

## CENTRAL IDEMPOTENT MEASURES ON COMPACT GROUPS

BY

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**ABSTRACT.** Let  $G$  be a compact group with dual object  $\Gamma = \Gamma(G)$  and let  $M(G)$  be the convolution algebra of regular finite Borel measures on  $G$ . The author has characterized the central idempotent measures on certain  $G$ , including the unitary groups, in terms of the hypercoset structure of  $\Gamma$ . The characterization also says that, on certain  $G$ , a central idempotent measure is a sum of such measures each of which is absolutely continuous with respect to the Haar measure of a closed normal subgroup. The main result of this paper is an extension of this characterization to products of certain groups. The known structure of connected groups and a recent result of Ragozin on connected simple Lie groups will then show that the characterization is valid for connected groups. On the other hand, a simple example will show it is false in general for non-connected groups. This characterization was done by Cohen for abelian groups and the proof borrows extensively from Amemiya and Itô's simplified proof of Cohen's result.

1. **Canonical measures.** Throughout the paper  $G$  will be a compact group. The dual object  $\Gamma$  of  $G$  is the set of equivalence classes of irreducible unitary representations of  $G$ . For  $\alpha \in \Gamma$ ,  $\chi_\alpha$  will denote the character of the class and  $d(\alpha)$  its degree. For ease of notation we define  $\Psi_\alpha = \chi_\alpha/d(\alpha)$ . A measure  $\mu \in M^Z(G)$ , the center of  $M(G)$ , has a Fourier-Stieltjes transform

$$\hat{\mu}(\alpha) = \int \bar{\Psi}_\alpha d\mu \quad (\alpha \in \Gamma).$$

$\mu$  is idempotent, that is  $\mu * \mu = \mu$ , provided  $\hat{\mu}(\alpha)$  is always 0 or 1.  $J(G)$  will denote the class of central idempotent measures on  $G$ .

If  $H$  is a closed subgroup of  $G$  let  $\mathfrak{M}_H$  denote the normalized Haar measure of  $H$ .  $\mathfrak{M}_H$  is idempotent;  $\mathfrak{M}_H \in J(G)$  provided  $H$  is normal.

It is convenient to consider a larger class

$$F(G) = \{\mu \in M^Z(G) : \hat{\mu}(\alpha) \text{ is an integer}\}.$$

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$F(G)$  consists then of those central measures  $\mu$  that satisfy  $P(\mu) = 0$  for some polynomial  $P$  with integral roots (for a noncompact  $G$ ,  $F(G)$  will be defined in this way). For  $\mu \in F(G)$  we let  $E(\mu) = \{\alpha: \hat{\mu}(\alpha) \neq 0\}$ .

The hypercoset structure of  $\Gamma$  is described in [5]. If  $H$  is a closed normal subgroup of  $G$  then  $E(\mathbb{M}_H) = \{\alpha: \Psi_\alpha|_H \equiv 1\} = H^\perp$  is a normal subhypergroup of  $\Gamma$ . If  $\beta \in \Gamma$  the hypercoset  $\beta H^\perp$  consists of the  $\alpha \in \Gamma$  such that  $\chi_\alpha$  appears in the decomposition of  $\chi_\beta \chi_\gamma$  for some  $\gamma \in H^\perp$ . Also  $\beta H^\perp = \{\alpha: \Psi_\alpha|_H = \Psi_\beta|_H\}$ . The hypercoset ring of  $\Gamma$  is the smallest ring of sets containing all hypercosets.

There are two ways to attempt to characterize the measures in  $F(G)$ . First, for  $\mu \in F(G)$  and  $n$  an integer let  $E_n(\mu) = \{\alpha: \hat{\mu}(\alpha) = n\}$ . If  $\mu \in J(G)$  then  $E_1(\mu) = E(\mu)$ . It is shown in [5] that every set in the hypercoset ring of  $\Gamma$  is  $E(\mu)$  for some  $\mu \in J(G)$  and that for certain groups the converse is also true. This implies, for such groups, that for  $\mu \in F(G)$  each  $E_n(\mu)$  is in the hypercoset ring.

Second, some measures in  $F(G)$  arise naturally from well-known measures on  $G$ .

**Definition 1.1.** *A measure  $\mu$  is canonical if*

$$\mu = \sum_{\alpha} n_{\alpha} c_{\alpha} d(\alpha) \chi_{\alpha} \mathbb{M}_{H_{\alpha}}$$

where the sum is finite,  $n_{\alpha}$  is an integer,  $H_{\alpha}$  is a closed normal subgroup and

$$\frac{1}{c_{\alpha}} = \int |\chi_{\alpha}|^2 d\mathbb{M}_{H_{\alpha}}.$$

The following lemma, which connects the two concepts above, is an immediate consequence of Theorem 1 of [1] and the fact that the intersection of two hypercosets is a finite union of hypercosets.

**Lemma 1.2.** (a) *Every canonical measure is in  $F(G)$ .*

(b) *A measure  $\mu \in F(G)$  is canonical if and only if each  $E_n(\mu)$  belongs to the hypercoset ring of  $\Gamma$ .*

We will use the usual notation  $\mu \ll \nu$  to indicate that  $\mu$  is absolutely continuous with respect to  $\nu$ . It is easy to see

**Lemma 1.3.**  *$\mu \in F(G)$  is canonical if and only if there are finitely many closed normal subgroups  $H_i$  and  $\mu = \sum \mu_i$  with  $\mu_i \ll \mathbb{M}_{H_i}$ .*

2. **The main result.** Let  $\Gamma_1$  denote those  $\alpha \in \Gamma$  with  $d(\alpha) = 1$ .  $\Gamma_1$  consists of the complex homomorphisms of  $G$  and is the dual group of the abelian group  $G/G'$  where  $G'$  is the commutator subgroup of  $G$ . For  $\alpha \in \Gamma_1$  we can identify  $\alpha$  and  $\chi_{\alpha}$ . Now if  $\alpha, \beta \in \Gamma$  it may happen that the tensor product  $\alpha \otimes \beta$  is irreducible. If it is we let  $\alpha\beta$  denote  $\alpha \otimes \beta$  so that  $\chi_{\alpha}\chi_{\beta} = \chi_{\alpha\beta}$ . If  $\alpha \in \Gamma_1$

this is always the case. We also let  $Z = Z(G)$  denote the center of  $G$ .

**Definition 2.1.**  $G$  is said to satisfy condition I provided

$$\lim_{d(\alpha) \rightarrow \infty} \Psi_\alpha(x) = 0$$

for all  $x \notin Z$ .

$G$  is said to satisfy condition II provided that for each positive integer  $t$  there are finitely many irreducible representations  $\beta_1, \dots, \beta_s$  of degree  $t$  such that if  $d(\beta) = t$  then  $\beta = \alpha\beta_i$  for some  $i$  and some  $\alpha \in \Gamma_1$ .

It should be noted that groups having open centers, in particular abelian groups and finite groups, satisfy both conditions. In [5] it is shown that unitary groups do also. In §9 it will be shown, using a result of Ragozin [4], that every compact connected simple Lie group satisfies both conditions.

We can now state the main result of the paper.

**Theorem 2.2.** Let  $G_i$  ( $i \in A$ ) be compact groups satisfying conditions I and II and let  $G = \prod_A G_i$ . Then every measure in  $F(G)$  is canonical.

Together with Lemma 1.2 this then gives

**Corollary 2.3.** If  $G$  is as above then  $E \subset \Gamma$  is  $E(\mu)$  for some  $\mu \in J(G)$  if and only if  $E$  belongs to the hypercoset ring of  $\Gamma$ .

It is well known (cf. [3, Theorem 2.1.4]) that an idempotent measure of norm 1 on a locally compact group is of the form  $\gamma \mathfrak{M}_H$  for some compact subgroup  $H$  and some  $\gamma \in \Gamma_1(H)$ . It follows that, for any compact group  $G$ , the elements of  $F(G)$  of norm 1 are canonical.

**Definition 2.4.** A measure  $\mu \in F(G)$  is irreducible if it cannot be written as the sum of two mutually singular nonzero measures in  $F(G)$ .

**Definition 2.5.** The support group  $L(\mu)$  of a measure  $\mu \in M(G)$  is the smallest closed subgroup that carries  $\mu$ .

Clearly if  $\mu \in M^Z(G)$  then  $L(\mu)$  is normal. A rough idea of the proof of Theorem 2.2 is to show that if  $\mu \in F(G)$  is irreducible then  $\mu \ll \mathfrak{M}_{L(\mu)}$ .

The proof of Theorem 2.2 is in §8. §3 deals with projections of  $M(G)$  onto the measures carried by the cosets of a normal Borel subgroup. §4 contains results concerning  $F(G)$  for an arbitrary compact group  $G$ . In §5 it is shown that if  $\mu$  is canonical and  $\|\mu\| > 1$  then  $\|\mu\| > 1 + 1/700$ . This generalizes a well-known result on abelian groups and is perhaps of independent interest. §§6 and 7 contain results about  $F(G)$  for  $G$  as in the hypotheses of Theorem 2.2. They are an attempt to use the methods of Amemiya and Itô [1] for abelian groups in this more general setting. In §9 a result of Ragozin [4] is used to show that the conclusions of Theorem 2.2 and Corollary 2.3 are valid for connected groups. An example of where Theorem 2.2 fails is given in §10.

3. **Projections.** Let  $H$  be a normal Borel subgroup of  $G$ . For  $\mu \in M(G)$  define

$$\Pi_H \mu(E) = \sum_x \mu(E \cap Hx),$$

the sum being over distinct coset representatives of  $H$ .

- Lemma 3.1.** (a)  $\Pi_H$  is a homomorphism of the algebra  $M(G)$  into itself.  
 (b)  $\Pi_H$  maps  $F(G)$  into itself.

**Proof.** (a) is proved in [7, Theorem 3.4.1] for  $H$  a closed subgroup and  $G$  abelian. The proof works equally well for this more general situation. Since  $H$  is normal it is easily seen that  $\Pi_H$  maps  $M^Z(G)$  into itself and (b) then follows since  $\Pi_H$  is a homomorphism.

The following theorem gives the first indication that an irreducible measure in  $F(G)$  is absolutely continuous with respect to the Haar measure of its support group.

**Theorem 3.2.** Let  $\mu \in F(G)$  be irreducible and have support group  $L$ . If  $H$  is a closed normal subgroup of  $G$  and  $\Pi_H \mu \neq 0$  then  $H \cap L$  is open in  $L$ .

**Proof.** Write  $\mu = \Pi_H \mu + (\mu - \Pi_H \mu)$ . By Lemma 3.1 these last two measures are in  $F(G)$ . Since they are singular and  $\mu$  is irreducible we must have  $\mu = \Pi_H \mu$ . Also  $\mu = \Pi_L \mu$  and it is then easily seen that  $\mu = \Pi_{H \cap L} \mu$ .

Retopologize  $G$  so that the closed subgroup  $H \cap L$  is open; let  $G$  with this new topology be denoted by  $G_0$ . Since  $\mu$  is supported on countably many cosets of  $H \cap L$  we have then that  $\mu \in F(G_0)$ . Now  $G_0$  is a locally compact group and, since  $H \cap L$  is a compact open normal subgroup,  $G_0$  has small invariant neighborhoods. It follows from [6, Theorem 1] that  $\mu$  is supported on a compact subgroup of  $G_0$ . Thus, as an element of  $F(G_0)$ ,  $\mu$  is supported on a finite extension  $P$  of  $H \cap L$ .  $P$  is then a closed subgroup of  $G$  that carries  $\mu$ . Hence  $L \subset P$  and so  $L$  is a finite extension of  $H \cap L$ ; that is  $H \cap L$  is open in  $L$ .

4.  $F(G)$  for arbitrary  $G$ . This section contains some lemmas concerning  $F(G)$  for an arbitrary compact group  $G$ .

**Lemma 4.1.** If  $\mu \in F(G)$  has support group  $L$  and  $T = \{\Psi_\alpha|_L : \alpha \in E(\mu)\}$  is finite then  $\mu$  is canonical.

**Proof.** Let  $\Psi_{\alpha_1}|_L, \dots, \Psi_{\alpha_t}|_L$  be the distinct elements of  $T$ . Then  $E(\mu) = \bigcup_1^t \alpha_i L^\perp$ . These hypercosets are disjoint and, since  $L$  carries  $\mu$ ,  $\hat{\mu}$  is constant on each  $\alpha_i L^\perp$ . Thus  $\mu = \sum_1^t a_i \chi_{\alpha_i} \upharpoonright_L$  for some constants  $a_i$  so that  $\mu$  is canonical by Lemma 1.3.

**Lemma 4.2.** *Let  $H$  be a closed normal subgroup of  $G$ . If  $\mu \in F(G)$  and  $|\mu|(H^c) < 1/2$  then  $|\mu|(H^c) = 0$ .*

**Proof.** Let  $\alpha$  belong to the hypercoset  $\beta H^\perp$ . This implies that  $\Psi_\alpha|_H = \Psi_\beta|_H$  so that

$$|\hat{\mu}(\alpha) - \hat{\mu}(\beta)| \leq \int |\bar{\Psi}_\alpha - \bar{\Psi}_\beta| d|\mu| \leq 2|\mu|(H^c) < 1.$$

Since  $\hat{\mu}$  is integer valued this gives that  $\hat{\mu}$  is constant on  $\beta H^\perp$ . It follows easily that  $\mu$  is carried by  $H$ ; that is  $|\mu|(H^c) = 0$ .

If  $\alpha \in \Gamma$  and  $\mu \in F(G)$  then it does not follow that  $\Psi_\alpha \mu \in F(G)$ . The problem of course is that the product of two irreducible characters decomposes, in general, into a sum of several irreducible characters. However a sequence in  $\Gamma$  may have the following property.

**Definition 4.3.** *A sequence  $\{\alpha\} \subset \Gamma$  is an irreducible sequence if for each  $\beta \in \Gamma$  there is  $\alpha(\beta)$  such that  $\alpha \otimes \beta$  is irreducible whenever  $\alpha > \alpha(\beta)$ .*

For example if  $G = \prod_1^\infty G_i$  and  $\alpha_i \in \Gamma(G)$  is given by some  $\alpha_i \in \Gamma(G_i)$  then  $\{\alpha_i\}$  is an irreducible sequence. This example will be useful in the proof of Theorem 2.2 because if  $\{\alpha\}$  is an irreducible sequence and  $\mu \in F(G)$  then every weak limit point of  $\{\Psi_\alpha \mu\}$  also belongs to  $F(G)$ .

The following lemma is a generalization of Helson's translation lemma [7, Lemma 3.5.1].

**Lemma 4.4.** *Let  $L$  be a closed normal subgroup of  $G$ . Suppose  $\{\alpha\}$  is a sequence in  $\Gamma$  with  $\Psi_\alpha|_L$  being distinct. Let  $\mu \in M^Z(G)$  be carried by  $L$ . If  $\lambda$  is a weak limit point of  $\{\Psi_\alpha \mu\}$  then  $\lambda$  and  $\mathfrak{M}_L$  are mutually singular.*

**Proof.**  $\chi_\alpha|_L$  decomposes into a sum of irreducible characters on  $L$ . Since the  $\Psi_\alpha|_L$  are distinct any character of  $\Gamma(L)$  which appears in  $\chi_\alpha|_L$  does not appear in  $\chi_\beta|_L$  (for  $\alpha \neq \beta$ ). The remainder of the proof follows that of [7, Lemma 3.5.1] exactly.

**Lemma 4.5.** *Let  $\mu \in F(G)$  be irreducible. If  $\{\alpha\}$  is an irreducible sequence and  $\Psi_\alpha \mu$  converges weakly to a nonzero canonical measure  $\lambda$  then  $\mu$  is also canonical.*

**Proof.** Let  $L$  be the support group of  $\mu$ .  $\lambda$  is then also carried by  $L$ . Now if  $H$  is a closed normal subgroup with  $\prod_H \lambda \neq 0$  then  $\prod_H \mu \neq 0$ . By Theorem 3.2 this implies that  $H \cap L$  is open in  $L$ . Thus the Haar measures that appear in the canonical measure  $\lambda$  are all absolutely continuous with respect to  $\mathfrak{M}_L$ . By Lemma 4.4 we then have that  $\Psi_\alpha|_L$  are the same for  $\alpha \geq \alpha_0$  so that

$$(1) \quad \Psi_{\alpha_0} \mu = \lambda,$$

and

$$(2) \quad |\Psi_{\alpha_0}|_L^2 = \overline{\Psi_{\alpha_0}} \Psi_{\alpha_0}|_L \quad \text{for } \alpha \geq \alpha_0.$$

We will now show that

$$(3) \quad |\Psi_{\alpha_0}|_L \equiv 1.$$

This gives, by (1), that  $\mu \ll \lambda$  and so  $\mu \ll \mathfrak{M}_L$  which implies, by Lemma 1.3, that  $\mu$  is canonical.

Since  $\{\alpha\}$  is an irreducible sequence there is  $\alpha \geq \alpha_0$  so that  $\beta = \overline{\alpha_0} \alpha$  is irreducible. Because of (2),  $\int_L \Psi_{\beta} d\mathfrak{M}_L \neq 0$  so that  $\beta \in L^{\perp}$ ; that is  $\Psi_{\beta}|_L \equiv 1$ . (3) then holds because of (2).

The example of §10 will show that we can have  $\mu \in F(G)$  being carried by a closed normal subgroup  $H$  with  $\mu$  canonical when considered as an element of  $F(H)$  but not canonical as an element of  $F(G)$ . However we do have the following.

**Lemma 4.6.** *Let  $\mu \in F(G_1 \times G_2)$  have support group  $L$ . Suppose that  $L \subset G_1 \times K$  where  $K$  is a finite normal extension of  $Z = Z(G_2)$ . Then if  $\mu$  is canonical with respect to  $G_1 \times K$  it is also canonical with respect to  $G_1 \times G_2$ .*

**Proof.** Without loss of generality we can assume that  $\mu$  is irreducible as an element of  $F(G_1 \times G_2)$ . Now  $\mu$  is the sum of finitely many nonzero measures  $\mu_i$  each of which is absolutely continuous with respect to  $\mathfrak{M}_{H_i}$ , where  $H_i$  is a closed normal subgroup of  $G_1 \times K$ . Now each  $H_i \subset L$  so that  $K_i = H_i \cap (G_1 \times Z)$  is open in  $H_i$ . Thus  $\prod_{K_i} \mu_i = \mu$ . But also  $K_i$  is normal in  $G_1 \times G_2$  and  $\prod_{K_i} \mu_i \neq 0$  so that, by Theorem 3.2,  $K_i = K_i \cap L$  is open in  $L$ . Thus  $H_i$  is also open in  $L_i$  so that  $\mu_i \ll \mathfrak{M}_{H_i} \ll \mathfrak{M}_L$ . This implies  $\mu \ll \mathfrak{M}_L$  which, by Lemma 1.3, makes  $\mu$  canonical on  $G_1 \times G_2$ .

The previous lemma can be generalized to infinite products.

**Lemma 4.7.** *Let  $G = \prod_1^{\infty} G_i$  and let  $\mu \in F(G)$  have support group  $L$ . Suppose that  $L \subset G_1 \times \prod_2^{\infty} K_i$  where each  $K_i$  is a finite normal extension of  $Z_i = Z(G_i)$ . Then if  $\mu$  is canonical with respect to  $G_1 \times \prod_2^{\infty} K_i$  it is also canonical with respect to  $G$ .*

**Proof.** Without loss of generality we can assume  $\mu$  is irreducible as an element of  $F(G)$ . By Lemma 4.6 we have that, for each  $n$ ,  $\mu$  is canonical with respect to  $A_n = \prod_1^n G_i \times \prod_{n+1}^{\infty} K_i$ . Thus for each  $n$  we can write

$$(4) \quad \mu = \sum_{j=1}^{a(n)} \mu_{j,n}$$

where each  $\mu_{j,n}$  is absolutely continuous with respect to  $\mathfrak{M}_{j,n}$ , the Haar measure of  $H_{j,n}$ , a closed normal subgroup of  $A_n$ . Also  $H_{j,n} \subset L$ . We can also assume, for each fixed  $n$ , that the  $\mathfrak{M}_{j,n}$  and hence the  $\mu_{j,n}$  are mutually singular. Now  $\|\mu_{j,n}\| \geq 1$  so that  $a(n) \leq \|\mu\|$ . Thus we can assume that  $a(n) = a$  for  $n \geq n_0$ . For each  $n$  let the  $\mathfrak{M}_{j,n}$  be ordered so that

$$(5) \quad \prod_{H_{j,n}} \mathfrak{M}_{i,n} = 0 \quad \text{for } j < i.$$

Then

$$(6) \quad \prod_{H_{1,n}} \mu = \mu_{1,n} = \sum_j \prod_{H_{1,n}} \mu_{j,n+1}.$$

Now  $\prod_{H_{1,n}} \mu_{j,n+1} = \mu_{j,n+1}$  or 0. Pick the smallest  $j$  so that it is not 0 and apply  $\prod_{H_{j,n+1}}$  to (6). Using the fact that  $H_{j,n+1}$  is normal in  $A_n$  as well as  $A_{n+1}$  (5) gives  $\mu_{1,n} = \mu_{j,n+1}$ . By repeating this process, and also reordering the  $\mathfrak{M}_{j,n}$ , we obtain that

$$(7) \quad \mu_{j,n} = \mu_{j,n+1} \quad (1 \leq j \leq a; n \geq n_0).$$

Now let  $\mu_j = \mu_{j,n}$ . Then, for  $n \geq n_0$ ,  $\mu_j \in F(A_n)$ . Since  $\bigcup_{n_0}^\infty A_n$  is dense in  $G$  we must then have that  $\mu_j \in M^Z(G)$  and so  $\mu_j \in F(G)$ . But the  $\mu_j$  are mutually singular and  $\mu$  is irreducible so that  $a = 1$ . Hence  $\mu$  itself can be written, for  $n \geq n_0$ , as  $\mu = \mu_{1,n}$ . Thus  $\mu \ll \mathfrak{M}_{1,n}$  and so  $\mu$  is carried by  $H_{1,n} \subset L$ . But since  $L$  is the support group of  $\mu$  and  $H_{1,n}$  is closed we must have  $H_{1,n} = L$ . Thus  $\mu$ , being absolutely continuous with respect to the Haar measure of  $L$ , a closed normal subgroup of  $G$ , is canonical with respect to  $G$ .

5. Norms of canonical measures. It is known [7, Theorem 3.7.2] that if  $\mu$  is an idempotent measure on an abelian group and  $\|\mu\| > 1$  then  $\|\mu\| > \sqrt{5/2}$ . This section contains a generalization of this to canonical measures on compact groups.

Lemma 5.1. Let  $P = \sum a_\alpha \chi_\alpha$  be a polynomial on  $G$ . Suppose  $a_\alpha \geq 0$  and  $\|P\|_\infty = P(e) = 1$ . If

$$(1) \quad |P(g_i) - z_i| \leq \delta_i \quad (1 \leq i \leq p)$$

where  $|z_i| = 1$  then

$$(2) \quad \left| P(g_1 \cdots g_p) - \prod_1^p z_i \right| \leq \left( \sum \delta_i^{1/2} \right)^2.$$

**Proof.** The lemma needs only to be proved for  $p = 2$  as the general case then easily follows by induction. If  $T_\alpha$  is a representation affording  $\chi_\alpha$  we can assume  $T_\alpha(g_1)$  is a diagonal unitary matrix with diagonal entries  $b_{\alpha,i}$  ( $1 \leq i \leq d_\alpha$ ) and  $T_\alpha(g_2)$  is a unitary matrix with diagonal entries  $c_{\alpha,i}$ . Then

$$\begin{aligned} \chi_\alpha(g_1) &= \sum_i b_{\alpha,i}, & \chi_\alpha(g_2) &= \sum_i c_{\alpha,i} \quad \text{and} \\ \chi_\alpha(g_1g_2) &= \sum_i b_{\alpha,i}c_{\alpha,i}. \end{aligned}$$

Since  $\sum_\alpha d(\alpha) = 1$ , (2) follows directly from (1).

**Lemma 5.2.** *Let  $Q = \sum_E d(\alpha)\chi_\alpha$  be a central idempotent polynomial on  $G$ . If  $\|Q\|_1 > 1$  then*

$$(3) \quad \|Q\|_1 > 1 + 1/300.$$

**Proof.** Suppose

$$(4) \quad \|Q\|_1 \leq 1 + 1/300;$$

we will show  $\|Q\|_1 = 1$ . It can be assumed that  $G$  is the support group of  $Q$ . Write  $|Q|^2 = \sum_\alpha a_\alpha \chi_\alpha$  and  $|Q|^4 = \sum_\alpha b_\alpha \chi_\alpha$ . The  $a_\alpha$  and  $b_\alpha$  are nonnegative integers and

$$(5) \quad a_\alpha = \int |Q|^2 \bar{\chi}_\alpha \leq M d(\alpha), \quad b_\alpha = \int |Q|^4 \bar{\chi}_\alpha \leq M^3 d(\alpha)$$

where

$$(6) \quad M = \sum_E d^2(\alpha) = \|Q\|_\infty = Q(e) = \|Q\|_2^2.$$

Also, by Hölder's inequality,

$$(7) \quad \sum a_\alpha b_\alpha = \int |Q|^6 \geq M^5 \|Q\|_1^{-4}.$$

Define  $A_1(g) = M^{-4} \sum_\alpha b_\alpha (1 - a_\alpha (M d(\alpha))^{-1}) \chi_\alpha(g)$  and

$$A_2(g) = M^{-2} \sum_\alpha a_\alpha (1 - b_\alpha (M^3 d(\alpha))^{-1}) \chi_\alpha(g).$$

It follows from (4)–(7) that

$$(8) \quad \|A_i\|_\infty = A_i(e) \leq 1 - \|Q\|_1^{-4} < 1/60 \quad (i = 1, 2).$$

Thus

$$(9) \quad 0 \leq |Q|^2/M^2 - |Q|^4/M^4 = A_2 - A_1 < 1/30.$$



Hence, for all  $g$ , either

$$(10) \quad |Q(g)| < M/5 \quad \text{or}$$

$$(11) \quad |Q(g)| > 4M/5.$$

If  $g_1$  and  $g_2$  satisfy (11) then applying Lemma 5.1 to  $P = Q/M$  shows  $|Q(g_1g_2)| \geq M/5$  so that  $g_1g_2$  must also satisfy (11). Hence the  $g$  for which (11) holds form a closed normal subgroup  $H$ . Since (10) holds on  $H^c$  we have from (4) and (6) that

$$(12) \quad \frac{4}{5} \int_{H^c} |Q| \leq \int_{H^c} |Q| \left(1 - \frac{|Q|}{M}\right) \leq \int |Q| \left(1 - \frac{|Q|}{M}\right) \leq \frac{1}{300}.$$

It follows from (12) and Lemma 4.2 that the idempotent measure  $Q\mathfrak{M}_G$  is carried on  $H$ ; that is  $Q$  vanishes on  $H^c$ . It was assumed however that  $G$  is the support group of  $Q$ . Hence (11) holds for all  $g$ . But then  $4M/5 \leq \|Q\|_1 \leq 1 + 1/300$  so that  $M = 1$ . This then implies that  $Q$  is just a character of degree 1 and the proof is complete.

**Theorem 5.3.** *If  $\mu$  is a canonical measure and  $\|\mu\| > 1$  then  $\|\mu\| > 1 + 1/700$ .*

**Proof.** If  $\mu$  is reducible then  $\|\mu\| \geq 2$ . An irreducible canonical measure is absolutely continuous with respect to  $\mathfrak{M}_H$  for some closed normal subgroup  $H$ . Also  $\|\mu\| \geq |\hat{\mu}(\alpha)|$ . Thus we can assume that  $\mu$  is given by a polynomial  $Q = \sum(\pm 1)d(\alpha)\chi_\alpha$ . Suppose  $\|Q\|_1 \leq 1 + 1/700$ . If either  $\pm Q$  is idempotent then  $\|Q\|_1 = 1$  by Lemma 5.2. Otherwise  $Q * Q$  and  $(Q * Q \pm Q)/2$  are nonzero idempotents with norms less than  $1 + 1/300$  and so, by Lemma 5.2, they all have norm 1. It is easily seen from this that  $Q = \gamma_1 - \gamma_2$  where  $\gamma_1$  and  $\gamma_2$  are distinct characters of degree 1 with  $\gamma_1^2 = \gamma_2^2$ . It follows that  $|Q| = 0$  on a subgroup of index 2 and  $|Q| = 2$  on the complement of this subgroup. Thus  $\|\mu\| = \|Q\|_1 = 1$ .

The estimates in 5.2 and 5.3 can easily be improved. It would be interesting to know the best ones. A special case of 5.2 is that if  $\|d(\alpha)\chi_\alpha\|_1 > 1$  then  $\|d(\alpha)\chi_\alpha\|_1 > 1 + 1/300$ , whenever  $\chi_\alpha$  is an irreducible character. It should be noted that a group can be constructed having a sequence  $\{\alpha\} \subset \Gamma$  with  $d(\alpha) \rightarrow \infty$  and  $\|d(\alpha)\chi_\alpha\|_1 = 1$  and having another sequence  $\{\beta\}$  with  $d(\beta) \rightarrow \infty$  and  $1 < \|d(\beta)\chi_\beta\|_1 \leq 2$ .

6.  $F(G)$  for special  $G$ . This section contains some rather technical results which are necessary for the proof of Theorem 2.2. We will first prove a special case of that theorem.

**Theorem 6.1.** *Let  $G_i$  ( $1 \leq i \leq n$ ) be compact groups satisfying conditions I and II and let  $G = \prod G_i$ . Then every measure in  $F(G)$  is canonical.*

**Proof.** The proof will be by induction on  $\|\mu\|$ . If  $\|\mu\| = 1$  then it is known [3, Theorem 2.1.4] that  $\mu$  is canonical. Suppose that for such  $G$  every measure in  $F(G)$  with norm less than  $A$  is canonical and let  $\|\mu\| < A + 1$ . We can assume  $\mu$  is irreducible.

Say that  $\mu$  is of *bounded representation type* (b.r.t.) if there is  $M < \infty$  such that  $\hat{\mu}(\alpha) = 0$  whenever  $d(\alpha) > M$ . If  $\mu$  is not of b.r.t. then, for some  $j \neq 0$ ,  $E_j(\mu)$  contains a sequence  $\{\alpha\}$  with  $d(\alpha) \rightarrow \infty$ . Now  $\alpha$  can be written as  $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$  where  $\alpha_i \in \Gamma(G_i)$  and  $d(\alpha) = d(\alpha_1) \dots d(\alpha_n)$ . For some  $i$ , say  $i = 1$ ,  $d(\alpha_1) \rightarrow \infty$ . Since  $G_1$  satisfies condition I,  $\Psi_{\alpha_1} \rightarrow 0$  off  $Z_1 = Z(G_1)$ . Thus since

$$(1) \quad j = \hat{\mu}(\alpha) = \int \bar{\Psi}_{\alpha} d\mu$$

we having, using the Lebesgue dominated convergence theorem, that  $|\mu|(Z_1 \times \prod_2^n G_i) \neq 0$ . Since  $\mu$  is irreducible it follows from Theorem 3.2 that  $L \cap (Z_1 \times \prod_2^n G_i)$  is open in  $L$  ( $L$  is the support group of  $\mu$ ). Thus  $L \subset K_1 \times \prod_2^n G_i = G_i^*$  where  $K_1$  is a finite normal extension of  $Z_1$ .

There are now three possibilities:

- (a)  $\mu$  is irreducible but not of b.r.t. on  $G^*$ .
- (b)  $\mu$  is irreducible and of b.r.t. on  $G^*$ .
- (c)  $\mu$  is reducible on  $G^*$ .

For  $\beta \in \Gamma(K_1)$ ,  $d(\beta) \leq [K_1 : Z_1]^{1/2} < \infty$ . Thus in case (a) we can repeat the above process and obtain that  $L \subset K_1 \times K_2 \times \prod_3^n G_i$  where  $K_2$  is a finite normal extension of  $Z(G_2)$ . With respect to this new group, one of the above cases holds.

After a finite number of such steps we obtain that, for some rearrangement of the  $G_i$  and some  $m$ ,  $L \subset H_1 \times H_2 = G^*$  where  $H_1$  is a finite normal extension of  $Z(\prod_1^m G_i)$  and  $H_2 = \prod_{m+1}^n G_i$  and either (b) or (c) holds for this  $G^*$ . Since  $H_1$  has an open center it satisfies both conditions I and II.  $G^*$  also satisfies condition II because it is the product of groups that do.

If case (b) holds then by Theorem 6 of [5] and Lemma 1.1 we have that  $\mu$  is canonical on  $G^*$ . (Theorem 6 of [5] is stated for  $J$  but it applies equally well to  $F$ .) If case (c) holds then  $\mu$  is a sum of measures in  $F(G^*)$  each of norm less than  $A$  and so by the induction hypothesis  $\mu$  is canonical on  $G^*$ . Lemma 4.6 then gives that  $\mu$  is canonical on  $G$ .

**Lemma 6.2.** *Let  $G_i$  ( $1 \leq i < \infty$ ) be compact groups satisfying conditions I and II and let  $G = \prod G_i$ . Suppose  $\mu \in J(G)$  satisfies*

$$(2) \quad \alpha \in E(\mu) \text{ if and only if } \|(\Psi_{\alpha} - 1)\mu\| < 1/300.$$

Then  $\mu$  is canonical; in fact  $\mu = \mathfrak{M}_H$  for some closed normal subgroup  $H$ .

**Remark.** For abelian groups this is obvious. The problem in the nonabelian case is that  $\Psi_\alpha \mu$  need not belong to  $F(G)$ . It seems likely that the restriction on  $G$  can be lifted but I have been unable to do so. It does hold, however, if  $G$  is totally disconnected.

**Proof.** Let  $\mathfrak{M}_\tau$  be the Haar measure of  $\Pi_{\tau+1}^{\infty} G_i \subset G$  and let  $\mu_\tau = \mu * \mathfrak{M}_\tau$ . Then  $\mu_\tau \rightarrow \mu$  weakly and  $\mu_\tau$  can be considered as an element of  $J(\Pi_1^r G_i)$ . (2) still holds for  $\mu_\tau$  which is canonical by Theorem 6.1. We will show that  $\|\mu_\tau\| = 1$  for all  $\tau$ . It then follows that  $\|\mu\| = 1$  so that, by [3, Theorem 2.1.4],  $\mu$  is canonical. Since  $1 \in E(\mu)$ , where 1 is the trivial representation,  $\mu$  is a Haar measure.

Since  $\mu_\tau$  is canonical it can be written as  $\mu_\tau = \sum_1^s \nu_j$  where  $\nu_j \in F(G)$  and  $\nu_j \ll \lambda_j$ , the Haar measure of a closed normal subgroup  $H_j$ . We show first that  $s = 1$ . We can assume the  $\lambda_j$  are mutually singular and that  $\Pi_{H_1} \lambda_j = 0$  for  $j > 1$ . Now if  $\alpha \in E(\mu_\tau)$  then

$$(3) \quad \|(\Psi_\alpha - 1)\nu_j\| \leq \|(\Psi_\alpha - 1)\mu_\tau\| < 1/300.$$

Fix  $\gamma \in E(\nu_j)$  then by (3)

$$(4) \quad \left| \int (\bar{\Psi}_\alpha - 1)\bar{\Psi}_\gamma d\nu_j \right| = \left| \int \bar{\Psi}_\alpha \bar{\Psi}_\gamma d\nu_j - \hat{\nu}_j(\gamma) \right| < 1/300.$$

Since  $\hat{\nu}_j(\gamma)$  is a nonzero integer, (4) implies that the decomposition of  $\alpha \otimes \gamma$  contains an element of  $E(\nu_j)$ . Thus  $E(\mu_\tau) \subset \bar{\gamma}E(\nu_j)$  where  $\bar{\gamma}$  is the representation conjugate to  $\gamma$  and  $\bar{\gamma}E(\nu_j)$  consists of the  $\beta \in \Gamma$  that appear in the decomposition of  $\bar{\gamma} \otimes \theta$  for some  $\theta \in E(\nu_j)$ . Now  $\nu_j \ll \lambda_j$  which implies that  $E(\nu_j)$  is the union of finitely many hypercosets of  $H_j^\perp$ . Thus  $E(\mu_\tau)$  is contained in the union of finitely many hypercosets of  $H_j^\perp$  so that there is  $\omega_j \in J(G)$  with  $\omega_j \ll \lambda_j$  and  $\mu_\tau * \omega_j = \mu_\tau$ . Let  $j > 1$  and apply  $\Pi_{H_1}$  to this last equality. Since  $\Pi_{H_1} \lambda_j = 0$  it follows that  $0 = \Pi_{H_1} \mu_\tau = \lambda_1$ . This is a contradiction so that  $s = 1$  and  $\mu_\tau = \lambda_1$ .

We can thus write  $\mu_\tau = Q\lambda_1$  where  $Q$  is a central idempotent polynomial on  $H_1$  and also  $Q = \sum_E a_\alpha \chi_\alpha|_{H_1}$  where  $a_\alpha > 0$  and  $E \subset E(\mu_\tau)$ . It follows from (2) that

$$(5) \quad \int |(Q/Q(e) - 1)Q| d\lambda_1 \leq \sum_E \frac{a_\alpha d_{(\alpha)} \|(\Psi_\alpha - 1)\mu_\tau\|}{\sum_E a_\alpha d_{(\alpha)}} < \frac{1}{300}.$$

Thus since  $\int |Q|^2 d\lambda_1 = Q(e)$  it follows from (5) that

$$(6) \quad \|\mu_\tau\| = \int |Q| d\lambda_1 < 1 + 1/300$$

so that, by Lemma 5.2,  $\|\mu_r\| = 1$  and the proof is complete.

**Lemma 6.3.** *Let  $p$  be a positive integer and  $M < \infty$ . There is  $\delta = \delta(p, M) > 0$  such that if  $G$  is as in 6.2 and  $\mu \in F(G)$  satisfies the following:*

- (a)  $\|\mu\| \leq M$ .
- (b) *There is a normal Borel subgroup  $T$  of  $G$  with  $|\mu|(T^c) = 0$ .*
- (c) *There are Borel homomorphisms  $f_1, f_2, \dots, f_p$  of  $T$  into the unit circle with  $f_1 \equiv 1, f_i \mu \neq f_j \mu$  for  $i \neq j$  and for each  $i$  there is an irreducible sequence  $\{\alpha\} \subset E(\mu)$  with  $\Psi_\alpha \rightarrow f_i$  pointwise on  $T$ .*
- (d)  $\alpha \in E(\mu)$  *if and only if*  $\|(\Psi_\alpha - \bar{f}_i)\mu\| < \delta$  *for some  $i$ ,*  
*then  $\omega = f_1 \mu * f_2 \mu * \dots * f_p \mu = A \mathbb{M}_H$  for some integer  $A \neq 0$  and some closed normal subgroup  $H$ .*

**Proof.**  $\delta$  is chosen so that

$$(7) \quad p\delta^{1/2} M^{p-1} (pM + 2) < 1/300.$$

Clearly each  $f_i \mu \in F(G)$  so that  $\omega \in F(G)$ . Also, by (c),

$$\int f_i d\mu = \lim \int \bar{\Psi}_\alpha d\mu \neq 0 \text{ so that } \hat{\omega}(1) = \prod (f_i \mu)^\wedge(1) \neq 0.$$

Let  $\beta$  be a fixed element of  $E(\omega)$ . Then  $\beta \in E(f_i \mu)$  for all  $i$  so that, by using (c),  $\lim \int \bar{\Psi}_\beta \bar{\Psi}_\alpha d\mu = \int \bar{\Psi}_\beta f_i d\mu \neq 0$ . Since  $\{\alpha\}$  is an irreducible sequence we must have  $\beta \otimes \alpha$  irreducible eventually and  $\beta\alpha \in E(\mu)$ . Thus (d) implies that

$$(8) \quad \|(\Psi_\beta \Psi_\alpha - \bar{f}_j)\mu\| < \delta \text{ for some } j.$$

Since  $\mu$  is carried by  $T$  it follows from (c) that

$$(9) \quad \|(\Psi_\beta - f_i \bar{f}_j)\mu\| = \|(\Psi_\beta \bar{f}_i - \bar{f}_j)\mu\| \leq \delta.$$

Now since  $f_1 \equiv 1, E(\omega) \subset E(\mu)$  so that there is  $k$  with

$$(10) \quad \|(\Psi_\beta - \bar{f}_k)\mu\| < \delta.$$

We will show for such a  $k$  that

$$(11) \quad f_k \omega = \omega.$$

Since  $\bar{f}_k \mu$  and  $f_i \bar{f}_j \mu$  are both in  $F(G)$  it follows from (9) and (10) that (since  $\delta < 1/2$ )  $\bar{f}_k \mu = f_i \bar{f}_j \mu$ . Hence for this  $k$  and all  $i$  there is  $j$  with  $f_i f_k \mu = f_j \mu$ . Since  $\{f_i \mu\}$  are distinct it follows that

$$(12) \quad \{f_i f_k \mu : 1 \leq i \leq p\} = \{f_i \mu : 1 \leq i \leq p\}.$$

Now the  $f_i \mu$  are central measures so that if  $\sigma$  is a permutation of  $(1, 2, \dots, p)$  then

$$(13) \quad \omega \approx f_{\sigma(1)}\mu * \dots * f_{\sigma(p)}\mu.$$

Let  $\lambda = \mu \times \mu \times \dots \times \mu$  on  $G \times G \times \dots \times G$ . Then, for  $\gamma \in \Gamma$ , it follows from (12) and (13) that for some permutation  $\sigma$

$$\begin{aligned} (f_k\omega)^\wedge(\gamma) &= \int \bar{\Psi}_\gamma f_k d\omega \\ (14) \quad &= \int \dots \int \bar{\Psi}_\gamma(g_1 \dots g_p) f_k(g_1 \dots g_p) \prod_1^p f_i(g_i) d\lambda \\ &= \int \dots \int \bar{\Psi}_\gamma(g_1 \dots g_p) \prod f_{\sigma(i)}(g_i) d\lambda = \int \bar{\Psi}_\gamma d\omega = \hat{\omega}(\gamma). \end{aligned}$$

(11) then follows from (14).

Now let  $R = \{g \in T: |\Psi_\beta(g) - \bar{f}_k(g)| < \delta^{1/2}\}$ . It follows from (10) that

$$(15) \quad |\mu|(R^c) \leq \delta^{1/2}.$$

It also follows from Lemma 5.1 that if  $g_i \in R$  ( $1 \leq i \leq p$ )

$$(16) \quad |\Psi_\beta(g_1 \dots g_p) - \bar{f}_k(g_1 \dots g_p)| < p^2 \delta^{1/2}.$$

Then by (7), (11), (15), (16) and (a)

$$\begin{aligned} (17) \quad \|\Psi_\beta - 1\| \omega &= \|(\Psi_\beta - \bar{f}_k)\omega\| \\ &\leq \int \dots \int |\Psi_\beta(g_1 \dots g_p) - \bar{f}_k(g_1 \dots g_p)| d(|\mu| \times \dots \times |\mu|) \\ &\leq p^2 \delta^{1/2} M^p + 2pM^{p-1} \delta^{1/2} < 1/300. \end{aligned}$$

The last inequality is obtained by integrating over  $S = R \times \dots \times R$  and  $S^c$  separately.

(17) holds for all  $\beta \in E(\omega)$ . Lemma 6.2 can then be applied to show that  $\omega/\hat{\omega}(1)$  is a Haar measure.

7. Some more technical lemmas. The main purpose of this section is to prove Lemma 7.6.

Lemma 7.1. Let  $H$  be a closed normal subgroup of  $G$ . Suppose  $\{\alpha\}$  is a sequence in  $\Gamma$  such that

$$(1) \quad |\Psi_\alpha(g)| \rightarrow 1 \quad \text{a.e. } (\mathfrak{M}_H).$$

Then, for a large enough,  $|\Psi_\alpha(g)| \equiv 1$  on  $H$ .

Proof. Since  $H$  is closed and normal we can write  $\chi_\alpha|_H = a \sum_1^p \chi_{\beta_i}$  where  $\beta_i \in \Gamma(H)$  and  $ap d(\beta_i) = d(\alpha)$  for all  $i$ . Then

$$(2) \quad \int_H |\Psi_\alpha|^2 d\mathfrak{M}_H = a^2 p(d(\alpha))^{-2} = (pd^2(\beta_1))^{-1}.$$

By (1) and the Lebesgue dominated convergence theorem  $(pd^2(\beta_1))^{-1} \rightarrow 1$ . Thus eventually  $p = d(\beta_1) = 1$ . That is  $\chi_\alpha|_H = d(\alpha)\gamma$  where  $\gamma \in \Gamma(H)$  and  $d(\gamma) = 1$  so that  $|\Psi_\alpha| \equiv 1$  on  $H$ .

**Lemma 7.2.** *Let  $H$  and  $K$  be closed normal subgroups of  $G$  with  $H$  open in  $K$ . Let  $f$  be the characteristic function of  $H$ . Suppose  $\{\alpha\}$  is a sequence in  $\Gamma$  such that*

$$(3) \quad |\Psi_\alpha(g)| \rightarrow f(g) \text{ a.e. } (\mathfrak{M}_K).$$

*Then, for  $\alpha$  large enough,  $|\Psi_\alpha| \equiv f$  on  $K$ .*

**Proof.** From Lemma 7.1 we can assume  $|\Psi_\alpha| \equiv 1$  on  $H$ . Write  $\chi_\alpha|_K = a \sum_1^p \chi_{\beta_i}$  where  $\beta_i \in \Gamma(K)$ . It follows as in the previous proof that

$$(4) \quad (pd^2(\beta_1))^{-1} = \int_K |\Psi_\alpha|^2 d\mathfrak{M}_K = \mathfrak{M}_K(H) + o(1).$$

Thus eventually  $(pd^2(\beta_1))^{-1} = \mathfrak{M}_K(H)$  and so  $\int_{K-H} |\Psi_\alpha|^2 d\mathfrak{M}_K = 0$ ; that is  $\Psi_\alpha = 0$  on  $K - H$ .

**Lemma 7.3.** *Let  $H$  and  $K$  be closed normal subgroups of  $G$  with  $H$  open in  $K$ . Let  $E \subset \Gamma$  be such that  $\{\Psi_\alpha|_H : \alpha \in E\}$  is finite. Then  $\{\Psi_\alpha|_K : \alpha \in E\}$  is also finite.*

**Proof.** If the lemma is false there is a sequence  $\{\alpha\} \subset \Gamma$  such that  $\Psi_\alpha|_K$  are distinct and  $\Psi_\alpha|_H$  are all the same. By Lemma 4.4  $\Psi_{\alpha_0}|_H$ , being a weak limit point of  $\{\Psi_\alpha|_H\}$ , is singular to  $\mathfrak{M}_K$ . But  $\mathfrak{M}_H \ll \mathfrak{M}_K$  which gives a contradiction.

The following lemma was proved by Amemiya and Itô [1] although not stated in this form:

**Lemma 7.4.** *Let  $\mu$  and  $\nu$  be nonzero regular Borel measures on some space such that  $f_n \mu \rightarrow \nu$  weakly where  $\|f_n\|_\infty \leq 1$ . Then given  $k < \|\nu\|/\|\mu\| \leq 1$  there is  $N$  such that*

$$(5) \quad \|(f_n - f_m)\mu\| < 2\|\mu\|(1 - k + (1 - k^2)^{1/2}) \text{ for } n, m \geq N.$$

**Lemma 7.5.** *If, in the above,  $\|\mu\| = \|\nu\|$  then  $f_n \mu \rightarrow \nu$  in norm.*

**Proof.** Letting  $k$  be close to 1 in (5) shows that  $\{f_n \mu\}$  is a Cauchy sequence.

It also follows immediately from Lemma 7.4 that if  $1 \leq A \leq \|\nu\| \leq \|\mu\| < A + 1/100A$  then

$$(6) \quad \|(f_n - f_m)\mu\| < 1/4 \text{ for large } n, m.$$

**Lemma 7.6.** *Let  $G_i$  ( $1 \leq i < \infty$ ) be compact groups satisfying condition II and let  $G = \prod G_i$ . Let  $\mu \in F(G)$  have support group  $L$ . If  $\mu$  satisfies the following two conditions then  $\mu$  is canonical.*

(a) *There is an integer  $N$  so that if  $P$  is a normal Borel subgroup of  $G$  and  $\prod_1^N G'_i \subset P$  ( $G'_i$  is the commutator subgroup of  $G_i$ ) then  $\prod_p \mu \neq 0$  implies  $P \cap L$  is open in  $L$ .*

(b) *For each  $i$  there is  $M_i < \infty$  so that if  $\alpha \in E(\mu)$  and  $\alpha = \alpha_1 \dots \alpha_n$  where  $\alpha_i \in \Gamma(G_i)$  then  $d(\alpha_i) < M_i$ .*

**Proof.** The proof is by induction on  $\|\mu\|$ . Assume it is true for  $1 \leq \|\mu\| < A$  and let  $A \leq \|\mu\| < A + 1/100A$ . We will show that  $T = \{\Psi_\alpha|_L : \alpha \in E(\mu)\}$  is finite;  $\mu$  is then canonical by Lemma 4.1.

We will assume  $T$  is infinite in order to obtain a contradiction. By using a diagonal process and the fact that each  $G_i$  satisfies condition II it follows from (b) that there is a sequence  $\{\alpha_n\} \subset E(\mu)$  such that

- (i)  $\alpha_n = \alpha_{n,1} \alpha_{n,2} \dots$  where  $\alpha_{n,i} \in \Gamma(G_i)$ ,
- (ii) there are  $\beta_i \in \Gamma(G_i)$  such that  $\alpha_{n,i} = \beta_i \gamma_{n,i}$  for  $n \geq n(i)$  where  $\gamma_{n,i} \in \Gamma(G_i)$  and  $d(\gamma_{n,i}) = 1$ , and
- (iii)  $\Psi_{\alpha_n}|_L$  are distinct.

By taking a subsequence of  $\{\alpha_n\}$  we can write

$$(7) \quad \alpha_n = \beta_1 \dots \beta_n \gamma_n \lambda_n$$

where  $d(\gamma_n) = 1$  and  $\lambda_n \in \Gamma(\prod_{n+1}^\infty G_i)$ .

We will first show that there is  $i_0 > N$  such that

$$(8) \quad |\Psi_{\beta_i}|_L \equiv 1 \quad \text{for } i \geq i_0.$$

Let  $P = \{g : \lim_i \dots \infty |\Psi_{\beta_i}(g)| = 1\}$ .  $P$  is clearly a normal Borel subgroup and by (7)  $\Psi_{\alpha_n} \rightarrow 0$  on  $P^c$  as  $n \rightarrow \infty$ . Since  $\hat{\mu}(\alpha_n) \neq 0$  it follows that  $|\mu|(P) \neq 0$ . Thus since  $\prod_1^N G'_i \subset P$  it follows from (a) that  $P \cap L$  is open in  $L$ . In particular  $P \cap L$  is closed so that by Lemma 7.1 there is  $i_0 > N$  such that  $|\Psi_{\beta_i}| \equiv 1$  on  $P \cap L$  for  $i \geq i_0$ . We will show  $P \cap L = L$ .

If  $P \cap L \neq L$  then, by Lemma 4.2,

$$(9) \quad |\mu|(L - P \cap L) \geq 1/2.$$

Also  $\lim_p \dots \infty |\Psi_{\beta_{i_0}} \dots \Psi_{\beta_p}| = 0$  on  $L - P \cap L$  so that if  $p$  is large enough it follows from Lemma 7.2 that

$$(10) \quad \Psi_{\beta_{i_0}} \dots \Psi_{\beta_p} \equiv 0 \quad \text{on } P \cap L.$$

Let  $\theta = \beta_{i_0} \dots \beta_p$  and let  $K = \prod_1^p G'_i$ . Define  $\phi = \bar{\Psi}_\theta \mu * \mathfrak{M}_K$ . It is easily seen that  $\phi \neq 0$ ,  $\phi \in F(G)$  and

$$(11) \quad \sigma_n = \beta_{p+1} \cdots \beta_n \gamma_n \lambda_n \in E(\phi) \text{ for } n > p.$$

Also since  $\Psi_\theta$  vanishes on  $L - (L \cap P)$  it follows from (9) that  $\|\phi\| \leq \|\mu\| - \frac{1}{2} < A$ . Also  $\phi$  satisfies (a) and (b). (b) is easy. To see (a) (with  $p$  in place of  $N$ ) suppose  $K \subset Q$  and  $\prod_Q \phi \neq 0$ . Then  $\prod_Q \mu \neq 0$  so that  $Q \cap L$  is open in  $L$ . Now the support group  $L^* = L(\phi) \subset (L \cap P)K$ . Thus  $L^*Q \subset LQ$ .  $Q$  is open in  $LQ$  and thus is also open in  $L^*Q$ ; that is  $L^* \cap Q$  is open in  $L^*$ . It follows from the induction hypothesis that  $\phi$  is canonical.

Since  $\phi$  is canonical it follows that for some closed normal subgroup  $H$  with  $\prod_H \phi \neq 0$  there is an infinite set  $I$  such that  $\sigma_n \in \sigma_{n_0} H^\perp$  for all  $n \in I$ ; that is  $\Psi_{\sigma_n}|_H$  are the same for  $n \in I$ . Since  $\phi * \mathfrak{M}_K = \phi$  we can assume  $K \subset H$  so that  $H^n \cap L$  is open in  $L$ . By Lemma 7.3 this implies that  $\{\Psi_{\sigma_n}|_L : n \in I\}$  is finite. But  $\alpha_n = \theta \sigma_n$  so that we have a contradiction to (iii). Hence  $P \cap L = L$  and (8) is proved.

Now let  $\theta_n = \beta_{i_0} \cdots \beta_n \gamma_n \lambda_n$ . A subsequence of  $\{\bar{\Psi}_\theta \mu\}$  converges weakly to some  $\omega$ . Now  $\omega \neq 0$  and  $\omega \in F(G)$ ; this is because the  $\Psi_{\beta_i}$  are multiplicative on  $L$  for  $i \geq i_0$  and because  $\{\gamma_n \lambda_n\}$  is an irreducible sequence. Now  $\omega$  clearly satisfies (b). To see (a) suppose  $\prod_P \omega \neq 0$ ; since  $\omega \ll \mu$  it follows that  $\prod_P \mu \neq 0$ . Hence if  $\prod_1^N G'_i \subset P$  we have  $P \cap L$  open in  $L$ . But  $L(\omega) \subset L$  so that  $P \cap L(\omega)$  is open in  $L(\omega)$ . The remainder of the proof is divided into two cases.

Case 1.  $\|\omega\| < A$ . Then  $\omega$  is canonical by the induction hypothesis. Let  $H$  be a closed normal subgroup of  $G$  such that  $0 \neq \prod_H \omega \ll \mathfrak{M}_H$ . Since  $\bar{\Psi}_\theta \mu|_H \rightarrow \omega|_H$  weakly it follows from Lemma 4.4 that  $\{\Psi_{\theta_n}|_H\}$  is finite. Now  $\Psi_\theta \mu \equiv 1$  on  $\prod_1^N G'_i$  so that  $\{\Psi_\theta|_S\}$  is finite where  $S = H \cdot \prod_1^N G'_i$ . Also  $\prod_S \mu \neq 0$  so that  $S \cap L$  is open in  $L$ . It follows from Lemma 7.3 that  $\{\Psi_\theta|_L\}$  is finite and since  $\alpha_n = \beta_1 \cdots \beta_{i_0-1} \theta_n$  this contradicts (iii).

Case 2.  $\|\omega\| \geq A$ . It follows here from the remark (6) after Lemma 7.5 that, if  $n, m$  are large enough,

$$(12) \quad \|(\bar{\Psi}_\theta - \bar{\Psi}_{\theta_m})\mu\| < 1/4.$$

Fixing  $n$  and letting  $m \rightarrow \infty$  in (12) then gives

$$(13) \quad \|\bar{\Psi}_\theta \mu - \omega\| \leq 1/4$$

and

$$(14) \quad \|\bar{\Psi}_\theta \mu\| \geq \|\omega\| - 1/4 \geq A - 1/4.$$

By (13) and (14), using that  $|\Psi_\theta| \leq 1$ ,



$$\begin{aligned}
 \|\mu - \Psi_{\theta_n} \omega\| &\leq \|(1 - |\Psi_{\theta_n}|^2)\mu\| + \|\Psi_{\theta_n}^2 \mu - \Psi_{\theta_n} \omega\| \\
 (15) \qquad \qquad &\leq 2[\|\mu\| - \|\bar{\Psi}_{\theta_n} \mu\|] + \|\bar{\Psi}_{\theta_n} \mu - \omega\| \\
 &\leq 3/4 + 1/50A.
 \end{aligned}$$

Since  $\omega$  is carried by  $L$  it follows that (a subsequence of)  $\{\Psi_{\theta_n} \omega\}$  converges weakly to some  $\lambda \in F(G)$ . (15) shows that  $\|\mu - \lambda\| < 1$  so that  $\mu = \lambda$ . Since  $\|\lambda\| \leq \|\omega\|$  we must then have that  $\|\omega\| = \|\mu\|$  so that by Lemma 7.5  $\bar{\Psi}_{\theta_n} \mu \rightarrow \omega$  in norm. It then follows that, for a subsequence,

$$(16) \qquad \qquad |\Psi_{\theta_n}| \rightarrow 1 \quad \text{a.e. } (|\mu|).$$

Now (16) occurs on a normal Borel subgroup  $S$  which contains  $\prod_1^N G_i'$ . Thus, by (a),  $S \cap L$  is open in  $L$ .  $S \cap L$  is then closed and carries  $\mu$  so that  $S \cap L = L$ . This gives, by Lemma 7.1, that

$$(17) \qquad \qquad |\Psi_{\theta_n}| \equiv 1 \quad \text{on } L \text{ for large } n.$$

But then  $\bar{\Psi}_{\theta_n} \mu \in F(G)$  so that, because of (12),

$$(18) \qquad \qquad \bar{\Psi}_{\theta_n} \mu = \bar{\Psi}_{\theta_n} \mu.$$

Finally (17) and (18) show that  $\Psi_{\theta_n}|_L = \Psi_{\theta_n}|_L$  which implies that  $\Psi_{\alpha_n}|_L = \Psi_{\alpha_n}|_L$  and this contradicts (iii).

8. Proof of Theorem 2.2.

**Theorem 2.2.** *Let  $G_i$  ( $i \in A$ ) be compact groups satisfying conditions I and II and let  $G = \prod_A G_i$ . Then every measure in  $F(G)$  is canonical.*

**Proposition 8.1.** *If the theorem is true for countable products it is true for any  $A$ .*

**Proof.** Let  $\mu \in F(G)$ . We can assume  $\mu$  is irreducible and has support group  $L$ . If  $\mu$  is not canonical then by Lemma 4.1 there is a sequence  $\{\alpha_n\} \subset E(\mu)$  such that  $\Psi_{\alpha_n}|_L$  are distinct. There is a countable set  $B \subset A$  such that  $\alpha_n \in \Gamma(\prod_B G_i)$  for all  $n$ . Let  $K = \prod_{A-B} G_i$  and  $\nu = \mu * \mathbb{1}_K$ . Then  $E(\nu)$  contains  $\{\alpha_n\}$  and  $\nu$  is canonical since it can be considered as a measure on  $\prod_B G_i$ . It is then easily seen that  $\nu$  is irreducible so that  $\{\Psi_{\alpha_n}|_{L(\nu)}\}$  is finite. But  $L(\nu) \cap L$  is open in  $L$  so that  $\{\Psi_{\alpha_n}|_L\}$  is finite by Lemma 7.3 which is a contradiction.

It remains to prove the theorem when  $A$  is countable. The proof is by induction on  $\|\mu\|$ . Assume it is true for  $\|\mu\| < C$  and let  $\|\mu\| < C + 1/100C$ . We can also assume that  $\mu$  is irreducible.

**Proposition 8.2.** *We can assume  $\mu$  satisfies condition (b) of Lemma 7.6 and that  $1 \in E(\mu)$ .*

**Proof.** Suppose  $E(\mu)$  contains a sequence  $\{\alpha\}$  where  $\alpha = \alpha_1 \alpha_2 \dots$  with  $\alpha_i \in \Gamma(G_i)$  and suppose, for some  $j$ , that  $d(\alpha_j) \rightarrow \infty$  as  $\alpha \rightarrow \infty$ . Since  $\bar{\mu}(\alpha) \neq 0$  and  $G_j$  satisfies condition I it follows that

$$(1) \quad |\mu| \left( \prod_{i \neq j} G_i \times Z(G_j) \right) \neq 0.$$

By Theorem 3.2 we then have that  $L = L(\mu) \subset \prod_{i \neq j} G_i \times K_j$  where  $K_j$  is a finite normal extension of  $Z(G_j)$ . Doing this for all such  $j$  we obtain that  $L \subset \prod K_i = G^*$  where  $K_i = G_i$  or  $K_i$  is a finite normal extension of  $Z(G_i)$ . Now each  $K_i$  satisfies conditions I and II and  $\mu$  satisfies condition (b) with respect to  $G^*$ . If  $\mu$  is canonical with respect to  $G^*$  then by Lemma 4.7 it is canonical with respect to  $G$ . It can still be assumed that  $\mu$  is irreducible; if  $\mu$  is reducible as an element of  $F(G^*)$  then it is canonical on  $G^*$  by the induction hypothesis and then it is canonical on  $G$ .

Let  $\alpha = \alpha_1 \dots \alpha_N \in E(\mu)$  and let  $K = \prod_1^N G'_i$ . Then  $\bar{\Psi}_\alpha \mu * \mathfrak{M}_K = \lambda \in F(G)$ ,  $1 \in E(\lambda)$  and  $\|\lambda\| \leq \|\mu\|$ . Also  $\lambda$  satisfies condition (b) since we can now assume that  $\mu$  does. We will show that if  $\lambda$  is canonical then  $\mu$  is also. Suppose  $\lambda = \sum \nu_j$  where  $\nu_j < \mathfrak{M}_{H_j}$  for some closed normal subgroup  $H_j$ . Let  $P$  be a normal Borel subgroup with  $K \subset P$  and  $\prod_P \mu \neq 0$ . Then since  $\mu$  is irreducible  $\prod_P \mu = \mu$ ; also  $\prod_P \mathfrak{M}_K = \mathfrak{M}_K$  since  $K \subset P$ . Thus  $\prod_P \lambda = \lambda$  so that  $\prod_P \mathfrak{M}_{H_j} = \mathfrak{M}_{H_j}$  for all  $j$ . This implies that  $P \cap H_j$  is open in  $H_j$ . But  $\prod_{H_j} \mu \neq 0$  so that  $H_j \cap L$  is open in  $L$ . Hence  $P \cap L$  is open in  $L$ . By Lemma 7.6 then  $\mu$  is also canonical. We can thus assume  $\mu$ , like  $\lambda$ , has  $1 \in E(\mu)$ .

The remainder of the proof involves using Lemma 6.3 to show that  $\mu$  satisfies condition (a) of Lemma 7.6.

Call a sequence  $\{\alpha_n\} \subset E(\mu)$  an *F-sequence* if  $\alpha_n = \alpha_{n,1} \alpha_{n,2} \dots$  with  $\alpha_{n,i} \in \Gamma(G_i)$  and  $d(\alpha_{n,i}) = 1$  for  $i \leq n$ . An *F-sequence* is an irreducible sequence so that a subsequence of  $\{\bar{\Psi}_{\alpha_n} \mu\}$  will converge weakly to some nonzero  $\omega \in F(G)$ . Let  $B$  be the collection of all such  $\omega$ .  $\mu \in B$  since  $\alpha_n = 1$  is an *F-sequence*. Now if  $\omega \in B$  and  $\|\omega\| < C$  then  $\omega$  is canonical by induction and  $\mu$  is then canonical by Lemma 4.5. We can thus assume that  $\omega \in B$  implies

$$(2) \quad C \leq \|\omega\| \leq \|\mu\| < C + 1/100C.$$

It then follows, for  $\omega \in B$ , that, as in Case 2 at the end of the proof of Lemma 7.6,  $\|\omega\| = \|\mu\|$  and

$$(3) \quad \|\bar{\Psi}_{\alpha_n} \mu - \omega\| \rightarrow 0 \text{ for some } F\text{-sequence.}$$

Then  $\omega = f\mu$  for some  $f \in L(\mu)$  with  $|f| = 1$  a.e.  $|\mu|$ . We will identify  $B$  with the collection of such  $f$ .

Now another subsequence has

$$(4) \quad \bar{\Psi}_{\alpha_n} \rightarrow f \text{ a.e. } |\mu|.$$

Let  $S_f$  be where (4) holds and  $|f| = 1$ . Then  $S_f$  is a normal Borel subgroup,  $|\mu|(S_f^c) = 0$  and  $f$  is a Borel homomorphism of  $S_f$  into the unit circle.

Now  $B$  is finite. Otherwise we would have  $\omega_i \in B$  with  $\omega_i \rightarrow \omega$  weakly for some  $\omega \neq 0$ . Using (3) and a diagonal process on the sequences converging to  $\omega_i$  we could then find an  $F$ -sequence  $\{\alpha_i\}$  with

$$(5) \quad \|\bar{\Psi}_{\alpha_i}\mu - \omega_i\| < 1/2 \quad \text{and} \quad \|\bar{\Psi}_{\alpha_i}\mu - \omega\| < 1/2.$$

Since  $\omega_i \in F(G)$  this would give  $\omega_i = \omega$ .

Let  $p = \text{card } B$ ,  $M = \|\mu\|$  and  $\delta = \delta(p, M)$  be the constant in Lemma 6.3.

By using (3) there is an integer  $N$  such that if  $\alpha = \alpha_1\alpha_2 \cdots \in E(\mu)$  and  $d(\alpha_i) = 1$  for  $i \leq N$  then

$$(6) \quad \|(\bar{\Psi}_\alpha - f)\mu\| < \delta \quad \text{for some } f \in B.$$

Now let  $T = \bigcap S_f$  over  $f$  with  $f \in B$ . Each  $f$  is a homomorphism on  $T$  and  $|\mu|(T^c) = 0$ . Let  $K = \prod_1^N G_i'$  and  $\lambda = \mu * \mathfrak{M}_K$ . Then  $\lambda \neq 0$ ,  $\|\lambda\| \leq M$  and, since  $K \subset T$ ,  $|\lambda|(T^c) = 0$ . It also follows from (6) that  $\alpha \in E(\lambda)$  if and only if

$$(7) \quad \|(\bar{\Psi}_\alpha - f)\lambda\| < \delta \quad \text{for some } f \mu \in B.$$

We can now apply Lemma 6.3 to  $\lambda$  to obtain

$$(8) \quad f_1\lambda * f_2\lambda * \cdots * f_q\lambda = A\mathfrak{M}_H$$

where the  $f_i\lambda$  are the distinct elements of  $\{f\lambda : f \in B\}$ ,  $A \neq 0$  and  $H$  is a closed normal subgroup.

Now  $\prod_H \mu \neq 0$  so that  $H \cap L$  is open in  $L$  by Theorem 3.2. On the other hand if  $P$  is a normal Borel subgroup,  $K \subset P$  and  $\prod_P \mu \neq 0$  then  $\mu = \prod_P \mu$  and so  $\lambda = \prod_P \lambda$ . Hence  $\prod_P \mathfrak{M}_H = \mathfrak{M}_H$  so that  $P \cap H$  is open in  $H$ . This then implies  $P \cap L$  is open in  $L$  so that  $\mu$  is canonical by Lemma 7.6.

**9. Connected groups.** We can now use Theorem 2.2 to characterize  $F(G)$  for connected  $G$ .

**Lemma 9.1.** *If  $G$  is a compact connected simple Lie group then  $G$  satisfies conditions I and II.*

**Proof.** Ragozin [4, Theorem 2.2] has shown that if  $n = \text{dimension } G$  and  $\mu \in M^Z(G)$  is continuous then  $\mu^n \in L_1(G)$ . For  $g \notin Z(G)$  let  $\mu$  be given implicitly by

$$(1) \quad \int f d\mu = \int f(xgx^{-1}) d\mathfrak{M}_G(x).$$

It is easily seen that  $\mu$  is a central continuous measure and  $\hat{\mu}(\alpha) = \overline{\Psi}_\alpha(g)$  so that  $(\mu^n)^\wedge(\alpha) = (\overline{\Psi}_\alpha(g))^n$ . By Ragozin's result and the Riemann-Lebesgue lemma it follows that

$$(2) \quad \Psi_\alpha(g) \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty \quad (g \notin Z).$$

Condition I follows immediately. Also if  $\{\alpha\}$  is a sequence with  $d(\alpha) = t$  then (2) and the dominated convergence theorem show

$$(3) \quad 1 = \int |\chi_\alpha|^2 d\mathfrak{M}_G = t^2 \mathfrak{M}_G(Z) + o(1) \quad \text{as } \alpha \rightarrow \infty.$$

But then  $\mathfrak{M}_G(Z) \neq 0$  so that  $Z$  is open which is not possible since  $G$  is connected. Thus there are only finitely many  $\alpha$  with  $d(\alpha) = t$  which implies condition II.

**Theorem 9.2.** *If  $G$  is a connected compact group then every measure in  $F(G)$  is canonical.*

**Proof.** It is known (cf. [8, Chapitre VI]) that a connected compact group  $G$  is a factor group of a group  $G^* = \prod G_i \times A$  where  $A$  is abelian and the  $G_i$  are connected simple Lie groups. By Lemma 9.1 and Theorem 2.2 every measure in  $F(G^*)$  is canonical. Since a measure in  $F(G)$  can be considered as a measure in  $F(G^*)$  the theorem follows.

**10. An example.** Unfortunately the characterization of  $F(G)$  does not hold for all  $G$  as the following simple example shows. Let  $T \times T$  be the two dimensional torus and let  $G$  be the semidirect product of  $T \times T$  and  $Z_2$  where  $\delta(t_1, t_2) = (t_2, t_1)\delta$  for  $\delta \in Z_2, \delta \neq e$ . Let  $\mu_1$  (resp.  $\mu_2$ ) be the Haar measure of  $T \times e$  (resp.  $e \times T$ ). Then  $\mu = \mu_1 + \mu_2 \in F(G)$  but  $\mu$  is not canonical. That  $\mu$  is not canonical is seen by noting that  $\mu$  is singular to  $\mathfrak{M}_H$  for every closed normal subgroup  $H$  of  $G$ .

In this example  $\mu$  is a sum of (noncentral) idempotents each of which is canonical with respect to its support group. It seems reasonable to conjecture that for any  $G$  this is always the case.

Also if  $\mu$  is canonical or as in the example then the conclusion of Lemma 4.2 holds for any Borel subgroup  $H$  (whether or not it is closed and normal). It would be helpful to know whether this is true for any  $\mu \in F(G)$ .

It also seems likely that there is  $\delta > 0$  so that if  $\mu \in F(G)$  and  $\|\mu\| > 1$  then  $\|\mu\| > 1 + \delta$  (cf. Theorem 5.3). Otherwise some strange elements of  $F$  could be obtained by taking infinite products of measures on product groups.

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