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CENTRAL LIMIT THEORY FOR THE NUMBER OF SEEDS IN A GROWTH MODEL IN $\mathbb{R}^d$ WITH INHOMOGENEOUS POISSON ARRIVALS

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A Poisson point process Ψ in d-dimensional Euclidean space and in time is used to generate a birth–growth model: *seeds* are *born* randomly at locations x_i in $\mathbb{R}^d$ at times $t_i \in [0, \infty)$. Once a seed is born, it begins to create a cell by growing radially in all directions with speed $v > 0$. Points of Ψ contained in such cells are discarded, that is, *thinned*. We study the asymptotic distribution of the number of seeds in a region, as the volume of the region tends to infinity. When $d = 1$, we establish conditions under which the evolution over time of the number of seeds in a region is approximated by a Wiener process. When $d \geq 1$, we give conditions for asymptotic normality. Rates of convergence are given in all cases.

1. Introduction. Consider the following spatial birth–growth model in $\mathbb{R}^d$. *Seeds* are *born* (or *formed*) randomly at locations x_i at time t_i, $i = 1, 2, \dots,$ according to a spatial–temporal point process $\Psi \equiv \{(x_i, t_i) \in \mathbb{R}^d \times [0, \infty)\}$. Once a seed is born, it immediately generates a cell by growing radially in all directions with a constant speed $v > 0$. The space occupied by cells is regarded as *covered*. Cells and new seeds continue to grow and form, respectively, only in uncovered space in $\mathbb{R}^d$.

The point process Ψ is assumed to be a Poisson process with intensity measure $l \times \Lambda$, where l is the Lebesgue measure in $\mathbb{R}^d$, while Λ is an arbitrary locally finite measure on $[0, \infty)$ such that $\Lambda([0, \infty)) > 0$ and

$$\mu \equiv \int_0^\infty \exp\left\{-\int_0^t \omega_d v^d (t-u)^d \Lambda(du)\right\}\Lambda(dt) < \infty, \tag{1.1}$$

where $\omega_d = \sqrt{\pi^d}/\Gamma(1 + d/2)$ is the volume of a unit ball in $\mathbb{R}^d$. It will be shown in the next section that μ is the intensity of the seeds formed in $\mathbb{R}^d$. Throughout the paper we use $\Lambda(t)$ to denote $\Lambda([0, t])$.

Such a birth–growth process was first suggested and studied by Kolmogorov (1937) in the case $d = 2$ to model crystal growth [see Chiu (1995, 1996) for details of subsequent developments]. Interestingly, special cases of this birth–growth process when $d = 1$ have found applications in several different biological contexts [see Holst, Quine and Robinson (1996) and the references therein].

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Denote by Φ the spatial–temporal point process of the seeds formed, which is a dependently thinned version of the Poisson process Ψ. For ease of presentation, we consider Ψ and Φ as both random sets of points in $\mathbb{R}^d \times [0, \infty)$ and random measures defined on the Borel σ-algebra of $\mathbb{R}^d \times [0, \infty)$. Denote by ξ_z the random variables $\Phi(\{z + [0, 1]^d\} \times [0, \infty))$, where $z \in \mathbb{Z}^d$ and $\{z + [0, 1]^d\} = \{z + x\colon x \in [0, 1]^d\}$. Then $\{\xi_z\colon z \in \mathbb{Z}^d\}$ is a real-valued stationary random field. It is stationary because Ψ is spatially homogeneous, and so is Φ.

For z_1 and z_2 in $\mathbb{Z}^d$, let $d(z_1, z_2) = \max_{1\le i\le d} |z_1(i) - z_2(i)|$, where $z(i)$, $1 \le i \le d$, are the components of z. For $\Gamma \subset \mathbb{Z}^d$, denote by $\#(\Gamma)$ the number of elements in Γ and by $\partial\Gamma$ the set $\{z \in \Gamma\colon$ there exists $z' \notin \Gamma$ such that $d(z, z') = 1\}$. Let $\Gamma_n \uparrow \mathbb{Z}^d$ be a fixed sequence of finite subsets of $\mathbb{Z}^d$ satisfying the regularity condition that $\lim_{n\to\infty} \#(\partial\Gamma_n)/\#(\Gamma_n) = 0$. It implies that the sequence $\{\Gamma_n\}$ does not increase in only one direction, except in the case $d = 1$. Define S_n to be $\sum_{z\in\Gamma_n}(\xi_z - \mu)$ for each $n \in \mathbb{N}$. Let $S_0 = 0$.

Quine and Robinson (1990) established asymptotic normality for the number of seeds in the case $d = 1$ with a homogeneous arrival rate. Their method was extended to cover more general arrival regimes by Chiu (1996). Holst, Quine and Robinson (1996) proved results similar to Chiu's by considering an associated Markov process. In this paper we use a completely different method, based on mixing properties, to establish asymptotic normality in an arbitrary dimension $d \ge 1$ for a very general class of Λ. In particular, when $d = 1$, we prove the functional central limit theorem for S_n; that is, after suitable normalization and linear interpolation, S_n behaves asymptotically like a Brownian motion. Rates of convergence are also discussed.

2. Moments. Let $\Xi(\Psi, t)$ denote the random region in $\mathbb{R}^d \times [0, \infty)$ which is covered just before time t by the Ψ-generated birth–growth process.

For each point (x, t) in Ψ,

$$\{(x, t) \notin \Xi(\Psi, t)\} = \{(x, t) \notin \Xi(\Psi \setminus \{(x, t)\}, t)\} = \{(x, t) \in \Phi\},$$

because the first two events imply that at time t the position x has not yet been covered by the Ψ-generated birth–growth process, and consequently a seed is formed at (x, t). Therefore, we have

$$\mathbf{E}[\xi_z] = \mathbf{E}\Bigg[\sum_{(x,\, t)\in\Psi(\{z+[0,\,1]^d\}\times[0,\,\infty))} \mathbf{1}((x, t) \notin \Xi(\Psi, t))\Bigg],$$

where $\mathbf{1}(\cdot)$ denotes the indicator function. By Mecke [(1967), Satz 3.1] or Møller [(1992), equation (3.1)],

$$\begin{aligned}\mathbf{E}[\xi_z] &= \int_0^\infty \int_{z+[0,1]^d} \mathbf{E}\big[\mathbf{1}((x, t) \notin \Xi(\Psi \cup \{(x, t)\}, t))\big] l(dx)\Lambda(dt)\\ &= \int_0^\infty \int_{[0,\,1]^d} \mathbf{E}\big[\mathbf{1}((x, t) \notin \Xi(\Psi, t))\big] l(dx)\Lambda(dt).\end{aligned}$$

Note that each $(x, t) \in \Psi$ does not belong to $\Xi(\Psi, t)$ if and only if

$$\Psi(\{(y, u): \|y - x\| \le v(t - u),\ 0 \le u \le t\}) = 0, \tag{2.1}$$

where $\|\cdot\|$ is the Euclidean distance. Thus,

$$\mathbf{E}[\xi_z] = \int_0^\infty \exp\left\{-\int_0^t \omega_d v^d (t-u)^d \Lambda(du)\right\}\Lambda(dt) = \mu,$$

where μ has been assumed to be finite in condition (1.1).

By observing that

$$\{(x_1, t_1) \notin \Xi(\Psi \cup \{(x_1, t_1), (x_2, t_2)\}, t)\} \subseteq \{(x_1, t_1) \notin \Xi(\Psi, t)\}$$

and using Møller [(1992), equation (3.1)], we obtain

$$\mathbf{E}[\xi_z(\xi_z - 1)\cdots(\xi_z - j)] \le \mu^{j+1} < \infty \quad \text{for } j = 1, 2, 3, \dots. \tag{2.2}$$

Thus, $\mathbf{E}[\xi_z^j] < \infty$ for each positive integer j. Let $\Gamma_n + [0, 1]^d = \{z + [0, 1]^d: z \in \Gamma_n\}$. Using Møller [(1992), equation (3.1)] again, we have

$$\begin{aligned}
&\mathbf{E}\left[\sum_{z\in\Gamma_n} \xi_z\left(\sum_{z\in\Gamma_n}\xi_z - 1\right)\right] \\
&\quad = \mathbf{E} \sum_{(x_i, t_i)\in\Psi(\{\Gamma_n+[0,1]^d\}\times[0,\infty)),\, i=1,2,\, x_1\ne x_2} \mathbf{1}((x_1, t_1) \notin \Xi(\Psi, t_1)) \\
&\qquad\qquad \times \mathbf{1}((x_2, t_2) \notin \Xi(\Psi, t_2)) \\
&\quad = \int_0^\infty \int_{\Gamma_n+[0,1]^d} \int_0^\infty \int_{\|x_1-x_2\|>v|t_2-t_1|,\, x_2\in\Gamma_n+[0,1]^d} \exp\{-\triangle(t_1) - \triangle(t_2)\} \\
&\qquad\qquad \times \exp\left\{\triangle\left(\frac{v(t_1+t_2) - \|x_1 - x_2\|}{2v}\right)\right\} l(dx_2)\Lambda(dt_2) l(dx_1)\Lambda(dt_1),
\end{aligned} \tag{2.3}$$

where $\triangle(t) = \int_0^{t\vee 0} \omega_d v^d (t-u)^d \Lambda(du)$ and $x \vee y = \max(x, y)$.

Suppose X_1 and X_2 are two independent uniformly distributed points in $\Gamma_n + [0, 1]^d$. Denote by f_n the density of $Y \equiv \|X_1 - X_2\|$ and let $r_n = \sup\{y: f_n(y) > 0\}$. From (2.3), we have

$$\begin{aligned}
&\mathbf{E}[S_n(S_n - 1)] + \#(\Gamma_n)^2\mu^2 - \#(\Gamma_n)\mu \\
&\quad = \#(\Gamma_n)^2 \int_0^\infty \int_0^\infty \int_{y>v|t_1-t_2|} \exp\left\{-\triangle(t_1) - \triangle(t_2) + \triangle\left(\frac{v(t_1+t_2) - y}{2v}\right)\right\} \\
&\qquad\qquad \times f_n(y)\, dy \Lambda(dt_2)\Lambda(dt_1) \\
&\quad = \#(\Gamma_n)^2 \int_0^\infty \int_0^{r_n} \int_{t_1-y/v}^{t_1+y/v} \exp\left\{-\triangle(t_1) - \triangle(t_2) + \triangle\left(\frac{v(t_1+t_2) - y}{2v}\right)\right\} \\
&\qquad\qquad \times f_n(y)\Lambda(dt_2)\, dy\Lambda(dt_1)
\end{aligned}$$

$$
\begin{aligned}
&= \#(\Gamma_n)^2 \int_0^\infty \exp\{-\triangle(t_1)\} \int_0^{r_n} f_n(y) \\
&\quad \times \biggl\{ \int_0^\infty \exp\{-\triangle(t_2)\}\Lambda(dt_2) - \int_{(y/v-t_1)\vee 0}^\infty \exp\{-\triangle(t_2)\}\Lambda(dt_2) \\
&\qquad + \int_{(y/v-t_1)\vee(t_1-y/v)}^{t_1+y/v} \exp\biggl\{-\triangle(t_2) + \triangle\biggl(\frac{v(t_1+t_2)-y}{2v}\biggr)\biggr\}\Lambda(dt_2)\biggr\}\, dy\Lambda(dt_1) \\
&= \#(\Gamma_n)^2\mu^2 - \#(\Gamma_n)^2 \int_0^\infty \exp\{-\triangle(t_1)\} \int_0^\infty \exp\{-\triangle(t_2)\} \\
&\qquad\qquad \times \int_0^{v(t_1+t_2)\wedge r_n} f_n(y)\, dy\, \Lambda(dt_2)\Lambda(dt_1) \\
&\quad + \#(\Gamma_n)^2 \int_0^\infty \exp\{-\triangle(t_1)\} \int_0^{r_n} f_n(y) \\
&\qquad \times \int_{|t_1-y/v|}^{t_1+y/v} \exp\biggl\{-\triangle(t_2) + \triangle\biggl(\frac{v(t_1+t_2)-y}{2v}\biggr)\biggr\}\Lambda(dt_2)\, dy\Lambda(dt_1),
\end{aligned}
$$

where $x \wedge y = \min(x, y)$. The density f_n depends on the shape of $\Gamma_n + [0,1]^d$ but $\sigma^2 \equiv \lim_{n\to\infty} \operatorname{var}[S_n]/\#(\Gamma_n)$ does not. We can derive σ^2 by evaluating the above integrals with $\Gamma_n + [0,1]^d$ and $\#(\Gamma_n)$ replaced by a ball of large radius R and volume $\omega_d R^d$, respectively. The density of the distance between two independent uniformly distributed points in this ball is $f(y) = dR^{-d}y^{d-1}B_{(d+1)/2,\,1/2}(1 - y^2/(4R^2))$ where $B_{a,b}(\cdot)$ is the distribution function of the beta distribution with parameters a and b [Kendall and Moran (1963), equation (2.122)]. Therefore

$$
\begin{aligned}
\sigma^2 &= \mu - \int_0^\infty \exp\{-\triangle(t_1)\} \int_0^\infty \omega_d v^d (t_1+t_2)^d \exp\{-\triangle(t_2)\}\Lambda(dt_2)\Lambda(dt_1) \\
&\quad + \int_0^\infty \exp\{-\triangle(t_1)\} \int_0^{t_1} \exp\{\triangle(y)\} \int_y^\infty 2\,d\,\omega_d v^d (t_1+t_2-2y)^{d-1} \\
&\qquad\qquad \times \exp\{-\triangle(t_2)\}\Lambda(dt_2)\, du\Lambda(dt_1).
\end{aligned}
$$

In particular, if $\Lambda(dt) = \lambda\, dt$, where $0 < \lambda < \infty$, then writing $\gamma_d = \lambda\omega_d v^d/(d+1)$,

$$
\begin{aligned}
\mu &= \frac{\lambda}{(d+1)\,\gamma_d^{1/(d+1)}}\Gamma\biggl(\frac{1}{d+1}\biggr), \\
\sigma^2 &= \mu - I_1 + I_2,
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
I_1 &= \frac{\lambda}{(d+1)\,\gamma_d^{1/(d+1)}} \sum_{j=0}^{d} \binom{d}{j} \Gamma\biggl(\frac{j+1}{d+1}\biggr)\Gamma\biggl(\frac{d+1-j}{d+1}\biggr), \\
I_2 &= \int_0^\infty \lambda \exp\{-\gamma_d t_1^{d+1}\} \int_0^{t_1} \exp\{\gamma_d y^{d+1}\} \int_y^\infty 2d(d+1)\gamma_d \\
&\qquad \times (t_1+t_2-2y)^{d-1} \exp\{-\gamma_d t_2^{d+1}\}\, dt_2\, dy\, dt_1.
\end{aligned}
$$

When $d = 1$, we can obtain an analytic solution by means of the transformation $u = (t_2 - y)/\sqrt{2}, w = (t_2 + y)/\sqrt{2}$ and a series expansion, giving

$$\begin{aligned} I_2 &= -\int_0^\infty \lambda \exp(-\lambda v t_1^2) \sum_{j=1}^{\infty} (-2)^j (\lambda v t_1^2)^{j/2} \frac{\Gamma(j/2)}{j!}\, dt_1 \\ &= \sqrt{\frac{\pi\lambda}{v}} \log 2. \end{aligned}$$

For $d \geq 2$ we can write I_2 in a form suitable for numerical integration as follows.

Put $u = \gamma_d^{1/(d+1)}(t_1 - y)$, $w = \gamma_d^{1/(d+1)}(t_2 - y)$ and $x = \gamma_d^{1/(d+1)} y$. Then

$$I_2 = \frac{2d(d+1)\lambda}{\gamma_d^{1/(d+1)}} K_d,$$

where

$$K_d = \int_0^\infty \int_0^\infty \int_0^\infty (u+w)^{d-1} \exp\{-(u+x)^{d+1} + x^{d+1} - (w+x)^{d+1}\}\, du\, dw\, dx$$

and (2.4) gives

$$\sigma^2 = \frac{\lambda}{\gamma_d^{1/(d+1)}} \left\{ 2d(d+1)K_d - \frac{1}{d+1} \sum_{j=1}^{d} \binom{d}{j} \Gamma\left(\frac{j+1}{d+1}\right) \Gamma\left(\frac{d+1-j}{d+1}\right) \right\}.$$

By means of substitutions like $\alpha = (u + x)^{d+1}$, K_d can be reduced to an integral of the variable x alone, the integral containing distribution functions of gamma variables. In this form the integral can be readily evaluated using an S-Plus program. The numerical values to three decimal places for $d = 1$, 2, 3 and 4 are as follows:

d	1	2	3	4
K_d	0.307	0.213	0.195	0.207
$\sigma^2 \gamma_d^{1/(d+1)}/\lambda$	0.342	0.439	0.515	0.579.

Hereafter we consider only the class of Λ with $\sigma^2 > 0$.

3. Mixing coefficients. Denote by $(\Omega, \mathscr{A}, \mathbf{P})$ the probability space induced by $\{\xi_z: z \in \mathbb{Z}^d\}$. For $\Gamma^{(1)}, \Gamma^{(2)} \subset \mathbb{Z}^d$, let $d(\Gamma^{(1)}, \Gamma^{(2)}) = \inf\{d(z_1, z_2): z_i \in \Gamma^{(i)},\ i = 1, 2\}$. Define the mixing coefficients to be

$$\begin{aligned} \alpha_{a,b}(k) \equiv \sup\big\{ |\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)| : A_i \in \sigma(\xi_z: z \in \Gamma^{(i)}), \#(\Gamma^{(1)}) \leq a, \\ \#(\Gamma^{(2)}) \leq b,\ d(\Gamma^{(1)}, \Gamma^{(2)}) \geq k \big\}, \end{aligned}$$

where $k \in \mathbb{N}$, $a, b \in \mathbb{N} \cup \{\infty\}$ and $\sigma(\xi_z: z \in \Gamma)$ is the σ-algebra generated by $\{\xi_z: z \in \Gamma\}$.

We impose the following condition on Λ to govern how fast it goes to infinity.

CONDITION 3.1. There exists a constant $M < \infty$ such that

$$\{\Lambda(t+c)-\Lambda(t)+1\}\{\Lambda(s+c)-\Lambda(s)+1\}\exp\left\{-\int_0^t \omega_d v^d (t-u)^d \Lambda(du)\right\} \le M$$

for all $0 \le s \le t < \infty$, where $c = \sqrt[d]{d}/v$.

In this section we derive an upper bound only for $\alpha_{1,1}(k)$.

Consider ξ_{z_1} and ξ_{z_2} such that $d(\xi_{z_1}, \xi_{z_2}) \ge k$. For each $A_i \in \sigma(\xi_{z_i})$, there exists an index set J_i of nonnegative integers such that $A_i = \bigcup_{j \in J_i} A_i^{(j)}$ where $A_i^{(j)} = \{\xi_{z_i} = j\}$ and $i = 1$ or 2. Let

$$\left|\mathbf{P}\left(A_1^{(n)} \cap A_2^{(m)}\right) - \mathbf{P}\left(A_1^{(n)}\right)\mathbf{P}\left(A_2^{(m)}\right)\right| = \beta_{n,m}(k).$$

Then, for any $A_i \in \sigma(\xi_{z_i})$, $i = 1$ and 2,

$$\left|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)\right| \le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \beta_{n,m}(k). \tag{3.1}$$

Note that

$$\begin{aligned}&\mathbf{P}\left(A_1^{(0)} \cap A_2^{(m)}\right) - \mathbf{P}\left(A_1^{(0)}\right)\mathbf{P}\left(A_2^{(m)}\right)\\ &\quad = \mathbf{P}\left(\bigcup_{n\ge 1} A_1^{(n)}\right)\mathbf{P}\left(A_2^{(m)}\right) - \mathbf{P}\left(\bigcup_{n \ge 1} A_1^{(n)} \cap A_2^{(m)}\right).\end{aligned}$$

Hence we obtain $\beta_{0,m}(k) \le \sum_{n=1}^{\infty} \beta_{n,m}(k)$, $\beta_{n,0}(k) \le \sum_{m=1}^{\infty}\beta_{n,m}(k)$ and $\beta_{0,0}(k) \le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{n,m}(k)$. Consequently, it suffices to consider only $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{n,m}(k)$ because

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\beta_{n,m}(k) \le 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{n,m}(k). \tag{3.2}$$

Let $T_i = \inf\{t\colon (x,t) \in \Psi \cap \{\{z_i + [0,1]^d\} \times [0,\infty)\}\}$ and let X_i be the position of the seed corresponding to the birth-time T_i, for $i = 1$ and 2. Because Ψ is a Poisson process which is spatially homogeneous, T_1 and T_2 are independent whenever $z_1 \ne z_2$. They have the same distribution function F which is given by

$$F(t) = 1 - \exp\{-\Lambda(t)\} \quad \text{for } t \ge 0 \tag{3.3}$$

and zero otherwise. The random positions X_i are uniformly distributed in $z_i + [0,1]^d$.

Recall that for each $(x,t) \in \Psi$, $(x,t) \in \Phi$ if and only if (2.1) holds. That means for each $(x,t) \in \Phi$ there is a *forbidden region* $R(x,t)$ in which no points of Ψ exist. For $d = 1$ and 2, $R(x,t)$ is a triangle and a cone in $\mathbb{R}^d \times [0,\infty)$, respectively. For $\{(x^{(j)}, t^{(j)}) \in \Phi\colon j = 1,\dots,n\}$, the forbidden region is just the union $\bigcup_{j=1}^{n} R(x^{(j)}, t^{(j)})$. Since Ψ is a Poisson process, for $n \ge 1$ and $m \ge 1$,

$$\mathbf{P}\left(A_1^{(n)} \cap A_2^{(m)} | T_i = t_i,\ i = 1,2\right) \ne \mathbf{P}\left(A_1^{(n)} | T_1 = t_1\right)\mathbf{P}\left(A_2^{(m)} | T_2 = t_2\right)$$

only if conditional on $\{T_i = t_i, i = 1, 2\}$ the forbidden regions for $\{A_1^{(n)}\}$ and $\{A_2^{(m)}\}$ have a nonempty intersection. This can happen only if $v(t_1+t_2)+2\sqrt[d]{d} > k-1$. Hence,

$$\begin{aligned} \beta_{n,m}(k) \le \Bigg| &\iint_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} \mathbf{P}\big(A_1^{(n)} \cap A_2^{(m)} | T_i = t_i,\ i = 1, 2\big) \\ &\quad \times dF(t_1)\, dF(t_2) \\ &- \iint_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} \mathbf{P}\big(A_1^{(n)} | T_1 = t_1\big) \\ &\quad \times \mathbf{P}\big(A_2^{(m)} | T_2 = t_2\big)\, dF(t_1)\, dF(t_2)\Bigg|. \end{aligned} \tag{3.4}$$

Consider $\mathbf{P}(A_i^{(n)} | T_i = t_i)$, $i = 1$ and 2. Conditional on $\{(X_i, T_i) = (x_i, t_i)\}$, $i = 1$ or 2, there are n seeds formed in $z_i + [0, 1]^d$ only if $(x_i, t_i) \notin \Xi(\Psi, t_i)$ and at least $n-1$ more points of Ψ exist in $z_i + [0, 1]^d$ after t but before the cell generated by the seed at (x_i, t_i) covers $z_i + [0, 1]^d$, which will occur before $t_i + \sqrt[d]{d}/v$. Thus,

$$\begin{aligned} &\mathbf{P}\big(A_i^{(n)} | T_i = t_i\big) \\ &\quad \le \exp\left\{-\int_0^{t_i} \omega_d v^d (t_i - u)^d \Lambda(du)\right\} \\ &\qquad \times \sum_{j \ge n-1} \frac{\{\Lambda(t_i + \sqrt[d]{d}/v) - \Lambda(t_i)\}^j \exp\{-(\Lambda(t_i + \sqrt[d]{d}/v) - \Lambda(t_i))\}}{j!} \end{aligned}$$

for $i = 1$ and 2. Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} &\iint_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} \mathbf{P}\big(A_1^{(n)} | T_1 = t_1\big) \\ &\quad \times \mathbf{P}\big(A_2^{(m)} | T_2 = t_2\big)\, dF(t_1)\, dF(t_2) \\ &\le \iint_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} \big\{\Lambda(t_1 + \sqrt[d]{d}/v) - \Lambda(t_1) + 1\big\} \\ &\quad \times \big\{\Lambda(t_2 + \sqrt[d]{d}/v) - \Lambda(t_2) + 1\big\} \\ &\quad \times \exp\left\{-\int_0^{t_1} \omega_d v^d (t_1 - u)^d \Lambda(du) \right. \\ &\qquad \left. - \int_0^{t_2} \omega_d v^d (t_2 - u)^d \Lambda(du)\right\} dF(t_1)\, dF(t_2). \end{aligned} \tag{3.5}$$

Similarly, consider $\mathbf{P}(A_1^{(n)} \cap A_2^{(m)} | T_i = t_i,\ i = 1, 2)$. Conditional on $\{(X_i, T_i) = (x_i, t_i),\ i = 1, 2\}$, there are n and m seeds formed in $z_1 + [0, 1]^d$ and $z_2 + [0, 1]^d$, respectively, only if at least $n-1$ and $m-1$ more points of Ψ exist in $\{z_1 + [0, 1]^d\} \times [t_1, t_1 + \sqrt[d]{d}/v]$ and $\{z_2 + [0, 1]^d\} \times [t_2, t_2 + \sqrt[d]{d}/v]$, respectively, and $\{\Psi(R(x_1, t_1) \cup R(x_2, t_2)) = 0\}$. The probability of the latter is at most $\exp\{-\int_0^{t_{\max}} \omega_d v^d (t_{\max} - u)^d \Lambda(du)\}$ where $t_{\max} = \max(t_1, t_2)$. Therefore,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int\int_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} \mathbf{P}\big(A_1^{(n)} \cap A_2^{(m)} | T_i = t_i,\ i = 1, 2\big)\, dF(t_1)\, dF(t_2) \\
&\qquad \le \int\int_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} \{\Lambda\big(t_1 + \sqrt[d]{d}/v\big) - \Lambda(t_1) + 1\} \\
&\qquad\qquad \times \{\Lambda\big(t_2 + \sqrt[d]{d}/v\big) - \Lambda(t_2) + 1\} \\
&\qquad\qquad \times \exp\bigg\{-\int_0^{t_{\max}} \omega_d v^d (t_{\max} - u)^d \Lambda(du)\bigg\}\, dF(t_1)\, dF(t_2) \\
&\qquad = \int\int_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} I(t_1, t_2)\, dF(t_1)\, dF(t_2), \text{ say.}
\end{aligned}
\tag{3.6}
$$

Under Condition 3.1, there exists a constant M such that $I(t_1, t_2) \le M$ for all $t_1, t_2 \ge 0$. From (3.4), (3.5) and (3.6), we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{n,m}(k) &\le 2 \int\int_{v(t_1+t_2)+2\sqrt[d]{d}>k-1} I(t_1, t_2)\, dF(t_1)\, dF(t_2) \\
&\le 4M \int\int_{v(t_1+t_2)+2\sqrt[d]{d}>k-1,\ t_1 \ge t_2} dF(t_1)\, dF(t_2) \\
&\le 4M \int_{[k-1-2\sqrt[d]{d}]_+/(2v)}^{\infty} dF(t_1),
\end{aligned}
\tag{3.7}
$$

where $[x]_+ = \max(x, 0)$. Thus, by the stationarity of $\{\xi_z : z \in \mathbb{Z}^d\}$, (3.1), (3.2), (3.3) and (3.7),

$$
\alpha_{1,1}(k) \le 16M\bigg(\exp\bigg\{-\Lambda\bigg(\frac{[k-1-2\sqrt[d]{d}]_+}{2v}\bigg)\bigg\} - \exp\{-\Lambda(\infty)\}\bigg) = \alpha'(k),
\tag{3.8}
$$

which tends to zero as k tends to infinity.

4. Central limit theorem.

We prove the central limit theorem for S_n in an arbitrary dimension $d \ge 1$ in this section.

LEMMA 4.1 [Bolthausen (1982)]. *Suppose that $\{\xi_z : z \in \mathbb{Z}^d\}$ is stationary. If $\sum_{k=1}^{\infty} k^{d-1}\alpha_{a,b}(k) < \infty$ for $a + b \le 4$, $\alpha_{1,\infty}(k) = o(k^{-d})$, and $\mathbf{E}|\xi_z|^{2+\delta} < \infty$ and $\sum_{k=1}^{\infty} k^{d-1}\alpha_{1,1}(k)^{\delta/(2+\delta)} < \infty$ for some $\delta > 0$, then $\sum_{z\in\mathbb{Z}^d} |\operatorname{cov}(\xi_{z_0}, \xi_z)| < \infty$ and if $\sigma^2 = \sum_{z\in\mathbb{Z}^d} \operatorname{cov}(\xi_{z_0}, \xi_z) > 0$, then the distribution of $S_n/\sqrt{\#(\Gamma_n)\sigma^2}$ converges weakly to the standard normal distribution as $n \to \infty$.*

In order to use this lemma to show the asymptotic normality of S_n, we have to know upper bounds of $\alpha_{1,\infty}(k)$ and $\alpha_{a,b}(k)$ for $a + b \le 4$.

LEMMA 4.2. *Under Condition* 3.1, *for all* $k, a, b \in \mathbb{N}$,

$$\alpha_{a,b}(k) \le ab\alpha'(k).$$

PROOF. Consider $\Gamma^{(i)} = \{z_j\colon j \in J_i\}$ for $i = 1$ and 2 such that $d(\Gamma^{(1)}, \Gamma^{(2)}) \ge k$, where $J_1 = \{2j-1\colon j = 1, \dots, a'\}$, $J_2 = \{2j\colon j = 1, \dots, b'\}$, $a', b' \in \mathbb{N}$, $a' \le a$, $b' \le b$ and all z_j are distinct. Let $A_i^{(n)} = \{\xi_{z_i} = n\}$, where n is a nonnegative integer and $i = 1$ and 2. For each $A_i \in \sigma(\xi_z\colon z \in \Gamma^{(i)})$, $A_i = \bigcup_{n=0}^{\infty}(A_i^{(n)} \cap B_i^{(n)})$ for some $B_i^{(n)} \in \sigma(\xi_z\colon z \in \Gamma^{(i)} \setminus \{z_i\})$. Let

$$\left|\mathbf{P}\left(A_1^{(n)} \cap A_2^{(m)} \cap B_1^{(n)} \cap B_2^{(m)}\right) - \mathbf{P}\left(A_1^{(n)} \cap B_1^{(n)}\right)\mathbf{P}\left(A_2^{(m)} \cap B_2^{(m)}\right)\right| = \gamma_{n,m}(k).$$

Then, in view of (3.2), (3.7) and (3.8), it suffices to show that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \gamma_{n,m}(k) \le a'b' \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_{n,m}(k).$$

Let $T_j = \inf\{t\colon (x,t) \in \Psi \cap \{\{z_j + [0,1]^d\} \times [0,\infty)\}\}$ where $j \in J_1 \cup J_2$. Similar to the argument in Section 3, for $n \ge 1$ and $m \ge 1$, $\mathbf{P}(A_1^{(n)} \cap A_2^{(m)} \cap B_1^{(n)} \cap B_2^{(m)} | T_j = t_j,\ j \in J_1 \cup J_2) \ne \mathbf{P}(A_1^{(n)} \cap B_1^{(n)} | T_j = t_j,\ j \in J_1)\mathbf{P}(A_2^{(m)} \cap B_2^{(m)} | T_j = t_j,\ j \in J_2)$ only if the forbidden regions intersect, that is, if $v(t_{j_1} + t_{j_2}) + 2\sqrt[d]{d} > k-1$ for some $j_1 \in J_1$ and $j_2 \in J_2$. This pair (j_1, j_2) can be any one of the $a'b'$ elements in the set $\{(j_1, j_2)\colon j_i \in J_i,\ i = 1, 2\}$. Since $\mathbf{P}(A_1^{(n)} \cap A_2^{(m)} \cap B_1^{(n)} \cap B_2^{(m)} | T_j = t_j,\ j \in J_1 \cup J_2) \le \mathbf{P}(A_1^{(n)} \cap A_2^{(m)} | T_j = t_j,\ j \in J_1 \cup J_2)$ and $\mathbf{P}(A_i^{(n)} \cap B_i^{(n)} | T_j = t_j,\ j \in J_i) \le \mathbf{P}(A_i^{(n)} | T_j = t_j,\ j \in J_i)$ for $i = 1$ and 2, from (3.5), (3.6) and (3.7), the result follows. □

LEMMA 4.3. *Under Condition* 3.1, *for all* $k \in \mathbb{N}$,

$$\alpha_{1,\infty}(k) \le \sum_{h=k}^{\infty} 2^{d^2-1} h^{d-1} \alpha'(h).$$

PROOF. We use the same argument and notation as in the proof of Lemma 4.2 except that $b = \infty$. Now $J_1 = \{1\}$ and $J_2 = \{2, 4, 6, \dots\}$. Let $J_2^{(h)} = \{j\colon d(z_1, z_j) = h\}$ for all integers $h \ge k$. Then the number of elements in $J_2^{(h)}$ is $(2h+1)^d - (2h-1)^d$, which is less than $2^{d^2-1}h^{d-1}$. The forbidden regions intersect only when $v(t_1 + t_j) + 2\sqrt[d]{d} > h-1$ for some $t_j \in J_j^{(h)}$ and $h \ge k$. Therefore, from (3.5), (3.6) and (3.7), $\sum\sum \gamma_{n,m}(k) \le \sum_{h=k}^{\infty}\{2^{d^2-1}h^{d-1}\sum\sum \beta_{n,m}(h)\}$, and the result follows. □

REMARK. Lemmas 4.2 and 4.3 are quite similar to Bradley (1981), Lemma 8. However, in our context Bradley's lemma is not applicable because his condition, that the σ-algebras $\sigma(\xi_{z_j}\colon j \in J_2^{(h)})$ be independent, is not fulfilled.

Now we impose one more condition on Λ.

CONDITION 4.1. For sufficiently large $k \in \mathbb{N}$,

$$\sum_{h=k}^{\infty} h^{d-1}\alpha'(h) = o(k^{-d-\tau})$$

for some $\tau \geq 0$.

From (2.2) and Lemmas 4.2 and 4.3, if Condition 4.1 holds, which implies that $\alpha'(k) = o(k^{-2d+1-\tau})$, then all the requirements of Lemma 4.1 are met when (1) $\tau \geq 0$ and $\delta = 5$ if $d \geq 2$ or (2) $\tau = \varepsilon$ for some $\varepsilon > 0$ and $\delta > 2/\varepsilon$ if $d = 1$. Thus, the following central limit theorem is obtained.

THEOREM 4.1. *Under Conditions* 3.1 *and* 4.1 *where* $\tau \geq 0$ *if* $d \geq 2$ *or* $\tau > 0$ *if* $d = 1$, *the distribution of* $S_n/\sqrt{\#(\Gamma_n)\sigma^2}$ *converges weakly to the standard normal distribution as* $n \to \infty$.

Conditions 3.1 and 4.1 are fulfilled (for any τ) when, for example, $\Lambda(t) \sim Kt^j$ for some positive K and $1 \leq j < \infty$. If $\Lambda(\infty) < \infty$, then Condition 3.1 holds, but Condition 4.1 requires a fast convergence of $\Lambda(t) \to \Lambda(\infty)$. Consider, for example, $\Lambda(t) = \lambda\Gamma(\alpha)^{-1}\int_0^t y^{\alpha-1}e^{-y}\,dy$ for some positive finite α and λ so that $\Lambda(\infty) = \lambda$. Then there exists a t_o such that

$$\begin{aligned}\exp\{-\Lambda(t)\} - \exp\{-\lambda\} &= \exp\{-\lambda\}(\exp\{\lambda - \Lambda(t)\} - 1)\\ &\leq 2\exp\{-\lambda\}(\lambda - \Lambda(t)) \text{ for } t > t_o\\ &= O\big(t^{\alpha-1}\exp\{-t\}\big).\end{aligned}$$

Thus, by (3.8), this Λ satisfies Conditions 3.1 and 4.1 for any τ.

5. Functional central limit theorem. In particular, we consider $d = 1$ in this section, and so $\sigma^2 = \sum_{z\in\mathbb{Z}} \operatorname{cov}(\xi_0, \xi_z)$. For each $n \in \mathbb{N}$, for ease of presentation we assume $\#(\Gamma_n) = n$ and define

$$W_n(t, \omega) = S_{\lfloor nt\rfloor}(\omega)/\sqrt{\sigma^2 n} \quad \text{for } t \in [0, 1] \text{ and } \omega \in \Omega,$$

where $\lfloor x\rfloor$ is the greatest integer not exceeding x. The function $\omega \mapsto W_n(\cdot, \omega)$ is a measurable mapping from $(\Omega, \mathscr{A})$ into $(D, \mathscr{D})$, where D is the space of functions on $[0, 1]$ that are right continuous and have left-hand limits, and $\mathscr{D}$ denotes the Borel σ-algebra induced by the Skorokhod topology [see, e.g., Billingsley (1968)]. Let

$$\begin{aligned}\alpha(k) \equiv \sup_{n\in\mathbb{Z}}\big\{\big|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)\big|: A_1 &\in \sigma(\xi_z: z \leq n),\\ A_2 &\in \sigma(\xi_z: z \geq n + k)\big\}\end{aligned}$$

for $k \in \mathbb{N}$. Note that $\alpha(k) \leq \alpha_{\infty,\infty}(k)$ for all k.

LEMMA 5.1 [Herrndorf (1984), Corollary 1]. *If there exists some* $\delta > 0$ *such that* $\sum_{k=1}^{\infty} \alpha(k)^{\delta/(2+\delta)} < \infty$ *and* $\mathbf{E}|\xi_z|^{2+\delta} < \infty$ *for all* $z \in \mathbb{Z}$, *and* $\mathrm{var}[S_n]/n \to \sigma^2$, *where* $0 < \sigma^2 < \infty$, *then* W_n *converges in distribution to the standard Wiener measure on* D *as* $n \to \infty$.

In view of this lemma, we should find an upper bound for $\alpha(k)$.

LEMMA 5.2. *Under Condition* 3.1, *for each* $k \in \mathbb{N}$,

$$\alpha(k) \le \sum_{r=0}^{\infty}(r+1)\alpha'(k+r) = \sum_{r=k}^{\infty}\sum_{h=r}^{\infty}\alpha'(h).$$

PROOF. We use again the same argument and notation as in the proof of Lemma 4.2 except that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ have to be in the form $\{z \in \mathbb{Z}: z \le n\}$ and $\{z \in \mathbb{Z}: z \ge n+k\}$, respectively, for some $n \in \mathbb{Z}$. Now $J_1 = \{1, 3, 5, \ldots\}$ and $J_2 = \{2, 4, 6, \ldots\}$. Conditional on $\{T_j = t_j: j \in J_1 \cup J_2\}$, the forbidden regions intersect only when $v(t_{j_1} + t_{j_2}) + 2\sqrt{2} > k + r - 1$ where $d(z_{j_1}, z_{j_2}) = k + r$ for some $r \in \mathbb{N} \cup \{0\}$ and $j_i \in J_i$, $i = 1$ and 2. For each such r, the number of elements in the set $\{(j_1, j_2): d(z_{j_1}, z_{j_2}) = k + r,\ j_i \in J_i,\ i = 1, 2\}$ is at most $r + 1$. The statement is now obvious. □

If Conditions 3.1 and 4.1 hold for $\tau = 1 + \varepsilon$ for some $\varepsilon > 0$, then by Lemma 4.1, $\mathrm{var}[S_n]/n \to \sigma^2 < \infty$. Moreover, by Lemma 5.2, $\alpha(k) = \sum_{r=k}^{\infty} o(r^{-2-\varepsilon}) = o(k^{-1-\varepsilon/2})$. Thus, the requirements of Lemma 5.1 are met whenever $\delta > 4/\varepsilon$. Hence, we have proved the functional central limit theorem for S_n in one dimension.

THEOREM 5.1. *For* $d = 1$, *under Conditions* 3.1 *and* 4.1 *where* $\tau > 1$, W_n *converges in distribution to the standard Wiener measure on* D *as* $n \to \infty$.

6. Rates of convergence. In this section we assume that

$$\Lambda(t) \sim Kt^j \quad \text{for some positive } K \text{ and } 1 \le j < \infty, \tag{6.1}$$

or

$$\Lambda(t) = \lambda \int_0^t \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dy \quad \text{for some positive finite } \alpha \text{ and } \lambda. \tag{6.2}$$

Either (6.1) or (6.2) implies that $\alpha'(k) = O(e^{-\rho k})$ for some positive finite ρ. Thus, by Lemma 5.2, when $d = 1$, $\alpha(k) = O(e^{-\rho k})$.

Denote by G_n the distribution function of $S_n/\sqrt{\#(\Gamma_n)\sigma^2}$ and by G the standard normal distribution.

THEOREM 6.1. *If* (6.1) *or* (6.2) *holds, then for* $d \ge 1$,

$$\sup\left|G_n(x) - G(x): x \in \mathbb{R}\right| = O\big(\#(\Gamma_n)^{-1/2} \log^d \#(\Gamma_n)\big). \tag{6.3}$$

Furthermore, when $d = 1$,

$$|G_n(x) - G(x)| = O\left(\frac{\log^3 \#(\Gamma_n)}{\sqrt{\#(\Gamma_n)}(1+|x|)^4}\right) \quad \textit{for each } x \in \mathbb{R}. \tag{6.4}$$

PROOF. For $d \geq 2$, (6.3) follows from (2.2), Lemma 4.2 and Takahata (1983), Theorem 1, whereas for $d = 1$, (6.3) and (6.4) follow from (2.2) and Tikhomirov (1980), Theorem 4. □

In order to obtain a rate of convergence for the functional central limit theorem, we need to consider a smoothed version of W_n. For each $n \in \mathbb{N}$ we assume $\#(\Gamma_n) = n$ and define

$$W'_n(t, \omega) = \frac{S_{\lfloor nt \rfloor}(\omega)}{\sqrt{\sigma^2 n}} + \frac{nt - \lfloor nt \rfloor}{\sqrt{\sigma^2 n}}\big(S_{\lfloor nt \rfloor + 1}(\omega) - S_{\lfloor nt \rfloor}(\omega)\big)$$

for $t \in [0, 1]$ and $\omega \in \Omega$. That means W'_n is the random polygonal line with nodes at $(j/n, S_j/\sqrt{\sigma^2 n})$, $j = 0, \dots, n$. Thus, W'_n belongs not only to D but also to C, the space of bounded, continuous, real-valued functions defined on $[0, 1]$.

Let P_n and W be the distributions of W'_n and the standard Wiener process on D. Denote by $L(\cdot, \cdot)$ the Lévy–Prokhorov distance between two probability measures defined on the Borel σ-algebra of the metric space C with the sup-norm. The following theorem follows from (2.2) and Utev (1985), Corollary 7.2.

THEOREM 6.2. *If* (6.1) *or* (6.2) *holds, then*

$$L(P_n, W) = O\big(n^{-1/4+\varepsilon}\big),$$

where $\varepsilon > 0$.

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