CENTRAL LIMITS OF SQUEEZING OPERATORS

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Contents

1 Introduction

In [1] we have proved a quantum De Moivre-Laplace theorem based on a modification of the Giri-von Waldenfels quantum central limit theorem. In [2] P.A. Meyer outlined a method based on direct calculations which, taking advantage of the explicit structure of the algebra of 2×2 matrices, allows a drastic simplification of the proof of the main result of the first part of our paper and relates it with a similar result obtained, independently and simultaneusly, by Parthasarathy [5]. In the first part of the present note we simplify the Parthasarathy-Meyer method and extend it to deal with arbitrary d-dimensional Bernoulli processes, where d is a natural integer (cfr. Sections (3),(4). We also prove another statement in Meyer's note (cf. Theorem (5.1)). Finally (Section (6)) we show that the method of proof used in [1] allows, with minor modifications, to solve the problem of the central limit approximation of the squeezing states - a problem left open in [1] and to which , due to the nonlinearity of the coupling, Parthasarathy-Meyer direct computational method cannot be applied.

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2 The De Moivre-Laplace theorem in the 2-dimensional case

Throughout this paper we adopt the notations of [1] with the only exception that, in the first four secions, we use Meyer' s normalization for the N-coherent vectors and the N-Weyl opertors . Thus, in particular, $\varphi_N(0)$ denotes the vacuum state in $\otimes^n \mathbf{C}^2$; $W_0(z) = \exp(zs^+ - \overline{z}s)$ the Weyl operator on $\mathbf{C}^2$; $W_N(z) = \otimes^N W_0(z/\sqrt{N})$ the $N - Weyl$ operator on $\otimes^N \mathbf{C^2}$; while $\Phi(0)$, $W(z) = \exp(za^+ - \overline{z}a)$ denotes the corresponding objects for the harmonic oscillator. As usual

$$s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \quad n_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \tag{1}$$

$$\varphi_N(0) = \otimes^N e_1 \qquad ; \qquad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The main result of the first part of [1] is:

Theorem 1 *For every natural integer k and for every $k \in, z_1 \dots, z_R \in \mathbf{C}$*

$$\lim_{N\to\infty} < \varphi_N(0),\, w_N(z_1)\dots w_N(z_R)\varphi_N(0) > = < \Phi(0), W(z_1)\dots W(z_R)\Phi(0) > \tag{2}$$

uniformly for $z_1, \dots, z_R$ in a bounded set.

Proof. For $z_j \in \mathbf{C}$ one has

$$< \varphi_N(0),\, W_N(z_1)\dots W_N(z_k)\varphi_N(0) > = \tag{3}$$

$$= < \otimes^N e_i, \otimes^N W_o\left(\frac{z_1}{\sqrt{N}}\right)\dots\otimes^N W_o\left(\frac{z_k}{\sqrt{N}}\right)\otimes^N e_i > = < e_i, W_o\left(\frac{z_i}{\sqrt{N}}\right) e \dots W_o\left(\frac{z_k}{\sqrt{N}}\right) e_1 >^N$$

$$= \left\{ \sum_{\substack{(j_1,\dots j_k)\\ \in N^k}} \frac{1}{j_1!\dots j_k!} < e_i, \prod_{i=1,\dots k}^{\rightarrow} \left(\frac{z_i}{\sqrt{N}} s^+ - \frac{\overline{z}_i}{\sqrt{N}} s^-\right)^{j_i} e_1 > \right\}^N$$

$$= \left\{ \sum_{n=0}^{\infty} N^{-n/2} \sum_{\substack{(j_1\dots j_k)\in \mathbf{N}^k \\ \sum_{i=k}^k j_i = n}} \frac{1}{j_1!\dots j_k!} < e_1, \prod_{i=1,\dots,k}^{\rightarrow} (z_i s^+ - \overline{z}_i s^-)^{j_i} e_1 > \right\}^N$$

Denote R_N the right hand side of (3) with the sum in n starting from 3 rather than 0. Then

$$|R_N| \leq \sum_{n=3}^{\infty} N^{-n/2} \frac{\sum_{j_1,\dots,j_k}}{\sum_i j_i = n} \frac{1}{j_1!\dots j_k!} \parallel \prod_{i=1,\dots,k}^{\rightarrow} (z_i s^+ - \overline{z}_i s^-)^{j_i} \parallel \leq \tag{4}$$

$$\leq \sum_{n=3}^{\infty} N^{-n/2} \sum \frac{1}{j_i!\dots j_k!} \mid z_1 \mid^{j_1} \dots \mid z_k \mid^{j_k} \leq \frac{2e^{|z_1|}\dots e^{|z_k|}}{N^{3/2}} = o\left(N^{-1/2}\right)$$

For $n = 0$ there is only one term equal to 1; for $n = 1$ the contribution is zero, since $< e_1, S^{\pm} e_i >= 0$. The term $n = 2$ is equal to

$$\frac{1}{N} \frac{\sum_{(j_1,\dots,j_k)\in \mathbf{N}^k}}{\sum_i j_i = 2} \frac{1}{j_1! \dots j_k!} < e_1, \prod_{\substack{i=1\\ \dots k}}^{\rightarrow} (z_i s^+ - \overline{z}_i s^-)^{j_i} e_1 >=$$

$$= \frac{1}{N} \Big\{ \sum_{i=1}^{k} \frac{1}{2} < e_1, \left(z_i s^+ - \overline{z}_i s^-\right)^2 e_1 > + \sum_{i<j} < e_1, \left(z_i s^+ - \overline{z}_i s^-\right) \left(z_j s^+ - \overline{z}_j s^-\right) e_1 > \Big\} =$$

$$= \frac{1}{N} \left(\frac{1}{2} \sum_{i=1}^{k} -\overline{z}_i z_i + \sum_{i<j} -z_j \overline{z}_i \right)$$

In conclusion we obtain

$$< \varphi_N(0), W_N(z_1), \dots W_n(z_k) \varphi_N(0) >$$

$$= \left\{ 1 + \frac{1}{N} \left[-\frac{1}{2} \sum_{i=1}^{k} |z_i|^2 - \sum_{i<j} \overline{z}_i z_j \right] + o(N^{-1/2}) \right\}^N \tag{5}$$

and the algebraic identity

$$-\frac{1}{2} \sum_i |z_i|^2 - \sum_{i<j} \overline{z}_i z_j = -\frac{1}{2} \left| \sum_i z_i \right|^2 - i\, Im \sum_{i<j} z_i \overline{z}_j$$

shows that the right hand (side of (5) converges to the right hand side of (2). The uniformity of the convergence is obvious in view of (4).

3 d-level systems: 1 copy

Our strategy to deal with d-level systems is to exploit the isomorphism between the symmetric tensor product $(\otimes^{d-1}\mathbf{C})_+$ of d-1 copies of $\mathbf{C}^2$ and $\mathbf{C}^d$, which intertwines the *Lie* algebra representation of $SO(3; R)$ given by

$$S^{\pm} = \sum_{k=1}^{d-1} j_k(s^{\pm}) \qquad , \qquad \sum_{k=1}^{d-1} j_k(s_3) = S_3$$

with the one of the $\frac{d-1}{2} - th$ irreducible representation. This allows to reduce the d-dimensional case to the 2-dimensional one. This isomorphism is described as follows: let

$$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then the symmetric tensor product $\left(\otimes^{d-1}\mathbf{C}^2\right)_+$is a d-dimensional complex vector space with basis given by the $(d-1)$-number vectors:

$$\varphi_{(d-1)}(k) = \binom{d-1}{k}^{1/2} \text{symm}\left(\otimes^{d-1-k} e_1 \otimes^k e_2\right); \quad k = 0, \dots, d-1 \tag{6}$$

where symm (.) denotes the projection onto the symmetric tensor product. The action of the operators $S^{\pm}$, S_3 on this basis is given by:

$$S^+\varphi_{1-1}(j) = \sqrt{(j+1)(d-1-j)}\varphi_{d-1}(j+1) \tag{7}$$

$$S^-\varphi_{d-1}(j) = \sqrt{j(d-j)}\varphi_{d-1} \tag{8}$$

$$S_3\varphi_{d-1}(j) = (d-1-j)\varphi_{d-1}(j) \tag{9}$$

Since, by obvious dimensional reasons, the representation is irreducible, it follows that it is isomorphic to the $(d-1)/2$-th representation of $SO(3)$ (cf. [2],[3]).

Now, for $W_0(z)$ as in section (2) ($z =\mid z \mid e^{i\alpha} \in \mathbf{C}$), we denote

$$W_{d-1}(z) = \otimes^{d-1} W_o(z)$$

and

$$\begin{aligned} \psi(z) = W_{d-1}(z)\varphi_{d-1}(0) = \otimes_{j=1}^{d-1}\left[\exp\left(zs^+ - \overline{z}s^-\right)e_1\right] = \\ = \otimes^{d-1}\left(e_1 \cos \mid z \mid + e_2 e^{i\alpha} \sin \mid z \mid\right) = \\ = \sum_{j=0}^{d-1} \binom{d-1}{j}^{1/2} \left(\sin \mid z \mid e^{i\alpha}\right)^j \left(\cos \mid z \mid\right)^{d-j-1} \varphi_{d-1}(j) \end{aligned} \tag{10}$$

From this one easily deduces that, uniformly for z, z' in a bounded set :

$$< \psi\left(\frac{z}{\sqrt{n}}\right), \psi\left(\frac{z'}{\sqrt{n}}\right) >= \left(1 + 0\left(\frac{1}{n}\right)\right)^{d-1} \tag{11}$$

Lemma 1 *Uniformly for $z, z' \in$ a bounded subset of* $\mathbf{C}$*:*

$$\sqrt{N} < \psi\left(\frac{z}{\sqrt{N}}\right), S^{+}\psi\left(\frac{z'}{\sqrt{N}}\right) >\to (d-1)\,\overline{z} \tag{12}$$

$$\sqrt{N} < \psi(z/\sqrt{N}), S^{-}\psi(z'/\sqrt{N}) >\to (d-1)z' \tag{13}$$

Proof. From (7), (8) one deduces that the left hand side of (12) is equal to:

$$\sqrt{N}\sum_{j,j'}\binom{d-1}{j}^{1/2}\binom{d-1}{j'}^{1/2}\left(\sin\frac{|\,z\,|}{\sqrt{N}}e^{i\alpha}\right)^{j}\left(\cos\frac{|\,z\,|}{\sqrt{N}}\right)^{d-1-j}$$

$$\left(\sin\frac{|\,z'\,|}{\sqrt{N}}e^{i\alpha'}\right)^{j'}\left(\frac{\cos|\,z'\,|}{\sqrt{N}}\right)^{d-1-j'}\sqrt{(j'+1)\,(d-1-j')}\delta_{j,j'+1} \tag{14}$$

and the left hand side of (13) is equal to:

$$\sqrt{N}\sum_{j,j'}\binom{d-1}{j}^{1/2}\binom{d-1}{j'}^{1/2}\left(\sin\frac{|\,z\,|}{\sqrt{N}}e^{i\alpha}\right)^{j}\left(\cos\frac{|\,z\,|}{\sqrt{N}}\right)^{d-1-j}$$

$$\left(\sin\frac{|\,z'\,|}{\sqrt{N}}e^{i\alpha'}\right)^{j'}\left(\frac{\cos|\,z'\,|}{\sqrt{N}}\right)^{d-1-j'}\sqrt{j'\,(d-j)}\delta_{j,j'-1} \tag{15}$$

therefore, in the limit $N \to \infty$, the only surviving term in (**??**) is the one with $j = 1$, i.e.

$$\binom{d-1}{1}^{1/2}\sqrt{d-1}\,|\,z\,|\,e^{-i\alpha} = (d-1)\overline{z}$$

while in (**??**) the only surviving term is the one with $j = 0$ and this gives

$$\binom{d-1}{1}^{1/2}\sqrt{d-1}\,|\,z'\,|\,e^{j\alpha'} = (d-1)z$$

4 d-level systems: N copies

Define the normalized $N-$coherent vectors by:

$$W_N(z) = \otimes^N W_o\left(\frac{z}{\sqrt{N}}\right)$$

$$\psi_N(z) = [\otimes^N W_o\left(\frac{z}{\sqrt{N}}\right)] \otimes^N \varphi_{d-1}(0) = \otimes^N \psi\left(\frac{z}{\sqrt{N}}\right)$$

Lemma 2

$$< \psi_N(z),\, \psi_N(z') > \to < \otimes^{d-1}\psi(z), \otimes^{d-1}\psi(z') > \tag{16}$$

Proof.

$$< \psi_N(z), \psi_N(z') > = < \psi\left(\frac{z}{\sqrt{N}}\right), \psi\left(\frac{z'}{\sqrt{N}}\right) >^N =$$

$$= \left\{\sum_{j=0}^{d-1}\binom{d-1}{j}\left(\sin\frac{|\,z\,|}{\sqrt{N}}e^{-i\alpha}\sin\frac{|\,z'\,|}{\sqrt{N}}e^{-i\alpha'}\right)^j\left(\cos\frac{|\,z\,|}{\sqrt{N}}\cos\frac{|\,z'\,|}{\sqrt{N}}\right)^{d-1-j}\right\}^N =$$

$$= \left\{\sum_{j=0}^{d-1}\binom{d-1}{j}\left(\frac{\overline{z}z}{N} + o(N)\right)^j\left(1 - \frac{|z|^2 + |z'|^2}{2N} + o(N)\right)^{d-1-j}\right\}^N =$$

$$= \left\{1 - \frac{|z|^2 + |z'|^2}{2N} + \frac{\overline{z}z}{N} + o(N)\right\}^{N(d-1)} \to < \psi(z), \psi(z') >^{d-1}$$

For $u, v \in \mathbf{C}$, we set

$$E_N : M(d;\mathbf{C}) \to \mathbf{C}$$

$$E_N(b) = < \psi\left(\frac{v}{\sqrt{N}}\right), \psi\left(\frac{u}{\sqrt{N}}\right) >^{-1} < \psi\left(\frac{u}{\sqrt{N}}\right), b\psi\left(\frac{v}{\sqrt{N}}\right) > \tag{17}$$

From (11) we deduce that

$$E_N(b) \to < \varphi_{d-1}(0), b\varphi_{d-1}(0) >$$

Theorem 2 *For any $u, v, z \in \mathbf{C}$*

$$< \psi_N(u), \exp\left(z\frac{S_N(S^+)}{\sqrt{N}} - z\frac{S_N(S^-)}{\sqrt{N}}\right)\psi_N(v) >$$

$$\to < \psi(u), W(z)\psi(v) >^{d-1}$$

$$= \left\{\exp\left(-\frac{|\,z\,|^2}{2}\right)\exp(z\overline{u} - \overline{z}v)\right\}^{d-1} \cdot \left(\exp\left(-\frac{|\,u\,|^2 + |\,v\,|^2}{2} + \overline{u}v\right)\right)^{d-1}$$

Proof

$$< \psi_N(u), \exp\left(z\frac{S_N(S^+)}{\sqrt{N}} - z\frac{S_N(S^-)}{\sqrt{N}}\right)\psi_N(v) >$$

$$=< \psi_N(u), \exp\left[z\frac{1}{\sqrt{N}}\left(\otimes^N E_N\right)(S_N(S^+)) - z\frac{1}{\sqrt{N}}\left(\otimes^N E_N\right)(S_N(S^-))\right]$$

$$\exp\left[z\frac{\hat{S}(S^+)}{\sqrt{N}} - z\frac{\hat{S}(S^-)}{\sqrt{N}}\right]\psi_N(v) >$$

$$=< \psi_N(u), \psi_N(v) > \cdot$$

$$\cdot \exp\left\{\frac{z}{\sqrt{N}}\left(\otimes^N E_N\right)(S_N(S^+)) - \frac{\overline{z}}{\sqrt{N}}\left(\otimes^N E_N\right)(S_N(S^-))\right\}$$

$$\cdot \left(\otimes^N E_N\right)\left\{\exp\left(z\frac{\hat{S}_N(S^+)}{\sqrt{N}} - z\frac{\hat{S}_N(S^-)}{\sqrt{N}}\right)\right\} = I \cdot II \cdot III$$

We know from Lemma 2:

$$I =< \psi_N(u), \psi_N(v) > \to < \psi(u), \psi(v) >^{d-1}$$

Moreover

$$\frac{z}{\sqrt{N}}\left(\otimes^N E_N\right)(S_N(S^+)) - \frac{\overline{z}}{\sqrt{N}}\left(\otimes^N E_N\right)(S_N(S^-))$$

$$= z\sqrt{N} < \psi\left(\frac{u}{\sqrt{N}}\right), \psi\left(\frac{v}{\sqrt{N}}\right) >^{-1} < \psi\left(\frac{u}{\sqrt{N}}\right), S^+\psi\left(\frac{v}{\sqrt{N}}\right) >$$

$$-\overline{z}\sqrt{N} < \psi\left(\frac{u}{\sqrt{N}}\right), \psi\left(\frac{v}{\sqrt{N}}\right) >^{-1} < \psi\left(\frac{u}{\sqrt{N}}\right), S^-\psi\left(\frac{v}{\sqrt{N}}\right) >$$

$$\to (z\overline{u} - \overline{z}v)(d-1)$$

and, because of Lemma 1 and 2

$$II \to (\exp(z\overline{u} - \overline{z}v))^{d-1}$$

For III we have (cf. Theorem (5.1) of [1])

$$\left(\otimes^N E_N\right)\exp\left(z\frac{\hat{S}_N(S^+)}{\sqrt{N}} - \overline{z}\frac{\hat{S}_N(S^-)}{\sqrt{N}}\right)$$

$$= \sum_k \frac{1}{k!} \sum_\varepsilon z^{\varepsilon(1)} \dots z^{\varepsilon(k)} \left(\otimes^N E_N\right) \left(\frac{\hat{S}_N(S^{\varepsilon(1)})}{\sqrt{N}} \dots \frac{\hat{S}_N(\varepsilon^{(k)})}{\sqrt{N}} \right)$$

and as,

$$< \varphi_{d-1}(0),\ S^+ \varphi_{d-1}(0) >= 0$$
$$< \varphi_{d-1}(0),\ S^- \varphi_{d-1}(0) >= 0$$

we have:

$$< \varphi_{d-1}(0), \hat{S}^+ \hat{S}^+ \varphi_{d-1} >= 0$$
$$< \varphi_{d-1}(0), \hat{S}^+ \hat{S}^- \varphi_{d-1} >= 0$$
$$< \varphi_{d-1}(0), \hat{S}^- \hat{S}^- \varphi_{d-1} >= 0$$
$$< \varphi_{d-1}(0), \hat{S}^- \hat{S}^+ \varphi_{d-1} >=< \varphi_{d-1}(0), \hat{S}^- \hat{S}^+ \varphi_{d-1}(0) >=| S^+ \varphi_{d-1}(0) |^2$$
$$=| \sqrt{(d-1)} \varphi_{d-1}(1) |^2 = (d-1)$$

Therefore:

$$III \to \sum_{p=0} \frac{1}{(2p)!} \frac{(2p)!}{p! 2^p} (- | z |^2 (d-1))^p = \left(\exp\left(-\frac{| z |^2}{2} \right) \right)^{d-1} \qquad (18)$$

5 Inclusion of the number operator

Theorem 3 *For any $u, v, z, \in \mathbf{C}$ and any $\lambda \in \mathbf{R}$*

$$\lim_{N\to\infty} < \psi_N \left(\frac{u}{\sqrt{N}} \right), \exp\left\{ z \frac{S_N(S^+)}{\sqrt{N}} - \overline{z} \frac{S_N(S^-)}{\sqrt{N}} + i2\lambda S_N(n^+) \right\} \psi_N \left(\frac{u}{\sqrt{N}} \right) >$$
$$= \psi(u), \exp(za^+ - \overline{z}a + i2\lambda a^+ a) \psi(v) > \qquad (19)$$

Proof. We first evaluate the right hand side of equation (**??**). For $\lambda = 0$ we have

$$< \psi(u),\ W(z),\ \psi(v) >= \exp\left(-\frac{\overline{z}v}{z} + \frac{z\overline{v}}{z} \right) < \psi(u),\ \psi(z+u) >=$$
$$= \exp\left(-\frac{| u |^2}{2} - \frac{| v |^2}{2} - \frac{| z |^2}{2} - \frac{\overline{z}v}{2} - \frac{z\overline{v}}{2} + \overline{u}z + \overline{u}v - \frac{\overline{z}v}{2} - \frac{z\overline{v}}{2} \right) =$$
$$= \exp\left(-\frac{| u |^2}{2} - \frac{| v |^2}{2} - \frac{| z |^2}{2} - \overline{z}v + \overline{u}z + \overline{u}v \right) \qquad (20)$$

Assume now $\lambda \neq 0$, then

$$\exp\left(za^{+} - \overline{z}a + i2\lambda a^{+}a\right)$$

$$= \exp\left(i2\lambda\left(aa^{+} + \frac{z}{2i\lambda}a^{+} - \frac{\overline{z}}{2i\lambda}a\right)\right)$$

$$= \exp\left\{i2\lambda\left(a^{+} - \frac{\overline{z}}{2i\lambda}\right)\left(a + \frac{z}{2i\lambda}\right) - i2\lambda\frac{|z|^2}{4\lambda 2}\right\}$$

$$= \exp\left(-i2\lambda|\zeta|^2\right)\exp\left\{i2\lambda\left(a^{+} + \overline{\zeta}\right)(a + \zeta)\right\} \tag{21}$$

where

$$\zeta := \frac{z}{i2\lambda}, \qquad \overline{\zeta} = -\frac{\overline{z}}{i2\lambda}, \qquad |\zeta|^2 = \frac{|z|^2}{4\lambda^2} \tag{22}$$

therefore,

$$\exp\left(za^{+} - \overline{z}a + i2\lambda a^{+}a\right) = \exp\left(-i2\lambda \mid \zeta \mid^2\right) W^{+}(\zeta) \exp\left(i2\lambda a^{+}a\right) W(\zeta) \tag{23}$$

and for the matrix elements we obtain

$$< \psi(u), \exp\left(za^{+} - \overline{z}a + i2\lambda a^{+}a\right)\psi(v) >= \tag{24}$$

$$= \exp(-i2\lambda \mid \zeta \mid^2)\exp\left(-\frac{\zeta\overline{u}}{2} + \frac{\overline{\zeta}u}{2} - \frac{\overline{\zeta}v}{2} + \frac{\zeta\overline{v}}{2}\right) < \psi(u+S), \exp(i2\lambda a^{+}a)\psi(v+\zeta) >$$

Using the identity

$$\exp(\alpha a^{+}a) = \sum_k \frac{(e^{\alpha} - 1)^k}{k!} a^{+k}a^k \tag{25}$$

we obtain

$$< \psi(u), \exp\left(za + i2\lambda a^{+}a\right)\psi(v) > \tag{26}$$

$$\exp\left(-i2\lambda \mid \zeta \mid^2 - \zeta\overline{u} - \overline{\zeta}v - \frac{\mid v \mid^2}{2} - \frac{\mid u \mid^2}{2} - \mid \zeta \mid^2 + e^{2i\lambda}\left(\overline{u} + \overline{\zeta}\right)(v + \zeta)\right)$$

$$= \exp\Big(-i2\lambda \mid \zeta \mid^2 - \mid \zeta \mid^2 + e^{-i2\lambda} \mid \zeta \mid^2 - \frac{\mid u \mid^2}{2} - \frac{\mid v \mid^2}{2} - \zeta\overline{u} - \overline{\zeta}v + e^{2i\lambda}\overline{u}\zeta + e^{2i\lambda}\overline{\zeta}v\Big)$$

Introducing

$$w = ve^{2i\lambda}$$

we have

$$< \psi(u), \exp\left(za + -\overline{z}a + i2\lambda a^+ a\right) \psi(v) >= \tag{27}$$

$$= \exp\Big\{ \left(e^{2i\lambda} - 2i\lambda - 1\right) \mid \zeta \mid^2 - \frac{\mid u \mid^2}{2} - \mid w \mid^2 + \overline{u}w + \left(e^{2i\lambda} - 1\right) \overline{u}\zeta + \left(1 - e^{-2i\lambda}\right) \overline{\zeta} w \Big\}$$

Transforming back to $z, \overline{z}$ we obtain

$$< \psi(u), \exp(za^+ - \overline{z}a + 2i\lambda a^+ a)\psi(v) >= \tag{28}$$

$$= \exp\left\{ \frac{e^{2i\lambda} - 2i\lambda - 1}{4\lambda^2} \mid z \mid^2 - \frac{\mid u \mid^2}{2} - \frac{\mid w \mid^2}{2} + \overline{u}w + \frac{e^{2i\lambda} - 1}{i2\lambda}\overline{u}z - \frac{e^{2i\lambda} - 1}{i2\lambda}\overline{z}w \right\}$$

In the limit $\lambda \to 0$ we have

$$w = ve^{i2\lambda} \to v \ ; \ \frac{e^{2i\lambda} - 1}{i2\lambda} \to 1 \ , \ \frac{e^{-2i\lambda} - 1}{-i2\lambda} \to 1 \ , \ \frac{e^{2i\lambda} - 2i\lambda - 1}{4\lambda^2} \to \frac{1}{2}$$

so that (**??**) is valid for all values of $\lambda \in \mathbf{R}$. In order to evaluate the left hand side of (**??**), we notice that

$$< \psi_N \left(\frac{u}{\sqrt{N}}\right), \exp\left(z\frac{S_N(s^+)}{\sqrt{N}} - \overline{z}\frac{S_N(s^-)}{\sqrt{N}} + 2i\lambda\, S_N(n^+)\right) \psi_N \left(\frac{v}{\sqrt{N}}\right) >=$$

$$=< \psi \left(\frac{u}{\sqrt{N}}\right), \exp\left(\frac{z}{\sqrt{N}}s^+ - \frac{\overline{z}}{\sqrt{N}}s^- + 2i\lambda n^+\right) \psi \left(\frac{v}{\sqrt{N}}\right) >^N \tag{29}$$

We may use $n^+ = \frac{1}{2}(\sigma^3 + 1)$ and, denoting

$$\underline{t} = \left(Im\frac{z}{\sqrt{N}}, Re\frac{z}{\sqrt{N}}, \lambda\right) \in \mathbf{R}^3$$

we obtain

$$\exp(i\underline{t} \times \underline{\sigma}) = \cos(\mid t \mid)\underline{u} + i\frac{\underline{t} \cdot \underline{\sigma}}{\mid t \mid}\sin(\mid t \mid) \tag{30}$$

$$\exp\left(\frac{z}{\sqrt{N}}s^+ - \frac{\overline{z}}{\sqrt{N}}s^- + 2i\lambda n^+\right) = \exp(i\lambda)\exp(i\underline{t} \cdot \underline{\sigma}) \tag{31}$$

Since

$$\mid t \mid^2 = \frac{\mid z \mid^2}{N} + \lambda^2 \qquad , \qquad \mid t \mid = \lambda\sqrt{\frac{\mid z \mid^2}{\lambda^2 N} + 1} \tag{32}$$

We can collect the leading terms in (**??**) finding (in obvious notations)

$$\underline{1}: \quad \cos(\mid t \mid) = \cos\lambda - \frac{\mid z \mid^2}{2\lambda N}\sin\lambda + O\left(\frac{1}{N^2}\right) \tag{33}$$

$$\sigma_3: \quad \lambda\frac{\sin(\mid t \mid)}{\mid t \mid} = \sin\lambda + \frac{\mid z \mid^2}{2\lambda N}\cos\lambda - \frac{\mid z \mid^2}{2\lambda^2 N}\sin\lambda + O\left(\frac{1}{N^2}\right) \tag{34}$$

$$\sigma_1: \quad \Im\frac{z}{\sqrt{N}}\cdot\frac{\sin(\mid t \mid)}{\mid t \mid} = \left(\Im\frac{z}{\sqrt{N}}\right)\frac{\sin\lambda}{\lambda} + O\left(N^{-3/2}\right) \tag{35}$$

$$\sigma_2: \quad Re\frac{z}{\sqrt{N}}\cdot\frac{\sin(\mid t \mid)}{\mid t \mid} = \left(Re\frac{z}{\sqrt{N}}\right)\frac{\sin\lambda}{\lambda} + O\left(N^{-3/2}\right) \tag{36}$$

Therefore,

$$\exp\left(\frac{z}{\sqrt{N}}s^+ - \frac{\overline{z}}{\sqrt{N}}s^- + 2i\lambda n^+\right) \cong$$

$$\cong e^{i\lambda}\begin{pmatrix} e^{i\lambda} + i\frac{|z|^2}{2\lambda N}\left(e^{i\lambda} - \frac{\sin\lambda}{\lambda}\right) & \frac{\sin\lambda}{\lambda}\frac{z}{\sqrt{n}} \\ -\frac{\sin\lambda}{\lambda}\frac{\overline{z}}{\sqrt{N}} & e^{i\lambda} - i\frac{|z|^2}{2\lambda N}\left(e^{i\lambda} - \frac{\sin\lambda}{\lambda}\right) \end{pmatrix} \tag{37}$$

Moreover,

$$\psi\left(\frac{v}{\sqrt{N}}\right) = \begin{pmatrix} \frac{v}{\sqrt{N}} \\ \sqrt{1 - \frac{|v|^2}{N}} \end{pmatrix} \cong \begin{pmatrix} \frac{v}{\sqrt{N}} \\ 1 - \frac{1}{2}\frac{|v|^2}{N} \end{pmatrix} \tag{38}$$

and analogously, for $\psi(v/\sqrt{N})$. In the limit $\lambda \to 0$ we have

$$\frac{e^{\pm i\lambda} - \frac{\sin\lambda}{\lambda}}{\lambda} \to \pm i \tag{39}$$

so that the matrix (**??**) is reduced (up to terms of order $o(1/N)$) to

$$\begin{pmatrix} 1 - \frac{|z|^2}{2N} & \frac{z}{\sqrt{N}} \\ -\frac{\overline{z}}{\sqrt{N}} & 1 - \frac{|z|^2}{2N} \end{pmatrix} \cong \exp\left(\frac{z}{\sqrt{N}}s^+ - \frac{\overline{z}}{\sqrt{N}}s^-\right) \tag{40}$$

For the matrix elements we obtain in the limit $\lambda \to 0$

$$< \psi\left(\frac{u}{\sqrt{N}}\right), \exp\left(\frac{z}{\sqrt{N}}S^+ - \frac{\overline{z}}{\sqrt{N}}S^-\right)\psi\left(\frac{v}{\sqrt{N}}\right) >$$

$$1 - \frac{\mid z \mid^2}{2N} - \frac{\mid v \mid^2}{2N} - \frac{\mid u \mid^2}{2N} + \frac{\bar{u}v}{N} + \frac{\bar{u}z}{N} - \frac{\bar{z}v}{N} \tag{41}$$

which , as $N \to \infty$, converges to

$$< \psi(u), \exp(za^+ - \bar{z}a)\psi(v) >= \exp\left(-\frac{\mid z \mid^2}{2} - \frac{\mid u \mid^2}{2} - \frac{\mid v \mid^2}{2} + \bar{u}v + \bar{u}z - \bar{z}v\right) \tag{42}$$

We now evaluate the matrix elements of (**??**). This gives

$$\frac{\bar{u}v}{N}e^{2i\lambda} + \frac{\sin\lambda}{\lambda}\frac{z\bar{u}}{N}e^{i\lambda} - \frac{\sin\lambda}{\lambda}\frac{\bar{z}v}{N}e^{i\lambda} + 1$$

$$-i\frac{|\,z\,|^2}{2N\lambda}e^{i\lambda}\left(e^{-i\lambda} - \frac{\sin\lambda}{\lambda}\right) - \frac{|\,v\,|^2}{2N} - \frac{|\,u\,|^2}{2N} \tag{43}$$

$$\cong < \psi\left(\frac{u}{\sqrt{N}}\right)\exp\left(\frac{z}{\sqrt{N}}s^+ - \frac{\overline{z}}{\sqrt{N}}s^- + 2i\lambda n^+\right)\psi\left(\frac{v}{\sqrt{N}}\right) >$$

Therefore

$$< \psi\left(\frac{u}{\sqrt{N}}\right)\exp\left(z\frac{S_N(s^+)}{\sqrt{N}} - \overline{z}\frac{S_N(s^-)}{\sqrt{N}} + 2i\lambda S_N(n^+)\right)\psi\left(\frac{v}{\sqrt{N}}\right) >$$

converges to

$$\exp\left\{-\frac{|\,u\,|^2}{2} - \frac{|\,v\,|^2}{2} - i\frac{|\,z\,|^2}{2}e^{i\lambda}\frac{e^{i\lambda} - \frac{\sin\lambda}{\lambda}}{\lambda} + \overline{u}ve^{2i\lambda} + \frac{\sin\lambda}{\lambda}z + \overline{u}e^{i\lambda} - \frac{\sin\lambda}{\lambda}\overline{z}ve^{i\lambda}\right\} = \tag{44}$$

$$= \exp\left\{-\frac{|\,u\,|^2}{2} - \frac{|\,w\,|^2}{2} - i\frac{|\,z\,|^2}{2}\frac{e^{i\lambda}}{\lambda}\left(e^{-i\lambda} - \frac{\sin\lambda}{\lambda}\right) + \overline{u}w + \overline{u}ze^{i\lambda}\frac{\sin\lambda}{\lambda} - \overline{z}e^{-i\lambda}\frac{\sin\lambda}{\lambda}\right) \tag{45}$$

Finally comparing

$$\frac{e^{2i\lambda} - 1}{2i\lambda} = e^{i\lambda}\frac{e^{i\lambda} - e^{-i\lambda}}{2i\lambda} = e^{i\lambda}\frac{\sin\lambda}{\lambda}$$

$$\frac{e^{-2i\lambda} - 1}{-2i\lambda} = e^{-i\lambda}\frac{e^{-i\lambda} - e^{i\lambda}}{-2i\lambda} = e^{-i\lambda}\frac{\sin\lambda}{\lambda}$$

$$\frac{e^{2i\lambda} - 2i\lambda - 1}{4\lambda^2} = \frac{e^{i\lambda}}{2i\lambda}\left\{e^{i\lambda} - 2i\lambda e^{-i\lambda} - \frac{e^{-i\lambda}}{-2i\lambda}\right\} = -i\frac{e^{i\lambda}}{2\lambda}\left(e^{i\lambda} - \frac{\sin\lambda}{\lambda}\right)$$

with (**??**), we obtain the thesis.

6 Squeezing states

Let a, a^+ be a representation of the CCR on a certain invariant domain D in a Hilbert space H and vacuum vector Φ.

Definition 1 *A vector* χ *is called a squeezing vector for the representation* $\{a^{\pm}, D \subseteq H, \Phi\}$ *if it can be represented in the form*

$$\chi = e^{(za^{+2} - \overline{z}a^{2})} W(z')\Phi \tag{46}$$

The unitary operator $\exp(za^{+2} - \overline{z}a^{2})$ *is called the squeezing operator The exponential in (46) is defined by its power series on the number vectors, which are analytic for* $(a^{\pm})^2$*.*

Theorem 4 *For any* $u, v, z, \in, \mathbf{C}$

$$\lim < \psi_N\left(\frac{u}{\sqrt{N}}\right), \exp\left\{z\left(\frac{S_N(s^+)}{\sqrt{N}}\right) - \overline{z}\left(\frac{S_N(s^-)}{\sqrt{N}}\right)^2\right\}\psi_N\left(\frac{v}{\sqrt{N}}\right) >$$

$$=< \psi(u), \exp(za^{+2} - \overline{z}a^{2})\psi(v) > \tag{47}$$

The proof of the theorem consists of two steps first to compute explicitly the matrix elements of the right hand side of (47) in the coherent vectors; then to prove that the limit on the left hand side exists and the equality holds. We shall only outline the first step, while we give the full proof of the second one.

The explicit computation of the right hand side of (47) is based on the formula

$$e^{\frac{1}{2}(za^{+2} - \overline{z}a^{2})} = e^{\frac{1}{2}e^{i\alpha}\tan|z|a^{+2}} \cdot e^{-\log(\cos|z|)(a^{+}a+1/2} \cdot e^{-\frac{1}{2}e^{-i\alpha}\tan|z|a^{2}} \tag{48}$$

which implies:

$$< \psi(u), e^{\frac{1}{2}(za^{+2} - \overline{z}a^{2})}\psi(v) >= \tag{49}$$

$$= \exp\left(-\frac{| u |^2 + | v |^2}{2} + \frac{1}{2}e^{i\alpha}\overline{u}^2 \tanh | z | - \frac{1}{2}e^{-i\alpha}\overline{v}^2 \tanh | z | - (\frac{1}{2} + \bar{u}v)\log\cosh | z |\right)$$

To control the limit on the left hand side of (47), we expand the exponential and show, in the following lemma, that the limit and the summation sign can be interchanged. To this goal we introduce the following notation: for any $u, v, z, \in \mathbf{C}, N \in \mathbf{N}$ we set

$$a_{k,N} = \frac{1}{k!} < \psi_N(\frac{u}{\sqrt{N}}), \left\{z\frac{S_N^2(s^+)}{N} - \overline{z}\frac{S_N^2(s^-)}{N}\right\}^k \psi_N\left(\frac{v}{\sqrt{N}}\right) >$$

Lemma 3 *For any $\varepsilon > 0$ there exists N_ε and h_ε such that for any $N > N_\varepsilon$ and any $h > h_\varepsilon$*

$$\sum_{k=h}^{\infty} |a_{k,N}| < \varepsilon$$

Proof. We set

$$f_N : b \in M(2;\mathbf{C}) \to f_N(b) = < \psi\left(\frac{u}{\sqrt{N}}\right), b\psi\left(\frac{v}{\sqrt{N}}\right) > \in \mathbf{C}$$

and

$$E_N : b \in \otimes^N M(2;\mathbf{C}) \to E_N(b) = \frac{\left(\otimes^N f_N\right)(b)}{\left(\otimes^N f_N\right)(1_N)} \in \mathbf{C}$$

With this notation

$$a_{k,N} = \frac{1}{k!}\left(\otimes^N f_N\right)(1_N) E_N \left(z\frac{S_N^2(s^+)}{N} - \overline{z}\frac{S_N^2(s^-)}{N}\right)^k$$

$$= \frac{1}{k!}\left(\otimes^N f_N\right)(1_N) E_N \left(\sum_{\varepsilon=1}^{6} \gamma_N^{(\varepsilon)} J_N^{(\varepsilon)}\right)^k$$

where

$$\gamma_N^{(1)} = z\frac{E_N^2(S_N(s^+))}{N}\ , \ J_N^{(1)} = 1_N$$

$$\gamma_N^{(2)} = -\overline{z}\frac{E_N^2(S_N(s^-))}{N}\ , J_N^{(2)} = 1_N$$

$$\gamma_N^{(3)} = 2z\frac{E_N(S_N(s^+))}{\sqrt{N}}\ , J_N^{(3)} = \frac{S_N^{\wedge}(s^+)}{\sqrt{N}}$$

$$\gamma_N^{(4)} = -2\overline{z}\frac{E_N(S_N(s^-))}{\sqrt{N}}\ , \ J_N^{(4)} = \frac{S_N^{\wedge}(s^-)}{\sqrt{N}}$$

$$\gamma_N^{(5)} = z \qquad , \ J_N^{(5)} = \frac{S_N^{\wedge 2}(s^+)}{N}$$

$$\gamma_N^{(6)} = -\overline{z} \qquad , \ J_N^{(6)} = \frac{S_N^{\wedge 2}(s^-)}{N}$$

and where we have used the notation (cf [1])

$$\hat{X} = X - E_N(X)$$

With these notations:

$$a_{k,n} = \frac{1}{k!}\left(\otimes^N f_N\right)(1_N)$$

$$\sum_{(\varepsilon_1,\dots,\varepsilon_k)\in\{1,\dots,6\}^k} \gamma_N^{(\varepsilon_1)}\dots\gamma_N^{\varepsilon_k} E_N\left(J_N^{(\varepsilon_1)}\dots J_N^{(\varepsilon_k)}\right)$$

We know that

$$\left|\left(\otimes^N f_N\right)(1_N)\right| \le 1 \qquad ; \qquad \left|\frac{E_N(S_N(s^{\pm}))}{\sqrt{N}}\right| \le C_1 + \frac{C_2}{N} \tag{50}$$

Therefore, for some $(\varepsilon_1,\dots,\varepsilon_k)\in\{1,\dots,6\}^k$ one has

$$|a_{k,n}| \le \frac{1}{k!}6^k C_N^k |E_N\left(J_N^{(\varepsilon_1)}\dots J_N^{(\varepsilon_k)}\right)|$$

where, because of (50)

$$C_N = C_3 + \frac{C_4}{N}$$

and C_3 depends upon $\overline{u}, v, z, \overline{z}$. The products $J_N^{(\varepsilon_1)}\dots J_N^{(\varepsilon_k)}$ consist of a scalar times j $(0 \le j \in 2k)$, terms of the form $\frac{\hat{S}_N(s^{\pm})}{\sqrt{N}}$. From the central limit theorem estimate of [1] we know that for any $j \in \mathbf{N}$

$$|E_N\left(\frac{\hat{S}_N(b_i)}{\sqrt{N}}\dots\frac{\hat{S}_N(b_j)}{\sqrt{N}}\right)| \le C_5\binom{j}{2} + C_6^j\frac{1}{\sqrt{N}} \le C_5\binom{2k}{2} + C_6^{2k}\frac{1}{\sqrt{N}}$$

In conclusion we obtain

$$| a_{k,n} | \le \frac{1}{k!}6^k\left(C_3 + \frac{C_4}{N}\right)^k\left\{C_5\binom{2k}{2} + C_6^{2k}\frac{1}{\sqrt{N}}\right\}$$

$$\le \frac{1}{k!}\left(C_7 + \frac{C_8}{N}\right)^k\frac{1}{\sqrt{N}} + \frac{C_5}{k!}\left(C_9 + \frac{C_{10}}{N}\right)^k\frac{(2k)!}{2(2k-2)!}$$

$$\le \frac{1}{k!}\left(C_7 + \frac{C_8}{N}\right)^k\frac{1}{\sqrt{N}} + \frac{(2k)(2k-1)}{k!}C_{11}\left(C_9 + \frac{C_{10}}{N}\right)^k$$

from wich the thesis follows.

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