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CENTRAL LIMIT THEOREMS FOR ASSOCIATED RANDOM VARIABLES AND THE PERCOLATION MODEL

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We prove a central limit theorem for families $\{X_{\mathbf{n}}(N): \mathbf{n} \in \Lambda(N)\}$ of associated random variables indexed by subsets $\Lambda(N)$ of $\mathbb{Z}^d$, as $N \to \infty$; this is an extension of the Newman-Wright invariance principle for associated stationary sequences $\{X_n: n \geq 1\}$ satisfying $\sum_n \text{cov}(X_1, X_n) < \infty$, but with the stationarity property replaced by conditions on the moments of the X's. The theorem has applications to the voter model and the percolation model. In the latter case, it provides an extension of a central limit theorem of the authors [4], by reducing the severity of the moment conditions.

Also, we prove a central limit theorem for certain non-stationary non-associated families of random variables which arise in percolation theory. This includes, for example, a central limit theorem for the number of open clusters contained within the circuit $\gamma(n)$ of $\mathbb{Z}^2$, where $\{\gamma(n)\}$ is a sequence of circuits which satisfy a regularity condition and whose interiors $\{\mathring{\gamma}(n)\}$ satisfy $|\mathring{\gamma}(n)| \to \infty$ as $n \to \infty$.

1. Introduction. Many recent papers have been concerned with central limit theorems for random variables indexed by $\mathbb{Z}^d$, where $d \geq 1$. Typically, such results deal with the partial sums $S(N) = \sum_{\mathbf{i} \in \Lambda(N)} X_{\mathbf{i}}$ of stationary processes over "regular boxes" $\Lambda(N)$, and impose suitable conditions on the rate at which $\text{cov}(X_0, X_{\mathbf{i}})$ decays when $|\mathbf{i}|$ is large (see [1], [4], [7], [9]–[11] for examples). The purpose of the present paper is twofold: to present a central limit theorem for associated random variables (which need not be stationary) subject to certain conditions on their moments and covariances, and to prove a special central limit theorem dealing with the number of open clusters of the percolation model which are contained within a large circuit $\gamma(n)$ of $\mathbb{Z}^2$.

If $S = (S_1, S_2, \cdots)$ is a sequence of random variables, then we say that *S satisfies the central limit theorem* (CLT) if $(S_n - ES_n)/(\text{var } S_n)^{1/2}$ is asymptotically normally distributed as $n \to \infty$. This notation is slightly at odds with common practice since it involves the elements themselves rather than the partial sums of S.

A collection $\{X_i: i \in I\}$ of random variables is called *associated* if for every finite subcollection $X_1, \cdots, X_m$ and every pair of coordinatewise nondecreasing functions $f_1, f_2: \mathbb{R}^m \to \mathbb{R}$, we have that the random variables $\tilde{f}_j = f_j(X_1, \cdots, X_m)$, $j = 1, 2$, satisfy

$$\text{cov}(\tilde{f}_1, \tilde{f}_2) \geq 0$$

whenever they are such that $E(\tilde{f}_j^2) < \infty$ for $j = 1, 2$. Associated families occur

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frequently in probabilistic models in statistical mechanics, including the percolation model and models for ferromagnetism (see the discussion in [12]). Newman and Wright [12] have shown that a nondegenerate, strongly stationary sequence $X_1, X_2, \cdots$, which is associated and has the property that

$$\sum_{j=1}^{\infty} \operatorname{cov}(X_1, X_j) < \infty, \tag{1.1}$$

satisfies an invariance principle in that, suitably normalized, the partial sums of the X's converge to a standard Wiener process. The implicit CLT for these partial sums may be generalized (see [11] for example) to apply to certain strictly stationary families $\{X_i: i \in \mathbb{Z}^d\}$ for $d \geq 2$, and these results find applications to such processes as the percolation model. Two drawbacks of this approach for $\mathbb{Z}^d$ are that it requires strong stationarity, and that the method may be applied only to partial sums

$$S(\Lambda) = \sum_{i \in \Lambda} X_i$$

where Λ is a particularly regular subset of $\mathbb{Z}^d$, such as a cube $\{\mathbf{x} = (x_1, \cdots, x_d): 0 \leq x_i \leq n\}$. See [9] and [10] for another approach to theorems of this type.

In our first result we show that the assumption of strict stationarity may be relaxed and replaced by certain conditions on the moments of the X's. We are also able to relax slightly the strict regularity requirements on the index regions Λ, but are unable to remove them completely.

Here is some notation. If $\mathbf{x} \in \mathbb{Z}^d$, we write x_i for the ith coordinate of $\mathbf{x}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, we write $\mathbf{x} \leq \mathbf{y}$ (respectively $\mathbf{x} < \mathbf{y}$) if $x_i \leq y_i$ (respectively $x_i < y_i$) for all i. We define

$$|\mathbf{x} - \mathbf{y}| = \sup\{|x_i - y_i|: i = 1, 2, \cdots, d\}$$

and write $\mathbf{1}$ for a vector with unit entries. If $\mathbf{k} \in \mathbb{Z}^d$, we denote by $\Lambda(\mathbf{k})$ the box

$$\Lambda(\mathbf{k}) = \{\mathbf{x} \in \mathbb{Z}^d: \mathbf{1} \leq \mathbf{x} \leq \mathbf{k}\}$$

and let $\|\mathbf{k}\| = k_1 k_2 \cdots k_d$ represent the number of points in this box.

THEOREM 1.2. *Let $\{\mathbf{k}(N): N = 1, 2, \cdots\}$ be a sequence in $\mathbb{Z}^d$ such that $k_i(N) \to \infty$ as $N \to \infty$ for $i = 1, 2, \cdots, d$. Suppose that, for each N, $\{X_{\mathbf{n}}(N): \mathbf{n} \in \Lambda(\mathbf{k}(N))\}$ is a family of associated random variables satisfying*:

(i) *there are strictly positive, finite constants c_1, c_2 such that*

$$\operatorname{var}(X_{\mathbf{n}}(N)) \geq c_1 \text{ and } E(|X_{\mathbf{n}}(N)|^3) \leq c_2 \text{ for all } n \text{ and } N,$$

(ii) *there is a function $u: \{0, 1, 2, \cdots\} \to \mathbb{R}$ such that $u(r) \to 0$ as $r \to \infty$ and*

$$\sum_{\mathbf{j}: |\mathbf{n}-\mathbf{j}| \geq r} \operatorname{cov}(X_{\mathbf{j}}(N), X_{\mathbf{n}}(N)) \leq u(r) \text{ for all } n, N \text{ and } r \geq 0.$$

Then the sequence $\{S(N) = \sum_{\mathbf{n} \in \Lambda(\mathbf{k}(N))} X_{\mathbf{n}}(N): N = 1, 2, \cdots\}$ satisfies the CLT.

For $d = 1$, $\mathbf{k}(N) = N$ and the X's stationary, this is the CLT of Newman and Wright [12], but subject to a superfluous third moment condition. For $d \geq 2$, the sets $\Lambda(\mathbf{k}(N))$ are d-dimensional "rectangles", but their dimensions need not grow at equal rates.

We will present two applications of this theorem, deferring the proof of the theorem to Section 2. Consider the d-dimensional voter model; this is a Markov process $\{\eta_t: t \geq 0\}$ taking values in $\{0, 1\}^{\mathbb{Z}^d}$. The transitions are

$$\eta_t(\mathbf{x}) \to 1 - \eta_t(\mathbf{x}) \text{ at rate } \frac{1}{2d} \,|\{\mathbf{y}: |\mathbf{y} - \mathbf{x}| = 1 \text{ and } \eta_t(\mathbf{x}) \neq \eta_t(\mathbf{y})\}|.$$

Here $|A|$ denotes the cardinality of A. See Liggett [8] for a rigorous account of the voter model. We suppose the initial distribution of the process is product measure with density θ where $0 < \theta < 1$. That is, we assume the family $\{\eta_0(\mathbf{x}): \mathbf{x} \in \mathbb{Z}^d\}$ is a family of independent, identically distributed Bernoulli random variables with parameter θ. We define the occupation time variables

$$T_t = \int_0^t \eta_s(\mathbf{0})\, ds, \quad t \geq 0.$$

A result of Cox and Griffeath [3] is that the family $\{T_t: t \geq 0\}$ satisfies the CLT for $d \geq 2$. This result, for $d \geq 5$, can also be proved by an application of Theorem 1.2. First, Harris's correlation inequality [5] implies that the (nonstationary) family $\{\eta_t(\mathbf{0}): t \geq 0\}$ is *associated*, and so is the family $\{X_n: n = 0, 1, 2, \cdots\}$ where $X_n = \int_n^{n+1} \eta_s(\mathbf{0})\, ds$. Next, calculations in [3] show that conditions (i) and (ii) of Theorem 1.2 hold (for $d \geq 5$). Consequently, $\{T_t: t \geq 0\}$ satisfies the CLT. This approach fails for $d \leq 4$ since

$$\int_0^\infty \operatorname{cov}(\eta_s(\mathbf{0}), \eta_t(\mathbf{0}))\, ds = \infty, \quad t \geq 0,$$

making condition (ii) of Theorem 1.2 false.

For the second application consider bond percolation on the square lattice $\mathscr{L}$, with sites $\mathbb{Z}^2 = \{(x, y): x, y = 0, \pm 1, \pm 2, \cdots\}$ and bonds joining all pairs of sites which are unit distance apart. The *distance* $d(\mathbf{x}, \mathbf{y})$ between two sites $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is defined to be

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

A *path* from $\mathbf{x}$ to $\mathbf{y}$ is an alternating sequence $\{\mathbf{x}_0, e_1, \mathbf{x}_1, \cdots, e_n, \mathbf{x}_n\}$ of sites and bonds such that $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_n = \mathbf{y}$, and e_i is a bond joining $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$ (for $i = 1, 2, \cdots, n$). A *circuit* γ is a path $\{\mathbf{x}_0, e_1, \cdots, e_n, \mathbf{x}_n\}$ such that $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}$ are distinct and $\mathbf{x}_n = \mathbf{x}_0$. The *interior* $\mathring{\gamma}$ of a circuit γ is the subgraph of $\mathscr{L}$ induced by the sites contained strictly within γ. A *cluster* of a graph G is a connected subgraph of G.

Let $0 < p = 1 - q < 1$, and declare each bond of $\mathscr{L}$ to be either *open* or *closed* with respective probabilities p or $1 - p$, independently of all other bonds. Paths, circuits and clusters of $\mathscr{L}$ are called *open* (respectively *closed*) if all their bonds are open (respectively closed). Each site $\mathbf{x}$ of $\mathscr{L}$ belongs to some unique open cluster of $\mathscr{L}$, and we denote the set of sites of this cluster by $W_{\mathbf{x}}$, subject to the convention that $W_{\mathbf{x}} = \phi$ if $\mathbf{x}$ is incident to *no* open bond. Sometimes we may use $W_{\mathbf{x}}$ to denote the cluster itself, being the appropriate set of sites together with all incident open bonds, but we reserve the quantity $|W_{\mathbf{x}}|$ to denote the number of sites (including $\mathbf{x}$) joined to $\mathbf{x}$ by open paths, with the convention that $|W_{\mathbf{x}}| = 0$ if $W_{\mathbf{x}} = \phi$.

Let $\{\mathbf{k}(n)\colon n = 1, 2, \cdots\}$ be a sequence in $\mathbb{Z}^2$ such that $k_i(n) \to \infty$ as $n \to \infty$ for $i = 1, 2$. Let $\gamma(n)$ be such that $\mathring{\gamma}(n) = \Lambda(\mathbf{k}(n))$. A function f is said to be *increasing* (respectively *decreasing*) on the subsets of $\mathbb{Z}^2$ if $f(W_1) \le f(W_2)$ (respectively $f(W_1) \ge f(W_2)$) whenever $W_1 \subset W_2$. Let $\mathcal{T}$ be the set of (finite) real-valued functions defined on the connected subsets of $\mathbb{Z}^2$, which are either increasing or decreasing and which are *constant* on infinite sets (in that $f(W_1) = f(W_2)$ if $|W_1| = |W_2| = \infty$).

THEOREM 1.3. *Assume $p \neq 1/2$ and $\{f_n\colon n = 1, 2, \cdots\}$ is a sequence of functions satisfying the following*:

$$f_n \in \mathcal{T} \quad \textit{for each } n,$$

$$\sup_n \max_{\mathbf{x}\in\mathring{\gamma}(n)} E(|f_n(W_{\mathbf{x}})|^3) = c_3 < \infty,$$

$$\inf_n \min_{\mathbf{x}\in\mathring{\gamma}(n)} \mathrm{var}(f_n(W_{\mathbf{x}})) = \sigma^2 > 0.$$

If $S(n) = \sum_{\mathbf{x}\in\mathring{\gamma}(n)} f_n(W_{\mathbf{x}})$, then $\{S(n)\colon n = 1, 2, \cdots\}$ satisfies the CLT.

This result is a consequence of Theorem 1.2 and some well-known facts from percolation theory. First, the FKG inequality (see [6]) implies that the families $\{f_n(W_{\mathbf{x}})\colon \mathbf{x} \in \mathring{\gamma}(n)\}$ are *associated.* Secondly, the method of Lemma 2 of [4] provides the estimate

$$\mathrm{cov}(f_n(W_{\mathbf{x}}), f_n(W_{\mathbf{y}})) \le c\left\{P\left(\frac{1}{2}\, d(\mathbf{x}, \mathbf{y}) \le r(W_0) < \infty\right)\right\}^{1/3} \tag{1.4}$$

where $r(W_0) = \sup\{d(\mathbf{0}, \mathbf{x})\colon \mathbf{x} \in W_0\}$, and c is a finite constant depending only on c_3 and σ^2. Finally, estimates from [6] show that, for $p \neq 1/2$,

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} \left\{P\left(\frac{1}{2}\, d(\mathbf{0}, \mathbf{x}) \le r(W_0) < \infty\right)\right\}^{1/3} < \infty, \tag{1.5}$$

and so all the conditions of Theorem 1.2 are satisfied.

Theorem 1.3 should be compared with Theorem 2 of [4], which requires finite moments of the $f_n(W_{\mathbf{x}})$ of all orders, but allows arbitrary $\{\gamma(n)\}$ such that $|\mathring{\gamma}(n)| \to \infty$. The exponent 1/3 in (1.4) is an artifact of our method; however, we can only improve it at the cost of assuming that higher moments of the $f_n(W_{\mathbf{x}})$ exist.

As a final remark on this subject, we note that Theorem 1.3 has an obvious extension to bond percolation on $\mathbb{Z}^d$. The only change is that the condition "$p \neq 1/2$" be replaced by the assumption that p is such that

$$\sum_{\mathbf{x}\in\mathbb{Z}^d} \left\{P\left(\frac{1}{2}\, d(\mathbf{0}, \mathbf{x}) \le r(W_0) < \infty\right)\right\}^{1/3} < \infty. \tag{1.6}$$

Known estimates (see [6]) can be used to verify (1.6) for p sufficiently close to 0 or 1, and it may be that (1.6) holds for all $p \neq p_c(d)$, where $p_c(d) = \sup\{p: P(|W_0| < \infty) = 1\}$. As noted above, a weaker condition than (1.6) is sufficient for the CLT to hold, if more is assumed about the f_n's. For example, if the f_n's are uniformly bounded in modulus by a constant, then (1.6) may be replaced by

$$E(r(W_0)1(|W_0| < \infty)) < \infty,$$

where $1(A)$ is the indicator function of the event A. It is not currently known whether this holds for $d = 3$ and $p \neq p_c(3)$.

Our second main result deals with some random variables which are neither stationary nor associated, and which arise in the percolation model. For any circuit γ in $\mathscr{L}$ let C_γ be the number of open clusters of $\mathring{\gamma}$:

$$C_\gamma = \sum_{\mathbf{x} \in \mathring{\gamma}} \frac{1}{|W_{\mathbf{x},\gamma}|} 1(|W_{\mathbf{x},\gamma}| > 0)$$

where $W_{\mathbf{x},\gamma} = \{\mathbf{y} \in \mathring{\gamma}: \mathbf{x}$ and $\mathbf{y}$ are connected by an open path r contained in $\mathring{\gamma}\}$. We set $W_{\mathbf{x},\gamma} = \phi$ if all the bonds of $\mathring{\gamma}$ incident to $\mathbf{x}$ are closed. Theorem 1.8 involves a technical condition on a sequence of circuits $\{\gamma(n)\}$, and we call this condition the *cut condition.* We defer to Section 3 the lengthy precise statement of this condition (3.4), but note here that the rectangles of Theorem 1.3, with $\mathring{\gamma}(n) = \Lambda(\mathbf{k}(n))$, satisfy the cut condition provided that

$$\text{(1.7)} \qquad \liminf_{n\to\infty} \frac{k_1(n)}{\log(k_1(n)k_2(n))} = \liminf_{n\to\infty} \frac{k_2(n)}{\log(k_1(n)k_2(n))} = \infty.$$

THEOREM 1.8. *Assume $p \neq 1/2$, and let $\Gamma = \{\gamma(n): n = 1, 2, \cdots\}$ be a sequence of circuits such that $\mathring{\gamma}$ is connected for each $\gamma \in \Gamma$ and $|\mathring{\gamma}(n)| \to \infty$ as $n \to \infty$. If $p > 1/2$, assume in addition that Γ satisfies the cut condition. Then $\{C_{\gamma(n)}: n = 1, 2, \cdots\}$ satisfies the* CLT.

We will prove Theorem 1.8, as well as several related results, in Section 3. The proof of Theorem 1.2 is in Section 2.

2. Proof of Theorem 1.2. We may assume $E(X_\mathbf{n}(N)) = 0$ for all $\mathbf{n}$ and N. Following Newman and Wright [12], we let ℓ be a positive integer and define

$$\mathbf{m} = (\lfloor k_1(N)/\ell \rfloor, \lfloor k_2(N)/\ell \rfloor, \cdots, \lfloor k_d(N)/\ell \rfloor),$$

$$Y_\mathbf{j}(N) = \sum_{(\mathbf{j}-\mathbf{1})\ell < \mathbf{n} \le \mathbf{j}\ell} X_\mathbf{n}(N), \quad \text{for } \mathbf{1} \le \mathbf{j} \le \mathbf{m},$$

$$S(N, \ell) = \sum_{\mathbf{1} \le \mathbf{j} \le \mathbf{m}} Y_\mathbf{j}(N), \quad Z(N) = S(N) - S(N, \ell),$$

$$\sigma(N, \ell)^2 = \operatorname{var}(S(N, \ell)), \quad s(N, \ell)^2 = \sum_{\mathbf{1} \le \mathbf{j} \le \mathbf{m}} \operatorname{var}(Y_\mathbf{j}(N)),$$

$$\sigma(N)^2 = \operatorname{var}(S(N)),$$

$$\psi_N(t) = E(\exp(itS(N))), \quad \psi_{n,\ell}(t) = E(\exp(itS(N, \ell))),$$

$$\psi_{N,\mathbf{j}}(t) = E(\exp(itY_\mathbf{j}(N))).$$

Note that $Y_\mathbf{j}(N)$ and $Z(N)$ depend upon the choice of ℓ, and that $\mathbf{m} = \mathbf{m}(N) \to$

∞ as $N \to \infty$. We shall prove the theorem in three steps, dealing with the following statements.

(2.1) $$\limsup_{N\to\infty} \left| \psi_N\left(\frac{t}{\sigma(N)}\right) - \psi_{N,\ell}\left(\frac{t}{s(N,\ell)}\right) \right| \le |t| \frac{2d}{\ell c_1} \sum_{r=1}^{\ell} u(r),$$

(2.2) $$\limsup_{N\to\infty} \left| \psi_{N,\ell}\left(\frac{t}{s(N,\ell)}\right) - \prod_{1\le \mathbf{j}\le \mathbf{m}} \psi_{N,\mathbf{j}}\left(\frac{t}{s(N,\ell)}\right) \right| \le \frac{dt^2}{\ell c_1} \sum_{r=1}^{\ell} u(r),$$

(2.3) $$\limsup_{N\to\infty} \left| \prod_{1\le \mathbf{j}\le \mathbf{m}} \psi_{N,\mathbf{j}}\left(\frac{t}{s(N,\ell)}\right) - \exp\left(-\frac{1}{2}t^2\right) \right| = 0.$$

The theorem follows immediately from these three equations, since

$$\frac{1}{\ell} \sum_{r=1}^{\ell} u(r) \to 0 \quad \text{as} \quad \ell \to \infty.$$

We prove a preliminary lemma.

LEMMA 2.4.

$$\limsup_{N\to\infty} \frac{\sigma(N)^2}{s(N,\ell)^2} \le 1 + \frac{2d}{\ell c_1} \sum_{r=1}^{\ell} u(r).$$

PROOF. Note first that

(2.5) $$c_1 \|\mathbf{k}(N)\| \le \sigma(N)^2 \le c_3 \|\mathbf{k}(N)\|$$

where $c_3 = u(0)$. The left-hand side of (2.5) follows from

$$\sigma(N)^2 = \sum_{\mathbf{n}} \operatorname{var}(X_{\mathbf{n}}(N)) + \sum_{\mathbf{i}\ne\mathbf{j}} \operatorname{cov}(X_{\mathbf{i}}(N), X_{\mathbf{j}}(N)) \ge c_1 \|\mathbf{k}(N)\|$$

since the $X_{\mathbf{n}}(N)$ are associated. For the right-hand side,

$$\sum_{\mathbf{i},\mathbf{j}} \operatorname{cov}(X_{\mathbf{i}}(N), X_{\mathbf{j}}(N)) \le \sum_{\mathbf{i}} u(0) = u(0)\ \|\mathbf{k}(N)\|,$$

where the second sum is over $\mathbf{i} \in \Lambda(\mathbf{k}(N))$. Similar arguments show that

(2.6) $$\|\mathbf{m}\|\ \ell^d c_1 \le s(N,\ell)^2 \le \sigma(N,\ell)^2 \le \|\mathbf{m}\|\ \ell^d c_3$$

and

(2.7) $$\sigma(N,\ell)^2 \le \sigma(N)^2.$$

Expand $S(N)$ in terms of the Y's to find that

(2.8) $$\sigma(N)^2 = \operatorname{var}(Z(N)) + 2\operatorname{cov}(Z(N), S(N,\ell)) + \operatorname{var}(S(N,\ell))$$

and note that

(2.9) $$\operatorname{var}(Z(N)) \le c_3 \ell^d(\|\mathbf{m}+\mathbf{1}\| - \|\mathbf{m}\|),$$
$$\operatorname{cov}(Z(N), S(N,\ell)) \le c_3 \ell^d(\|\mathbf{m}+\mathbf{1}\| - \|\mathbf{m}\|),$$

giving by (2.6) that

$$\text{(2.10)}\qquad \operatorname{var}(Z(N)) + 2\operatorname{cov}(Z(N), S(N, \ell)) = o(s(N, \ell)^2) \text{ as } N \to \infty.$$

Consider the final term in (2.8):

$$\operatorname{var}(S(N, \ell)) = s(N, \ell)^2 + \Sigma(N)$$

where

$$\text{(2.11)}\qquad \Sigma(N) = \sum_{1\le \mathbf{i}\le \mathbf{m}} \sum_{1\le \mathbf{j}\le \mathbf{m}, \mathbf{j}\ne \mathbf{i}} \operatorname{cov}(Y_{\mathbf{i}}(N), Y_{\mathbf{j}}(N))$$

$$\le \sum_{1\le \mathbf{i}\le \mathbf{m}} \sum_{\alpha\in\Lambda_{\mathbf{i}}} \sum_{\beta\notin\Lambda_{\mathbf{i}}} \operatorname{cov}(X_\alpha(N), X_\beta(N))$$

where $\Lambda_{\mathbf{i}} = \{\alpha \in \mathbb{Z}^d \colon (\mathbf{i} - \mathbf{1})\ell < \alpha \le \mathbf{i}\ell\}$. But

$$\text{(2.12)}\qquad \sum_{\alpha\in\Lambda_{\mathbf{i}}} \sum_{\beta\notin\Lambda_{\mathbf{i}}} \operatorname{cov}(X_\alpha(N), X_\beta(N)) \le 2d\ell^{d-1} \sum_{r=1}^{\ell} u(r)$$

since there are at most $2d\ell^{d-1}$ points of $\Lambda_{\mathbf{i}}$ which are within distance r of some point outside $\Lambda_{\mathbf{i}}$. Combine (2.6), (2.8) and (2.10)–(2.12) to find that

$$\limsup_{N\to\infty} \frac{\sigma(N)^2}{s(N, \ell)^2} \le \frac{2d}{\ell c_1} \sum_{r=1}^{\ell} u(r)$$

as required.

We return to the proofs of (2.1), (2.2) and (2.3).

PROOF OF (2.1). By a standard inequality, we have that

$$\left| \psi_N\left(\frac{t}{\sigma(N)}\right) - \psi_{N,\ell}\left(\frac{t}{s(N, \ell)}\right) \right|$$

$$\le |t| \left(\operatorname{var}\left(\frac{S(N)}{\sigma(N)} - \frac{S(N, \ell)}{s(N, \ell)}\right)\right)^{1/2}$$

$$\le |t| \left(\sigma(N, \ell)\left(\frac{1}{s(N, \ell)} - \frac{1}{\sigma(N)}\right) + \frac{1}{\sigma(N)} (\operatorname{var}(Z(N)))^{1/2}\right)$$

$$\le |t| \left(\frac{\sigma(N)}{s(N, \ell)} - 1 + \left(\frac{c_3(\|\mathbf{m} + \mathbf{1}\| - \|\mathbf{m}\|)}{c_1\|\mathbf{m}\|}\right)^{1/2}\right)$$

by (2.6), (2.7) and (2.9); (2.1) follows from Lemma 2.4.

PROOF OF (2.2). We use Theorem 1 of [12] to find that

$$\left| \psi_{N,\ell}\left(\frac{t}{s(N,\ell)}\right) - \prod_j \psi_{N,j}\left(\frac{t}{s(N,\ell)}\right) \right| \le \frac{t^2}{2s(N,\ell)^2} \sum_{i\ne j} \operatorname{cov}(Y_i(N), Y_j(N))$$

$$= \frac{t^2}{2s(N,\ell)^2} (\sigma(N,\ell)^2 - s(N,\ell)^2)$$

$$\le \frac{1}{2} t^2 \left(\frac{\sigma(N)^2}{s(N,\ell)^2} - 1\right);$$

now use Lemma 2.4.

PROOF OF (2.3). This follows from Lyapunov's Theorem (see Theorem 7.1.2 of [2], for example). Just note that

$$E(|Y_j(N)|^3) \le \sum_{\alpha,\beta,\gamma\in\Lambda_j} E|X_\alpha(N)X_\beta(N)X_\gamma(N)|$$

$$\le \ell^{3d} c_2 \quad \text{by Hölder's inequality,}$$

and so, by (2.6),

$$\frac{1}{s(N,\ell)^3} \sum_{1\le j\le \mathbf{m}} E(|Y_j(N)|^3) \le \frac{\|\mathbf{m}\| \ell^{3d} c_2}{(\|\mathbf{m}\| \ell^d c_1)^{3/2}}$$

$$\to 0 \text{ as } N \to \infty.$$

3. Number of open clusters in the percolation model. Consider bond percolation on the square lattice $\mathscr{L}$, and let γ be a circuit in $\mathscr{L}$. Recall that the number of open clusters of $\mathring{\gamma}$ is C_γ, given by

$$C_\gamma = \sum_{\mathbf{x}\in\mathring{\gamma}} \frac{1}{|W_{\mathbf{x},\gamma}|} 1(|W_{\mathbf{x},\gamma}| > 0)$$

where $W_{\mathbf{x},\gamma} = \{\mathbf{y}$: $\mathbf{x}$ and $\mathbf{y}$ are connected by an open path r in $\mathring{\gamma}\}$. In place of C_γ, we might have studied the random variable

$$C'_\gamma = \sum_{\mathbf{x}\in\mathring{\gamma}} \frac{1}{1 \vee |W_{\mathbf{x},\gamma}|} 1(|W_\mathbf{x}| > 0),$$

where $a \vee b = \max\{a, b\}$; C'_γ is the number of open clusters of $\mathring{\gamma}$ together with the collection of sites in $\mathring{\gamma}$ which are in open clusters of $\mathscr{L}$ but are connected to sites in $\mathring{\gamma}$ by closed edges only. We choose to study C_γ rather than C'_γ, but note that similar techniques should apply to C'_γ.

We will also consider the random variables

$$I_\gamma = \sum_{\mathbf{x}\in\mathring{\gamma}} \frac{1}{|W_\mathbf{x} \cap \mathring{\gamma}|} 1(|W_\mathbf{x}| > 0)$$

and

$$J_\gamma = \sum_{\mathbf{x}\in\mathring{\gamma}} \frac{1}{|W_{\mathbf{x},\gamma}|} 1(0 < |W_{\mathbf{x},\gamma}| \le |W_\mathbf{x}| < \infty).$$

A few remarks about these random variables are in order. The quantity I_γ is the number of open clusters of $\mathscr{L}$ which intersect $\mathring{\gamma}$. The quantity J_γ is the number of open clusters of $\mathring{\gamma}$ which do not belong to an infinite open cluster of $\mathscr{L}$. In comparing I_γ with C'_γ, note that two clusters contributing to C'_γ may be parts of a single cluster contributing to I_γ, and thus

$$I_\gamma \le C'_\gamma.$$

Note that J_γ discounts those sites $\mathbf{x} \in \mathring{\gamma}$ for which $|W_\mathbf{x}| = \infty$, and thus

$$J_\gamma \le C_\gamma$$

and, furthermore, $J_\gamma = C_\gamma$ a.s. if $p < 1/2$, since if $p < 1/2$ then all open clusters are a.s. finite.

In addition to Theorem 1.8, we will prove the following.

THEOREM 3.1. *Assume $p \neq 1/2$, and let $\{\gamma(n)\}$ be a sequence of circuits such that $|\mathring{\gamma}(n)| \to \infty$ as $n \to \infty$. Then $\{I_{\gamma(n)}\}$ satisfies the* CLT.

THEOREM 3.2. *Assume $p \neq 1/2$, and let $\{\gamma(n)\}$ be a sequence of circuits such that each $\mathring{\gamma}(n)$ is connected, and $|\mathring{\gamma}(n)| \to \infty$ as $n \to \infty$. Then $\{J_{\gamma(n)}\}$ satisfies the* CLT.

The connectedness condition is imposed in order to avoid the following type of problem: if $\mathring{\gamma}$ is the union of isolated sites, then $J_\gamma = 0$ a.s.

Theorems 1.8, 3.1 and 3.2 bear a strong resemblance to Theorem 2 of [4], but they do not follow from that theorem since the functions involved are either not monotone or not constant on infinite sets, or both.

We may find estimates for the means and variances of the quantities C_γ, I_γ, J_γ discussed above, in the manner of page 238 of [4]. If $p \neq 1/2$, then there exist positive constants $c_1(p)$, $c_2(p)$ such that, for any circuit γ of $\mathscr{L}$ such that $\mathring{\gamma}$ is connected,

$$c_1(p)\,|\mathring{\gamma}| \le E(\theta_\gamma) \le c_2(p)\,|\mathring{\gamma}|, \quad c_1(p)\,|\mathring{\gamma}| \le \operatorname{var}(\psi_\gamma) \le c_2(p)\,|\mathring{\gamma}|,$$

where θ may represent C, I or J, and ψ may represent I or J. Furthermore $\operatorname{var}(C_\gamma) \ge c_1(p)\,|\mathring{\gamma}|$; we are not able to show the corresponding upper bound on $\operatorname{var}(C_\gamma)$. We do not prove these remarks here; they are consequences of the arguments of [4] and the forthcoming proofs.

Before turning to these proofs we must state our *cut condition.* To do this, we need the *dual lattice* $\mathscr{L}^*$, constructed from $\mathscr{L}$ in the usual way by placing a site in the centre of each square face of $\mathscr{L}$ and joining pairs of these sites whenever the corresponding faces have a common bond of $\mathscr{L}$. Each bond of $\mathscr{L}^*$ is declared open (respectively closed) if the corresponding bond of $\mathscr{L}$ is open (respectively closed). If G_1 and G_2 are site-disjoint clusters of $\mathring{\gamma}$, a *cutset in γ between G_1 and*

G_2 is a set B of bonds contained strictly within γ with the properties that

(i) every path which joins some site in G_1 to some site in G_2, and which lies strictly within γ, contains some bond in B, and

(ii) B is minimal with the above property, in that (i) holds for no strict subset of B.

We shall make use of the following observation, which is a consequence of Whitney's Theorem (see [6]): every nonempty cutset in γ between G_1 and G_2 corresponds to a path or a circuit in the dual lattice $\mathscr{L}^*$.

DEFINITION 3.3. For $\varepsilon, c > 0$ and for any circuit γ and site $\mathbf{x} \in \mathring{\gamma}$, we say that the *cut condition* $\kappa(\gamma, \mathbf{x}; \varepsilon, c)$ *holds* if, for all distinct triples $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of sites in $\mathring{\gamma}$ such that

$$d(\mathbf{x}, \mathbf{u}) \geq |\mathring{\gamma}|^{\varepsilon} \text{ and } d(\mathbf{v}, \mathbf{w}) \geq |\mathring{\gamma}|^{\varepsilon},$$

then, for any site-disjoint pair π_1, π_2 of paths in $\mathring{\gamma}$ joining $\mathbf{x}$ to $\mathbf{u}$, and $\mathbf{v}$ to $\mathbf{w}$, respectively, and for any cutset B in γ between π_1 and π_2, we have that B contains at least $c \log |\mathring{\gamma}|$ bonds.

We shall be interested in circuits γ which contain only few sites $\mathbf{x} \in \mathring{\gamma}$ for which $\kappa(\gamma, \mathbf{x}; \varepsilon, c)$ fails.

DEFINITION 3.4. We say that the sequence $\{\gamma(n)\}$ of circuits *satisfies the cut condition* if, for some ε satisfying $0 < \varepsilon < 1/14$ and for every $c > 0$, we have that

$$|B(n)|^2 = o(|\mathring{\gamma}(n)|) \text{ as } n \to \infty$$

where

$$B(n) = \{\mathbf{x} \in \mathring{\gamma}(n): \kappa(\gamma(n), \mathbf{x}; \varepsilon, c) \text{ does not hold}\}.$$

There are many circuit sequences which satisfy the cut condition by satisfying the stronger condition "$|B(n)| = 0$ for all large n". These include the rectangles $\Lambda(\mathbf{k}(N))$ which satisfy (1.7). Examples of circuit sequences which fail to satisfy the cut condition include dumb-bell shapes, and other circuits which contain an isthmus whose removal leaves two or more simply connected regions of $\mathbb{Z}^2$ whose contents are large in comparison with the width of the isthmus itself. Long thin rectangles fall into such a category; CLTs hold for some such circuits with bottlenecks, but we have been unable to find a unified approach to all such problems, and so we omit such rather special results.

PROOF OF THEOREM 3.1. We adopt the notation of [4] wherever appropriate. For any set W of sites of $\mathscr{L}$, define

$$f_n(W) = \frac{1}{|W \cap \mathring{\gamma}(n)|} 1(0 < |W| < \infty)$$

and

$$F_{\gamma(n)} = \textstyle\sum_{\mathbf{x} \in \mathring{\gamma}} f_n(W_{\mathbf{x}}).$$

Then $F_{\gamma(n)}$ is the number of finite open clusters of $\mathscr{L}$ which intersect $\gamma(n)$; note that

$$F_{\gamma(n)} \le I_{\gamma(n)} \le F_{\gamma(n)} + 1 \quad \text{a.s.},$$

since there is at most one infinite open cluster in $\mathscr{L}$, for any value of p (of course, $F_{\gamma(n)} = I_{\gamma(n)}$ a.s. if $p \le 1/2$). Hence it suffices to show that $\{F_{\gamma(n)}\}$ satisfies the CLT. Note that $\{f_n\}$ is a sequence of functions which satisfies all the conditions of Theorem 2 of [4] except the requirement of monotonicity; thus Theorem 2 may not be applied directly to the family $\{F_{\gamma(n)}\}$. However, the monotonicity assumption was important only in that it implied that there exists a positive constant σ^2 such that

$$\text{var}(F_{\gamma(n)}) \ge \sigma^2 |\mathring{\gamma}(n)| \text{ for all } n, \tag{3.5}$$

and so the asymptotic normality of $\{F_{\gamma(n)}\}$, suitably normalized, will follow as soon as we have shown (3.5). Let $D_k(\mathbf{x}) = \{\mathbf{y} \in \mathbb{Z}^2 \colon d(\mathbf{x}, \mathbf{y}) = k\}$ and $\bar{D}_k(\mathbf{x}) = \{\mathbf{y} \in \mathbb{Z}^2 \colon d(\mathbf{x}, \mathbf{y}) \le k\}$. Let $V(\mathbf{x})$ denote the set of all bonds of $\mathscr{L}$ which join a site in $D_1(\mathbf{x})$ to a site in $D_2(\mathbf{x})$. We call $V(\mathbf{x})$ *closed* if all its bonds are closed. Note that

$$\pi = P(V(\mathbf{x}) \text{ is closed}) = (1-p)^{12} > 0 \text{ if } p < 1. \tag{3.6}$$

Let γ be any circuit in $\mathscr{L}$. It is not difficult to check that there exists a subset $G(\gamma)$ of $\mathring{\gamma}$ such that
(i) $\bar{D}_2(\mathbf{x}) \cap \bar{D}_2(\mathbf{y}) = \phi$ if $\mathbf{x}, \mathbf{y} \in G(\gamma)$ and $\mathbf{x} \ne \mathbf{y}$,
(ii) $|G(\gamma)| \ge |\mathring{\gamma}|/61$.
We will show that $\text{var}(F_\gamma)$ is large by considering the contribution to F_γ from open clusters in

$$D(\gamma) = \cup_{\mathbf{x} \in G(\gamma)} \bar{D}_1(\mathbf{x}).$$

Suppose that $G(\gamma)$ contains N sites. For each $A \subseteq G(\gamma)$, let $1(A)$ be the indicator function of the event that "for all $\mathbf{x} \in G(\gamma)$, $V(\mathbf{x})$ is closed if and only if $\mathbf{x} \in A$", and note that $E(1(A)) = \pi^{|A|}(1-\pi)^{N-|A|}$. For each $\mathbf{x} \in \mathring{\gamma}$, define

$$\xi(\mathbf{x}) = \begin{cases} 0 & \text{if all bonds incident to } \mathbf{x} \text{ are closed} \\ 1 & \text{otherwise;} \end{cases}$$

the variables $\{\xi(\mathbf{x}) \colon \mathbf{x} \in G(\gamma)\}$ are independent and identically distributed with common variance $s^2 > 0$. Finally, for $A \subseteq G(\gamma)$, define

$$F_\gamma(A) = \sum_{\mathbf{x} \in \mathring{\gamma} \setminus D(A)} \frac{1}{|W_\mathbf{x} \cap \mathring{\gamma}|} 1(0 < |W_\mathbf{x}| < \infty)$$

where

$$D(A) = \cup_{\mathbf{x} \in A} \bar{D}_1(\mathbf{x}).$$

Then, if $1(A) = 1$, we have that

$$F_\gamma = F_\gamma(A) + \sum_{\mathbf{x} \in A} \xi(\mathbf{x}), \tag{3.7}$$

and

$$\text{var}(F_\gamma) = \sum_{A \subseteq G(\gamma)} E((F_\gamma - EF_\gamma)^2 1(A)). \tag{3.8}$$

Expand a typical summand by (3.7), to obtain

$$\begin{aligned} E((F_\gamma - EF_\gamma)^2 1(A)) &= E((F_\gamma(A) - EF_\gamma(A))^2 1(A)) \\ &\quad + 2E((F_\gamma(A) - EF_\gamma(A))(\textstyle\sum_{\mathbf{x}\in A} (\xi(\mathbf{x}) - E\xi(\mathbf{x})) 1(A)) \\ &\quad + E((\textstyle\sum_{\mathbf{x}\in A} (\xi(\mathbf{x}) - E\xi(\mathbf{x})))^2 1(A)) \\ &\geq \operatorname{var}(\textstyle\sum_{\mathbf{x}\in A} \xi(\mathbf{x})) E(1(A)) = |A| s^2 E(1(A)), \end{aligned}$$

since first, if $1(A) = 1$ then $F_\gamma(A)$ and $\sum_{\mathbf{x}\in A} \xi(\mathbf{x})$ are independent (they depend only on bonds separated by the $V(\mathbf{x})$, $\mathbf{x} \in A$), and secondly, $\sum_{\mathbf{x}\in A} \xi(\mathbf{x})$ and $1(A)$ are independent. From (3.8),

$$\operatorname{var}(F_\gamma) \geq s^2 \sum_{A\subseteq G(\gamma)} |A| \pi^{|A|}(1-\pi)^{N-|A|} = s^2 N\pi \geq \sigma^2 |\mathring{\gamma}|$$

where $\sigma^2 = s^2\pi/61 > 0$. This shows (3.5), and the proof is complete.

PROOF OF THEOREM 3.2. For each site $\mathbf{x}$ and subgraph W of $\mathscr{L}$ containing $\mathbf{x}$, define

$$f_n(W, \mathbf{x}) = \frac{1}{|W_{\mathbf{x},n}|} 1(0 < |W_{\mathbf{x},n}| \leq |W| < \infty).$$

The functions $\{f_n\}$ do not fall quite into the context of Theorem 2 of Cox and Grimmett [4], but all their arguments apply to the random variables

$$f_n(W_{\mathbf{x}}) = f_n(W_{\mathbf{x}}, \mathbf{x}),$$

where $W_{\mathbf{x}}$ is the open cluster of $\mathscr{L}$ which contains $\mathbf{x}$, except inasmuch as the f's are not monotone functions of the site set of W. This difficulty is circumvented in exactly the same way as in the previous proof, with the following change. For each $\mathbf{x} \in \mathring{\gamma}$ we define

$$\xi(\mathbf{x}) = \begin{cases} 0 & \text{if all edges of } \mathring{\gamma}\text{, incident to } \mathbf{x}\text{, are closed} \\ 1 & \text{otherwise.} \end{cases}$$

By the connectedness of $\mathring{\gamma}$, there exists $s^2 > 0$ such that $\operatorname{var}(\xi(\mathbf{x})) \geq s^2$ for all $\mathbf{x} \in \mathring{\gamma}$.

PROOF OF THEOREM 1.8. We consider the case $p > 1/2$ only, since if $p < 1/2$ then $C_\gamma = J_\gamma$ a.s. and the result follows from Theorem 3.2. C_γ is defined as the sum of functions which are neither constant on infinite sets nor monotone, and so Theorem 2 of [4] may not be applied. However, the method of proof of that theorem may be adapted to deal with these difficulties. In place of monotonicity, we use the argument of the proof of Theorem 3.1 to show that there exists $\sigma^2 > 0$ such that $C_n = C_{\gamma(n)}$ satisfies

$$\operatorname{var}(C_n) \geq \sigma^2 |\mathring{\gamma}(n)| \text{ for all } n. \tag{3.9}$$

We shall see that the cut condition may be used in place of constantness on infinite sets. Suppose that $\{\gamma(n)\}$ satisfies the cut condition and fix $c > 0$; let $B(n, c)$ be the set of sites $\mathbf{x} \in \mathring{\gamma}(n)$ such that $\kappa(\gamma(n), \mathbf{x}; \varepsilon, c)$ does not hold. For any cluster G of $\mathscr{L}$, we define the *diameter* of G as

$$\delta(G) = \sup\{d(\mathbf{x}, \mathbf{y}): \mathbf{x}, \mathbf{y} \in G\},$$

where the supremum is over all pairs $\mathbf{x}$, $\mathbf{y}$ of sites of G. We write

$$C_n = C_n(1) + C_n(2) + C_n(3)$$

where

$$C_n(1) = \sum_{\mathbf{x}\in\mathring{\gamma}(n)} \frac{1}{|W_{\mathbf{x},n}|} 1(|W_{\mathbf{x},n}| > 0,\ \delta(W_{\mathbf{x},n}) \le |\mathring{\gamma}(n)|^{\epsilon})$$

is the number of open clusters of $\mathring{\gamma}(n)$ with diameter not exceeding $|\mathring{\gamma}(n)|^{\epsilon}$, $C_n(2)$ is the number of open clusters with diameter exceeding $|\mathring{\gamma}(n)|^{\epsilon}$ which intersect $B(n, c)$, and $C_n(3)$ is the number of open clusters remaining. We show that $C_n(2) + C_n(3)$ is negligible in comparison with $|\mathring{\gamma}(n)|^{1/2}$, and that $\{C_n(1)\}$ satisfies the CLT.

LEMMA 3.10. *$E((C_n(2) + C_n(3))^2) = o(|\mathring{\gamma}(n)|)$ as $n \to \infty$.*

PROOF. First note that

$$|C_n(2)| \le |B(n, c)| = o(|\mathring{\gamma}(n)|^{1/2}) \text{ for any } c > 0,$$

by the cut condition, and thus

$$E(C_n(2)^2) = o(|\mathring{\gamma}(n)|). \tag{3.11}$$

Next, consider $C_n(3)$. By the cut condition, we have that

$$\begin{aligned} P(C_n(3) \ge 2) &\le P(\exists\ \mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathring{\gamma}(n)\backslash B(n, c) \text{ such that} \\ &\qquad d(\mathbf{x}, \mathbf{u}) \ge |\mathring{\gamma}(n)|^{\epsilon},\ d(\mathbf{v}, \mathbf{w}) \ge |\mathring{\gamma}(n)|^{\epsilon}, \\ &\qquad \mathbf{u} \in W_{\mathbf{x},n},\ \mathbf{w} \in W_{\mathbf{v},n},\ W_{\mathbf{x},n} \cap W_{\mathbf{v},n} = \phi) \\ &\le P(\exists \text{ closed cluster in dual of } \mathring{\gamma}(n) \text{ of size exceeding} \\ &\qquad c \log |\mathring{\gamma}(n)| \text{ bonds}) \\ &\le \textstyle\sum_{e\in\mathring{\gamma}(n)} P(\text{in dual of } \mathscr{L}, e \text{ is in closed cluster of} \\ &\qquad \text{size exceeding } c \log |\mathring{\gamma}(n)| \text{ bonds}) \end{aligned}$$

where the sum is over all bonds e in $\mathring{\gamma}(n)$. There are at most $4|\mathring{\gamma}(n)|$ bonds in $\mathring{\gamma}(n)$, and so

$$P(C_n(3) \ge 2) \le 4|\mathring{\gamma}(n)| P(|Y| \ge c \log |\mathring{\gamma}(n)|)$$

where Y is the set of bonds in the closed cluster of $\mathscr{L}^*$ containing e. By results of [6],

$$P(|Y| \ge k) \le c_1 \exp(-c_2 k) \text{ for all } k$$

where $c_1(q), c_2(q) > 0$. Thus

$$P(C_n(3) \ge 2) \le 4c_1 |\mathring{\gamma}(n)|^{1-cc_2}.$$

The crude bound

$$E(C_n(3)^2) \leq P(C_n(3) \leq 1) + |\mathring{\gamma}(n)|^2 P(C_n(3) \geq 2)$$

yields

$$E(C_n(3)^2) = o(|\mathring{\gamma}(n)|) \quad \text{as } n \to \infty \tag{3.12}$$

so long as

$$cc_2 > 2. \tag{3.13}$$

An application of Minkowski's inequality completes the proof of Lemma 3.10.

LEMMA 3.14. *$\{C_n(1)\}$ satisfies the* CLT.

PROOF. Write

$$C_n(1) = \sum_{\mathbf{x}\in\mathring{\gamma}(n)} f_n(W_\mathbf{x})$$

where

$$f_n(W_\mathbf{x}) = \frac{1}{|W_{\mathbf{x},n}|} 1(|W_{\mathbf{x},n}| > 0,\ \delta(W_{\mathbf{x},n}) \leq |\mathring{\gamma}(n)|^\varepsilon).$$

The random variable $f_n(W_\mathbf{x})$ is defined in terms of bonds of $\mathscr{L}$ within distance $|\mathring{\gamma}(n)|^\varepsilon + 1$ of site $\mathbf{x}$, and so $f_n(W_\mathbf{x})$ and $f_n(W_\mathbf{y})$ are independent if

$$d(\mathbf{x}, \mathbf{y}) \geq 2|\mathring{\gamma}(n)|^\varepsilon + 2.$$

The proof of Lemma 1 of [4] now yields

$$|E(\rho_n(A)\rho_n(B)) - E\rho_n(A)E\rho_n(B)| \leq c_3 1(d(A, B) < 2|\mathring{\gamma}(n)|^\varepsilon + 2)$$

where

$$\rho_n(A) = \prod_{\mathbf{x}\in A} f_n(W_\mathbf{x})$$

and c_3 is a positive constant, depending on $|A|$, $|B|$ and p. The proof of Theorem 2 of [4] is valid until page 244, line 6, whence it is replaced by the following. The kth semi-invariant $\nu_k(n)$ of $Z_n = (C_n(1) - EC_n(1))(\mathrm{var}(C_n(1)))^{-1/2}$ satisfies

$$\begin{aligned}\nu_k(n) &\leq \frac{1}{(\sigma^2|\mathring{\gamma}(n)|)^{k/2}} \sum_{m=0}^{2|\mathring{\gamma}(n)|^\varepsilon+2} A(k)(2m+1)^{2k}|\mathring{\gamma}(n)| \\ &\leq B(k)|\mathring{\gamma}(n)|^{1-k/2}(|\mathring{\gamma}(n)|^\varepsilon|\mathring{\gamma}(n)|^{2k\varepsilon})\end{aligned}$$

for constants $A(k)$, $B(k)$, depending on k alone. Thus

$$\nu_k(n) \leq B(k)\ |\mathring{\gamma}(n)|^{1+(2k+1)\varepsilon-k/2} \to 0 \text{ as } n \to \infty, \text{ if } k \geq 3,$$

since $\varepsilon < 1/14$ and the power of $|\mathring{\gamma}(n)|$ satisfies

$$1 + (2k+1)\varepsilon - \frac{1}{2}k < \frac{15 - 5k}{14} \leq 0 \text{ if } k \geq 3;$$

this completes the proof of Lemma 3.14. (Note that the term $(2m + 1)^k$ on page 244, line 7 of [4] should be $(2m + 1)^{2k}$, there and later.)

The theorem follows immediately by way of equation (3.9). By (3.9) and Lemma 3.10, as $n \to \infty$,

$$r_n^2 = \frac{\operatorname{var}(C_n(1))}{\operatorname{var}(C_n)} \to 1$$

and

$$D_n = \frac{C_n(2) + C_n(3)}{\sqrt{\operatorname{var}(C_n)}} \to 0 \text{ in mean square,}$$

giving that

$$\frac{C_n - EC_n}{\sqrt{\operatorname{var}(C_n)}} = r_n\left(\frac{C_n(1) - EC_n(1)}{\sqrt{\operatorname{var}(C_n(1))}}\right) + (D_n - ED_n)$$

is asymptotically normally distributed.

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