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CENTRAL LIMIT THEOREMS FOR INFINITE SERIES OF QUEUES AND APPLICATIONS TO SIMPLE EXCLUSION

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We prove that a tagged particle in asymmetric simple exclusion satisfies a central limit theorem when properly rescaled. To obtain this result we derive several results of positive (or negative) correlation for occupation times of a server in a series of queues which imply various central limit theorems.

0. Introduction. Recently several papers have appeared on the movement of a tagged particle in infinite particle systems, partially stimulated by the importance of the problem for physics [13]. Let me briefly review what is known for the simple exclusion process. Intuitively this system consists of infinitely many particles moving on $\mathbb{Z}^d$, each particle attempting a jump of size z with probability $p(z)$ after an exponential mean one waiting time. If the site where the particle has chosen to jump is already occupied, the transition is suppressed, hence the interaction merely consists of exclusion.

Distinguish now among these particles one, which we call the tagged particle, and denote by x_t its displacement at time t, from its initial position, which we may assume to be the origin.

Of course if there were no other particles present we would be looking at a (continuous time) random walk and we would then have:

$$\lim_{t\to\infty} \frac{x_t}{t} = \sum z p(z) := v_0$$

and also provided p has a second moment

$$\frac{x_t - v_0 t}{\sqrt{t}} \to_{\mathscr{D}} N(0, \sigma^2) \quad \text{with } \sigma^2 = \sum z^2 p(z) - v_0^2.$$

Now if we have other particles present we want to see what is the effect of the interaction on the movement of the tagged particle. Since Bernoulli distribution of particles has been proved to be extremal invariant distribution for the simple exclusion process, we consider stationary situations by placing initially the particles according to a Bernoulli distribution with density ρ at all sites different from the origin.

In dimension $d \geq 2$ nothing spectacular is expected and indeed it has been proved [9] that for symmetric p's [i.e., if $p(z) = p(-z)$ for all z's] that $x_t/\sqrt{t} \to N(0, \bar{\sigma}^2)$ where $\bar{\sigma}^2$ is strictly positive although its exact dependence on the density is unknown but conjectured to be a decreasing function of the

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density. For nonsymmetric laws, a behaviour similar to that of an isolated particle is expected.

On the contrary for $d = 1$ the situation is more intricate. If p is symmetric and has a support of at least 4 points, then, $x_t/\sqrt{t}$ converges to a nondegenerate Gaussian distribution [9]. On the other hand if $p(x, x+1) = 1 - p(x, x-1) = p$, Arratia [2] showed that for $p = \frac{1}{2}$ (and all densities $1 > \rho > 0$!) $x_t/t^{1/4}$ converges to a normal distribution. Contrasting with this result Kesten made the following remark: if $p = 1$ (total asymmetry), then x_t itself is a Markov process with exponential holding time [12]; therefore if $\rho < 1$:

$$\frac{x_t}{t} \to (1-\rho) \quad \text{and} \quad \frac{x_t - t(1-\rho)}{\sqrt{t}} \to N\big(0, (1-\rho)^2\big)$$

(as we will see later this property of x_t is a direct consequence of Burke's theorem on series of queues).

What is then the behaviour of x_t for $\frac{1}{2} < p < 1$?

In this paper I prove (Theorem 4) that if $\frac{1}{2} < p < 1$ and $\rho < 1$ then:

$$\frac{x_t}{t} \to (1-\rho)(2p-1) = v(p, \rho) \quad \text{a.s.}$$

and

$$\frac{x_t - tv(p,\rho)}{\sqrt{t}} \to_{\mathscr{L}} N(0, \hat{\sigma}^2),$$

where

$$\hat{\sigma} \geq (1-\rho)\big(\sqrt{2p} - 1\big) > 0.$$

Arratia's result implies that $\varepsilon x_{t/\varepsilon^2} \to 0$ for all t's, and this shows how rigid our system is. Something remains in the case $p \neq \frac{1}{2}$ of this rigidity. Since if y_t^ε is another tagged particle such that $\varepsilon y_0^\varepsilon \to u > 0$ then

$$\lim_{\varepsilon \to 0} \varepsilon\big(x_{t/\varepsilon^2} - y^\varepsilon_{t/\varepsilon^2}\big) = u \quad \text{(in probability)}.$$

Therefore, on this scale, all particles move parallel. This is in sharp contrast with the result for p symmetric and not nearest-neighbour where two tagged particles move according to independent Brownian motions in the limit $\varepsilon \to 0$ [9].

At this point some words are in order on the techniques I use to prove Theorem 4. The first fact is a relation between *one-dimensional nearest-neighbour* simple exclusion process and an infinite series of queues, called in the jargon of infinite particle systems the zero-range process with constant rate: Because of nearest-neighbour jumps and exclusion, notice that in dimension one the order of the particles does not change. Label them initially $x_{-1} < x_0 < x_1 < \cdots$ and look at their positions at time t, $\cdots < x_{-1}(t) < x_0(t) < x_1(t) < \cdots$. Consider the random variables $\eta_i(t)$ equal to the number of empty sites between $x_i(t)$ and $x_{i+1}(t)$. Notice at this point that when the ith particle jumps (one unit) forward, $\eta_t(i)$ is changed into $\eta_t(i) - 1$ and $\eta_t(i+1)$ is changed into $\eta_t(i+1) + 1$. We see that this process is a zero-range process with jump law $(1-p, p)$, or equivalently that customer in line i is served at exponential rate (mean one) and joins the line $i-1$ with probability p, and line $i+1$ with probability $1-p$ [also $\eta_t(i)=0$ corresponds to $x_i(t)$ and $x_{i+1}(t)$ being nearest-neighbours]. Hence the invariance of the Bernoulli distribution corresponds to Jackson's theorem for product of geometric distributions of customers. Also note that x_t equals the total (algebraic) increase of customers in all the negative lines. Hence, if $p=1$, the fact that the number of customers that have crossed the bond between 0 and -1 is a Poisson process (Burke's theorem) is equivalent to Kesten's remark.

Last, the central limit theorems are proved using a very nice invariance principle for associated random variables due to Newman and Wright [11].

The paper is organized as follows: In Section 1, we briefly recall some results of the zero-range process. After having proved correlation inequalities in Section 2, we prove central limit theorems for certain functions of the zero-range process. Section 3 is devoted to the proofs of the results on the tagged particle for $p \neq \frac{1}{2}$.

1. Some results on the zero-range process. The zero-range process (see [12]) has state space $X = \mathbb{N}^{\mathbb{Z}}$ and formal pregenerator L defined on cylindrical functions by the formula

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} 1_{(\eta(x)\geq 1)}\left[pf(\eta^{x,x-1}) + (1-p)f(\eta^{x,x+1}) - f(\eta)\right],$$

where

$$\eta^{x,y}(u) = \begin{cases} \eta(u) & \text{for } u \neq x \text{ or } y, \\ \eta(x)-1 & \text{for } u = x, \\ \eta(y)+1 & \text{for } u = y. \end{cases}$$

This pregenerator defines a unique Markov process [10] and the product measures μ_ρ with marginals equal for all x's to:

$$\mu_\rho\{\eta(x)=k\} = \rho^k(1-\rho)$$

are extremal invariant measures [1]. This in particular implies that if we denote $\Omega = D(\mathbb{R}_+, X)$, then $(\Omega, \eta_t, \mathbb{P}_{\mu_\rho})$ is stationary and ergodic.

With respect to any μ_ρ, the process can be reversed since for all f and g cylindrical we have:

$$\int \mu_\rho(d\eta) f(\eta) Lg(\eta) = \int \mu_\rho(d\eta)\hat{L}f(\eta)g(\eta),$$

where $\hat{L}$ is the same as L except that p is replaced by $(1-p)$, from which it follows, denoting by $\hat{E}$ the expectation with respect to the Markov process of generator $\hat{L}$, that for all f, g, and t

$$\int \mu_\rho(d\eta) E_\eta(f(\eta_t)g(\eta_0)) = \int \mu_\rho(d\eta)\hat{E}_\eta(f(\eta_0)g(\eta_t)).$$

The above identity between L and $\hat{L}$ is easy to check from the following identity

$$\int \mu_\rho(d\eta) f(\eta) g(\eta^{x,x+1}) 1_{(\eta(x)\geq 1)} = \int \mu_\rho(d\eta) f(\eta^{x+1,x}) g(\eta) 1_{(\eta(x+1)\geq 1)}.$$

The zero-range process has also the property of being attractive as can be seen from the definition, but also from the following coupling of two copies of the process

$$\begin{aligned} \bar{L}f(\eta,\xi) = {} & \sum_x 1_{(\eta(x)\geq 1;\, \xi(x)\geq 1)} \big[pf(\eta^{x,x-1}, \xi^{x,x-1}) \\ & \qquad + (1-p) f(\eta^{x,x+1}, \xi^{x,x+1}) - f(\eta,\xi) \big] \\ & + \sum_x 1_{(\eta(x)\geq 1;\, \xi(x)=0)} \big[pf(\eta^{x,x-1}, \xi) + (1-p) f(\eta^{x,x+1}, \xi) - f(\eta,\xi) \big] \\ & + \sum_x 1_{(\eta(x)=0,\, \xi(x)\geq 1)} \big[pf(\eta, \xi^{x,x-1}) + (1-p) f(\eta, \xi^{x,x+1}) - f(\eta,\xi) \big]. \end{aligned}$$

This coupling shows clearly that if $\xi(x) \geq \eta(x)$ for all x, then this remains true for all times.

For those who are not familiar with reading the properties of Markov process from its generator, we point out that this coupling has to be understood as follows: two types of particles sit on the point of $\mathbb{Z}$, named η and ξ. When the bell rings at site x a particle of η *and* a particle of ξ are moved to the same point except of course if $\eta(x) = 0$ or $\xi(x) = 0$, and in this case only the configuration that satisfies $\eta(x) \geq 1$ or $\xi(x) \geq 1$ changes.

We establish the

LEMMA 1. *If $\xi(0) = \eta(0) + 1$ and $\xi(x) = \eta(x)$ for all $x \neq 0$, and η has initial distribution μ_ρ then for all z:*

$$\int_0^\infty \bar{P}(\xi_s(z) = 1;\, \eta_s(z) = 0)\, ds = G_p(0,z) \tag{1}$$

and for $p > \frac{1}{2}$

$$G_p(0,z) = \begin{cases} \dfrac{1}{2p-1} & \text{for } z \geq 0, \\ \dfrac{1}{2p-1}\left(\dfrac{p}{1-p}\right)^z & \text{for } z < 0. \end{cases}$$

PROOF. We prove this by constructing a realization of our coupled process by the following recipe: Let $((T_i)_{i\geq 1}, (X_n)_{n\geq 0}, \eta_t)$ be all independent and T_i's are i.i.d. exponential mean one random variables, X_n is a discrete random walk $(1-p, p)$ starting at the origin, and η_t a zero-range process with initial distribution μ_ρ.

Define inductively:

$$\begin{cases} \tau_1 = \inf\left\{u;\ \int_0^u 1_{(\eta_s(0)=0)}\,ds \ge T_1\right\} \\ \zeta_t(x) = \begin{cases} 1 & \text{for } x = X_0 \\ 0 & \text{for } x \ne X_0 \end{cases} \quad \text{for every } 0 \le t < \tau_1 \end{cases}$$

$$\begin{cases} \tau_2 = \inf\left\{u:\ \int_{\tau_1}^u 1_{(\eta_s(X_1)=0)}\,ds \ge T_2\right\} \\ \zeta_t(x) = \begin{cases} 1 & \text{for } x = X_1 \\ 0 & \text{for } x \ne X_1 \end{cases} \quad \text{and for } \tau_1 \le t < \tau_2, \end{cases}$$

and so on.

Note that in this definition nothing excludes the possibility for one τ_i to be infinite. But because we have for all $x \in \mathbb{Z}$, and almost surely

$$\int_0^\infty 1_{(\eta_s(x) \ge 1)}\,ds = +\infty$$

we have of course that all τ_i's are a.s. finite.

Then indeed $(\eta_t, \eta_t + \zeta_t)$ is a realization of our coupled process and

$$\int_0^\infty 1_{(\eta_u(z)=0,\, \xi_u(z)=1)}\,du = \sum_{j=0}^\infty T_j 1_{\{z\}}(X_j).$$

Integrating both sides with respect to $\mathbb{P}_\mu$ we obtain that (1) equals $\sum_0^\infty P(X_n = z)$. Now this last quantity is the potential of the random walk which can be explicitly computed [7] and equals $G_p(0, z)$. □

2. Central limit theorems for infinite networks of queues. Let $X_1, \dots, X_n, \dots$ be a sequence of real random variables and consider the sums

$$S_n = \sum_1^n X_k.$$

We will say that S_n is weakly positively associated iff for every $k > n$ and increasing functions f and g:

$$E(f(S_n)g(S_k - S_n)) \ge E(f(S_n))E(g(S_k - S_n))$$

(resp. weakly negatively associated if the sense of the inequality is reversed).

The proof of Theorem 12 of [11] proves the following theorem.

THEOREM 1. *Let X_n be a strictly stationary, finite variance sequence, weakly positively (or negatively) associated such that*

$$\lim_{n\to\infty} \frac{V(S_n)}{n} = \sigma^2 < +\infty.$$

Then

$$\frac{1}{\sqrt{n}}(S_n - nE(X_1)) \to N(0, \sigma^2).$$

REMARK. The extension to continuous time is clear. Also notice that when we deal with negative association $\lim V(S_n)/n$ is always finite.

In view of the previous theorem, it is useful to be able to establish that for any increasing functions f and g, two variables U and V satisfy:

$$E(f(U)g(V)) \geq E(f(U))E(g(V))$$

(or the reverse inequality). The following is often useful:

LEMMA ([5], [8]). *Let $X_1, \dots, X_n$ be independent real-valued random variables and suppose that φ and ψ are two functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that for all coordinates φ and ψ are monotone (up or down) in the same direction. Then*

$$E(\varphi(X_1, \dots, X_n)\psi(X_1, \dots, X_n)) \geq E(\varphi(X_1, \dots, X_n))E(\psi(X_1, \dots, X_n)).$$

If in all coordinates φ and ψ are monotone in opposite directions the inequality is reversed.

In order to apply these results to our problem we first prove:

THEOREM 2. *Let $\{\eta_t\}_{t \geq 0}$ be a zero-range process with initial distribution μ_ρ. Then let*

$$\nu(t) = \int_0^t 1_{(\eta_u(0) \geq 1)}\, du$$

and $N^+(t)$ (resp. $N^-(t)$) denote the number of particles that jumped from site 0 to site 1 (resp. from site 1 to site 0) during the time interval $[0, t]$. These three processes are weakly positively associated, whereas the process defined by $N^+(t) - N^-(t)$ is weakly negatively associated.

PROOF.
First case: $\nu(t)$. We only need to prove that for f and g increasing

$$E_{\mu_\rho}(f(\nu(t))g(\nu(t+s) - \nu(t))) \geq E_{\mu_\rho}(f(\nu(t)))E_{\mu_\rho}(g(\nu(s))).$$

As was already noticed in Section 1, the process η_t can be reversed with respect to μ_ρ into another zero-range process $\hat{\eta}_t$, with jump law $(1-p, p)$. Therefore the left-hand side of equals

$$\int \mu_\rho(d\eta)E_\eta(g(\nu(s)))\hat{E}_\eta(f(\nu(t)))$$

(by reversing time at t).

But this expression is of the form $\int \mu_\rho(d\eta)\varphi(\eta)\psi(\eta)$ and since μ_ρ is a product measure, by Lemma 3 of Section 2 we only need to notice that both φ and ψ are increasing in each coordinate $\eta(x)$, which is clear by coupling.

Second case: $N^+(t)$ (resp. $N^-(t)$). The same conditioning as in the case of $\nu(t)$ shows that

$$E_{\mu_\rho}(f(N^+(t))g(N^+(t+s)-N^+(t))) = \int \mu_\rho(d\eta)E_\eta(g(N^+(s)))\hat{E}_\eta(f(N^-(t)))$$

(notice that N^+ has been replaced by N^-!) but each function of η appearing in this expression is increasing in each coordinate $\eta(x)$. This is obvious since adding one particle at site x, i.e., increasing $\eta(x)$, can only increase the number of jumps from 0 to 1 (or from 1 to 0).

Third case: $\xi(t) = N^+(t) - N^-(t)$. As previously we have

$$E_{\mu_\rho}(f(\xi(t))g(\xi(t+s)-\xi(t))) = \int \mu_\rho(d\eta)E_\eta(g(\xi(s)))\hat{E}_\eta(f(-\xi(t))).$$

Now let us analyse the monotonicity properties of

$$\varphi(\eta) = E_\eta[g(\xi(s))].$$

If we add one particle at site x, for $x \le 0$, then it contributes to $\xi(s)$ by

$$\begin{cases} +1 & \text{if its position at time } s \text{ is } > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly adding a particle at $x > 0$ can only contribute by -1 or 0. So our function is increasing in the occupation numbers of the negative sites and decreasing in the occupation numbers of the positive sites.

Similarly the function $\hat{E}_\eta(f(-\xi(t)))$ varies in the opposite sense (notice the minus sign). Therefore by Lemma 3 the total integral is less than

$$\int \mu_\rho(d\eta)E_\eta(g(\xi(s)))\int \mu_\rho(d\eta)\hat{E}_\eta(f(-\xi(t))). \square$$

We are now ready to prove:

THEOREM 3. *For a zero-range process $(p, 1-p)$ with initial distribution μ_ρ the following sequences of random variables (for any $t > 0$)*

$$\nu_1^\varepsilon = \varepsilon\left(\nu_{t/\varepsilon^2} - \rho t/\varepsilon^2\right),$$

$$\nu_2^\varepsilon = \varepsilon\left(N^+_{t/\varepsilon^2} - p\rho t/\varepsilon^2\right),$$

$$\nu_3^\varepsilon = \varepsilon\left(N^-_{t/\varepsilon^2} - (1-p)\rho t/\varepsilon^2\right)$$

converge in distribution as ε goes to zero to a normal random variable with finite variance σ_i $(i = 1,2,3)$.

Moreover

$$\sigma_1^2 = 2\rho\frac{1}{2p-1}t$$

and

$$\sigma_2 \le \sqrt{p\rho t} + p\sqrt{\frac{2}{2p-1} t\rho},$$

$$\sigma_3 \le \sqrt{\rho(1-p)t} + (1-p)\sqrt{\frac{2}{2p-1} t\rho}.$$

PROOF. As a consequence of Theorem 2 above and the Newman–Wright theorem we only need to prove that the variance of ν_i^ε remains bounded as ε goes to zero (for $i = 1, 2, 3$).

Case $i = 1$: Since $E(\nu_1^\varepsilon) = 0$, we compute

$$E\left[(\nu_1^\varepsilon)^2\right] = \varepsilon^2\left[E\left(\int_0^{t/\varepsilon^2} 1_{(\eta_u(0)\ge 1)}\,du \int_0^{t/\varepsilon^2} 1_{(\eta_v(0)\ge 1)}\,dv\right) - \int_0^{t/\varepsilon^2} E\left(1_{(\eta_u(0)\ge 1)}\right) du \int_0^{t/\varepsilon^2} E\left(1_{(\eta_v(0)\ge 1)}\right) dv\right].$$

Since all these expectations are with respect to a process with initial invariant distribution this is equal to

$$2\varepsilon^2 \int_0^{t/\varepsilon^2} du \int_u^{t/\varepsilon^2} \mathbb{P}[(\eta_0(0) \ge 1)(\eta_v(0) \ge 1)] - (\mathbb{P}[\eta_0(0) \ge 1])^2\, dv.$$

Hence it reduces to proving that

$$\int_0^\infty \mathbb{P}[(\eta_0(0) \ge 1)(\eta_v(0) \ge 1)] - (\mathbb{P}[\eta_0(0) \ge 1])^2\, dv < +\infty.$$

Note now that

$$\mathbb{P}[(\eta_0(0) \ge 1)(\eta_v(0) \ge 1)] / \mathbb{P}[\eta_0(0) \ge 1]$$

is also the probability of the set $(\eta_v(0) \ge 1)$ when the initial distribution is not μ_ρ but $\mu_\rho(\cdot \mid \eta(0) \ge 1)$. This new probability (denoted $\tilde{\mu}$) is easily seen to have independent occupation numbers at all sites and geometric distribution at all sites different from the origin while

$$\tilde{\mu}(\eta(0) = k) = (1-\rho)\rho^{k-1} \quad (\text{for } k \ge 1).$$

This measure can be realized by "adding a particle at the origin" to a configuration ξ chosen according to μ. Therefore it we couple in this way μ and $\tilde{\mu}$ and call Y_t the position of the "extra"-particle we have

$$\mathbb{P}_{\tilde{\mu}}(\eta_v(0) \ge 1) - \mathbb{P}_\mu(\eta_v(0) \ge 1) = \overline{\mathbb{P}}(Y_v = 0 \text{ and } \eta_v(0) = 0).$$

Integrating this equality we obtain by Lemma 1, the finiteness of the variance and the expression of σ_1.

Case $i = 2$ *or* 3: As before we only need to check that $\mathrm{Var}(\nu_i^\varepsilon)$ remains bounded as ε goes to zero.

We start from the fact that

$$M_t = N_t^+ - p\int_0^t 1_{(\eta_u(0)\ge 1)}\,du$$

is a martingale (compensated sum of jumps of size one) and also

$$M_t^2 - p\int_0^t 1_{(\eta_u(0)\geq 1)}\,du$$

(this last fact is true because we deal with a compensated sum of jumps of size *one*, see [3], [6]). Hence

$$E(M_t^2) = \rho pt.$$

On the other hand

$$\nu_2^\varepsilon = \varepsilon\left[\left(N_{t/\varepsilon^2}^+ - p\int_0^{t/\varepsilon^2} 1_{(\eta_u(0)\geq 1)}\,du\right) + p\left(\int^{t/\varepsilon^2} 1_{(\eta_u(0)\geq 1)}\,du - \rho t/\varepsilon^2\right)\right].$$

Hence by the Schwarz inequality

$$E\left((\nu_2^\varepsilon)^2\right) \leq \varepsilon^2\left[\rho pt/\varepsilon^2 + p^2\frac{E\left[(\nu_1^\varepsilon)^2\right]}{\varepsilon^2} + 2\sqrt{p^2\frac{E\left[(\nu_1^\varepsilon)^2\right]}{\varepsilon^2}pt/\varepsilon^2}\right]$$

$$\leq \left(\sqrt{\rho pt} + p\sqrt{E\left[(\nu_1^\varepsilon)^2\right]}\right)^2,$$

which, combined with the result for the case $i = 1$, concludes the proof. □

3. Position of a tagged particle in the simple exclusion process.

THEOREM 4. *Let x_t be the position of a tagged particle in the simple exclusion process $(1-p, p)$ with initial state a Bernoulli distribution with parameter β. Then $\varepsilon(x_{t/\varepsilon} - x_0)$ converges almost surely to $(1-2p)(1-\beta)t$ and $\varepsilon(x_{t/\varepsilon^2} - (1-2p)(1-\beta)t/\varepsilon^2)$ converges in law to a nondegenerate normal random variable for all $p \neq \frac{1}{2}$ and $t > 0$.*

PROOF. As was explained in the introduction we transform our problem by using the correspondence between simple exclusion and zero-range. In this transformation the initial distribution of the zero-range becomes $\mu_{1-\beta}$, which we know to be extremal invariant. Also we have the formula

$$x_t = -N_t^+ + N_t^-,$$

$$\varepsilon x_{t/\varepsilon} = -\varepsilon\left(N_{t/\varepsilon}^+ - p\int_0^{t/\varepsilon} 1_{(\eta_u(0)\geq 1)}\,du\right) - \varepsilon p\int_0^{t/\varepsilon} 1_{(\eta_u(0)\geq 1)}\,du$$

$$+\varepsilon\left(N_{t/\varepsilon}^- - (1-p)\int_0^{t/\varepsilon} 1_{(\eta_u(-1)\geq 1)}\,du\right) + \varepsilon(1-p)\int_0^{t/\varepsilon} 1_{(\eta_u(-1)\geq 1)}\,du.$$

Since $M_t = N_t^+ - p\int_0^t 1_{(\eta_u(0)\geq 1)}\,du$ is a martingale and $M_0 = 0$ and

$$\mathbb{E}_{\mu_{1-\beta}}(M_t^2) = p(1-\beta)t,$$

we have that $\varepsilon M_{t/\varepsilon}$ goes to zero. Because of the ergodicity of $\mathbb{P}_{\mu_{1-\beta}}$ we also have that

$$\varepsilon\int_0^{t/\varepsilon} 1_{(\eta_u(0)\geq 1)} \to t\mu_{1-\beta}(\eta(0)\geq 1) = (1-\beta)t.$$

Hence

$$\varepsilon x_{t/\varepsilon} \to (1-\beta)t(1-2p).$$

In order to prove the convergence to a nondegenerate normal distribution write:

$$\begin{aligned} Z^\varepsilon &= \varepsilon\left(x_{t/\varepsilon^2} - (1-\beta)(1-2p)t/\varepsilon^2\right) \\ &= -\varepsilon\left(N^+_{t/\varepsilon^2} - p\int_0^{t/\varepsilon^2} 1_{(\eta_u(0)\geq 1)}\,du\right) \\ &\quad -p\varepsilon\int_0^{t/\varepsilon^2}\left(1_{(\eta_u(0)\geq 1)} - (1-\beta)\right)du \\ &\quad +\varepsilon\left(N^-_{t/\varepsilon^2} - (1-p)\int_0^{t/\varepsilon^2} 1_{(\eta_u(1)\geq 1)}\,du\right) \\ &\quad +(1-p)\varepsilon\int_0^{t/\varepsilon^2}\left(1_{(\eta_u(1)\geq 1)} - (1-\beta)\right)du \\ &= (1)+(2)+(3)+(4). \end{aligned}$$

We already established in Theorem 3 that each of these four sequences of random variables is centered and has a bounded second moment. Hence Z^ε has a bounded variance as ε goes to zero.

By negative weak association of Theorem 2, we conclude that Z^ε converges in distribution to a normal random variable.

We are therefore left with proving that the variance of Z^ε does not go to zero as $\varepsilon \to 0$. Now (1) + (3) comes from the martingale

$$\left(N_t^- - (1-p)\int_0^t 1_{(\eta_u(1)\geq 1)}\,du\right) - \left(N_t^+ - p\int_0^t 1_{(\eta_u(0)\geq 1)}\,du\right),$$

which is the sum of two martingales which are orthogonal since they have no common jumps. Therefore the variance of (1) + (3) is the sum of their individual variances, i.e., equals $(1-\beta)t$. On the other hand the limiting variances of (2) and (4) were already computed.

Now to obtain their covariance we compute

$$\begin{aligned} &\lim_{t\to\infty}\frac{1}{t}E\left[\int_0^t\left(1_{(\eta_u(0)\geq 1)} - (1-\beta)\right)du\int_0^t\left(1_{(\eta_v(1)\geq 1)} - (1-\beta)\right)dv\right] \\ &\quad = \lim_{t\to\infty}\int_0^t\int_0^t\left[\mathbb{P}(\eta_u(0)\geq 1;\, \eta_v(1)\geq 1) - \mathbb{P}(\eta_u(0)\geq 1)\mathbb{P}(\eta_v(0)\geq 1)\right]du\,dv. \end{aligned}$$

Computing along the same lines as in Theorem 3 we obtain for $p > \frac{1}{2}$ that this quantity equals $(1-\beta)[G(0,1)+G(1,0)]$. Hence the total variance of (2) + (4) equals: $t(1-\beta)2p$. Therefore by Cauchy–Schwarz the variance of our random variable is larger than $t(1-\beta)(\sqrt{2p}-1)^2$. □

REMARK. The only point that remains is the rigidity of the system. Note that this is easily deduced from the analysis of the asymptotic behaviour for $a > 0$ of

$$\varepsilon\left(\sum_{k=0}^{[a/\varepsilon]}\eta(k) - \sum_{k=0}^{[a/\varepsilon]}\eta_{t/\varepsilon^2}(k)\right) = D_\varepsilon.$$

But in equilibrium we have that almost surely

$$\varepsilon\left(\sum_{k=0}^{[\alpha/\varepsilon]}\eta(k)\right) \to a\int\eta(0)\mu_{1-\beta}(d\eta) = a\frac{1-\beta}{\beta}.$$

Hence it converges to this constant in probability. Also η_{t/ε^2} has the same distribution as η_0 therefore $\varepsilon\sum_{k=0}^{a/\varepsilon}\eta_{t/\varepsilon^2}(k)$ converges in probability to $a(1-\beta)/\beta$. This implies that $D_\varepsilon \to 0$ in probability. At this point we conjecture, by analogy with Arratia's result, that in fact the correct order is $t^{1/4}$, i.e., that

$$\varepsilon\sum_{k=0}^{[a/\varepsilon]}\left(\eta_0(k) - \eta_{t/\varepsilon^4}(k)\right)$$

converges in distribution to a normal random variable.

Concluding remarks. We list here several problems that arise naturally in view of the preceding results:

(1) What is the behaviour of one tagged particle in simple exclusion when the dimension is larger than one? Of course the natural conjecture is that the behaviour is exactly the same as in Theorem 4.

(2) What are the exact values of the diffusion coefficients? Notice that we obtain the bound:

$$\sigma^2(p,\beta) \ge (1-\beta)\left(\sqrt{2p} - 1\right).$$

This is exact for $p = \frac{1}{2}$ and $p = 1$. This gives a lower estimate for the critical behaviour at $p = \frac{1}{2}$. What is the exact critical exponent? It has been shown by De Masi and Ferrari that the exact value of $\sigma^2(p,\beta)$ is $(1-\beta)|2p-1|$ [4].

(3) We proved the convergence to a normal distribution of the one-time distribution. We conjecture that in fact the following stronger statement holds: the *process* $\varepsilon(x_{t/\varepsilon^2} - (1-\beta)(1-2p)t/\varepsilon^2)$ converges in law to Brownian motion. Using the results of Newman and Wright it should be easy to obtain tightness, but we failed to prove convergence of the finite-dimensional distributions.

(4) In Theorem 3 all variances blow up when $p = \frac{1}{2}$. What is the correct renormalization for these processes?

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REFERENCES

[1] ANDJEL, E. (1982). Invariant measures for the zero-range process. *Ann. Probab.* **10** 525–547.

[2] ARRATIA, R. (1983). The motion of a tagged particle in the simple symmetric exclusion system on $\mathbb{Z}$. *Ann. Probab.* **11** 362–373.

[3] DELLACHERIE, C. and MEYER, P. A. (1980). *Probabilités et potential. Théorie des Martingales.* Hermann, Paris.
[4] DE MASI, A. and FERRARI, P. (1985). Self diffusion in one dimensional lattice gases in the presence of an external field. To appear.
[5] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040.
[6] ELLIOTT, R. J. (1982). *Stochastic Calculus and Applications.* Springer, New York.
[7] FELLER, W. (1957). *An Introduction to Probability Theory and Its Applications* **1**. Wiley, New York.
[8] KESTEN, H. (1982). *Percolation Theory for Mathematicians.* Birkhäuser, Boston.
[9] KIPNIS, C. and VARADHAN, S. R. S. (1985). Central limit theorem for certain non-Markovian processes and applications. To appear in *Comm. Math. Phys.*
[10] LIGGETT, T. (1973). An infinite particle systems with zero-range interactions. *Ann. Probab.* **1** 240–253.
[11] NEWMAN, C. M. (1984). Asymptotic independence and limit theorems for positively and negatively dependent random variables. In *Inequalities in Statistics and Probability, IMS Lecture Notes-Monograph Series* **5** 127–140.
[12] SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 246–290.
[13] SPOHN, H. and LEBOWITZ, J. (1982). A microscopical basis for Fick's law. *J. Stat. Phys.* **28** 539–556.

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