–principal bundle H^λ over G^3 (the three maps $d_0 d_0$, $d_2 d_0$ and $d_2 d_2$ are the simplicial maps living over the three projections from G^3 to G). These Lie groupoids fit into a diagram of smooth functors.

$$\begin{array}{ccccc} E^\lambda \times E^\lambda & \xleftarrow{\quad g=(d_0,d_2) \quad} & F^\lambda & \xrightarrow{\quad \mu \quad} & E^\lambda \\ E^\lambda \times E^\lambda \times E^\lambda & \xleftarrow{\quad h=(d_0 d_0, d_2 d_0, d_1 d_2) \quad} & H^\lambda & \xrightarrow{\quad f_1 \quad} & E^\lambda \\ E^\lambda \times E^\lambda \times E^\lambda & \xleftarrow{\quad h=(d_0 d_0, d_2 d_0, d_1 d_2) \quad} & H^\lambda & \xrightarrow{\quad f_2 \quad} & E^\lambda \end{array}$$

The left-pointing functors become equivalences after bundlization. The right-pointing functors are given explicitly by the following formulas:

$$\begin{aligned} \mu\colon F_1^\lambda = U_2^{[2]} \times A^2 &\to E_1^\lambda = U_1^{[2]} \times A \\ (v_0, v_1, a, b) &\mapsto (d_1(v_0), d_1(v_1), a + b + \lambda_2(v_0, v_1)) \\ f_1\colon H_1^\lambda = U_3^{[2]} \times A^3 &\to E_1^\lambda = U_1^{[2]} \times A \\ (w_0, w_1, a, b, c) &\mapsto (d_1 d_1(w_0), d_1 d_1(w_1), a + b + c \\ &\qquad\qquad + d_2^* \lambda_2(w_0, w_1) + d_0^* \lambda_2(w_0, w_1)) \\ f_2\colon H_1^\lambda = U_3^{[2]} \times A^3 &\to E_1^\lambda = U_1^{[2]} \times A \\ (w_0, w_1, a, b, c) &\mapsto (d_1 d_1(w_0), d_1 d_1(w_1), a + b + c \\ &\qquad\qquad + d_1^* \lambda_2(w_0, w_1) + d_3^* \lambda_2(w_0, w_1)) \end{aligned}$$

These are functors because the identity $\delta_v \lambda_1 = \delta_h \lambda_2$ holds. Moreover the identity $\delta_h \lambda_3 = \delta_v \lambda_2$ implies that $\lambda_3\colon U_3 \to A$ gives the components of a smooth natural transformation a from f_1 to f_2. Turning these into bibundles and inverting g will give us bibundle M from $E^\lambda \times E^\lambda$ to E^λ. Inverting h and composing with f_1 and f_2 yields two bibundles from $E^\lambda \times E^\lambda \times E^\lambda$ to E^λ. These are canonically identified with $M \circ (M \times 1)$ and $M \circ (1 \times M)$, respectively. The natural transformation a induces a natural isomorphism $\alpha\colon M\circ(M\times 1)\to M\circ(1\times M)$. The equation $\delta_v\lambda_3=0$ ensures that this associator satisfies the pentagon identity.

More concretely, consider the composition

$$m\colon U_1\times U_1\to G\times G\xrightarrow{m} G,$$

and the induced fiber product $U_1\times_G^m(U_1\times U_1)$. This space admits a covering by the space $V=U_1\times_G^{d_1} U_2{}^{(d_0,d_2)}\times_{G\times G}(U_1\times U_1)$. The data (λ_1,λ_2) defines a function ϕ given by the following formula:

$$\begin{aligned}\phi\colon U_1\times_G U_2^{[2]}\times_{G\times G}(U_1\times U_1)&\to A\\ (u_0,v_0,v_1,u_2,u_3)&\mapsto \lambda_2(v_0,v_1)-\lambda_1(u_0,d_1v_0,d_1v_1)\\ &\qquad -\lambda_1(d_0v_0,d_0v_1,u_1)\\ &\qquad -\lambda_1(d_2v_0,d_2v_1,u_2)\end{aligned}$$

This function defines a Čech cocycle $\check{C}^1_V(U_1\times_G(U_1\times U_1);A)$ and hence a corresponding A–bundle M over $U_1\times_G(U_1\times U_1)$. This is the total space of the bibundle M above. The necessary groupoid actions are easily constructed from this description. A compatible unit is straight forward to define and is determined up a contractible category of choices.

A direct calculation shows that the sequence of homomorphisms

$$[\mathrm{pt}/A]\to E^\lambda\to G$$

is a central extension of smooth 2–groups, and thus provides a construction of a central extension from a cocycle $\lambda\in Z^3_{\mathrm{SM}}(G;A)$. It remains to show that this construction can be extended to the entire cochain bicategory $Z_{\mathrm{SM}}(G;A)$.

Let $\lambda,\lambda'\in Z^3_{\mathrm{SM}}(G;A)$ be two cocycles. A 1–morphism in $Z_{\mathrm{SM}}(G;A)$ from λ to λ' is precisely a cochain $\theta\in C^2_{\mathrm{SM}}(G;A)$ such that $\delta\theta=\lambda-\lambda'$. This cochain has components $\theta_1\colon U_1^{[2]}\to A$ and $\theta_2\colon U_2\to A$. In components the equation $\delta\theta=\lambda-\lambda'$ becomes

$$\begin{aligned}\delta_h\theta_1&=\lambda_1-\lambda_1'\\ \delta_v\theta_1+\delta_h\theta_2&=\lambda_2-\lambda_2'\\ \delta_v\theta_2&=\lambda_3-\lambda_3'.\end{aligned}$$

This data gives rise to three functors

$$\begin{aligned}p_E\colon E^\lambda&\to E^{\lambda'}\\ p_F\colon F^\lambda&\to F^{\lambda'}\\ p_H\colon H^\lambda&\to H^{\lambda'}\end{aligned}$$

and a natural isomorphism of functors $b\colon \mu' \circ p_F \to p_E \circ \mu$, such that the following diagrams commute strictly:

$$\begin{array}{ccc} E^\lambda \times E^\lambda & \xleftarrow{\;g\;} & F^\lambda \\ {\scriptstyle p_E \times p_E}\downarrow & & \downarrow{\scriptstyle p_F} \\ E^{\lambda'} \times E^{\lambda'} & \xleftarrow[\;g'\;]{} & F^{\lambda'} \end{array} \qquad \begin{array}{ccc} E^\lambda \times E^\lambda \times E^\lambda & \xleftarrow{\quad h\quad} & H^\lambda \\ {\scriptstyle p_E \times p_E \times p_E}\downarrow & & \downarrow{\scriptstyle p_H} \\ E^{\lambda'} \times E^{\lambda'} \times E^{\lambda'} & \xleftarrow[\quad h'\quad]{} & H^{\lambda'} \end{array}$$

Explicitly these functors are defined as follows. Each of p_E, p_F, and p_H is the identity on objects, and on 1–morphisms they are given by the formulas:

$$\begin{aligned} p_{E,1}\colon E_1^\lambda = U_1^{[2]} \times A &\to U_1^{[2]} \times A = E_1^{\lambda'} \\ (u_0, u_1, a) &\mapsto (u_0, u_1, a + \theta_1(u_0, u_1)) \\ p_{F,1}\colon F_1^\lambda = U_2^{[2]} \times A \times A &\to U_2^{[2]} \times A \times A = F_1^{\lambda'} \\ (v_0, v_1, a, b) &\mapsto (v_0, v_1, a + d_0^*\theta_1(v_0, v_1), b + d_2^*\theta_1(v_0, v_1)) \\ p_{H,1}\colon H_1^\lambda = U_1^{[2]} \times A \times A \times A &\to U_1^{[2]} \times A \times A \times A = H_1^{\lambda'} \\ (w_0, w_1, a, b, c) &\mapsto (w_0, w_1, a + d_0^* d_0^* \theta_1(w_0, w_1), \\ &\qquad b + d_0^* d_2^* \theta_1(w_0, w_1), c + d_2^* d_2^* \theta_1(w_0, w_1)). \end{aligned}$$

The equation $\delta_h \theta_1 = \lambda_1 - \lambda_1'$ is equivalent to the statement that these formulas define functors. The natural transformation $b\colon \mu' \circ p_F \to p_E \circ \mu$ is given by $b = (\Delta \circ d_1, \theta_2)\colon U_2 \to U_1^{[2]} \times A$. The equation $\delta_v \theta_1 + \delta_h \theta_2 = \lambda_2 - \lambda_2'$ is equivalent to the naturality of this natural transformation.

We now turn each of these functors into bibundles. The functors p_E, p_F and p_H become equivalence bibundles, which are induced from the single bibundle $P\colon E^\lambda \to E^{\lambda'}$. The natural transformation b becomes a natural isomorphism of bibundles $\beta\colon M' \circ (P \times P) \to P \circ M$ from $E^\lambda \times E^\lambda$ to $E^{\lambda'}$. The final equation $\delta_v \theta_2 = \lambda_3 - \lambda_3'$ is equivalent to the commutativity of the diagram in Figure 3, which says that P and β are components of a 1–homomorphism from E^λ to $E^{\lambda'}$.

The remaining components of the 1–homomorphism (P, β), namely those involving units of E^λ and $E^{\lambda'}$, exist and are uniquely determined by the requirement that this be a homomorphism. The homomorphism $P\colon E^\lambda \to E^{\lambda'}$ is canonically a homomorphism over G (indeed any two 1–homomorphisms from $E^\lambda \to G$ which are isomorphic are uniquely isomorphic). Moreover there is a unique 2–homomorphism $i' \cong P \circ i$ making (P, β) into a morphism of central extensions, where $i\colon [\mathrm{pt}/A] \to E^\lambda$ and $i'\colon [\mathrm{pt}/A] \to E^{\lambda'}$ are the previously constructed inclusions. In this way we obtain from each 2–cochain $\theta \in Z_{\mathrm{SM}}(G;A)$ a homomorphism of central extensions which we denote P_θ.

If λ, λ', and λ'' are three cocycles in $Z^3_{\mathrm{SM}}(G;A)$ and θ and θ' are two cochains in $C^2_{\mathrm{SM}}(G;A)$ which represent 1–morphisms from λ to λ' and from λ' to λ'', respectively, then their composite in $Z_{\mathrm{SM}}(G;A)$ is given by the sum $\theta'+\theta$. A simple calculation shows that the construction of the functors p_E, p_F and p_H preserves this composition strictly. For example $p_E^{\theta'} \circ p_E^{\theta} = p_E^{\theta+\theta'}$, on the nose. The natural isomorphisms b^θ and $b^{\theta'}$ also obey a strict composition identity:

$$b^{\theta'+\theta} = (p_E^{\theta'} * b^\theta) \circ (b^{\theta'} * p_F^{\theta}).$$

After bundlization, these strict equalities become the natural isomorphisms $P^{\theta'} \circ P^{\theta} \cong P^{\theta'+\theta}$ of homomorphisms of central extensions. The natural isomorphisms induced by b^θ and $b^{\theta'}$ also obey the expected composition law. A similar calculation gives natural isomorphisms $P^0 \cong \mathrm{id}_{E^\lambda}$ for any cocycle $\lambda \in Z^3_{\mathrm{SM}}(G;A)$. These natural isomorphisms are part of the data of the functor $Z_{\mathrm{SM}}(G;A) \to \mathrm{Ext}(G;[\mathrm{pt}/A])$.

The rest of the data of this functor concerns 2–morphisms in $Z_{\mathrm{SM}}(G;A)$. Let $\lambda, \lambda' \in Z^3_{\mathrm{SM}}(G;A)$ be objects in $Z_{\mathrm{SM}}(G;A)$ and let $\theta, \theta' \in C^2_{\mathrm{SM}}(G;A)$ represent 1–morphisms from λ to λ'. Thus $\delta\theta = \delta\theta' = \lambda - \lambda'$. A 2–morphism from θ to θ' is represented by a 1–cochain $\omega \in C^1_{\mathrm{SM}}(G;A)$ such that $\delta\omega = \theta - \theta'$. Such a 1–cochain consists of a single function $\omega\colon U_1 \to A$ such that $-\delta_h\omega = \theta_1 - \theta_1'$ and $\delta_v\omega = \theta_2 - \theta_2'$. This gives rise to a natural isomorphism of functors $\eta\colon\ p_E^{\theta} \to p_E^{\theta'}$ whose components are $\eta(u) = (u, u, \omega(u))$. That this formula defines a natural isomorphism is equivalent to the equation $-\delta_h\omega = \theta_1 - \theta_1'$. This induces a natural isomorphism of homomorphisms $\eta\colon\ P^\theta \to P^{\theta'}$. The second equation, $\delta_v\omega = \theta_2 - \theta_2'$ is equivalent to the commutativity of the first diagram in Figure 6. The commutativity of the second diagram in that figure is automatic in this case. The 2–homomorphism η^ω is clearly compatible with the projection to G and is also compatible with the inclusion of $[\mathrm{pt}/A]$ into E^λ and $E^{\lambda'}$. Thus it defines a 2–morphism of central extensions. A similar calculation to before shows that at the level of natural transformations of functors, horizontal and vertical composition in $Z_{\mathrm{SM}}(G;A)$ is preserved strictly. After bibundlization this provides the remaining natural isomorphisms and equalities which show that our assignment $Z_{\mathrm{SM}}(G;A) \to \mathrm{Ext}(G;[\mathrm{pt}/A])$ is a functor between bicategories.

If the simplicial cover used to define the Segal–Mitchison cohomology is *good*, then this is an equivalence of bicategories, which is equivalent to showing that it is essentially surjective on objects, essentially full on 1–morphisms and fully faithful on 2–morphisms. To see this, recall that for any extension E, we know that there exists a sufficiently fine cover of G over which the principal bundle $E \to G$ may be trivialized, a sufficiently fine cover of $G \times G$ over which the principal bundle $E \times E$ and the bibundle M can both be trivialized, and a sufficiently fine cover over which the associator may be trivialized. In particular these may be trivialized over any good covering. Choosing explicit trivializations of these principal bundles and bibundles reproduces exactly the components of a Segal–Mitchison cocycle and applying the above construction reproduces (up to equivalence) the original extension. Similarly, any homomorphism $P\colon E^{\lambda} \to E^{\lambda'}$ between extensions arising from cocycles may be trivialized over a good cover and such a trivialization gives rise to an explicit cochain $\theta \in C^2_{\mathrm{SM}}(G; A)$ representing a morphism between the corresponding cocycles. Applying our previous construction yields a homomorphism of extensions isomorphic to the original P. Finally if E^{λ} and $E^{\lambda'}$ are extension arising from cocycles, $P^{\theta}, P^{\theta'}\colon E^{\lambda} \to E^{\lambda'}$ are morphisms of extensions arising from 2–cochains, then any 2–morphism $\omega\colon P^{\theta} \to P^{\theta'}$ arises from a unique 1–cochain ω. This last statement is essentially equivalent to the fact that an isomorphism between trivialized principal A–bundles is given by a unique function on the base. This completes our proof of Theorem 99. □

4.3 String(n) as an extension of smooth 2–groups

The model of Segal–Mitchison topological group cohomology that we used Theorem 99 computes this cohomology from the total complex associated to certain a double complex, and consequently gives us several calculational tools. In particular there are the edge homomorphisms

$$H^i_{\mathrm{smooth}}(G; A) \to H^i_{\mathrm{SM}}(G; A)$$
$$H^{i+1}_{\mathrm{SM}}(G; A) \to \check{H}^i(G; \mathcal{O}_A),$$

where $\mathcal{O}_A$ is the sheaf of smooth A–valued functions on the space G and $H^i_{\mathrm{smooth}}(G; A)$ denotes naive group cohomology with smooth cocycles. The construction in the previous section shows that in a central extension of smooth 2–groups

$$[\mathrm{pt}/A] \to E^{\lambda} \to G$$

coming from a cocycle $\lambda \in C^3(G; A)$, the underlying $[\mathrm{pt}/A]$–principal bundle of E^{λ} is classified by the image of $[\lambda_1]$ in $\check{H}^2(G; \mathcal{O}_A)$. This allows us to identify the homotopy type of the geometric realization of E^{λ}.

In fact we can realize the component λ_1 simplicially as a map of simplicial spaces:

$$\lambda_1\colon (G_{U_1})_{\bullet} \to K(A[2])_{\bullet}.$$

Here $K(A[2])_{\bullet}$ is the simplicial topological abelian group associated to the chain complex with no differentials and with A concentrated in degree two, and $(G_{U_1})_{\bullet}$ is the Čech simplicial manifold associated with the cover $U_1 \to G$. A direct calculation shows that the simplicial nerve of E^λ is the pullback in simplicial spaces:

$$\begin{array}{ccc} (E^\lambda)_\bullet & \longrightarrow & E(A[2])_\bullet \\ \big\downarrow \ulcorner & & \big\downarrow \\ (G_{U_1})_\bullet & \underset{\lambda_1}{\to} & K(A[2])_\bullet \end{array}$$

where $E(A[2])_\bullet$ is the simplicial topological abelian group corresponding to the two term chain complex of topological abelian groups with A in degrees one and two, and with differential the identity. This is a contractible chain complex and hence the geometric realization of the corresponding simplicial space is contractible. If fact it is the universal $K(A,1)$–bundle over the geometric realization $|K(A[2])_\bullet| \simeq K(A,2)$. Since geometric realization of simplicial spaces commutes with fiber products, we have

$$G \xrightarrow{\simeq} |G_U| \xrightarrow{|\lambda_1|} K(A,2)$$

is the classifying map in the long exact fibration sequence. This identifies the homotopy type of the space $|E^\lambda|$.

Theorem 100 *Let $n \geq 5$. Then $H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1) \cong H^4(B\mathrm{Spin}(n)) \cong \mathbb{Z}$ and the central extension of smooth 2–groups corresponding to a generator gives a model for* $\mathrm{String}(n)$ *as a smooth 2–group.*

Proof It is well known that $H^4(B\mathrm{Spin}(n)) \cong \mathbb{Z}$ for $n \geq 5$, and so we know by Corollary 97 that Segal cohomology $H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1) \cong \mathbb{Z}$ and by Theorem 99 that this classifies central extensions of smooth 2–groups, $[\mathrm{pt}/S^1] \to E \to \mathrm{Spin}(n)$. If $[\lambda] \in H^3_{\mathrm{SM}}(\mathrm{Spin}(n), S^1)$ is a class associated to a given central extensions, then by the above considerations we know that the topology of $|E|$ is determined by the image of $[\lambda]$ under the edge homomorphisms $H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1) \to \check{H}^2(\mathrm{Spin}(n); S^1) \cong \mathbb{Z}$, and moreover $|E^\lambda|$ will have the correct homotopy type precisely if $[\lambda]$ is mapped to a generator of $\check{H}^2(\mathrm{Spin}(n); S^1)$.

There are several ways to deduce that this edge homomorphism is surjective, and hence an isomorphism. For example, using the short exact sequence of smooth Lie groups $\mathbb{Z} \to \mathbb{R} \to S^1$ and the induced long exact sequence in Segal–Mitchison cohomology, we see that the edge homomorphism is the same as the one for integral coefficients:

$$H^4(BG) \cong H^4_{\mathrm{SM}}(G;\mathbb{Z}) \to \check{H}^3(G;\mathbb{Z}) \cong H^3(G).$$

Segal [56] identifies this map with the transfer map from the Serre spectral sequence, which is well known to be an isomorphism in these degrees for simply connected Lie groups like $G = \mathrm{Spin}(n)$. Alternatively, we may re-examine the double complex which computes Segal–Mitchison cohomology, and use it to extract slightly more information. Associated to this double complex is a spectral sequence with E_1–term

$$E_1^{p,q} = \check{H}^q(G^p; \mathcal{O}_A) \Rightarrow H_{\mathrm{SM}}^{p+q}(G; A).$$

In the case that $G = \mathrm{Spin}(n)$ and $A = S^1$, the E_1–term looks as follows:

$$\begin{array}{c|lllll}
 & 0 & \vdots & \vdots & & \\
 & 0 & \check{H}^2(G; S^1) & \check{H}^2(G^2; S^1) & \cdots & \\
q & 0 & 0 & 0 & 0 & \cdots \\
 & S^1 & C^\infty(G; S^1) & C^\infty(G^2; S^1) & C^\infty(G^3; S^1) & \cdots \\
\hline
 & & & p & &
\end{array}$$

The cohomology of the first row under the d^1–differential is precisely the smooth group cohomology of G, ie the cohomology computed using smooth S^1–valued group cocycles. Since G is compact and 1–connected, this is trivial in degrees larger then zero [30; 29; 17; 18; 25]. This yields the exact sequence

$$0 \to H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1) \to \check{H}^2(G; S^1) \xrightarrow{d^1} \check{H}^2(G^2; S^1).$$

The kernel of d^1 consists of the *primitive elements*. It is well known that for $G = \mathrm{Spin}(n)$ every element of $\check{H}^2(G; S^1)$ is primitive in this sense. □

Remark 101 There are two generators of this cohomology group and hence there are two associated central extensions. The corresponding smooth 2–groups which model String(n) are equivalent and this equivalence is a map of extensions over G which induces the order two automorphism of $[\mathrm{pt}/S^1]$.

The above considerations also give a new conceptual re-interpretation of the notion of *multiplicative bundle gerbe*. Carey, Johnson, Murray, Stevenson and Wang [15] introduced multiplicative S^1–bundle gerbes over G and showed they correspond to elements of $H^4(BG)$. We see from the above that a multiplicative bundle gerbe over G may instead be viewed as a central extension of smooth 2–groups.

4.4 Concluding remarks

Theorem 99 provides a construction which produces a central extension of smooth 2–groups from a given smooth Segal–Mitchison cocycle and Theorem 100 shows that for any choice of generator of $H^3_{\mathrm{SM}}(\mathrm{Spin}(n); S^1)$ the corresponding central extension gives a model for $\mathrm{String}(n)$. But in what sense is this a construction of $\mathrm{String}(n)$? One may worry about the choices involved in this construction. However the choices don't matter. Theorem 99 shows that the isomorphism classes of homomorphisms between any two extensions are in bijection with second Segal–Mitchison cohomology $H^2_{\mathrm{SM}}(G; A)$, and the 2–homomorphisms between any two such homomorphisms form a torsor for $H^1_{\mathrm{SM}}(G; A)$. In the case of $\mathrm{String}(n)$, where $G = \mathrm{Spin}(n)$ and $A = S^1$, both of these groups vanish, so that the bicategory of $\mathrm{String}(n)$–extensions forms a contractible bicategory, ie any two extensions are equivalent, and any two homomorphisms realizing this equivalence are isomorphic via a unique 2–isomorphism. This is the strongest possible uniqueness result one could hope for, and shows that the $\mathrm{String}(n)$ 2–group extension is unique in precise analogy with the unique $\mathrm{Spin}(n)$ extension of $\mathrm{SO}(n)$.

This brings us to the matter of other extensions. While it was sufficient to construct a model of $\mathrm{String}(n)$, Theorem 99 only classifies central extensions of smooth 2–groups of the particular form

$$[\mathrm{pt}/A] \to E \to G,$$

where G is an ordinary Lie group, and A an ordinary abelian Lie group, viewed as a trivial G–module. This can be generalized, and the construction presented here works with negligible modification when the action of G on A is nontrivial. In this case we get an extension of smooth 2–groups, but it will not be a *central* extension.

More generally, we would like to understand the bicategory of extensions of arbitrary smooth 2–groups $\mathrm{Ext}(\mathbb{G}; \mathbb{A})$, where $\mathbb{G}$ and $\mathbb{A}$ do not necessarily come from ordinary Lie groups. One could hope for some sort of cohomology theory which classifies theses extensions and which reduces to Segal–Mitchison cohomology when $\mathbb{G} = G$ is an ordinary Lie group and $\mathbb{A} = [\mathrm{pt}/A]$ for A an abelian Lie group.

This hypothetical cohomology should take short exact sequences of smooth abelian 2–groups to long exact sequences and have other nice homological properties. Indeed such a cohomology does in fact exist, and we may identify $\mathrm{Ext}(\mathbb{G}; \mathbb{A}) \simeq H^2(\mathbb{G}; \mathbb{A})$. Specializing to the case $\mathbb{G} = G$ and $\mathbb{A} = [\mathrm{pt}/A]$, and using the short exact sequence $[\mathrm{pt}/A] \to 0 \to A$ from Example 86, we have isomorphisms

$$H^2(G; [\mathrm{pt}/A]) \cong H^3(G; A) = H^3_{\mathrm{SM}}(G; A).$$

However the proper way to define this cohomology theory and deduce its properties requires developing the machinery of bicategorical homological algebra, in particular in a form that can be applied to the smooth setting. This would take us too far afield of current goals, but is a topic we take up in [53].

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