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Central fuzzy sets

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Let B be a collection of fuzzy sets. What are the fuzzy sets which are sufficiently similar to every fuzzy set from B, i.e. 'central' fuzzy sets for B? Such a question naturally arises if B is large and one wishes to replace B by a single fuzzy set – the representative of B. In this paper, we develop a framework which enables us to answer this question and related ones. We use complete residuated lattices as the structures of truth degrees, covering thus the real unit interval with left-continuous t-norm and its residuum as an important but particular case. We present results describing central fuzzy sets and optimal central fuzzy sets, provided similarity of fuzzy sets is assessed by Leibniz rule.

Keywords: fuzzy set; similarity; tolerance; central point

1. Problem setting

Suppose there is a collection of metal poles of different lengths. Suppose a person sees a picture of two poles from that collection and is asked to assess their similarity, i.e. the person is asked to tell a degree $\mathrm{p}_1 \approx \mathrm{p}_2$ to which the poles are similar. The degree has to be a value between 0 and 1, $\mathrm{p}_1 \approx \mathrm{p}_2 = 0$ and $\mathrm{p}_1 \approx \mathrm{p}_2 = 1$ indicate that the poles are not similar at all and that the poles are indistinguishable, respectively. Since the poles are narrow, the person assesses their similarity based solely on their lengths. The picture does not show a scale, i.e. the person does not know the actual lengths of the poles. An obvious way to assess the similarity s of poles p_1 and p_2 of lengths $l(\mathrm{p}_1)$ and $l(\mathrm{p}_2)$ is to put

$$\mathrm{p}_1 \approx \mathrm{p}_2 = \min\left(\frac{l(\mathrm{p}_1)}{l(\mathrm{p}_2)}, \frac{l(\mathrm{p}_2)}{l(\mathrm{p}_1)}\right), \tag{1}$$

i.e. to make the similarity judgement based on the ratio of the lengths. Namely, the ratio does not depend on the actual lengths, i.e.

$$\mathrm{p}_1 \approx \mathrm{p}_2 = \min\left(\frac{c\cdot l(\mathrm{p}_1)}{c\cdot l(\mathrm{p}_2)}, \frac{c\cdot l(\mathrm{p}_2)}{c\cdot l(\mathrm{p}_1)}\right)$$

for any $c > 0$, so it can be assessed even when the person does not know the actual magnification coefficient $c > 0$, i.e. does not know the scale for the picture.

Given poles p_1 and p_2 with lengths $l(\mathrm{p}_1)$ and $l(\mathrm{p}_2)$, what is the length of the pole in the middle? That is, what is the length of the 'central pole' p for which

$$\mathrm{p} \approx \mathrm{p}_1 = \mathrm{p} \approx \mathrm{p}_2,$$

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i.e. for which the similarity to p_1 equals the similarity to p_2? An easy verification shows that the central pole p has length

$$l(\mathrm{p}) = \sqrt{l(\mathrm{p}_1)}\cdot\sqrt{l(\mathrm{p}_2)}. \tag{2}$$

Suppose now that the longest possible pole has the length normalised to 1 and the person knows the scale, i.e. knows the lengths $l(\mathrm{p}_1)$ and $l(\mathrm{p}_2)$. Then there is another, perhaps more natural, way to assess the similarity. Namely, one can put

$$\mathrm{p}_1 \approx \mathrm{p}_2 = 1 - \,|\, l(\mathrm{p}_1) - l(\mathrm{p}_2)\,|\,, \tag{3}$$

i.e. the similarity is proportional to the distance of the normalised lengths of p_1 and p_2. If such a measure of similarity is used, the length of the central pole p is

$$l(\mathrm{p}) = \frac{l(\mathrm{p}_1) + l(\mathrm{p}_2)}{2}. \tag{4}$$

Obviously, given a set $B = \{\mathrm{p}_1, \ldots, \mathrm{p}_n\}$ of poles, the length of the optimal central pole for B is

$$l(\mathrm{p}) = \sqrt{\min_i l(\mathrm{p}_i)}\cdot\sqrt{\max_i l(\mathrm{p}_i)}$$

for similarity given by (1) and

$$l(\mathrm{p}) = \frac{\min_i l(\mathrm{p}_i) + \max_i l(\mathrm{p}_i)}{2}$$

for similarity given by (3).

In this paper, we present theorems and algorithms motivated by the above types of problems. The first hint to a general framework for this kind of problems is the observation that in (1)

$$\mathrm{p}_1 \approx \mathrm{p}_2 = l(\mathrm{p}_1) \leftrightarrow l(\mathrm{p}_2), \tag{5}$$

with $\leftrightarrow$ being the biresiduum corresponding to product t-norm and that in (2)

$$l(\mathrm{p}) = m \otimes \sqrt{l(\mathrm{p}_1) \leftrightarrow l(\mathrm{p}_2)}, \tag{6}$$

with $m = \min\{l(\mathrm{p}_1), l(\mathrm{p}_2)\}$, $\otimes$ denoting the product t-norm and $\sqrt{\ }$ denoting its square root, as introduced by Höhle (1995). Likewise, (5) and (6) become (3) and (4) if $\leftrightarrow$ and $\otimes$ denote the Łukasiewicz biresiduum and t-norm. Henceforth, we consider the framework of left-continuous t-norms and their residua. In fact, we consider a more general framework of complete residuated lattices (Ward and Dilworth 1939).

In general, we assume that B is a subset of a set $\mathcal{S}$ of fixpoints of some fuzzy closure operator C in a universe set X and study the 'central' fuzzy sets for B, i.e. fuzzy sets from $\mathcal{S}$ which are sufficiently similar to any fuzzy set from B. If C is the identity, $\mathcal{S}$ is the set of all fuzzy sets in X, in which case no constraint is imposed, i.e. B as well as the central fuzzy sets may be arbitrary fuzzy sets in X. However, our setting with a general operator C allows us to consider only certain fuzzy sets (those which are the fixpoints of C) as the elements of B as well as the central fuzzy sets of B. Example 3.7 clarifies why we consider general operators C.

2. Preliminaries

2.1 *Tolerance relations*

A tolerance relation, see e.g. (Pogonowski 1981, Schreider 1975), in a set X is a binary relation T in X which is reflexive and symmetric, i.e. for every $x, y \in X$, T satisfies

$$\langle x, x \rangle \in T,$$
$$\langle x, y \rangle \in T \text{ implies } \langle y, x \rangle \in T.$$

The concept of a tolerance relation generalises the well-known concept of an equivalence relation. Namely, T is an equivalence relation if it is a tolerance relation which is, moreover, transitive, i.e. for every $x, y, z \in X$, if $\langle x, y \rangle \in T$ and $\langle y, z \rangle \in T$, then $\langle x, z \rangle \in T$.

Let T be a tolerance in X. A class of T given by $x \in X$ is the set $[x]_T = \{y \mid \langle x, y \rangle \in T\}$. A set $B \subseteq X$ is called a block of T if $B \times B \subseteq T$, i.e. if for every $x, y \in B$, $\langle x, y \rangle \in T$. A block B of T is called maximal if it is maximal with respect to set inclusion, i.e. if $B' \times B' \not\subseteq T$ for any $B' \supset B$. It is easy to see that if T is an equivalence relation, classes of T coincide with maximal blocks of T.

While equivalence relations serve as simple mathematical models of indistinguishability, tolerance relations serve as models of similarity. Namely, equivalence relations represent relationships defined by 'have same features', while tolerance relations represent relationships defined by 'have some features in common', see Schreider (1975).

2.2 *Fuzzy sets and fuzzy logic*

2.2.1 *Residuated lattices as structures of truth degrees*

In classical logic, the structure of truth degrees is the two-element Boolean algebra, i.e. a structure $\mathbf{L}$ which consists of a two-element set $L = \{0,1\}$ of truth degrees and is equipped with truth functions of logical connectives. In fuzzy logic, there are more options, both for the set L of truth degrees and for the functions of logical connectives. As the structures of truth degrees, we use complete residuated lattices. Complete residuated lattices, introduced to fuzzy logic by Goguen (1968–1969), and their variants are used in mathematical fuzzy logic (Hájek 1998, Gottwald 2008). Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$) $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. The fact that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice means that the infimum $\wedge_{i \in I} a_i$ and supremum $\vee_{i \in I} a_i$ exist for any subset $\{a_i \mid i \in I\} \subseteq L$. Elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\rightarrow$, called multiplication and residuum, are truth functions of logical connectives 'fuzzy conjunction' and 'fuzzy implication'. A biresiduum of $\mathbf{L}$ is a binary operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

We denote by $\leq$ the lattice order induced by $\mathbf{L}$. Examples of residuated lattices are well known. A generic one is: take a left-continuous t-norm $\otimes$. That is, $\otimes$ is binary operation on [0,1], which is left-continuous in its first argument (as a real function of two variables), commutative, associative, monotone and has 1 as its neutral element (Hájek 1998). Put $a \rightarrow b = \vee\{c \in L \mid a \otimes c \leq b\}$. Then $\langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1 \rangle$ is a complete residuated lattice. Three most important pairs of adjoint operations on [0,1] obtained this way are Łukasiewicz: $a \otimes b = \max(0, a + b - 1)$ and $a \rightarrow b = \min(1, 1 - a + b)$;

Gödel (minimum): $a \otimes b = a \wedge b$, $a \to b = b$ for $a > b$ and $a \to b = 1$ for $a \leq b$ and Goguen (product): $a \otimes b = a \cdot b$, $a \to b = b/a$ for $a > b$ and $a \to b = 1$ for $a \leq b$. In the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, denoted by $\mathbf{2}$. That is, the operations $\wedge, \vee, \otimes, \to$ of $\mathbf{2}$ are the truth functions (interpretations) of connectives of classical logic.

2.2.2 *Fuzzy sets and fuzzy relations*

Given $\mathbf{L}$, we define the usual notions regarding fuzzy sets and fuzzy relations: a fuzzy set (an $\mathbf{L}$-set) A in a universe X is a mapping $A : X \to L$, $A(x)$ being interpreted as 'the degree to which x belongs to A'. The set of all fuzzy sets in X is denoted by L^X. Operations with fuzzy sets are defined component-wise. For instance, the intersection of fuzzy sets $A, B \in L^X$ is a fuzzy set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. For fuzzy sets $A, B \in L^X$, put

$$S(A,B) = \bigwedge_{x \in X} (A(x) \to B(x)), \tag{7}$$

$$A \approx B = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \tag{8}$$

$S(A,B)$ and $A \approx B$ are called the degree of subsethood of A in B and the degree of equality of A and B, respectively. Note that $S(A,B)$ can be seen as a truth degree of 'for each $x \in X$: if x belongs to A, then x belongs to B'. Similarly, $A \approx B$ can be seen as a truth degree of 'for each $x \in X$: x belongs to A if and only if x belongs to B'. $\approx$ is a fuzzy equivalence relation, i.e. $A \approx A = 1$ (reflexivity), $A \approx B = B \approx A$ (symmetry) and $(A \approx B) \otimes (B \approx C) \leq A \approx C$ (transitivity), which is called a Leibniz similarity. We denote the fact that $S(A,B) = 1$ by $A \subseteq B$ (A is fully contained in B). Hence, we have

$$A \subseteq B \quad \text{if and only if for each } x \in X : A(x) \leq B(x). \tag{9}$$

For more details we refer to (Belohlavek 2002, Hájek 1998).

3. Central points

3.1 *Fuzzy closure operators*

Suppose $\mathcal{S}$ is a system of fuzzy sets in X, i.e. $\mathcal{S} \subseteq L^X$. We consider the following type of problems. Given $B \subseteq \mathcal{S}$, what are the fuzzy sets $A \in \mathcal{S}$ which are similar to every $A' \in B$ to a degree at least ε? To assess similarity of A and A', we use $\approx$ defined by (8). That is, A being similar to A' to a degree at least ε means $A \approx A' \geq \varepsilon$. Furthermore, we assume that $\mathcal{S}$ is a system of fixpoints of an $\mathbf{L}$-closure operator (fuzzy closure operator) C in X, see Examples 3.1, 3.6 and 3.7 for particular examples.

Recall (Belohlavek 2001, 2002, Rodríguez *et al.* 2003) that an $\mathbf{L}$-closure operator C in X is a mapping $C : L^X \to L^X$ satisfying

$$A \subseteq C(A), \tag{10}$$

$$S(A_1, A_2) \leq S(C(A_1), C(A_2)), \tag{11}$$

$$C(A) = C(C(A)), \tag{12}$$

for every $A, A_1, A_2 \in L^X$. As a consequence, we also have

$$(A_1 \approx A_2) \leq (C(A_1) \approx C(A_2)). \tag{13}$$

The set $\mathrm{fix}(C)$ of all fixpoints of C is defined by

$$\mathrm{fix}(C) = \{A \in L^X \mid C(A) = A\}.$$

$\langle \mathrm{fix}(C), \subseteq \rangle$ is a complete lattice in which the infima $\bigwedge$ and suprema $\bigvee$ are given by

$$\bigwedge_{j\in J} A_j = \bigcap_{j\in J} A_j, \bigvee_{j\in J} A_j = C\left(\bigcup_{j\in J} A_j\right),$$

for every $\{A_j \mid j \in J\} \subseteq \mathrm{fix}(C)$. In this paper, we often denote subsets of $\mathrm{fix}(C)$ by B. Correspondingly, we denote the infimum and the supremum of B by $\bigwedge B$ and $\bigvee B$, respectively.

Example 3.1. Clearly, the identity mapping $C : L^X \to L^X$, i.e. $C(A) = A$ for every $A \in L^X$, is an **L**-closure operator in X. In this case, $\mathrm{fix}(C) = L^X$.

Remark 1. The concept of an **L**-closure operator generalises the well-known concept of a closure operator. Namely, for $L = \{0, 1\}$, **L**-closure operators coincide with ordinary closure operators.

3.2 *Central points, closed balls and blocks*

Definition 3.2. Let $B \subseteq \mathrm{fix}(C)$. Given a threshold $\varepsilon \in L$, let

$$C_\varepsilon(B) = \{A \in \mathrm{fix}(C) \mid \text{for every } A' \in B : A \approx A' \geq \varepsilon\}.$$

We call the elements of $C_\varepsilon(B)$ ε-central points of B.

That is, $C_\varepsilon(B)$ is the set of all fixpoints of C for which the degree of equality to every $A' \in B$ is at least ε. In a sense, $C_\varepsilon(B)$ contains all fixpoints which are ε-similar to every fixpoint from B.

Example 3.3. If B is empty or $\varepsilon = 0$, then $C_\varepsilon(B) = \mathrm{fix}(C)$.

Definition 3.4. Let $A \in \mathrm{fix}(C)$. Given a threshold $\varepsilon \in L$, let

$$B_\varepsilon(A) = \{A' \in \mathrm{fix}(C) \mid A \approx A' \geq \varepsilon\}.$$

We call the set $B_\varepsilon(A)$ a closed ε-ball with centre A.

Example 3.5. If $\varepsilon = 0$, then $B_\varepsilon(A) = \mathrm{fix}(C)$.

Note that it follows immediately from the definitions that

$$B_\varepsilon(A) = C_\varepsilon(\{A\}). \tag{14}$$

Remark 2. The concept of similarity can be regarded as dual to the concept of a distance. A simple way to illustrate this correspondence is the following one. For any metric space M with a distance function d, there can be introduced an **L**-equivalence $\approx$ on M, with **L** being the unit real interval [0,1] with Goguen (product) structure, by putting

$$(x \approx y) = \mathrm{e}^{-d(x,y)},$$

where $d(x,y)$ is the distance of the points x and y. On the other hand, for any **L**-equivalence $\approx$ on M satisfying

$$x \approx y = 1 \quad \text{iff} \;\; x = y,$$

we can define a metric on M by

$$d(x,y) = -\lg(x \approx y).$$

Note that the above relationship is a special case of a general relationship between metric distances and fuzzy equivalences which are transitive w.r.t. a continuous Archimedean t-norm, such as the Goguen (product) t-norm, which is described by De Baets and Mesiar (2002).

Now, any closed ε-ball with centre A in $\text{fix}(C)$ coincides with the closed ball with centre A and radius $-\lg\varepsilon$ in the metric space $\langle \text{fix}(C), d\rangle$. This illustrates the fact that the concept of a closed ball has its well-known counterpart in the theory of metric spaces. However, let us emphasise that such a counterpart is available only for $L = [0,1]$, equipped with a continuous Archimedean t-norm $\otimes$.

The notion of ε-central point seems to have no counterpart in the theory of metric spaces.

Example 3.6. The notions of ε-central points and closed ε-balls generalise those studied by Belohlavek and Krupka (2008a). Namely, Belohlavek and Krupka (2008a) introduced the following concepts. Let $\mathbf{L}$ be a complete residuated lattice with a support set L. For $B \subseteq L$ and $\varepsilon \in L$, the set $C_\varepsilon(B)$ of central points and the closed ε-ball with centre $c \in L$ were defined by

$$C_\varepsilon(B) = \{a \in L \,|\, \text{for each}\, b \in B : a \leftrightarrow b \geq \varepsilon\}, B_\varepsilon(c) = \{a \in L \,|\, a \leftrightarrow c \geq \varepsilon\}.$$

Clearly, if we let $X = \{x\}$ and identify the **L**-sets in X with truth degrees from L, i.e. identify $A \in \mathbf{L}^X$ s.t. $A(x) = a$ with a, then the notions of ε-central points and closed ε-balls are particular examples of the corresponding notions introduced in this paper.

Example 3.7. Another example in which central points and closed balls naturally appear comes from concept analysis of data with fuzzy attributes (Belohlavek 2002, 2004), see also (Ganter and Wille 1999) for formal concept analysis of data with binary attributes. Let $\langle X, Y, I\rangle$ be a formal fuzzy context, i.e. X and Y are sets of objects and attributes, and $I : X \times Y \to L$ is a fuzzy relation between X and Y. For $x \in X$ and $y \in Y$, $I(x,y)$ is interpreted as the degree to which object x has attribute y. Let $\uparrow : L^X \to L^Y$ and $\downarrow : L^Y \to L^X$ denote the associated operators, i.e.

$$A^\uparrow(y) = \bigwedge_{x\in X}(A(x) \to I(x,y)),\; B^\downarrow(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y)).$$

Let $\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \,|\, A^\uparrow = B, B^\downarrow = A\}$ denotes the associated concept lattice. Elements $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ are called formal concepts and represent particular clusters in the data described by $\langle X,Y,I\rangle$. A and B are called the extent and the intent of $\langle A,B\rangle$ and represent the collection of all objects and attributes covered by the formal concept $\langle A,B\rangle$. Consider the set

$$\text{Ext}(X,Y,I) = \{A \,|\, \langle A,B\rangle \in \mathcal{B}(X,Y,I) \quad \text{for some} \;\; B \in L^Y\}$$

of all extents of $\langle X, Y, I\rangle$. It can be easily shown that $\mathrm{Ext}(X, Y, I) = \mathrm{fix}(C)$ for the **L**-closure operator $C : L^X \longrightarrow L^X$ defined by $C(A) = A^{\uparrow\downarrow}$.

Since $\mathcal{B}(X, Y, I) = \{\langle A, A^{\uparrow}\rangle \mid A \in \mathrm{Ext}(X, Y, I)\}$, $\mathcal{B}(X,Y,I)$ can be identified with Ext(X,Y,I). Given a threshold $\varepsilon \in L$ and a set $B \subseteq \mathcal{B}(X, Y, I)$ of formal concepts, $C_\varepsilon(B)$, i.e. the set of ε-central points, is the set of all formal concepts which are similar to every formal concept from B to a degree at least ε. Such a set may be desirable particularly if B is large, and we need just a representative formal concept(s) instead of B. In such a case, it is particularly interesting to ask for the best representative formal concept, i.e. for which the similarity degree to every formal concept from B is the largest possible. We call such elements optimal central points and investigate them in Section 3.3.

Remark 3.

(a) Recall that for a binary relation T between sets U and V, the Galois connection (Ore 1944) induced by T is a pair of mappings ${}^{\uparrow_T} : 2^U \longrightarrow 2^V$ and ${}^{\uparrow_T} : 2^V \longrightarrow 2^U$ defined for $M \in 2^U$ and $N \in 2^V$ by

$$M^{\uparrow_T} = \{v \in V \mid \text{for each } u \in M : \langle u, v\rangle \in T\},$$
$$N^{\downarrow_T} = \{u \in U \mid \text{for each } v \in N : \langle u, v\rangle \in T\}.$$

If $U = V$ and T is symmetric, then ${}^{\uparrow_T}$ coincides with ${}^{\downarrow_T}$, and we write just M^T instead of $M^{\uparrow_T}$ or $M^{\downarrow_T}$.

(b) Consider the Galois connection induced by the ε-cut ${}^{\varepsilon}\!\approx$ of $\approx$, i.e. by the symmetric binary relation ${}^{\varepsilon}\!\approx$ between fix(C) and fix(C) defined for $A, A' \in \mathrm{fix}(C)$ by

$$\langle A, A'\rangle \in {}^{\varepsilon}\!\approx \quad \text{if and only if } A \approx A' \geq \varepsilon. \tag{15}$$

Clearly, ${}^{\varepsilon}\!\approx$ is a tolerance relation which need not be transitive. $\langle A, A'\rangle \in {}^{\varepsilon}\!\approx$ means that A and A' are similar to a degree at least ε. As a result of the definitions, for $B \subseteq \mathrm{fix}(C)$ and $A \in \mathrm{fix}(C)$, we have

$$C_\varepsilon(B) = B^{{}^{\varepsilon}\!\approx} \quad \text{and} \quad B_\varepsilon(A) = \{A\}^{{}^{\varepsilon}\!\approx}.$$

Note also that $B_\varepsilon(A)$ is just the class of tolerance ${}^{\varepsilon}\!\approx$ given by A.

From the basic properties of Galois connections, we get the following assertions.

LEMMA 3.8. For $B, B_1, B_2 \subseteq \mathrm{fix}(C)$,

$$B_1 \subseteq B_2 \text{ implies } C_\varepsilon(B_1) \supseteq C_\varepsilon(B_2), \tag{16}$$

$$B \subseteq C_\varepsilon(C_\varepsilon(B)), \tag{17}$$

$$C_\varepsilon(B) = C_\varepsilon(C_\varepsilon(C_\varepsilon(B))), \tag{18}$$

$$C_\varepsilon(B) = \bigcap_{A \in B} B_\varepsilon(A). \tag{19}$$

Note that (19) states that ε-central points of B are just the points common to all closed ε-balls with centres $A \in B$.

As a consequence, we get the following lemma.

LEMMA 3.9. For $B \subseteq \mathrm{fix}(C)$,

$$B \subseteq \bigcap_{A \in C_\varepsilon(B)} B_\varepsilon(A). \tag{20}$$

For $A \in \mathrm{fix}(C)$,

$$A \in C_\varepsilon(B_\varepsilon(A)). \tag{21}$$

Proof. (20) follows from (17) and (19). Due to (17), $\{A\} \subseteq C_\varepsilon(C_\varepsilon(\{A\})) = C_\varepsilon(B_\varepsilon(A))$, whence (20). $\square$

The following lemma is another direct consequence of the observation made in Remark 3 and the well-known properties of Galois connections.

LEMMA 3.10.

1. The mapping $cl_\varepsilon : 2^{\mathrm{fix}(C)} \to 2^{\mathrm{fix}(C)}$ defined for $D \subseteq \mathrm{fix}(C)$ by $cl_\varepsilon(D) = C_\varepsilon(C_\varepsilon(D))$ is an ordinary closure operator in $\mathrm{fix}(C)$.
2. The set $\mathrm{fix}(cl_\varepsilon) = \{D \subseteq \mathrm{fix}(C) \mid D = cl_\varepsilon(D)\}$ of all fixpoints of cl_ε equipped with $\subseteq$ is a complete lattice.
3. $D \in \mathrm{fix}(cl_\varepsilon)$ if and only if $D = C_\varepsilon(B)$ for some $B \subseteq \mathrm{fix}(C)$, i.e. $\mathrm{fix}(cl_\varepsilon)$ contains just sets of ε-central points.

We now present a description of the set $C_\varepsilon(B)$ of central points in our general setting. First, we need the following lemma.

LEMMA 3.11. $A \in C_\varepsilon(B)$ iff $S(A, \bigwedge B) \wedge S(\bigvee B, A) \geq \varepsilon$.

Proof. By definition, $A \in C_\varepsilon(B)$ means that for each $A' \in B$, $S(A, A') \geq \varepsilon$ and $S(A', A) \geq \varepsilon$. Hence, to prove the assertion, it suffices to check that (a) $S(A, A') \geq \varepsilon$ for each $A' \in B$ is equivalent to $S(A, \bigwedge B) \geq \varepsilon$ and (b) $S(A', A) \geq \varepsilon$ for each $A' \in B$ is equivalent to $S(\bigvee B, A) \geq \varepsilon$.

(a) By definition and using $(\bigwedge B)(x) = \bigwedge_{A' \in B} A'(x)$,

$$S(A, \bigwedge B) = \bigwedge_{x \in X} \left(A(x) \to \bigwedge_{A' \in B} A'(x) \right) = \bigwedge_{x \in X} \bigwedge_{A' \in B} (A(x) \to A'(x)).$$

Hence, $S(A, \bigwedge B) \geq a$ iff for every $A' \in B$, $S(A, A') \geq a$.

(b) Since $A \in \mathrm{fix}(C)$, we have

$$S(\bigvee B, A) = S\left(C\left(\bigcup B\right), C(A)\right) \geq S\left(\bigcup B, A\right)$$

by (11). On the other hand, (10) yields $\bigvee B \supseteq \bigcup B$, which implies $S(\bigvee B, A) \leq S(\bigcup B, A)$. Hence,

$$S\left(\bigvee B, A) = S\left(\bigcup B, A\right) = \bigwedge_{x \in X} \bigwedge_{A' \in B} (A'(x) \to A(x)) = \bigwedge_{A' \in B} S(A', A\right)$$

and thus $S(\bigvee B, A) \geq a$ iff for each $A' \in B$, $S(A', A) \geq a$. $\square$

The next theorem shows that central points form particular intervals in the lattice $\langle \mathrm{fix}(C), \subseteq \rangle$.

THEOREM 3.12. For any $B \subseteq \mathrm{fix}(C)$,

$$C_\varepsilon(B) = \left[C(\varepsilon \otimes \bigvee B), \varepsilon \rightarrow \bigwedge B\right].$$

Note that [_,_] denotes an interval in $\langle \mathrm{fix}(C), \subseteq \rangle$, i.e.

$$\left[C(\varepsilon \otimes \bigvee B), \varepsilon \rightarrow \bigwedge B\right] = \left\{A \in \mathrm{fix}(C) \mid C(\varepsilon \otimes \bigvee B) \subseteq A \subseteq \varepsilon \rightarrow \bigwedge B\right\},$$

and that fuzzy sets $\varepsilon \otimes \bigvee B$ and $\varepsilon \rightarrow \bigwedge B$ are defined by

$$\left(\varepsilon \otimes \bigvee B)(x) = \varepsilon \otimes (\bigvee B)(x) \quad \text{and} \quad (\varepsilon \rightarrow \bigwedge B)(x) = \varepsilon \rightarrow (\bigwedge B)(x\right).$$

Proof. By Lemma 3.11, A is a central point iff $S(A, \bigwedge B) \geq \varepsilon$ and $S(\bigvee B, A) \geq \varepsilon$, which is equivalent to $A \subseteq \varepsilon \rightarrow \bigwedge B$ and $\varepsilon \otimes \bigvee B \subseteq A$. Since fixpoints of C are closed under $\rightarrow$-shifts, see Belohlavek (2002), we get $\varepsilon \rightarrow \bigwedge B \in \mathrm{fix}(C)$. However, $\varepsilon \otimes \bigvee B$ need not be a fixpoint. The least fixpoint greater than or equal to $\varepsilon \otimes \bigvee B$ is $C(\varepsilon \otimes \bigvee B)$. This proves the theorem. □

The following theorem describes closed balls.

THEOREM 3.13. For any $A \in \mathrm{fix}(C)$,

$$B_\varepsilon(A) = [C(\varepsilon \otimes A), \varepsilon \rightarrow A].$$

Proof. Directly from Theorem 3.12 using (14). □

Consider now, in addition to ${}^{\varepsilon}{\approx}$, cf. (15), the binary relation ${}^{\varepsilon^2}{\approx}$ on $\mathrm{fix}(C)$ defined by

$$\langle A, A' \rangle \in {}^{\varepsilon^2}{\approx} \quad \text{if and only if} \quad A \approx A' \geq \varepsilon^2 = \varepsilon \otimes \varepsilon. \tag{22}$$

Since $\varepsilon \otimes \varepsilon \leq \varepsilon$, $\langle A, A' \rangle \in {}^{\varepsilon}{\approx}$ implies $\langle A, A' \rangle \in {}^{\varepsilon^2}{\approx}$. Hence, classes (i.e. closed balls, cf. Remark 3(b)) of ${}^{\varepsilon}{\approx}$ are contained in classes of ${}^{\varepsilon^2}{\approx}$, i.e. $B_\varepsilon(A) \subseteq B_{\varepsilon^2}(A)$. Likewise, blocks of ${}^{\varepsilon}{\approx}$ are blocks of ${}^{\varepsilon^2}{\approx}$. However, there is an interesting relationship between the closed balls $B_\varepsilon(A)$ and maximal blocks of ${}^{\varepsilon^2}{\approx}$ which we now investigate.

LEMMA 3.14. For each $A \in \mathrm{fix}(C)$, $B_\varepsilon(A)$ is a block of ${}^{\varepsilon^2}{\approx}$.

Proof. By Theorem 3.13, $B_\varepsilon(A) = [C(\varepsilon \otimes A), \varepsilon \rightarrow A]$. It follows from Belohlavek and Krupka (2008b, Theorem 2) that

$$B = \left[C(\varepsilon^2 \otimes (\varepsilon \rightarrow A)), \varepsilon^2 \rightarrow C(\varepsilon^2 \otimes (\varepsilon \rightarrow A))\right]$$

is a maximal block of ${}^{\varepsilon^2}{\approx}$ which contains the fixpoint $\varepsilon \rightarrow A$. Now, since $\varepsilon^2 \otimes (\varepsilon \rightarrow A) \subseteq \varepsilon \otimes A$, we get $C(\varepsilon^2 \otimes (\varepsilon \rightarrow A)) \subseteq C(\varepsilon \otimes A)$. Similarly, since $\varepsilon^2 \otimes (\varepsilon \rightarrow A) \subseteq C(\varepsilon^2 \otimes (\varepsilon \rightarrow A))$, we get $\varepsilon \rightarrow A \subseteq \varepsilon^2 \rightarrow C(\varepsilon^2 \otimes (\varepsilon \rightarrow A))$. We proved $B_\varepsilon(A) \subseteq B$ which completes the proof. □

LEMMA 3.15. For $B \subseteq \mathrm{fix}(C)$, $C_\varepsilon(B)$ is non-empty if and only if B is a block of ${}^{\varepsilon^2}{\approx}$.

Proof. Due to Theorem 3.12, $C_\varepsilon(B)$ is non-empty iff $C(\varepsilon \otimes \bigvee B) \leq \varepsilon \rightarrow \bigwedge B$. Furthermore, B is a block of ${}^{\varepsilon^2}{\approx}$ iff $\varepsilon^2 \leq S(\bigvee > B, \bigwedge B)$. Indeed, this follows by a slight modification of Ganter and Wille (1999, Proposition 54) by observing that $S(\bigvee B, \bigwedge B) = \bigvee B \approx \bigwedge B$ and that, due to Belohlavek and Krupka (2008b, Lemma 1), ${}^{\varepsilon^2}\!\!\approx$ is a complete tolerance relation on $\langle \mathrm{fix}(C), \subseteq \rangle$. To prove the lemma, we thus need to check that

$$C\left(\varepsilon \otimes \bigvee B\right) \subseteq \varepsilon \to \bigwedge B \quad \text{iff} \quad \varepsilon^2 \leq S\left(\bigvee B, \bigwedge B\right). \tag{23}$$

Let $C(\varepsilon \otimes \bigvee B) \leq \varepsilon \to \bigwedge B$. Since $\varepsilon \otimes \bigvee B \leq C(\varepsilon \otimes \bigvee B)$, we get $\varepsilon \otimes \bigvee B \leq \varepsilon \to \bigwedge B$ from which $\varepsilon^2 \leq S(\bigvee B, \bigwedge B)$ readily follows.

Conversely, if $\varepsilon^2 \leq S(\bigvee B, \bigwedge B)$, then $\varepsilon \otimes \bigvee B \leq \varepsilon \to \bigwedge B$, from which we get $C(\varepsilon \otimes \bigvee B) \leq C(\varepsilon \to \bigwedge B) = \varepsilon \to \bigwedge B$, because of monotony of C and the fact that $\varepsilon \to \bigwedge B$ is a fixpoint of C. The proof is completed. □

We say that a closed ε-ball $B_\varepsilon(A)$ is maximal if $B_\varepsilon(A) = B_\varepsilon(A')$ for every A' with $B_\varepsilon(A) \subseteq B_\varepsilon(A')$. The following theorem describes a relationship between closed balls and maximal blocks of ${}^{\varepsilon^2}\!\!\approx$.

THEOREM 3.16. For $B \subseteq \mathrm{fix}(C)$, B is a maximal closed ε-ball if and only if B is a maximal block of ${}^{\varepsilon^2}\!\!\approx$. In particular, if B is a maximal block of ${}^{\varepsilon^2}\!\!\approx$, then $C_\varepsilon(B) \neq 0$ and $B = B_\varepsilon(A)$ for every $A \in C_\varepsilon(B)$.

Proof. Let $B_\varepsilon(A)$ be maximal. Due to Lemma 3.14, $B_\varepsilon(A)$ is a block of ${}^{\varepsilon^2}\!\!\approx$. There exists a maximal block B of ${}^{\varepsilon^2}\!\!\approx$ for which $B_\varepsilon(A) \subseteq B$ (the existence of B follows from Zorn lemma). Due to Lemma 3.15, $C_\varepsilon(B) \neq \emptyset$. Take an arbitrary $A' \in C_\varepsilon(B)$. Due to (20), $B \subseteq B_\varepsilon(A')$. Therefore, $B_\varepsilon(A) \subseteq B \subseteq B_\varepsilon(A')$. Maximality of $B_\varepsilon(A)$ as a closed ε-ball yields $B_\varepsilon(A) = B$, i.e. $B_\varepsilon(A)$ is a maximal block of ${}^{\varepsilon^2}\!\!\approx$

Conversely, let B be a maximal block of ${}^{\varepsilon^2}\!\!\approx$. Observe first that if $B \subseteq B_\varepsilon(A)$, then $B = B_\varepsilon(A)$. Indeed, due to Lemma 3.15, $B_\varepsilon(A)$ is a block of ${}^{\varepsilon^2}\!\!\approx$ and hence $B = B_\varepsilon(A)$ follows from the fact that B is a maximal block of ${}^{\varepsilon^2}\!\!\approx$ Therefore, to prove the claim, it is sufficient to realise that $C_\varepsilon(B) \neq \emptyset$ (Lemma 3.15) and that for every $A \in C_\varepsilon(B)$, we have $B \subseteq B_\varepsilon(A)$ due to (20). □

3.3 *Optimal central points*

Consider now the following problem. Theorem 3.12 describes the set $C_\varepsilon(B)$ of ε-central points of B. Every $A \in C_\varepsilon(B)$ is good in the sense that the degree $A \approx A'$ of its similarity to any $A' \in B$ is at least ε. However, some central points from $C_\varepsilon(B)$ may be better than others. We call the best ones the optimal central points of B.

DEFINITION 3.17. Let $B \subseteq \mathrm{fix}(C)$. $A \in \mathrm{fix}(C)$ is called an optimal central point of B if and only if

$$\bigwedge_{A' \in B} (D \approx A') \leq \bigwedge_{A' \in B} (A \approx A') \tag{24}$$

for every $D \in \mathrm{fix}(C)$.

Remark 4. Note that according to the principles of fuzzy logic,

$$\bigwedge_{A' \in B} (D \approx A')$$

can be understood as the truth degree of 'for every $A' \in B$: D is similar to A''. Therefore, for an optimal central point of B, such a degree is the highest possible.

We now turn to a characterisation of optimal central points and their existence in terms of radii. We need the following concepts.

Definition 3.18. We say that $\varepsilon \in L$ is an admissible radius of $B \subseteq \mathrm{fix}(C)$ if $C_\varepsilon(B) \neq \emptyset$. We call ε the radius of B for A if ε is the largest radius for which $A \in C_\varepsilon(B)$.

Observe that for any B and A, the radius of B for A is $\bigwedge_{A' \in B}(A \approx A')$. This observation and (24) thus yield an alternative characterisation of optimal central points.

Lemma 3.19. *A is an optimal central point of B if and only if for every $D \in \mathrm{fix}(C)$, the radius of B for A is larger than or equal to the radius of B for D.*

The following theorem provides a characterisation of optimal central points of B.

Theorem 3.20. Conditions 1–3 are equivalent.

1. The set of all optimal central points of B is non-empty, and ε is the radius of B for some optimal central point A.
2. The set of all optimal central points of B is non-empty, and ε is the radius of B for any of the optimal central points.
3. ε is the largest admissible radius of B.

Any of the conditions 1–3 implies condition 4.

4. The set of all optimal central points is equal to $C_\varepsilon(B)$.

Proof. '1 $\Rightarrow$ 2': (24) implies that the radii of B for any two optimal central points A_1 and A_2 are equal.

'2 $\Rightarrow$ 3': Assume 2. Clearly, ε is an admissible radius of B. If ε' is an admissible radius of B, then for any $D \in C_{\varepsilon'}(B)$, we have $\varepsilon' \leq \bigwedge_{A' \in B}(D \approx A')$. Now, for any optimal central point A of B, (24) and the assumption $\bigwedge_{A' \in B}(A \approx A') = \varepsilon$ give $\bigwedge_{A' \in B}(D \approx A') \leq \varepsilon$, whence $\varepsilon' \leq \varepsilon$, proving 3.

'3 $\Rightarrow$ 1': For $A \in C_\varepsilon(B)$, $\varepsilon \leq \bigwedge_{A' \in B}(A \approx A')$. On the other hand, since $\bigwedge_{A' \in B}(A \approx A')$ is an admissible radius (the radius of B for A), we have $\bigwedge_{A' \in B}(A \approx A') \leq \varepsilon$, whence $\bigwedge_{A' \in B}(A \approx A') = \varepsilon$. Since for any D, $\bigwedge_{A' \in B}(D \approx A')$ is an admissible radius, we get $\bigwedge_{A' \in B}(D \approx A') \leq \varepsilon = \bigwedge_{A' \in B}(A \approx A')$, proving 1.

To complete the proof, we check '2 $\Rightarrow$ 4': Assume 2. Clearly, every optimal central point of B is in $C_\varepsilon(B)$. If A is not optimal, then $\bigwedge_{A' \in B}(A \approx A') < \varepsilon$ and hence $A \notin C_\varepsilon(B)$. □

Remark 5. Note that condition 4 of Theorem 3.20 nor the condition saying that the set of optimal central points of B is non-empty and is equal to $C_\varepsilon(B)$ implies conditions 1–3. Consider the following example (cf. Example 3.6). Let $\mathbf{L}$ be the Gödel algebra on the real unit interval $L = [0,1]$. Let $X = \{x\}$ (singleton). Then $\mathcal{S} = \{\{^0/x\}, \{^{0.5}/x\}, \{^1/x\}\}$ is a set of fixpoints of an $\mathbf{L}$-closure operator C. This claim follows from Belohlavek (2001) by verification of the fact that $\mathcal{S}$ is closed under intersections and that $a \to A \in \mathcal{S}$ for every $a \in L$ and $A \in \mathcal{S}$. Consider $B = \{\{^{0.5}/x\}, \{^1/x\}\}$. A moment's reflection shows that the set of optimal points of B is B. Now, $B = C_{0.4}(B)$, but the largest admissible radius of B is 0.5.

We now turn to the existence of optimal central points of B. We need the following lemma.

LEMMA 3.21.

1. ε is an admissible radius of B if and only if $\varepsilon \otimes \varepsilon \leq S(\bigvee B, \bigwedge B)$.
2. For every $z \in L$, $z \wedge (z \rightarrow S(\bigvee B, \bigwedge B))$ is an admissible radius of B.
3. ε is an admissible radius of B if and only if $\varepsilon = \varepsilon \bigwedge(\varepsilon \rightarrow S(\bigvee B, \bigwedge B))$.
4. The set
$$R = \left\{ z \wedge (z \rightarrow S(\bigvee B, \bigwedge B)) \mid z \in L \right\} \tag{25}$$
is the set of all admissible radii of B.

Proof. Denote $d = S(\bigvee B, \bigwedge B)$.

1. Using Theorem 3.12, $C_\varepsilon(B) \neq \emptyset$ iff $[C(\varepsilon \otimes \bigvee B), \varepsilon \rightarrow \bigwedge B] \neq \emptyset$ iff $C(\varepsilon \otimes \bigvee B) \subseteq \varepsilon \rightarrow \bigwedge B$ iff $\varepsilon \otimes \varepsilon \leq S(\bigvee B, \bigwedge B)$ (the last two conditions are equivalent due to (23)).
2. $(z \bigwedge(z \rightarrow d)) \otimes (z \bigwedge(z \rightarrow d)) \leq z \otimes (z \rightarrow d) \leq d$, hence the claim follows from (1).
3. Using (1), ε is an admissible radius of B iff $\varepsilon \leq \varepsilon \rightarrow d$ which is equivalent to $\varepsilon = \varepsilon \bigwedge(\varepsilon \rightarrow d)$.
4. A consequence of 2 and 3. □

The following theorem presents a necessary and sufficient condition for the existence of optimal central points of B.

THEOREM 3.22. A set $B \subseteq \mathrm{fix}(C)$ has optimal central points if and only if the set R from (25) has a largest element. This element is the largest admissible radius ε of B, and $C_\varepsilon(B)$ is the set of optimal central points of B.

Proof. It follows from 4 of Lemma 3.21 and from Theorem 3.20. □

For some of the well-known structures of truth degrees, the description of optimal central points can be made more particular. As an example, consider the setting of Example 3.6 and assume that the complete residuated lattice $\mathbf{L}$ is the real unit interval [0,1] equipped with Łukasiewicz t-norm and its residuum. Then if $B = [a, b]$, the largest admissible radius of B is $(a - b + 2)/2$ and the set of optimal central points of B contains just one $c \in [0, 1]$, namely $c = (a + b)/2$. In the rest of this paper, we show that such more particular descriptions are available if the complete residuated lattice $\mathbf{L}$ has square roots. According to Höhle (1995), a complete residuated lattice $\mathbf{L}$ has square roots if there is a function $\surd : L \rightarrow L$ satisfying

$$\sqrt{a} \otimes \sqrt{a} = a, \tag{26}$$

$$b \otimes b \leq a \quad \text{implies} \quad b \leq \sqrt{a}, \tag{27}$$

for every $a, b \in L$.

Example 3.23. (Höhle 1995) For Łukasiewicz, product, and Gödel algebras on [0,1] have square roots. They are given by

$$\sqrt{a} = \frac{a+1}{2} \quad \text{for Łukasiewicz,}$$
$$\sqrt{a} = \text{ordinary number-theoretic square root of } a \text{ for product,}$$
$$\sqrt{a} = a \text{ for Gödel.}$$

THEOREM 3.24. If **L** has square roots, then any subset $B \subseteq L$ has optimal central points. For the corresponding largest admissible radius ε, it holds

$$\varepsilon = \sqrt{S\left(\bigvee B, \bigwedge B\right)}. \tag{28}$$

Proof. According to 1 of Lemma 3.21 and (26), ε is the largest admissible radius of B. The rest follows from Theorem 3.20. □

COROLLARY 3.25. If **L** has square roots, then for any subset $B \subseteq L$, the set of optimal central points is equal to

$$\left[\sqrt{S\left(\bigvee B, \bigwedge B\right)} \otimes \bigvee B, \sqrt{S\left(\bigvee B, \bigwedge B\right)} \to \bigwedge B\right].$$

Example 3.26. Consider the setting of Example 3.6 and let $L = [0,1]$. In this case, for $a, b \in L$, we have $S(a,b) = a \to b$. Let $B \subseteq [0,1]$ and denote $[a,b] = [\bigwedge B, \bigvee B]$. For Łukasiewicz, product and Gödel algebras on [0,1], Theorem 3.24 gives the following description of the set $\mathcal{O}$ of optimal central points of B:

$$\mathcal{O} = \left\{\frac{a+b}{2}\right\} \quad \text{for Łukasiewicz,}$$

$$\mathcal{O} = \begin{cases} \{\sqrt{a}\cdot\sqrt{b}\} & \text{if } a > 0 \text{ or } a = b = 0, \\ [0,1] & \text{if } 0 = a < b, \end{cases} \quad \text{for product,}$$

$$\mathcal{O} = \begin{cases} [a,1] & \text{if } a < b, \\ \{a\} & \text{if } a = b, \end{cases} \quad \text{for Gödel.}$$

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