SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

TECHNICAL REPORT NO. 590

October 1983

CENTRAL LIMIT THEOREMS FOR QUADRATIC FORMS IN RANDOM VARIABLES HAVING LONG-RANGE DEPENDENCE

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Robert Fox
and
Murad S. Taqqu[1]

Typed by Kathy King

1 Research supported by the National Science Foundation grant ECS-80-15585 at Cornell University.

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1. Introduction

Let $f(x)$ and $g(x)$ be integrable real symmetric functions on $[-\pi,\pi]$ that are bounded on subintervals that exclude the origin. Let $X_1, X_2, \ldots$ be a mean zero stationary Gaussian sequence with spectral density $f(x)$, and let $\ldots, -a_1, a_0, a_1, \ldots$ be the Fourier coefficients of $g(x)$. We prove that the distribution of the normalized quadratic form

$$Z_N = \frac{1}{\sqrt{N}} \{ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j - E \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j \}$$

converges to a normal distribution if there exist constants $\alpha < 1$ and $\beta < 1$ with $\alpha + \beta < \frac{1}{2}$ such that for each $\delta > 0$, $f(x) = 0(|x|^{-\alpha-\delta})$ and $g(x) = 0(|x|^{-\beta-\delta})$ as $x \to 0$.

Of particular interest are the cases where $f(x) \sim x^{-\alpha} L_1(x)$ and $g(x) \sim x^{-\beta} L_2(x)$ as $x \to 0$ with L_1 and L_2 slowly varying. The exponents α and β are allowed to be positive, zero or negative. The sequence $\{X_j\}$ is said to exhibit a long-range dependence when $\alpha > 0$. When $\alpha < 0$, the covariances $r_k = EX_j X_{j+k}$ satisfy $\sum_{k=-\infty}^{+\infty} r_k = 0$.

Suppose $f(x) \sim x^{-\alpha} L_1(x)$ and $g(x) \sim x^{-\beta} L_2(x)$ as $x \to 0$. Rosenblatt (1961) showed that in the special case $\frac{1}{2} < \alpha < 1$ and $a_{i-j} = \delta_{ij}$, the quadratic form $\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i-j} X_i X_j$, adequately normalized, converges to a <u>non-normal</u> distribution. The assumption $a_{i-j} = \delta_{ij}$ implies $g(x)$ constant and

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thus $\beta = 0$. Our result shows that the normalized quadratic form Z_N converges to a <u>normal</u> distribution when $\frac{1}{2} < \alpha < 1$ and $\beta < \frac{1}{2} - \alpha < 0$. If $\alpha \leq \frac{1}{2}$, it is even possible to choose $\beta > 0$ as long as $\beta < \min(\frac{1}{2} - \alpha, 1)$.

Results of this type can be used in the study of the asymptotic behavior of maximum likelihood estimators related to the sequence $\{X_j\}$ (Fox and Taqqu, 1983). Example of sequences $\{X_j\}$ satisfying $f(x) \sim x^{-\alpha} L_1(x)$ that are of special interest include fractional Gaussian noise and fractional ARMA.

A sequence $\{X_j\}$ is <u>fractional Gaussian noise</u> (Mandelbrot and Van Ness, 1968) if its covariance satisfies

$$r(k) = EX_j X_{j+k} = \frac{\sigma^2}{2} \{(|k|-1)^{2H} - 2|k|^{2H} + (|k|+1)^{2H}\}$$

for $0 < H < 1$. In that case (Sinaï 1976)

$$f(x) = \frac{\sigma^2}{\int_{-\infty}^{+\infty} (1-\cos y)|y|^{-1-2H} dy} (1-\cos x) \sum_{k=-\infty}^{+\infty} |x+2k\pi|^{-1-2H},$$

so that $\alpha = 2H-1 \in (-1,1)$.

A sequence $\{X_j\}$ is <u>fractional ARMA</u> (Hoskings, 1981) if its spectral density is

$$f(x) = |e^{ix}-1|^{-d} \frac{|\psi(e^{ix})|^2}{|\phi(e^{ix})|^2}$$

where ψ and ϕ are polynomials having no zeroes on the unit circle and $d < 1$. In that case $\alpha = d$. Heuristically, fractional ARMA is the sequence, which, when differenced $d/2$ times, yields an autoregressive-moving average (ARMA) sequence with spectral density $|\psi(e^{ix})|^2/|\phi(e^{ix})|^2$.

Our main results are in Section 2. The remaining sections are devoted to the proof of Theorem 1. That proof uses "power counting" arguments in the sense of mathematical physics. In Section 3 we introduce the power counting set-up and state an extension of a power counting theorem of Lowenstein and Zimmerman (1975). Preliminary lemmas are proven in Section 4 and, together with the results of Section 5, they are used to establish Propositions 6.1 and 6.2 of Section 6. These propositions describe the asymptotic behavior of certain multiple integrals. Section 7 contains the proof of Theorem 1.

2. Main results

Let $f(x)$ and $g(x)$ be integrable real symmetric functions on $[-\pi,\pi]$, not necessarily non-negative. Define the Fourier coefficients

$$r_n = \int_{-\pi}^{\pi} e^{inx} f(x)dx$$

and

$$a_n = \int_{-\pi}^{\pi} e^{inx} g(x)dx.$$

Let R_N and A_N be the $N \times N$ matrices with entries $(R_N)_{j,k} = r_{j-k}$ and $(A_N)_{j,k} = a_{j-k}$, $0 \leq j,k \leq N-1$. Let $\operatorname{Tr} M$ denote the trace of a matrix M.

We say that f satisfies the <u>regularity condition</u> if the discontinuities of f have Lebesgue measure 0 and f is bounded on the interval $[\delta,\pi]$ for all $\delta > 0$.

<u>Theorem 1</u>. Suppose that f and g each satisfy the regularity condition. Suppose in addition that there exist $\alpha < 1$ and $\beta < 1$ such that for each $\delta > 0$

$$|f'x)| = 0(|x|^{-\alpha-\delta}) \quad \text{as} \quad x \to 0$$

and

$$|g(x)| = 0(|x|^{-\beta-\delta}) \quad \text{as} \quad x \to 0.$$

Then

a) If $p(\alpha+\beta) < 1$,

$$\lim_{N\to\infty} \frac{\operatorname{Tr}(R_N A_N)^p}{N} = (2\pi)^{2p-1} \int_{-\pi}^{\pi} [f(x)g(x)]^p dx.$$

b) If $p(\alpha+\beta) \geq 1$,

$$\operatorname{Tr}(R_N A_N)^p = o(N^{p(\alpha+\beta)+\varepsilon}) \quad \text{for every} \quad \varepsilon > 0.$$

The theorem is proven in Section 7.

Introduce now a stationary Gaussian sequence X_j, $j \geq 1$ with mean 0 and spectral density $f(x)$, so that

$$EX_jX_{j+k} = r_k = \int_{-\pi}^{\pi} e^{ikx} f(x)dx.$$

Let x_N denote the random vector $(X_1, X_2, \ldots, X_N)'$. Put $\mu_N = Ex_N' A_N x_N$.

<u>Theorem 2</u>. Suppose that f and g each satisfy the regularity condition. Suppose in addition that there exist $\alpha < 1$ and $\beta < 1$ such that $\alpha + \beta < \frac{1}{2}$ and such that for each $\delta > 0$

$$f(x) = 0(|x|^{-\alpha-\delta}) \quad \text{as} \quad x \to 0$$

$$g(x) = 0(|x|^{-\beta-\delta}) \quad \text{as} \quad x \to 0.$$

Then

$$\frac{x_N' A_N x_N - \mu_N}{\sqrt{N}}$$

tends in distribution to a normal random variable with mean 0 and variance $16\pi^3 \int_{-\pi}^{\pi} [f(x)g(x)]^2 dx$.

<u>Proof</u>. Since the sequence X_j is Gaussian, the pth cumulant of $x_N' A_N x_N$ is equal to $2^{p-1}(p-1)! Tr(R_N A_N)^p$. (See, for example, Grenander and Szego, 1958, page 218). Thus the pth cumulant of

$$\frac{x_N' A_N x_N - \mu_N}{\sqrt{N}}$$

is given by

$$c_p(N) = \begin{cases} 0 & \text{if} \quad p = 1 \\ 2^{p-1}(p-1)! \dfrac{Tr(R_N A_N)^p}{N^{p/2}} & \text{if} \quad p \geq 2. \end{cases}$$

An application of Theorem 1 yields

$$\lim_{N\to\infty} c_p(N) = \begin{cases} 0 & \text{if } p \neq 2 \\ 16\pi^3 \int_{-\pi}^{\pi} [f(x)g(x)]^2 dx & \text{if } p = 2. \end{cases}$$

This implies the conclusion of Theorem 2. □

The following is an immediate consequence of Theorem 2.

Theorem 3. Suppose that f and g each satisfy the regularity condition. Suppose in addition that there exist $\alpha < 1$ and $\beta < 1$ such that $\alpha+\beta < \frac{1}{2}$,

$$f(x) \sim |x|^{-\alpha} L_1(x) \quad \text{as} \quad x \to 0$$

and

$$g(x) \sim |x|^{-\beta} L_2(x) \quad \text{as} \quad x \to 0,$$

where L_1 and L_2 are slowly varying at 0. Then the conclusion of Theorem 2 holds.

3. Power counting theorems

Power counting methods can be used to verify the convergence of multiple integrals whose integrands are products of powers of linear functionals.

Let $L_1(x), L_2(x), \ldots, L_m(x)$ be m linear functionals on R^n and let $b_1, b_2, \ldots, b_m$ be real constants. Define the function $P: R^n \to R \cup \{\infty\}$ by

$$P(x) = |L_1(x)|^{b_1} |L_2(x)|^{b_2} \ldots |L_m(x)|^{b_m}.$$

We shall view $T = \{L_1, L_2, \ldots, L_m\}$ as a subset of the dual space of R^n. Let W be any subset of T. We use the notation $\text{span}\{W\}$ to denote the set of linear combinations of elements of W and we let $s(W)$ denote those linear combinations which coincide with elements of T. Thus

$$s(W) = T \cap \text{span}\{W\}.$$

For each $W \subset T$ we define the quantity

$$d(P,W) = |W| + \sum_{\{j:\, L_j \in s(W)\}} b_j,$$

where $|W|$ denotes the cardinality of W. We refer to $d(P,W)$ as the dimension of P with respect to W.

Let S be the set of those L_j in T that have exponents $b_j < 0$. Finally, for each $t > 0$, let

$$U_t = [-t,t]^n = \{x \in R^n: |x_i| \leq t, \quad i = 1,\ldots,n\}.$$

We can now state a basic result of Lowenstein and Zimmermann (1975).

Theorem 3.1. Suppose that $d(P,W) > 0$ for every independent set $W \subset S$. Then $\int_{U_t} P(x)dx < \infty$ for all $t > 0$.

To illustrate the application of the theorem, let $n = 3$ and define $P(x): R^3 \to R \cup \{\infty\}$ by

$$P(x) = |x_1+x_2|^{b_1} |x_1+x_2+x_3|^{b_2} |x_3|^{b_3},$$

where $b_1,b_2,b_3 < 0$. Define $L_1(x) = x_1+x_2$, $L_2(x) = x_1+x_2+x_3$ and $L_3(x) = x_3$. Then $S = T = \{L_1,L_2,L_3\}$. The independent subsets of S are $\{L_1\}$, $\{L_2\}$, $\{L_3\}$, $\{L_1,L_2\}$, $\{L_1,L_3\}$ and $\{L_2,L_3\}$. We have $d(P,\{L_j\}) = 1+b_j$, $j = 1,2,3$. The other three dimensions are all equal to $2+b_1+b_2+b_3$ because for example $s(\{L_1,L_2\}) = \{L_1,L_2,L_3\}$. Therefore $\int_{U_t} P(x)dx$ will be finite provided that $b_1+b_2+b_3 > -2$ and $b_1,b_2,b_3 > -1$.

The following theorem is more general than Theorem 3.1. (It will not be used in the sequel.)

<u>Theorem 3.2</u>. Let $\gamma > 0$, $\delta \geq 0$, and let S_1 and S_2 be a partition of S. Define

$$V_N = \{x \in U_t: |L| \geq \frac{1}{N^\gamma}, \quad L \in S_1\}.$$

If

a) $d(P,W) > 0$ for all independent sets $W \subset S_2$,

and

b) $d(P,W) > -\delta$ for all independent sets $W \subset S$,

then

$$\int_{V_N} P(x)dx = O(N^{\gamma\delta})$$

as $N \to \infty$.

The proof of Theorem 3.2 (as well as an alternate proof of Theorem 3.1) can be found at the end of Section 4. They are easy consequences of the setup necessary to prove Theorem 1.

We illustrate the application of Theorem 3.2 to the above example. In that example, suppose that $b_1 = b_2 = -\frac{1}{4}$ and $b_3 = -\frac{4}{3}$. Define $S_1 = \{L_3\}$, $S_2 = \{L_1, L_2\}$ and $V_N = \{x \in [-t,t]^3 : |L_3| > \frac{1}{N^\gamma}\}$. The independent subsets of S_2 are $\{L_1\}$, $\{L_2\}$ and $\{L_1, L_2\}$. We have $d(P,\{L_1\}) = d(P,\{L_2\}) = \frac{3}{4}$ and $d(P,\{L_1,L_2\}) = 2 - \frac{1}{4} - \frac{1}{4} - \frac{4}{3} = \frac{1}{6}$. Thus condition a of Theorem 3.2 is satisfied. We also have $d(P,\{L_1,L_3\}) = d(P,\{L_2,L_3\}) = \frac{1}{6}$ and $d(P,\{L_3\}) = -\frac{1}{3}$. Hence condition b of Theorem 3.2 is satisfied for any $\delta > \frac{1}{3}$. Theorem 3.2 implies that $\int_{V_N} P(x)dx = O(N^{\gamma/3 + \varepsilon})$ for any $\varepsilon > 0$.

Note that the result of Theorem 3.1 follows from Theorem 3.2 by setting $S_1 = \emptyset$, $S_2 = S$ and $\delta = 0$ in Theorem 3.2.

4. Preliminary lemmas

Retain the notation introduced at the beginning of Section 3. The functional $L_j(x)$ satisfies $|L_j(x)| \leq ||L_j|| \, |x|$ for all $x \in R^n$ where

$$||L_j|| = \max\{L_j(x): |x| = 1\}.$$

Since $|x| \leq \sqrt{n}\, t$ when $x \in U_t = [-t,t]^n$, we have

$$U_t \subset \{x \in R^n: |L_j(x)| \leq Mt, \quad j = 1,\ldots,m\}, \tag{4.1}$$

where $M = \sqrt{n} \max(||L_1||, ||L_2||,\ldots,||L_m||)$.

Partition the set $U_t = [-t,t]^n$ into subsets

$$E_\sigma = \{x \in U_t: |L_{\sigma_1}(x)| \leq |L_{\sigma_2}(x)| \leq \cdots \leq |L_{\sigma_m}(x)|\},$$

where $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_m)$ runs over all permutations of $\{1,2,\ldots,m\}$. Each E_σ depends on t.

Fix a permutation σ. Suppose that the subset S of $T = \{L_1,L_2,\ldots,L_m\}$ has q elements. Label these $L_{i_1},L_{i_2},\ldots,L_{i_q}$ so that

$$|L_{i_1}(x)| \leq |L_{i_2}(x)| \leq \cdots \leq |L_{i_q}(x)|$$

for $x \in E_\sigma$. (The labeling depends on the parameter σ.)

We now use the greedy algorithm to construct a basis B_σ for S. The greedy algorithm proceeds as follows. We put $L_{i_1} \in B_\sigma$. We put $L_{i_2} \in B_\sigma$ if L_{i_2} is not in the span of $\{L_{i_1}\}$. On the jth step we put $L_{i_j} \in B_\sigma$ if L_{i_j} is not in the span of $\{L_{i_1},\ldots,L_{i_{j-1}}\}$. It is well known that in this way we obtain a basis $B_\sigma = \{L_{\tau_1},\ldots,L_{\tau_r}\}$ for S, where r is the rank of S. We then have

$$|L_{\tau_1}| \leq |L_{\tau_2}| \leq \cdots \leq |L_{\tau_r}|, \quad x \in E_\sigma. \tag{4.2}$$

We can now use B_σ to construct a partition of T. Define

$$T_1 = s\{L_{\tau_1}\},$$

$$T_k = s\{L_{\tau_1},\ldots,L_{\tau_k}\}/s\{L_{\tau_1},\ldots,L_{\tau_{k-1}}\}, \quad k = 2,\ldots,r,$$

and

$$T_{r+1} = T\backslash s\{L_{\tau_1},\ldots,L_{\tau_r}\}.$$

Thus T_{r+1} consists of those elements of T which are not in the span of S. The sets $T_1,\ldots,T_{r+1}$ clearly form a partition of T.

Lemma 4.1. For each permutation σ there is a constant C_σ (independent of x and t) such that

a) If $L \in T_k$, $k \leq r$, then

$$|L| \leq C_\sigma |L_{\tau_k}|, \quad x \in E_\sigma.$$

b) If $L \in T_k \cap S$, $k \leq r$, then

$$|L_{\tau_k}| \leq |L|, \quad x \in E_\sigma.$$

Proof.

a) If $L \in T_k$ then $L = a_1 L_{\tau_1} + \cdots + a_k L_{\tau_k}$ for some constants $a_1,\ldots,a_k$. Therefore

$$|L| \leq |a_1||L_{\tau_1}| + \cdots + |a_k||L_{\tau_k}|, \quad x \in R^n.$$

Relation (4.2) implies that for $x \in E_\sigma$ the right hand side is less than $(|a_1| + \cdots + |a_k|)|L_{\tau_k}|$. It therefore suffices to put $C_\sigma \geq |a_1| + \cdots + |a_k|$.

b) Suppose that $L \in T_k \cap S$. We must have either $L = L_{\tau_k}$ or else L was rejected by the greedy algorithm. In proving b) we can thus assume that L was rejected by the greedy algorithm. Since $L \in T_k$ it follows that $L \notin s\{L_{\tau_1},\ldots,L_{\tau_{k-1}}\}$. Therefore it must be that L was considered by the greedy algorithm after L_{τ_k}. But the greedy algorithm considers candidates in order of increasing absolute value on E_σ. Thus we must have $|L_{\tau_k}| \leq |L|$, $x \in E_\sigma$. This completes the proof of Lemma 4.1. □

The next lemma provides a majorant for $P(x)$ involving only elements of B_σ.

Lemma 4.2. For each permutation σ there are constants C_1 and C_2 (independent of x and t) such that

$$P(x) \leq C_1(Mt)^{C_2}|L_{\tau_1}|^{\Delta_1}\cdots|L_{\tau_r}|^{\Delta_r}, \quad x \in E_\sigma,$$

where

$$\Delta_1 = d(P,\{L_{\tau_1}\}) - 1$$

$$\Delta_k = d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\}) - d(P,\{L_{\tau_1},\ldots,L_{\tau_{k-1}}\}) - 1, \quad k = 2,\ldots,r.$$

Proof. We have

$$P(x) = \prod_{k=1}^{r+1} F_k(x),$$

where

$$F_k(x) = \prod_{\{j:\ L_j\in T_k\}} |L_j|^{b_j} = \left[\prod_{\{j:\ L_j\in T_k\setminus S\}} |L_j|^{b_j}\right]\left[\prod_{\{j:\ L_j\in T_k\cap S\}} |L_j|^{b_j}\right].$$

Fix $k \leq r$ and consider the two products on the right hand side. In the first product all of the exponents are non-negative because the L_j's do not belong to S. Therefore Lemma 4.1.a implies that the first product is majorized on E_σ by

$$\prod_{\{j:\ L_j \in T_k \setminus S\}} C_\sigma^{b_j} \left|L_{\tau_k}\right|^{b_j}.$$

In the second product all of the exponents are negative. Thus Lemma 4.1.b implies that the second product is majorized on E_σ by

$$\prod_{\{j:\ L_j \in T_k \cap S\}} \left|L_{\tau_k}\right|^{b_j}.$$

Combining these facts we conclude that there is a constant C such that

$$F_k(x) \leq C\left|L_{\tau_k}\right|^{p_k}, \quad x \in E_\sigma, \quad k \leq r,$$

where

$$p_k = \sum_{\{j:\ L_j \in T_k\}} b_j.$$

Finally we consider $F_{r+1}(x)$. All of the exponents in the product defining $F_{r+1}(x)$ are non-negative because $S \subset \cup_{k=1}^{r} T_k$. Consequently (4.1) implies

$$F_{r+1}(x) \leq (Mt)^{p_{r+1}}, \quad x \in E_\sigma.$$

Lemma 4.2 will follow from the last two inequalities if we show that $\Delta_k = p_k$, $k = 1,\dots,r$. We have

$$d(P,\{L_{\tau_1}\}) = 1 + \sum_{\{j:\ L_j \in s(L_{\tau_1})\}} b_j = 1 + \sum_{\{j:\ L_j \in T_1\}} b_j = 1 + p_1.$$

Thus

$$\Delta_1 = d(P,\{L_{\tau_1}\}) - 1 = p_1.$$

If $k \geq 2$ then

$$d(P,(L_{\tau_1},\dots,L_{\tau_k})) = k + \sum_{\{j:\ L_j\in s(L_{\tau_1},\dots,L_{\tau_k})\}} b_j$$

$$= 1 + \left[(k-1) + \sum_{\{j:\ L_j\in s(L_{\tau_1},\dots,L_{\tau_{k-1}})\}} b_j\right] + \sum_{\{j:\ L_j\in T_k\}} b_j$$

$$= 1 + d(P\{L_{\tau_1},\dots,L_{\tau_{k-1}}\}) + p_k.$$

Thus

$$\Delta_k = d(P,\{L_{\tau_1},\dots,L_{\tau_k}\}) - d(P,\{L_{\tau_1},\dots,L_{\tau_{k-1}}\}) - 1 = p_k.$$

This completes the proof of Lemma 4.2. □

Lemma 4.3. Let $\phi_1,\phi_2,\dots,\phi_n$ be given real numbers. Then for all $t > 0$

$$\int_{|x_1|\le|x_2|\le\dots\le|x_n|\le t} |x_1|^{\phi_1}|x_2|^{\phi_2}\dots|x_n|^{\phi_n}\, dx_1dx_2\dots dx_n < \infty,$$

if $d_k = k + \sum_{j=1}^{k} \phi_j > 0$ for $k = 1,\dots,n$.

Proof. It clearly suffices to consider the case $t = 1$. We proceed by induction on n. The lemma is obviously true for $n = 1$. Now suppose that the lemma holds for $n-1$ and that we are given $\phi_1,\dots,\phi_n$ satisfying the hypotheses of the lemma. Choose $\delta \ge 0$ such that $d_n-\delta > 0$ and $\phi_n-\delta \ne -1$. (If $\phi_n \ne -1$ we can take $\delta = 0$). Then the above integral (with $t = 1$) is less than

$$\int_{|x_1|\le|x_2|\le\dots\le|x_n|\le 1} |x_1|^{\phi_1}\dots|x_{n-1}|^{\phi_{n-1}}|x_n|^{\phi_n-\delta}\, dx_1\dots dx_n$$

$$= \int_{|x_1|\le|x_2|\le\dots\le|x_{n-1}|} |x_1|^{\phi_1}\dots|x_{n-1}|^{\phi_{n-1}} \int_{|x_{n-1}|\le|x_n|\le 1} |x_n|^{\phi_n-\delta}\, dx_n dx_1\dots dx_{n-1}$$

After evaluating the integral over x_n, we obtain

$$\frac{2}{\phi_n-\delta+1}\left\{\int_{|x_1|<|x_2|<\cdots<|x_{n-1}|\leq 1} |x_1|^{\phi_1}\cdots|x_{n-1}|^{\phi_{n-1}}\,dx_1\cdots dx_{n-1}\right.$$

$$\left.-\int_{|x_1|\leq|x_2|\leq\cdots\leq|x_{n-1}|\leq 1} |x_1|^{\phi_1}\cdots|x_{n-2}|^{\phi_{n-2}}|x_{n-1}|^{\phi_{n-1}+\phi_n-\delta+1}\,dx_1\cdots dx_n\right\}.$$

The induction hypothesis implies that the first integral in the braces is finite. To apply the induction hypothesis to the second integral, note that

$$(n-1) + \phi_1 + \ldots + \phi_{n-2} + (\phi_{n-1}+\phi_n-\delta+1) = n + \phi_1 + \ldots + \phi_n - \delta = d_n - \delta > 0.$$

Thus the second integral is finite, which completes the proof of Lemma 4.3. □

Lemma 4.4. Let σ be a permutation of $\{1,\ldots,m\}$ such that $d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\}) > 0$, $k = 1,\ldots,r$. Then

$$\int_{E_\sigma} P(x)dx < \infty.$$

Proof. According to Lemma 4.2 it suffices to show

$$\int_{E_\sigma} |L_{\tau_1}|^{\Delta_1}\cdots|L_{\tau_r}|^{\Delta_r}\,dx < \infty,$$

where $\Delta_1,\ldots,\Delta_r$ are as defined in Lemma 4.2. Define

$$E'_\sigma = \{x \in U_t\colon |L_{\tau_1}| \leq |L_{\tau_2}| \leq \ldots \leq |L_{\tau_r}| \leq Mt\}.$$

Then (4.1) and (4.2) imply that $E_\sigma \subset E'_\sigma$. Therefore the last integral is majorized by

$$\int_{E'_\sigma} |L_{\tau_1}|^{\Delta_1} \cdots |L_{\tau_r}|^{\Delta_r} dx \le C_3 \int_{|y_1| \le |y_2| \le \cdots \le |y_r| \le Mt} |y_1|^{\Delta_1} \cdots |y_r|^{\Delta_r} dy_1 \cdots dy_r,$$

where C_3 is a constant obtained by integrating over $n-r$ extraneous variables. Note that $\Delta_1,\ldots,\Delta_r$ satisfy

$$k + \sum_{i=1}^{k} \Delta_i = d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\}) > 0, \quad k = 1,\ldots,r.$$

Hence Lemma 4.4 follows from Lemma 4.3. □

Proof of Theorem 3.1. Suppose that the conditions of Theorem 3.1 hold. Let σ be a permutation of $\{1,\ldots,m\}$. Since $\{L_{\tau_1},\ldots,L_{\tau_k}\} \subset S$, $k = 1,\ldots,r$, we have $d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\}) > 0$, $k = 1,\ldots,r$. Thus we can use Lemma 4.4 to conclude that $\int_{E_\sigma} P(x)dx < \infty$. Theorem 3.1 follows because U_t is the union over σ of the sets E_σ. □

Proof of Theorem 3.2. Let σ be a permutation of $\{1,\ldots,m\}$. Define $A_{N,\sigma} = V_N \cap E_\sigma$. It suffices to show that

$$\int_{A_{N,\sigma}} P(x)dx = O(N^{\gamma\delta}).$$

As before we use the greedy algorithm to obtain a basis $B_\sigma = \{L_{\tau_1}, L_{\tau_2},\ldots,L_{\tau_r}\}$ for S. We have two cases to consider.

Case I) $B_\sigma \subset S_2$. In this case it follows from Lemma 4.4 that $\int_{E_\sigma} P(x)dx < \infty$. Since $A_{N,\sigma} \subset E_\sigma$ for all N, we have $\int_{A_{N,\sigma}} P(x)dx = O(1)$.

Case II) $B_\sigma \not\subset S_2$.

Let $\ell = \min\{k: L_{\tau_k} \in S_1\}$. From the definition of V_N it follows that $1 \le N^{\gamma\delta}|L_{\tau_\ell}|^\delta$, $x \in V_N$. Combining this with the result of Lemma 4.2 we obtain

$$\int_{A_{N,\sigma}} P(x)dx \leq C_1(Mt)^{C_2} N^{\gamma\delta} \int_{E_\sigma} |L_{\tau_1}|^{\Delta_1} |L_{\tau_2}|^{\Delta_2} \cdots |L_{\tau_\ell}|^{\Delta_\ell+\delta} |L_{\tau_{\ell+1}}|^{\Delta_{\ell+1}} \cdots |L_{\tau_r}|^{\Delta_r},$$

where $\Delta_1,\ldots,\Delta_r$ are defined as in Lemma 4.2. The above integral is majorized by

$$C_4 \int_{|y_1|\leq|y_2|\leq\cdots\leq|y_r|} |y_1|^{\Delta_1} |y_2|^{\Delta_2} \cdots |y_\ell|^{\Delta_\ell+\delta} |y_{\ell+1}|^{\Delta_{\ell+1}} \cdots |y_r|^{\Delta_r} dy_1 \ldots dy_r,$$

where C_4 is obtained by integrating over $n-r$ extraneous variables. Next we verify that this integral satisfies the hypotheses of Lemma 4.3. For $k < \ell$ we have $\{L_{\tau_1},\ldots,L_{\tau_k}\} \subset S_2$. Hence condition a) of the theorem implies that $k + \sum_{j=1}^k \Delta_j = d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\}) > 0$. On the other hand, if $k \geq \ell$ it follows from condition b) of the theorem that $k + \sum_{j=1}^k \Delta_j + \delta = d(P,\{L_{\tau_1},\ldots,L_{\tau_k}\}) + \delta > 0$. Thus the last integral is finite, completing the proof of Theorem 3.2. □

5. Counting powers

This section is devoted to "counting powers" in the function $P_\eta: R^{2p} \to R$ given by

$$P_\eta(x) = |x_2 + \dots + x_{2p}|^{\eta-1}|x_2|^{\eta-1}\dots|x_{2p}|^{\eta-1}|x_1|^{-\alpha}|x_1+x_2|^{-\beta}|x_1+x_2+x_3|^{-\alpha}$$
$$\dots|x_1 + \dots + x_{2p-1}|^{-\alpha}|x_1 + \dots + x_{2p}|^{-\beta},$$

where $\alpha < 1$, $\beta < 1$ and $0 < \eta < 1$. The results are stated in Propositions 5.1 and 5.2. Introduce the set of linear functionals on R^{2p}

$$T = \{x_2 + \dots + x_{2p},\ x_2, x_3, \dots, x_{2p},\ x_1,\ x_1+x_2, \dots, x_1 + \dots + x_{2p}\}.$$

For each $W \subset T$ we define the set $s\{W\}$ and the quantity $d(P_\eta, W)$ as in Section 3.

Proposition 5.1. Let $\alpha < 1$, $\beta < 1$ and let η satisfy $0 < \eta < 1$ and $\eta > \frac{(\alpha+\beta)}{2}$. If $W \subset T$ is an independent set such that $|W| = 2p-1$ and $W \subset \{x_2 + \dots + x_{2p}, x_2, x_3, \dots, x_{2p}\}$, then $d(P_\eta, W) = 2p\eta-1$.

Proof. It is clear that if W satisfies the conditions of Proposition 5.1 then $s\{w\} = \{x_2 + \dots + x_{2p}, x_2, x_3, \dots, x_{2p}\}$. Therefore $d(P_\eta, w) = (2p-1) + 2p(\eta-1) = 2p\eta-1$. □

Proposition 5.2. Let $\alpha < 1$, $\beta < 1$ and let η satisfy $0 < \eta < 1$ and $\eta > \frac{(\alpha+\beta)}{2}$. If $W \subset T$ is an independent set such that either $|W| \neq 2p-1$ or $w \not\subset \{x_2 + \dots + x_{2p}, x_2, x_3, \dots, x_{2p}\}$, then $d(P_\eta, W) > 0$.

The rest of this section is devoted to the proof of Proposition 5.2.

In proving that proposition we can restrict ourselves to considering sets $W \subset T$ which do not contain $x_2 + \dots + x_{2p}$. To see this, assume that $x_2 + \dots + x_{2p} \in W$. Suppose first that the set $s\{W\} \backslash s\{W \backslash x_2 + \dots + x_{2p}\}$ contains some functional L other than $x_2 + \dots + x_{2p}$. Then we consider the set W' which is W with $x_2 + \dots + x_{2p}$ replaced by L, that is $W' = W \cup \{L\} \backslash \{x_2 + \dots + x_{2p}\}$. Clearly, $x_2 + \dots + x_{2p} \notin W'$. Furthermore, W' has the same span and cardinality as W. Therefore $d(P_\eta, W') = d(P_\eta, W)$. On the other hand, suppose that there is no such L. In this case we put $W' = W \backslash \{x_2 + \dots + x_{2p}\}$. We have $|W'| = |W| - 1$ and $s\{W'\} = s\{W\} \backslash \{x_2 + \dots + x_{2p}\}$. Hence $d(P_\eta, W') = d(P_\eta, W) - 1 - (\eta - 1) = d(P_\eta, W) - \eta < d(P_\eta, W)$. Thus in either case there is a set W' which does not contain $x_2 + \dots + x_{2p}$ and satisfies $d(P_\eta, W') \leq d(P_\eta, W)$. Hence we can assume that W does not contain $x_2 + \dots + x_{2p}$.

In proving Proposition 5.2 we can also restrict ourselves to sets $W \subset T$ which satisfy $\{x_k, x_1 + \dots + x_k\} \not\subset W$, $k = 2, \dots, 2p$. For suppose that T does not satisfy this restriction. Let j be the largest k for which $\{x_k, x_1 + \dots + x_k\} \subset W$. Let $W' = W \cup \{x_1 + \dots + x_{j-1}\} \backslash \{x_1 + \dots + x_j\}$. Since the sets $\{x_j, x_1 + \dots + x_{j-1}\}$ and $\{x_j, x_1 + \dots + x_j\}$ have the same span and cardinality, it follows that $d(P_\eta, W') = d(P_\eta, W)$. It is clear that the largest value of k for which $\{x_k, x_1 + \dots + x_k\} \subset W'$ is at most $j-1$. After repeating this process at most $j-2$ more times we obtain a set W'' satisfying $d(P_\eta, W'') = d(P_\eta, W)$ and $\{x_k, x_1 + \dots + x_k\} \not\subset W'$, $k = 2, \dots, 2p$. Thus we can restrict ourselves to sets W which do not contain both x_k and $x_1 + \dots + x_k$.

We will assume from now on that $W \subset T$ satisfies both of the above restrictions. To describe the sets W which we will be considering, it is helpful to think of the elements of $T\backslash\{x_2 + \dots + x_{2p}\}$ arranged in columns as follows:

$$\begin{array}{c|c|c|c|c|}
 & x_2 & x_3 & & x_{2p} \\
x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & \cdots & x_1 + \dots + x_{2p}
\end{array} .$$

In the rest of this section we consider sets W which contain at most one element from each column. For any set $T' \subset T$ we say that T' contains the kth column if $x_k \in T'$ or $x_1 + \dots + x_k \in T'$.

The proof of Proposition 5.2 involves three lemmas.

Lemma 5.3. Suppose that W does not contain the kth column. Then $s\{W\}$ does not contain the kth column.

Proof. We prove that neither x_k nor $x_1 + \dots + x_k$ is in $s\{W\}$. We distinguish two cases.

Case I. There is no $j > k$ such that $x_1 + \dots + x_j \in W$. In this case the conclusion of the lemma is clear since no element of W contains the summand x_k.

Case II. There exists $j > k$ such that $x_1 + \dots + x_j \in W$.

Suppose that j is the smallest index with this property. Then the only elements of W which contain the summand x_k are among $\{x_1 + \dots + x_j, x_1 + \dots + x_{j+1}, \dots, x_1 + \dots + x_{2p}\}$. Since $x_j \notin W$ these are also the only elements of W which contain the summand x_j. Thus in any linear combination of the elements of W the summands x_k and x_j appear with the same coefficient. Hence neither x_k nor $x_1 + \dots + x_k$ can be linear combinations of elements of W. This completes the proof of Lemma 5.3. □

We now partition W into blocks of contiguous columns. Any two blocks are separated by at least one column not in W. Formally, we will say that a set $B \subset W$ is a <u>block of columns</u>, if there exist $\ell_B < r_B$ such that

1) W contains neither column $\ell_B - 1$ nor column $r_B + 1$.

2) B contains column ℓ_B through r_B and no other columns. With this definition we obtain a partition $W = \cup_{j=1}^{n} B_j$, where each B_j is a block of columns. We will assume that B_j is to the left of B_{j+1} for each j.

Define the function $Q_\eta(x) = P_\eta(x) \cdot |x_2 + \dots + x_{2p}|^{1-\eta}$. It is clear that

$$d(P_\eta, W) = \begin{cases} d(Q_\eta, W) & \text{if } x_2 + \dots + x_{2p} \notin s\{W\} \\ \eta - 1 + d(Q_\eta, W) & \text{if } x_2 + \dots + x_{2p} \in s\{W\} \end{cases}$$

Furthermore Lemma 5.3 implies that $d(Q_\eta, W) = \sum_{j=1}^{n} d(Q_\eta, B_j)$. Thus we have

$$d(P_\eta, W) = \begin{cases} \sum_{j=1}^{n} d(Q_\eta, B_j), & \text{if } x_2 + \dots + x_{2p} \notin s(W) \\ \eta - 1 + \sum_{j=1}^{n} d(Q_\eta, B_j) & \text{if } x_2 + \dots + x_{2p} \in s(W). \end{cases} \tag{5.1}$$

The next lemma is useful in determining the quantities $d(Q_\eta, B_j)$. A block of columns will be called <u>nonsingular</u> if it contains $x_1 + \dots + x_k$ for some $k \geq 1$.

<u>Lemma 5.4</u>. Let B be a nonsingular block of columns. Put $\ell = \ell_B$ and $r = r_B$. Let m be the smallest k satisfying $x_1 + \dots + x_k \in B$.

1) If $\ell \leq j < m$, then $x_j \in s(B)$ and $x_1 + \ldots + x_j \notin s(B)$.

2) $x_m \notin s(B)$ and $x_1 + \ldots + x_m \in s(B)$.

3) If $m < j \leq r$, then $x_j \in s(B)$ and $x_1 + \ldots + x_j \in s(B)$.

Proof.

1) Let $\ell \leq j < m$. Since $j < m$ we have $\{x_\ell, x_{\ell+1}, \ldots, x_j\} \subset B$. Suppose that $x_1 + \ldots + x_j \in s(B)$. The identity $x_1 + \ldots + x_{\ell-1} = (x_1 + \ldots + x_j) - x_\ell - x_{\ell+1} - \ldots - x_j$ implies that $x_1 + \ldots + x_{\ell-1} \in s(B)$. This contradicts Lemma 5.3. We conclude that $x_1 + \ldots + x_j \notin s(B)$.

2) The definition of m implies that $x_1 + \ldots + x_m \in B$. Suppose that $x_m \in s(B)$. We have

$$x_1 + \ldots + x_{\ell-1} = (x_1 + \ldots + x_m) - x_\ell - x_{\ell+1} - \ldots - x_m,$$

again contradicting Lemma 5.3.

3) This is proven by induction. It is clear that if $x_1 + \ldots + x_j \in s(B)$ and B contains column $j+1$, then $\{x_{j+1}, x_1 + \ldots + x_{j+1}\} \subset s(B)$. To start the induction off, note that $x_1 + \ldots + x_m \in s(B)$ and B contains column $m+1$. This completes the proof of Lemma 5.4. □

If B is a singular block of columns, then $B \subset \{x_2, x_3, \ldots, x_{2p}\}$ and therefore

$$d(Q_\eta, B) = |B| + |B|(\eta - 1) = |B|\eta > 0. \tag{5.2}$$

To determine $d(Q_\eta, B)$ for a nonsingular block, we need to take into account the parities of the integers m and r introduced in the statement of Lemma 5.4. This is done in the next lemma. First define

$$C_1 = (m-\ell)\eta + (r-m)[\eta - \frac{(\alpha+\beta)}{2}]$$

and

$$C_2 = (m-\ell)\eta + (r-m+1)[\eta - \frac{(\alpha+\beta)}{2}].$$

Note that under the conditions of Proposition 5.2 we have $C_1 \geq 0$ and $C_2 > 0$.

Lemma 5.5. Suppose that the conditions of Proposition 5.2 hold. Let B be a nonsingular block of columns.

1) If m and r are both odd, then

$$d(Q_\eta, B) = (1-\alpha) + C_1 \geq 1-\alpha > 0.$$

2) If m and r are both even, then

$$d(Q_\eta, B) = (1-\beta) + C_1 \geq 1-\beta > 0.$$

3) If m and r have different parities, then

$$d(Q_\eta, B) = (1-\eta) + C_2 > 1-\eta > 0.$$

Proof. Note that $d(Q_\eta, B)$ is equal to the cardinality of B plus the sum of the powers of all the elements of $s(B)\backslash\{x_2 + \dots + x_{2p}\}$. The cardinality of B contributes $(r-\ell) + 1$ to $d(Q_\eta, B)$.

According to Lemma 5.4, the set $s(B)\backslash\{x_2 + \dots + x_{2p}\}$ is equal to $W_1 \cup W_2$, where

$$W_1 = \{x_\ell, x_{\ell+1}, \ldots, x_{m-1}, x_{m+1}, x_{m+2}, \ldots, x_r\}$$

and

$$W_2 = \{x_1 + \ldots + x_m,\ x_1 + \ldots + x_{m+1}, \ldots, x_1 + \ldots + x_r\}.$$

Counting the powers associated with W_1 we obtain a contribution $(r-\ell)(\eta-1) = -(r-\ell) + (m-\ell)\eta + (r-m)\eta$.

Counting the powers associated with W_2 we obtain a contribution

$$\begin{cases} -\alpha - \frac{(r-m)}{2}(\alpha+\beta) & \text{if } m,r \text{ are both odd} \\ -\beta - \frac{(r-m)}{2}(\alpha+\beta) & \text{if } m,r \text{ are both even} \\ -\frac{(r-m+1)}{2}(\alpha+\beta) & \text{if } m,r \text{ have different parities.} \end{cases}$$

Summing the appropriate contributions and using the inequalities $\alpha < 1$, $\beta < 1$, $C_1 \geq 0$ and $C_2 \geq 0$ we obtain the results of Lemma 5.5. □

Proof of Proposition 5.2. Suppose that the conditions of Proposition 5.2 hold and that the independent subset W of T also satisfies the restrictions described above. (Namely, W does not contain $x_2 + \ldots + x_{2p}$ and $\{x_k, x_1 + \ldots + x_k\} \not\subset W$, $k = 2,\ldots,2p$.) Relation (5.1), relation (5.2) and Lemma 5.5 imply that $d(P_\eta, W) > 0$ if $x_2 + \ldots + x_{2p} \notin s(W)$. To complete the proof, assume that $x_2 + \ldots + x_{2p} \in s(W)$. This implies that $r_{B_n} = 2p$ (where B_n is the rightmost block of W), because the summand x_{2p} appears only in the 2pth column.

First we will show that B_n is nonsingular, that is, it contains $x_1 + \ldots + x_k$ for some $k \geq 1$. Put $\ell = \ell_{B_n}$. If $\ell = 1$, then $x_1 \in B_n$ and so B_n is nonsingular. If $\ell = 2$ and B_n is singular, then $W = B_n = \{x_2, \ldots, x_{2p}\}$, contradicting the assumptions of the proposition. If $\ell > 2$ and B_n is singular, then no element of W contains the summand $x_{\ell-1}$, contradicting the assumption that $x_2 + \ldots + x_{2p} \in s(W)$. Thus B_n must be nonsingular.

Next we will show that $\ell_{B_1} = 1$. Since B_n is nonsingular, Lemma 5.4 shows that $x_1 + \ldots + x_{2p} \in s(W)$. Since we have assumed that $x_2 + \ldots + x_{2p} \in s(W)$, it follows that $x_1 \in s(W)$. Thus we must have $\ell_{B_1} = 1$ in order to avoid contradicting Lemma 5.3.

To complete the proof, we distinguish two cases, according to whether W consists of a single block or more than one block.

Case I. $n = 1$.

In this case we have only one block B_1 satisfying $\ell_{B_1} = m_{B_1} = 1$ and $r_{B_1} = 2p$. Lemma 5.5 implies that $d(Q_\eta, B_1) = 1 - \eta + C_2$. According to (5.1), $d(P_\eta, W) = d(Q_\eta, B_1) + \eta - 1 = C_2 > 0$.

Case II. $n > 1$.

We again have $\ell_{B_1} = m_{B_1} = 1$. Thus either Part 1 or Part 3 of Lemma 5.5 applies. Hence $d(Q_\eta, B_1) \geq 1-\alpha$ or $d(Q_\eta, B_1) \geq 1-\eta$.

Since $r_{B_n} = 2p$ and B_n is nonsingular, either Part 2 or Part 3 of Lemma 5.5 applies to B_n. Thus $d(Q_\eta, B_n) \geq 1-\beta$ or $d(Q_\eta, B_n) \geq 1-\eta$.

The proof can now be completed as follows. According to (5.2) and Lemma 5.5, we have $d(Q_\eta, B_k) > 0$, $k = 1, \ldots, n$. Thus by (5.1),

$$d(P_\eta, W) = \eta - 1 + \sum_{j=1}^{n} d(Q_\eta, B_j)$$

$$\geq \eta - 1 + d(Q_\eta, B_1) + d(Q_\eta, B_n).$$

If $d(Q_\eta, B_1) \geq 1-\eta$, then $d(P_\eta, W) \geq d(Q_\eta, B_n) > 0$. Similarly, $d(P_\eta, W) > 0$ if $d(Q_\eta, B_n) \geq 1-\eta$. Therefore we can assume that $d(Q_\eta, B_n) \geq 1-\alpha$ and $d(Q_\eta, B_n) \geq 1-\beta$. Then $d(P_\eta, W) \geq \eta - 1 + (1-\alpha) + (1-\beta)$ $= 1 - \eta + 2[\eta - \frac{(\alpha+\beta)}{2}] > 1-\eta > 0$. This completes the proof of Proposition 5.2. □

6. Applications of power counting

In this section, we establish Propositions 6.1 and 6.2 which will be used in the proof of Theorem 1.

For each integer $N \geq 1$, define the function $h_N: R \to R$ by $h_N(x) = \min(\frac{1}{|x|}, N)$ and the function $f_N: R^{2p} \to R$ by

$$f_N(x) = h_N(x_2 + \ldots + x_{2p})h_N(x_2)h_N(x_3)\ldots h_N(x_{2p})|x_1|^{-\alpha}|x_1+x_2|^{-\beta}|x_1+x_2+x_3|^{-\alpha} \ldots |x_1 + \ldots + x_{2p-1}|^{-\alpha}|x_1 + \ldots + x_{2p}|^{-\beta} \qquad (6.1)$$

where $\alpha < 1$ and $\beta < 1$. Fix $t > 0$ and put $U_t = [-t,t]^{2p}$ and $V = \{x \in R^{2p}: |x_1| \leq |x_2|\}$. The following results are useful in studying the behavior of $\int_{[-\pi,\pi]^{2p}} f_N(x)dx$ as $N \to \infty$.

Proposition 6.1. Let $\alpha < 1$ and $\beta < 1$.

a) If $\alpha + \beta > 0$, then as $N \to \infty$,

$$\int_{U_t \cap V} f_N(x)dx = O(N^{p(\alpha+\beta)+\varepsilon})$$

for every $\varepsilon > 0$.

b) If $\alpha + \beta \leq 0$, then as $N \to \infty$,

$$\int_{U_t \cap V} f_N(x)dx = O(N^{\varepsilon})$$

for every $\varepsilon > 0$.

Proposition 6.2. Let $\alpha < 1$ and $\beta < 1$.

a) If $p(\alpha+\beta) < 1$, then

$$\lim_{t\to 0} \limsup_{N\to\infty} \frac{\int_{U_t} f_N(x)dx}{N} = 0.$$

b) If $p(\alpha+\beta) \geq 1$, then for every $\varepsilon > 0$

$$\lim_{N\to\infty} \frac{\int_{U_t} f_N(x)dx}{N^{p(\alpha+\beta)+\varepsilon}} = 0.$$

In order to prove Propositions 6.1 and 6.2 we need to put the problem into the framework described in Section 4. Choose η satisfying $0 < \eta < 1$. If $x \in R$ satisfies $|x| \geq \frac{1}{N}$ then we have

$$h_N(x) = \frac{1}{|x|} \leq \frac{1}{|x|} N^{\eta}|x|^{\eta} = N^{\eta}|x|^{\eta-1}.$$

On the other hand, if $|x| < \frac{1}{N}$ we have

$$h_N(x) = N^{\eta}N^{1-\eta} \leq N^{\eta}|x|^{\eta-1}.$$

Thus we have shown

$$h_N(x) \leq N^{\eta}|x|^{\eta-1}, \quad x \in R, \quad 0 < \eta < 1.$$

This implies

$$f_N(x) \leq N^{2p\eta} P_{\eta}(x), \quad x \in R^{2p}, \quad 0 < \eta < 1, \tag{6.2}$$

where, as in Section 5,

$$P_{\eta}(x) = |x_2 + \ldots + x_{2p}|^{\eta-1}|x_2|^{\eta-1}\ldots|x_{2p}|^{\eta-1}|x_1|^{-\alpha}|x_1+x_2|^{-\beta}|x_1+x_2+x_3|^{-\alpha}$$
$$\ldots|x_1 + \ldots + x_{2p}|^{-\beta}. \tag{6.3}$$

We will apply the lemmas of Section 4 to the function $P_\eta(x)$. To this end we introduce the linear functionals $L_j: R^{2p} \to R$ as follows:

$$L_1(x) = x_2 + x_3 + \dots + x_{2p}$$

$$L_2(x) = x_2$$

$$L_3(x) = x_3$$

$$\vdots$$

$$L_{2p}(x) = x_{2p}$$

$$L_{2p+1}(x) = x_1$$

$$L_{2p+2}(x) = x_1 + x_2$$

$$\vdots$$

$$L_{4p}(x) = x_1 + x_2 + \dots + x_{2p}.$$

As in Section 3, we put $T = \{L_1, \dots, L_{4p}\}$. With these definitions

$$P_\eta(x) = |L_1|^{\eta-1}|L_2|^{\eta-1}\dots|L_{2p}|^{\eta-1}|L_{2p+1}|^{-\alpha}|L_{2p+2}|^{-\beta}|L_{2p+3}|^{-\alpha}\dots|L_{4p}|^{-\beta}.$$

For $W \subset T$ define $d(P_\eta, W)$ as in Section 3. In this context Propositions 5.1 and 5.2 become

<u>Proposition 6.3.</u> Let $\alpha, \beta < 1$ and let η satisfy $0 < \eta < 1$ and $\eta > \frac{(\alpha+\beta)}{2}$. Let $W \subset T$ be an independent set.

a) If $|W| = 2p-1$ and $W \subset \{L_1, L_2, \dots, L_{2p}\}$, then $d(P_\eta, W) = 2p\eta - 1$.

b) If $|W| \neq 2p-1$ or $W \not\subset \{L_1, L_2, \dots, L_{2p}\}$, then $d(P_\eta, W) > 0$.

Remark. If $\alpha \leq 0$ and $\beta \leq 0$ in Part b of Proposition 6.1, we can choose $\alpha_0 > 0$ and $\beta_0 > 0$ satisfying $p(\alpha_0+\beta_0) < \varepsilon$ so that for all $x \in U_t$

$$|x_1|^{-\alpha}|x_1+x_2|^{-\beta}\cdots|x_1 + \cdots + x_{2p}|^{-\beta} \leq C|x_1|^{-\alpha_0}|x_1+x_2|^{-\beta_0}\cdots|x_1 + \cdots + x_{2p}|^{-\beta_0}$$

for some constant C. Therefore if Proposition 6.1 holds for α_0 and β_0 it holds for α and β. Thus in proving the proposition we can assume that $\alpha > 0$ or $\beta > 0$. A similar argument shows that we can also make this assumption in proving Proposition 6.2.

In the rest of this section we will assume that $\alpha < 1$, $\beta < 1$ and either $\alpha > 0$ or $\beta > 0$. (It will not matter which one is positive.)

In the proofs of Propositions 6.1 and 6.2 we use the notation introduced in the beginning of Section 3. For example, S is the set of L_j's which appear with negative exponents in $P_\eta(x)$. Note that the rank of T is equal to 2p. Since $\alpha > 0$ or $\beta > 0$, the rank of S is also equal to 2p. As in Section 4, let M be a constant such that

$$|L_j| \leq Mt, \quad x \in U_t, \quad j = 1,\ldots,4p. \tag{6.4}$$

Let σ be any permutation of $\{1,\ldots,4p\}$. We define

$$E_\sigma = \{x \in U_t: |L_{\sigma_1}| \leq \cdots \leq |L_{\sigma_{4p}}|\}.$$

As described in Section 4, we use the greedy algorithm to obtain a basis $B_\sigma = \{L_{\tau_1},\ldots,L_{\tau_{2p}}\}$ for S satisfying

$$|L_{\tau_1}| \leq |L_{\tau_2}| \leq \cdots \leq |L_{\tau_{2p}}|, \quad x \in E_\sigma. \tag{6.5}$$

Proof of Proposition 6.1. Fix $\varepsilon > 0$. Fix η satisfying $0 < \eta < 1$ and $\eta > \frac{(\alpha+\beta)}{2}$. In view of (6.2),

$$\int_{U_t \cap V} f_N(x)dx \leq N^{2p\eta} \int_{U_t \cap V} P_\eta(x)dx. \tag{6.6}$$

Our aim is to show

$$\int_{U_t \cap V} P_\eta(x)dx < \infty. \tag{6.7}$$

Both parts of Proposition 6.1 follow from (6.6) and (6.7). Indeed, to obtain Part a we note that

$$N^{2p\eta} = N^{p(\alpha+\beta)+2p[\eta - \frac{(\alpha+\beta)}{2}]}.$$

Since (under the conditions of Part a) we have $0 < \frac{\alpha+\beta}{2} < 1$, we can choose η to make the second term in the exponent smaller than ε. To obtain Part b, suppose $\alpha+\beta \leq 0$. Then we can choose η small enough to satisfy $2p\eta < \varepsilon$, proving Part b.

It is thus sufficient to establish (6.7). To do this it suffices to show that for each permutation σ

$$\int_{E_\sigma \cap V} P_\eta(x)dx < \infty. \tag{6.8}$$

Fix a permutation σ. To show (6.8), we distinguish two cases.

Case I) $\{L_{\tau_1},\dots,L_{\tau_{2p-1}}\} \not\subset \{L_1,\dots,L_{2p}\}$.

In this case Proposition 6.3b and Lemma 4.4 imply $\int_{E_\sigma} P_\eta(x)dx < \infty$.

Case II) $\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\} \subset \{L_1,\ldots,L_{2p}\}$.

In this case we can describe the partition $T = T_1 \cup T_2 \cup \ldots \cup T_{2p+1}$ introduced in Section 4 before the statement of Lemma 4.1. To do this we determine the sets $s\{L_{\tau_1},\ldots,L_{\tau_k}\}$, $k = 1,\ldots,2p$. These are:

$$s\{L_{\tau_1},\ldots,L_{\tau_k}\} = \{L_{\tau_1},\ldots,L_{\tau_k}\}, \quad k = 1,\ldots,2p-2,$$

$$s\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\} = \{L_1,\ldots,L_{2p}\},$$

$$s\{L_{\tau_1},\ldots,L_{\tau_{2p}}\} = T.$$

The first relation follows from the assumption $\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\} \subset \{L_1,\ldots,L_{2p}\}$. The second follows from the relation $L_1 = L_2 + \ldots + L_{2p}$. The third follows from the fact that the rank of $\{L_{\tau_1},\ldots,L_{\tau_{2p}}\}$ is equal to the rank of S, which is $2p$. Define

$$L_q = \{L_1,\ldots,L_{2p}\}\backslash\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\}.$$

Then the sets T_k are as follows:

$$T_k = \{L_{\tau_k}\}, \quad k = 1,\ldots,2p-2,$$

$$T_{2p-1} = \{L_1,\ldots,L_{2p}\}\backslash\{L_{\tau_1},\ldots,L_{\tau_{2p-2}}\} = \{L_{\tau_{2p-1}},L_q\},$$

$$T_{2p} = \{L_1,\ldots,L_{4p}\}\backslash\{L_1,\ldots,L_{2p}\} = \{L_{2p+1},\ldots,L_{4p}\},$$

and

$$T_{2p+1} = \emptyset.$$

The next step is to use this to establish

$$|L_{2p+1}| \le |L_q|, \quad x \in E_\sigma \cap V. \tag{6.9}$$

Since $L_q \in T_{2p-1} \cap S$, Lemma 4.1.b implies that $|L_{\tau_{2p-1}}| \leq |L_q|$ for $x \in E_\sigma$. Combining this with (6.5) we see that $|L_{\tau_1}| \leq |L_{\tau_2}| \leq \cdots \leq |L_{\tau_{2p-1}}| \leq |L_q|$ for $x \in E_\sigma$. Therefore on E_σ we have $L_q = \max\{|L_1|,|L_2|,\ldots,|L_{2p}|\}$. In particular $|L_2| \leq |L_q|$ on E_σ. For $x \in V$ we have $|L_{2p+1}| = |x_1| \leq |x_2| = |L_2|$. These last two inequalities imply (6.9).

Since

$$P_\eta(x) = |L_{\tau_1}|^{\eta-1}\cdots|L_{\tau_{2p-1}}|^{\eta-1}|L_q|^{\eta-1}|L_{2p+1}|^{-\alpha}|L_{2p+1}|^{-\beta}\cdots|L_{4p}|^{-\beta},$$

relation (6.9) implies that $P_\eta(x) \leq P'_\eta(x)$, $x \in E_\sigma$, where

$$P'_\eta(x) = |L_{\tau_1}|^{\eta-1}\cdots|L_{\tau_{2p-1}}|^{\eta-1}|L_{2p+1}|^{-\alpha+\eta-1}|L_{2p+2}|^{-\beta}\cdots|L_{4p}|^{-\beta}.$$

Hence (6.8) will follow if we show

$$\int_{E_\sigma} P'_\eta(x)dx < \infty \tag{6.10}$$

To show (6.10) we use Lemma 4.4. For $k \leq 2p-2$ we have $s\{L_{\tau_1},\ldots,L_{\tau_k}\} = \{L_{\tau_1},\ldots,L_{\tau_k}\}$, from which it follows that

$$d(P'_\eta,\{L_{\tau_1},\ldots,L_{\tau_k}\}) = k + k(\eta-1) = k\eta > 0, \quad k \leq 2p-2.$$

Since $s\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\} = \{L_1,\ldots,L_{2p}\}$ we have

$$d(P'_\eta,\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\}) = (2p-1) + (2p-1)(\eta-1) = (2p-1)\eta > 0.$$

Finally

$$d(P'_\eta,\{L_{\tau_1},\ldots,L_{\tau_{2p}}\}) = 2p + 2p(\eta-1) - p\alpha - p\beta = 2p[\eta - \frac{(\alpha+\beta)}{2}] > 0.$$

Thus (6.10) follows from Lemma 4.4, completing the proof of Proposition 6.1. □

Proof of Proposition 6.2. Let σ be a permutation of $\{1,\ldots,2p\}$. It suffices to show that the conclusions of the proposition hold with U_t replaced by E_σ. As in the proof of Proposition 6.1 we distinguish two cases.

Case I) $\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\} \not\subset \{L_1,\ldots,L_{2p}\}$.
In this case both Part a and Part b follow from (6.6), Proposition 5.2 and Lemma 4.4 by choosing η appropriately.

Case II) $\{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\} \subset \{L_1,\ldots,L_{2p}\}$.
1. Proof of part a:

The partition $T = T_1 \cup T_2 \cup \ldots \cup T_{2p}$ was described in the proof of Proposition 6.1. We saw there that

$$|L_{\tau_1}| \leq |L_{\tau_2}| \leq \ldots \leq |L_{\tau_{2p-1}}| \leq |L_q|, \quad x \in E_\sigma, \tag{6.11}$$

where

$$L_q = \{L_1,\ldots,L_{2p}\} \setminus \{L_{\tau_1},\ldots,L_{\tau_{2p-1}}\}.$$

Since $h_N(x) \leq N$, we have

$$f_N(x) \leq N^{2p}|L_{2p+1}|^{-\alpha}|L_{2p+2}|^{-\beta}\ldots|L_{4p}|^{-\alpha}, \quad x \in R^{2p}.$$

Since $T_{2p} = \{L_{2p+1},\ldots,L_{4p}\}$, it follows from Lemma 4.1 that there is a constant C such that

$$|L_{2p+1}|^{-\alpha}|L_{2p+2}|^{-\beta}\ldots|L_{4p}|^{-\beta} \leq C|L_{\tau_{2p}}|^{-p(\alpha+\beta)}, \quad x \in E_\sigma. \tag{6.12}$$

From these last two inequalities we obtain

$$f_N(x) \leq CN^{2p}|L_{\tau_{2p}}|^{-p(\alpha+\beta)}, \quad x \in E_\sigma. \tag{6.13}$$

Define the sets

$$G_{N,0} = E_\sigma \cap \{\frac{1}{N} \le |L_{\tau_1}|\},$$

$$G_{N,j} = E_\sigma \cap \{|L_{\tau_j}| \le \frac{1}{N} |L_{\tau_{j+1}}|\}, \quad j = 2,\ldots,2p-2$$

and

$$G_{N,2p-1} = E_\sigma \cap \{|L_{\tau_{2p-1}}| \le \frac{1}{N}\}.$$

Because of (6.11) it is clear that

$$E_\sigma = G_{N,0} \cup G_{N,1} \cup \ldots \cup G_{N,2p-1}.$$

Thus it suffices to show that the conclusion of Part a holds with U_t replaced by $G_{N,j}$, $j = 0,\ldots,2p$. Define also the sets

$$K_{N,j} = \{|L_{\tau_k}| \le \frac{1}{N}, \quad k = 1,\ldots,j\} \cap \{\frac{1}{N} \le |L_{\tau_k}| \le Mt, \quad k = j+1,\ldots,2p-1\}$$
$$\cap \{|L_{\tau_{2p}}| \le Mt\}.$$

In view of (6.11) we have $G_{N,j} \subset K_{N,j}$. We now distinguish two subcases according to whether $j = 2p-1$ or not.

Subcase 1) $j = 2p-1$. From (6.13) we obtain

$$\int_{G_{N,2p-1}} f_N(x)dx \le CN^{2p} \int_{G_{N,2p-1}} |L_{\tau_{2p}}|^{-p(\alpha+\beta)} \le CN^{2p} \int_{K_{N,2p-1}} |L_{\tau_{2p}}|^{-p(\alpha+\beta)}.$$

where on $K_{N,2p-1}$, we have $|L_{\tau_k}| \le \frac{1}{N}$ for $k = 1,\ldots,2p-1$ and $|L_{\tau_{2p}}| \le Mt$. Since the vectors L_{τ_k} form a basis, we see, on making the appropriate change of variable, that the right hand side is majorized by

$$C'N^{2p}[\int_{-1/N}^{1/N} dy]^{2p-1} \int_{-Mt}^{Mt} |z|^{-p(\alpha+\beta)}dz = C''Nt^{1-p(\alpha+\beta)}.$$

Therefore

$$\limsup_N \frac{\int_{G_{N,2p-1}} f_N(x)dx}{N} \leq C''t^{1-p}(\alpha+\beta),$$

which implies the conclusion of Part a.

Subcase 2) $j < 2p-1$. For $x \in G_{N,j}$ we have

$$h_N(L_{\tau_k}) = N, \quad k = 1,\ldots,j,$$

$$h_N(L_{\tau_k}) = |L_{\tau_k}|^{-1}, \quad k = j+1,\ldots,2p-1,$$

and, by (6.11)

$$h_N(L_q) = |L_q|^{-1} \leq |L_{\tau_{2p-1}}|^{-1}.$$

These facts in combination with (6.12) yield

$$f_N(x) \leq CN^j |L_{\tau_{j+1}}|^{-1} \cdots |L_{\tau_{2p-2}}|^{-1} |L_{\tau_{2p-1}}|^{-2} |L_{\tau_{2p}}|^{-p(\alpha+\beta)}, \quad x \in G_{N,j}.$$

We do not yet integrate because $\int_{1/N}^{Mt} |L|^{-1} dL$ would yield a logarithm. We shall first redistribute powers. According to (6.11), we have $|L_{\tau_k}| \leq |L_{\tau_{2p-1}}|$, $k = j+1,\ldots,2p-1$, and thus $f_N(x) \leq g_N(x)$, where

$$g_N(x) = CN^j \left\{ \prod_{k=j+1}^{2p-1} |L_{\tau_k}|^{-1 - \frac{1}{2p-1-j}} \right\} |L_{\tau_{2p}}|^{-p(\alpha+\beta)}, \quad x \in G_{N,j}.$$

Hence

$$\int_{G_{N,j}} f_N(x)dx \leq \int_{K_{N,j}} g_N(x)dx.$$

This last integral is majorized by

$$C''N^j\Big[\int_{-1/N}^{1/N} dy\Big]^j \left[\int_{1/N\le|w|\le Mt} w^{-\{1+\frac{1}{2p-1-j}\}}\,dw\right]^{2p-1-j} \int_{-Mt}^{Mt} |z|^{-p(\alpha+\beta)}dz.$$

The first integral in brackets is $O(N^{-1})$. The second is $O(N^{\frac{1}{2p-1-j}})$. Under the hypothesis of Part a the whole expression is $O(N)t^{1-p(\alpha+\beta)}$. This concludes the proof of Part a.

2. Proof of Part b:

Fix $\varepsilon > 0$. Under the conditions of Part b we can choose η satisfying $0 < \eta < 1$, $\eta > \frac{(\alpha+\beta)}{2}$ and $1 < 2p\eta < p(\alpha+\beta) + \varepsilon$. According to Proposition 6.3 we have $d(P_\eta,W) > \min(2p\eta-1,0)$ for every independent set $W \subset T$. Since $2p\eta-1 > 0$, we conclude that $d(P_\eta,W) > 0$ for all independent sets $W \subset T$. Theorem 3.1 now implies that $\int_{U_t} P_\eta(x)dx < \infty$. From (6.2) we obtain $\int_{U_t} f_N(x)dx \le N^{2p\eta} \int_{U_t} P_\eta(x)dx$. Since $2p\eta < p(\alpha+\beta)+\varepsilon$, this is $O(N^{p(\alpha+\beta)+\varepsilon})$ as desired. This completes the proof of Proposition 6.2. □

7. Proof of Theorem 1

The proof requires a lemma in addition to Propositions 6.1 and 6.2. We use the notation introduced in Section 2 prior to the statement of Theorem 1. Fix $p \geq 1$ and note

$$\mathrm{Tr}(R_N A_N)^p = \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} r_{j_1-j_2} a_{j_2-j_3} r_{j_3-j_4} \cdots a_{j_{2p}-j_1}$$

$$= 2^{2p} \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} \Big[\int_0^\pi \cdots \int_0^\pi e^{i(j_1-j_2)y_1} e^{i(j_2-j_3)y_2} \cdots e^{i(j_{2p}-j_1)y_{2p}}$$

$$\cdot f(y_1)g(y_2)f(y_3)\cdots g(y_{2p})dy_1\cdots dy_{2p}\Big]$$

$$= 2^{2p} \int_{[0,\pi]^{2p}} P_N(y)Q(y)dy, \qquad (7.1)$$

where

$$P_N(y) = \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} e^{i(j_1-j_2)y_1} e^{i(j_2-j_3)y_2} \cdots e^{i(j_{2p}-j_1)y_{2p}}$$

and

$$Q(y) = f(y_1)g(y_2)f(y_3)\cdots g(y_{2p}).$$

To state the lemma, introduce the diagonal

$$D = \{y \in [0,\pi]^{2p}: y_1 = y_2 = \cdots = y_{2p}\}.$$

Let μ be the measure on $[0,\pi]^{2p}$ which is concentrated on D and satisfies $\mu\{y: a \leq y_1 = y_2 = \cdots = y_{2p} \leq b\} = b-a$ for all $0 \leq a \leq b \leq \pi$. Thus μ is Lebesgue measure on D, normalied so that $\mu(D) = \pi$.

Lemma 7.1. Define the measure μ_N on $[0,\pi]^{2p}$ by

$$\mu_N(E) = \frac{1}{\pi^{2p-1}N} \int_E P_N(y)dy, \quad E \subset [0,\pi]^{2p}.$$

Then μ_N converges weakly to μ as $N \to \infty$.

Proof. Since $[0,\pi]^{2p}$ is compact, it suffices to show that the Fourier coefficients of μ_N converge to those of μ. Fixing integers $n_1, n_2, \ldots, n_{2p}$, the corresponding Fourier coefficient of μ is

$$\int_{[0,\pi]^{2p}} e^{i(n_1 y_1 + \ldots + n_{2p} y_{2p})} d\mu(y) = \int_0^\pi e^{i(n_1 + \ldots + n_{2p})x} dx = \begin{cases} 0 & \text{if } \sum_{j=1}^{2p} n_j \neq 0 \\ \pi & \text{if } \sum_{j=1}^{2p} n_j = 0. \end{cases}$$

The corresponding Fourier coefficient of μ_N is

$$C_N = C_N(n_1, n_2, \ldots, n_p) = \int_{[0,\pi]^{2p}} e^{i(n_1 y_1 + \ldots + n_{2p} y_{2p}]} d\mu_N(y)$$

$$= \frac{1}{(\pi)^{2p-1}N} \sum_{j_1=0}^{N-1} \cdots \sum_{j_{2p}=0}^{N-1} \left\{ \int_0^\pi e^{i[n_1+j_1-j_2]y_1} dy_1 \right.$$

$$\left. \cdot \int_0^\pi e^{[n_2+j_2-j_3]y_2} dy_2 \cdots \int_0^\pi e^{i[n_{2p}+j_{2p}-j_1]y_{2p}} dy_{2p} \right\}. \quad (7.2)$$

Fix $j_1, \ldots, j_{2p}$. In order for the expression in braces to be nonzero we must have

$$\begin{aligned} n_1 &= -(j_1-j_2) \\ n_2 &= -(j_2-j_3) \\ &\vdots \\ n_{2p-1} &= -(j_{2p-1}-j_{2p}) \\ n_{2p} &= -(j_{2p}-j_1). \end{aligned} \tag{7.3}$$

But then $n_1 + \ldots + n_{2p} = 0$. Thus if $n_1 + \ldots + n_{2p} \neq 0$ each of the summands in (7.2) is equal to 0. Therefore $C_N = 0$ if $n_1 + \ldots + n_{2p} \neq 0$.

Suppose $n_1 + \ldots + n_{2p} = 0$. Then each summand in (7.2) equals 0 or π^{2p}. When the summand equals π^{2p}, the indices $j_1,\ldots,j_{2p}$ satisfy (7.3), which implies

$$\begin{aligned} j_2 &= j_1 + (n_1) \\ j_3 &= j_1 + (n_1+n_2) \\ &\vdots \\ j_{2p} &= j_1 + (n_1 + \ldots + n_{2p-1}). \end{aligned} \tag{7.4}$$

Define

$$M = \max\{n_1 + \ldots + n_k: \ k = 1,\ldots,2p-1\}, \quad M^+ = \max(M,0),$$

$$m = \min\{n_1 + \ldots + n_k: \ k = 1,\ldots,2p-1\}, \quad \text{and} \quad m^+ = \max(m,0).$$

Fix j_1 satisfying $0 \leq j_1 \leq N-1$ and determine $j_2,\ldots,j_{2p}$ according to (7.4). In order for the inequalities $0 \leq j_k \leq N-1$, $k = 2,\ldots,2p$ to be satisfied we must have $j_1 \leq N-1-M$ and $j_1 \geq m$. Thus the sum in (7.2) is equal to

$$\sum_{j_1=m^+}^{N-1-M^+} \pi^{2p} = (N - M^+ - m^+)\pi^{2p}.$$

Therefore

$$C_N = \frac{\pi(N - M^+ - m^+)}{N},$$

which tends to π as $N \to \infty$. This completes the proof of Lemma 7.1. □

Proof of Theorem 1. We must evaluate the asymptotic behavior of

$$\int_{[0,\pi]^{2p}} P_N(y)Q(y)dy.$$

Introduce the sets

$$W_k = \left\{y \in R^{2p}: |y_k| \leq \frac{|y_{k+1}|}{2}\right\}, \quad k = 1,\ldots,2p-1,$$

$$W_{2p} = \left\{y \in R^{2p}: |y_{2p}| \leq \frac{|y_1|}{2}\right\},$$

and

$$W = W_1 \cup W_2 \cup \ldots \cup W_{2p}.$$

For each $0 < t \leq \pi$ define

$$B_t = [0,t]^{2p}.$$

The singularities of Q occur on W and on B_t for small t. We shall divide the domain of integration B_π into three parts as follows. Let

$$E_t = B_\pi \backslash \{W \cup B_t\},$$

$$F_t = B_t \backslash W,$$

and

$$G = B_\pi \cap W.$$

For each $0 < t \leq \pi$, the sets E_t, F_t and G are disjoint and satisfy $B_\pi = E_t \cup F_t \cup G$.

According to (7.1), Part a of the theorem will be proven if we show that $p(\alpha+\beta) < 1$ implies

$$\lim_{N\to\infty} \frac{\int_{E_t} P_N Q}{N} = \pi^{2p-1} \int_t^\pi [f(x)g(x)]^p dx, \quad 0 < t \le 1, \tag{7.5}$$

$$\lim_{t\to 0} \limsup_{N\to\infty} \frac{\int_{F_t} P_N Q}{N} = 0, \tag{7.6}$$

and

$$\lim_{N\to\infty} \frac{\int_G P_N Q}{N} = 0. \tag{7.7}$$

To prove (7.6) it is enough to show that when $p(\alpha+\beta) < 1$

$$\lim_{t\to 0} \limsup_{N\to\infty} \frac{\int_{B_t} P_N Q}{N} = 0. \tag{7.8}$$

Since $G = \cup_{k=1}^{2p} [W_k \cap B_\pi]$, relation (7.7) will hold if $p(\alpha+\beta) < 1$ implies

$$\lim_{N\to\infty} \frac{\int_{B_\pi \cap W_k} P_N Q}{N} = 0, \quad k = 1,\ldots,2p. \tag{7.9}$$

From the definitions of P_N and Q it is clear that

$$\int_{B_\pi \cap W_1} P_N Q = \int_{B_\pi \cap W_3} P_N Q = \ldots = \int_{B_\pi \cap W_{2p-1}} P_N Q$$

and

$$\int_{B_\pi \cap W_2} P_N Q = \int_{B_\pi \cap W_4} P_N Q = \ldots = \int_{B_\pi \cap W_{2p}} P_N Q.$$

Because of the symmetry between α and β in the hypotheses of the theorem, it is clear that if we prove that $p(\alpha+\beta) < 1$ implies

$$\lim_{N\to\infty} \frac{\int_{B_\pi \cap W_1} P_N Q}{N} = 0, \tag{7.10}$$

we will have also established

$$\lim_{N\to\infty} \frac{\int_{B_\pi \cap W_2} P_N Q}{N} = 0.$$

Thus (7.9) will follow from (7.10).

In conclusion, Part a of the theorem will be proven if we show that $p(\alpha+\beta) < 1$ implies (7.5), (7.8) and (7.10).

To prove Part b, we must show that for $p(\alpha+\beta) \geq 1$

$$\lim_{N\to\infty} \int_{B_\pi} P_N Q = o(N^{p(\alpha+\beta)+\varepsilon}) \quad \text{for every } \varepsilon > 0. \tag{7.11}$$

We start with relation (7.5) and show that it holds in fact for all real values of α and β. We begin by showing that Q is bounded on E_t. Let $y \in E_t$. Since E_t is in the complement of B_t, there is some k such that $y_k > t$. Since E_t is also in the complement of W_j, $j = 1,\ldots,2p$, we have $y_j > \frac{y_{j+1}}{2}$, $j = 1,\ldots,2p-1$ and $y_{2p} > \frac{y_1}{2}$. Thus we have

$$y_{k+1} > \frac{y_{k+2}}{2} > \frac{y_{k+3}}{4} > \ldots > \frac{y_{2p}}{2^{2p-k}} > \frac{y_1}{2^{2p-k+1}} > \ldots > \frac{y_k}{2^{2p-1}} > \frac{t}{2^{2p-1}} .$$

Therefore $y_j > \frac{t}{2^{2p-1}}$, $j = 1,\ldots,2p$, for $y \in E_t$. Under the conditions of the theorem, f and g are bounded on $[\frac{t}{2^{2p-1}}, \pi]$. Hence Q is bounded on E_t. Since $E_t \cap D = \{y\colon t < y_1 = y_2 = \ldots = y_{2p} \leq \pi\}$, relation (7.5) follows from Lemma 7.1.

Before proving (7.8), (7.10) and (7.11) we need to obtain majorants for P_N and Q. We have

$$P_N(y) = \Big(\sum_{j_1=0}^{N-1} e^{i(y_1-y_{2p})j_1}\Big)\Big(\sum_{j_2=0}^{N-1} e^{i(y_2-y_1)j_2}\Big)\dots\Big(\sum_{j_{2p}=0}^{N-1} e^{i(y_{2p}-y_{2p-1})j_{2p}}\Big)$$

$$= h_N^*(y_1-y_{2p})h_N^*(y_2-y_1)\dots h_N^*(y_{2p}-y_{2p-1}),$$

where

$$h_N^*(x) = \sum_{j=0}^{N-1} e^{ixj}.$$

Since $h_N^*(x) = \frac{1-e^{iNx}}{1-e^{ix}}$ for $x \neq 0$, $|1-e^{iNx}| \leq 2$ and $|1-e^{ix}| \geq 2^{1/2}\pi^{-1}|x|^{-1}$ for $|x| \leq \pi$, we obtain $h_N^*(x) \leq 2^{1/2}\pi|x|^{-1}$ for $|x| \leq \pi$. Since $|h_N^*(x)| \leq N$, this implies that $h_N^*(x) \leq 2^{1/2}\pi h_N(x)$ for $|x| \leq \pi$ where, as in Section 6, $h_N(x) = \min(N, \frac{1}{|x|})$. Thus

$$P_N(y) \leq (\sqrt{2}\pi)^{2p}P_N'(y), \quad y \in B_\pi, \tag{7.12}$$

where

$$P_N'(y) = h_N(y_1-y_{2p})h_N(y_2-y_1)\dots h_N(y_{2p}-y_{2p-1}).$$

For fixed $\delta > 0$ let $\alpha_0 = \alpha + \delta$ and $\beta_0 = \beta + \delta$. It is clear that under the hypotheses of the theorem $Q(y)$ is at most a constant times $Q'(y)$, where

$$Q'(y) = |y_1|^{-\alpha_0}|y_2|^{-\beta_0}|y_3|^{-\alpha_0}\dots|y_{2p}|^{-\beta_0}.$$

Because of (7.12), the proof of the theorem can be completed by showing that (7.8), (7.10) and (7.11) hold with P_N replaced by P_N' and Q replaced by Q'. Now make the change of variable $x_1 = y_1$, $x_k = y_k - y_{k-1}$, $k = 2,\dots,2p$.

The Jacobian of this transformation is 1 and the integrand $P_N'(y)Q'(y)$ becomes

$$h_N(x_2 + \dots + x_{2p})h_N(x_2)h_N(x_3)\dots h_N(x_{2p})Q'(x_1, x_1+x_2, \dots, x_1 + \dots + x_{2p}) = f_n(x)$$

where $f_N(x)$ is defined by (6.1) with α and β replaced by α_0 and β_0.

If $y \in B_t$ then $x \in U_t = [-t,t]^{2p}$ because $x_1 = y_1$ and $|x_k| \le \max(y_k, y_{k-1})$, $k = 2,\dots,2p$. Therefore for each t,

$$\int_{B_t} P_N'Q' \le \int_{U_t} f_N(x)dx. \tag{7.13}$$

If $y \in B_\pi \cap W_1$, then $x \in V = \{x \in R^{2p} : |x_1| \le |x_2|\}$ and therefore

$$\int_{B_\pi \cap W_1} P_N'Q' \le \int_{U_\pi \cap V} f_N(x)dx. \tag{7.14}$$

We can now apply Propositions 6.1 and 6.2. Assume first that $p(\alpha+\beta) \ge 1$. Choose $\delta > 0$. Then $p(\alpha_0+\beta_0) > 1$. Therefore Part b of Proposition 6.2 implies that

$$\int_{U_\pi} f_N(x)dx = O(N^{p(\alpha_0+\beta_0)+\varepsilon}) = O(N^{p(\alpha+\beta)+2p\delta+\varepsilon}).$$

Since δ can be made arbitrarily small, (7.11) follows from (7.13).

Now suppose $p(\alpha+\beta) < 1$. To prove (7.8), choose $\delta > 0$ such that $p(\alpha_0+\beta_0) < 1$. Then (7.8) follows from (7.13) and Part a of Proposition 6.2. To prove (7.10) we need to consider two cases. If $\alpha+\beta < 0$ choose $\delta > 0$ such that $\alpha_0+\beta_0 < 0$. Then (7.10) is a consequence of (7.14) and Part b of Proposition 6.1. If $\alpha+\beta \ge 0$, choose δ such that $p(\alpha_0+\beta_0) < 1$ and use Part a of Proposition 6.1. This completes the proof of Theorem 1. □

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