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Centrality measure of complex networks using biased random walks

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Abstract. We propose a novel centrality measure based on the dynamical properties of a biased random walk to provide a general framework for the centrality of vertex and edge in scale-free networks (SFNs). The suggested centrality unifies various centralities such as betweenness centrality (BC), load centrality (LC) and random walk centrality (RWC) when the degree, k, is relatively large. The relation between our centrality and other centralities in SFNs is clearly shown by both analytic and numerical methods. Regarding to the edge centrality, there have been few established studies in complex networks. Thus, we also provide a systematic analysis for the edge BC (LC) in SFNs and show that the distribution of edge BC satisfies a power-law. Furthermore we also show that the suggested centrality measures on real networks work very well as on the SFNs.

PACS. 89.75.Hc Networks and genealogical trees – 05.40.Fb Random walks and Levy flights – 89.20.-a Interdisciplinary applications of physics

Many of recent studies on complex networks have uncovered the relationship between the underlying network topologies and the dynamical processes observed in diverse field such as physics, biology, engineering, and social sciences. These include the synchronization transition [1], epidemic spreading [2], and transmission of information [3]. One of the most significant findings is the existence of some important vertices or edges to characterize the dynamical properties on the networks. These are sometimes referred as dynamical "importance" or "centrality" of vertices and edges [4]. For example, in diffusive systems the vertices of large degree play a crucial role in determining the dynamical properties [5,6], which are decisive to resolve the traffic jam at a bottleneck [7]. Not only the dynamical properties, but also many interesting topological properties such as percolation transition [8] can be characterized by the important elements of a network. One of the substantial applications is the identification of the essential components of biological networks in developing a new drug [9–12]. Therefore, classifying the important vertices and edges is essential for both practical applications and deep understanding of many topological and dynamical phenomena on complex networks. However, the suggested definitions for the important elements of a network are completely different from each other depending on the physical properties under consideration [4,13–25]. Thus, we propose a unified centrality in this study to define important vertices and edges based on the dynamical properties of biased random walks (BRW). It also provide a clue to understand how we use the dynamical phenomena to reveal the topological properties of a underlying network.

There have been various studies on the centrality of vertices. Freeman introduced betweenness centrality (BC) based on the assumption that transmission of information spreads along the shortest paths [13]. BC of the vertex i is defined by the number of shortest paths that pass through i. More specifically, let $L_{h,j}$ be the total number of shortest paths from a vertex h to another vertex j and $L_{h,i,j}$ be the number of the shortest paths that pass through the vertex i. Then BC of vertex i, $b_v(i)$, is given by $b_v(i) = \frac{2}{N(N-1)} \sum_{h<j} \frac{L_{h,i,j}}{L_{h,j}}$, where N is the total number of vertices in the network. For practical measure of the centrality in traffic flow, load centrality (LC) of vertices was studied by measuring the load on each vertex [14]. If the traffic flows along the shortest paths, then BC and LC become equivalent. The distribution of BC or LC satisfies the same power-law [14–16]

$$P(b_v) \sim {b_v}^{-\delta_v}, \quad (1)$$

where the exponent δ_v is close to 2 [17]. Recently, Newman proposed another centrality of a vertex by net flow of random walker which does not flow along the shortest paths [18]. This centrality is known to be particularly useful for finding vertices of high centrality that do not happen to lie on the shortest paths and shown to have

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a strong correlation with degree and BC [18]. Noh et al. also studied the centrality based on the mean-first passage time of random walk and showed that the random walk centrality (RWC) of a vertex is related to the probability of finding a walker at a vertex [22]. Flow betweenness centrality (FBC) based on the maximum flow was studied [23]. FBC is defined as the amount of flow through a vertex. Thus, centrality of a given network is closely related to the network transport property.

Compared to the studies on the vertex centrality, there have been few studies on the edge centrality. The simplest edge centrality is the strength [24] of each edge which especially plays an important role in social systems [25]. Moreover, the edge centrality can be used in classifying the modular structure [26]. Despite such important roles of edge centrality, there have been no organized or comprehensive theories on the edge centrality of the network. In this study, we introduce a biased random walk centrality (BRWC) of a vertex (or an edge) as the number of traversals by the walker, n_v (or n_e). As we shall see, BRWC gives the comprehensive physical picture which unifies the various centralities for a vertex or an edge. Furthermore, BRWC gives an organized analytical tool to understand the statistical properties of the edge centrality.

Let us define the biased random walk. Initially, a walker is placed at a randomly chosen vertex on a given network. At each time step t, the walker at a vertex i jumps to one of the linked neighbors j with the probability $P_{i\to j} = k_j^\alpha / \sum_{j=1}^{k_i} k_j^\alpha$. Here k_j is the degree of the vertex j and α determines the bias of walker. BRWC is defined as the total number of traversals for a vertex n_v (or an edge n_e), i.e. how often a vertex (or an edge) is traversed by a walker. In the limit $\alpha \to \infty$, the walker is trapped at the hub with the maximal degree and its neighbors. Therefore, the hub or edges connected to the hub becomes the central part in the network. On the other hand, when $\alpha \to -\infty$ the walker visits dangling ends more frequently. When $\alpha = 0$, $P_{i\to j}$ does not depend on k and BRWC is simply proportional to the probability to find a walker at a vertex. The probability has been shown to be proportional to RWC [22]. This indicates that BRWC becomes RWC when $\alpha = 0$. Hence we expect that BRWC can determine the structural hierarchy of a network [27] by changing α.

To find out the scaling properties of BRWC, we use static scale-free network (SFN) model, whose degree distribution follows a power-law $P(k) \sim k^{-\gamma}$ with tunable γ [14]. The average degree of the network is fixed to be $\langle k\rangle = 4$ and the number of vertices is $N = 10^5$. We numerically find that the distribution of BRWC follows a power-law $P(n_v) \sim n_v{}^{-\sigma_v}$ (see Fig. 1a). Here, the exponent σ_v depends on α and γ. We find that BRWC scales exactly the same way with BC for a specific α which depends on γ. For such specific α, we confirm that $\sigma_v = \delta_v \simeq 2.2$ for $2 < \gamma \leq 3$ and $\sigma_v(=\delta_v)$ increases continuously for $\gamma > 3$. For example, $\sigma_v = \delta_v \simeq 2.2$ when $\alpha = 2/3$ for $\gamma = 3.0$ and $\sigma_v = \delta_v \simeq 3.2$ when $\alpha = 0.5$ for $\gamma = 4.3$ (Fig. 1a).

In order to find the relationship between BRWC and BC of a vertex, we note the role of k. It is known that

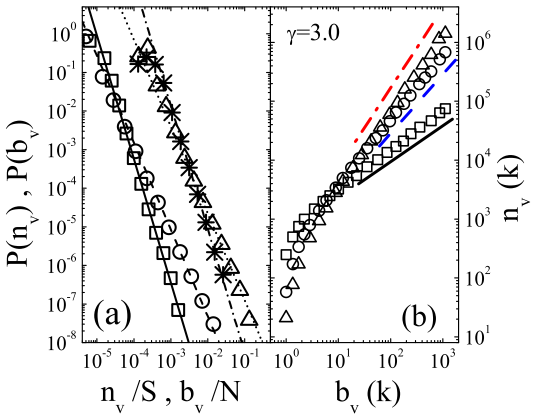

Fig. 1. (Color online) (a) $P(n_v)$ (◯ and □) and $P(b_v)$ (△ and ∗) for a vertex on SFNs with $\gamma = 3$ (◯ and △) and 4.3 (□ and ∗). We use $\alpha = 2/3$, and 0.5 for SFNs with $\gamma = 3.0$, and 4.3, respectively. Here S is the total number of steps of BRW. The obtained exponents are $\sigma_v = \delta_v = 2.2(1)$ (dashed and dotted lines) and 3.2(1) (solid and dashed-dotted lines). (b) $n_v(k)$ against $b_v(k)$ for $\alpha = 0$ (□), 2/3 (◯), and 1 (△) on SFN with $\gamma = 3.0$. The lines indicate $\beta_v = 0.7$ (solid line), 1.0 (dashed line), and 1.3 (dashed-dotted line).

the probability of finding a BRW at vertices with degree k is $P^\infty(k) \sim k^{\alpha+1-\gamma}$ [6]. Then BRWC of a vertex having degree k, $n_v(k)$, is

$$n_v(k) \sim P^\infty(k)/P(k) \sim k^{\alpha+1} \sim k^{\nu_v}, \quad (2)$$

when the degree-degree correlation is absent. The BC of a vertex having degree k, $b_v(k)$, is known to scale as

$$b_v(k) \sim k^{\eta_v} \quad (3)$$

with $\eta_v = (\gamma-1)/(\delta_v-1)$ for $2 < \gamma \leq 3$ [14]. From equations (2) and (3), we find a power-law relation between BRWC and BC;

$$n_v(k) \sim b_v(k)^{\beta_v}, \quad (4)$$

with

$$\beta_v = \nu_v/\eta_v = \frac{(\alpha+1)(\delta_v-1)}{(\gamma-1)}. \quad (5)$$

We numerically find that equation (5) holds for any γ when k is relatively large. Figure 1b shows $n_v(k)$ against $b_v(k)$ for $\alpha = 0$, 2/3, and 1 on SFN when $\gamma = 3.0$. Since $\delta_v = 2.2$ for $2 < \gamma \leq 3$ [14], the analytic expectation gives $\beta_v = 0.7$, 1.0, and 1.3 for $\alpha = 0$, 2/3, and 1, respectively, when $\gamma = 3$. The obtained values of β_v agree with the analytic expectations for relatively large k (see the lines in Fig. 1b). Equation (4) implies that BRWC and BC scale exactly in the same way when $\beta_v = 1$. Using equation (5) and the known exponent $\delta_v = 2.2$ [14–16], we can estimate α which gives $\beta_v = 1$ or the scaling of BRWC is exactly the same as that of BC for $2 < \gamma \leq 3$ [28]. When $\alpha = 0$ equation (2) recovers the simple RWC as $n_v(k) \sim k$ [22]. Therefore, BRWC of a vertex unifies centralities suggested so far including BC (LC) and RWC for relatively large k. For small k regime the deviation is inevitable, because

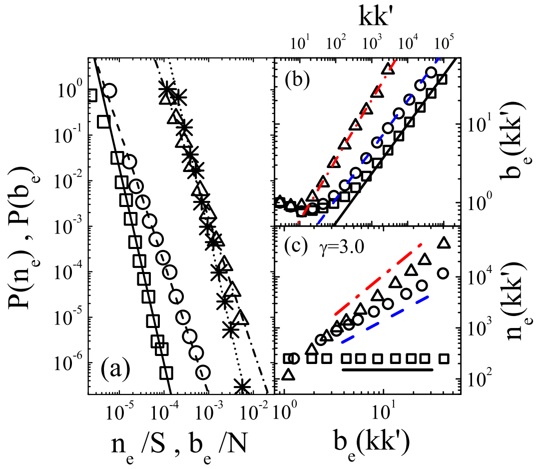

Fig. 2. (Color online) (a) Plots of $P(n_e)$ (◯ for $\gamma = 3.0$ and □ for $\gamma = 4.3$) and $P(b_e)$ (△ for $\gamma = 3.0$ and $*$ for $\gamma = 4.3$). We use $\alpha = 2/3$, and 0.75 for SFNs with $\gamma = 3.0$, and 4.3, respectively. S is the total number of steps of BRW. (b) Plot of $b_e(kk')$ against kk' for $\gamma = 2.75$ (□), 3.0 (◯), and 4.3 (△). (c) $n_e(kk')$ against $b_e(kk')$ for $\alpha = 0$ (□), 2/3 (◯), and 1 (△) on SFN with $\gamma = 3.0$. The lines indicate $\beta_e = 0$ (solid line), 1.0 (dashed line), and 1.5 (dashed-dotted line).

$P(k)$ does not satisfies the power-law for small k. Therefore, γ dependent relations are not valid for small k, for example equation (5).

Before discussing BRWC of an edge, we study the edge BC, b_e, which is defined and calculated in references [23,25]. The distribution of b_e, $P(b_e)$, in mobile phone call network (MPCN) is investigated [25]. Since $P(k)$ does not follow a power-law in MPCN, the measured $P(b_e)$ deviates from a power-law. In contrast to MPCN, we find that $P(b_e)$ in SFN follows a power-law

$$P(b_e) \sim b_e^{-\delta_e}. \tag{6}$$

Similar to the case of vertex BC, we find that $\delta_e \simeq 3.0$ for $2 < \gamma \leq 3$ and δ_e increases continuously for $\gamma > 3$. For example, we obtain that $\delta_e \simeq 3.0$ for $\gamma = 3$ and $\delta_e \simeq 4.3$ for $\gamma = 4.3$ from the data in Figure 2a. We also find that $b_e(kk')$ of an edge, whose ends have degrees k and k', scales as

$$b_e(kk') \sim (kk')^{\eta_e}, \tag{7}$$

where $\eta_e \simeq 0.66$ when $2 < \gamma \leq 3$ and η_e increases continuously when $\gamma > 3$, for example, we obtain $\eta_e \simeq 0.77$ when $\gamma = 4.3$ (see Fig. 2b). This relation is again valid for relatively large kk'. Next, we discuss the scaling properties of BRWC for an edge, n_e, which provides a generalized framework for the study on edge centralities in complex networks. The data in Figure 2a shows that the distribution of n_e in SFN follows a power-law

$$P(n_e) \sim n_e^{-\sigma_e}, \tag{8}$$

where σ_e depends on α and γ. We also find that BRWC of an edge scales exactly the same way with edge BC for a specific α, which varies with γ when $\gamma > 3$. However, in contrast to the case of vertex BRWC, such a specific value of α does not depend on γ when $2 < \gamma \leq 3$ (as shown in Fig. 2b) we obtain $\alpha \simeq 2/3$ for $2 < \gamma \leq 3$). For such specific α, we confirm that $\sigma_e = \delta_e \simeq 3.0$ for $2 < \gamma \leq 3$ and $\sigma_e(=\delta_e)$ increases continuously for $\gamma > 3$. For example, we obtain that $\sigma_e = \delta_e \simeq 3.0$ when $\alpha = 2/3$ for $\gamma = 3$ and $\sigma_e = \delta_e \simeq 4.3$ when $\alpha = 0.75$ for $\gamma = 4.3$ (Fig. 2a).

In order to find the relationship between n_e and b_e, we also note the power-law relation (7). If the degree-degree correlation is absent, then BRWC of an edge whose ends have degrees k and k', $n_e(kk')$, can be expressed as

$$n_e(kk') \sim \frac{1}{k}\frac{P^\infty(k)}{P(k)}\frac{1}{k'}\frac{P^\infty(k')}{P(k')} \sim (kk')^\alpha \sim (kk')^{\nu_e} \tag{9}$$

with $\nu_e = \alpha$. From equations (7) and (9), we find

$$n_e(kk') \sim b_e(kk')^{\beta_e}, \tag{10}$$

with

$$\beta_e = \nu_e/\eta_e = \alpha/\eta_e. \tag{11}$$

Equations (10) and (11) are verified by the numerical simulations. Figure 2c shows $n_e(kk')$ against $b_e(kk')$ for $\alpha = 0, 2/3$, and 1 when $\gamma = 3.0$. The analytic expectation gives $\beta_e = 0, 1.0$, and 1.5 for $\alpha = 0, 2/3$, and 1, respectively, when $\gamma = 3$. The obtained values of β_e agree with the analytic expectations for relatively large kk' (see the lines in Fig. 2c). Similar to the vertex centrality, equation (10) implies that BRWC and BC of an edge scale in the same way when $\beta_e = 1$. Using equation (11) and $\delta_e \simeq 0.66$, we can estimate α which gives $\beta_e = 1$ for $2 < \gamma \leq 3$. When $\alpha = 0$, the probability to find a walker at each vertex becomes the same (or $n_e(kk') \sim$ const.) for any edge. Therefore, BRWC of an edge gives a generalized framework for the study on edge centralities in complex networks when kk' is large.

As an application to a real network, we numerically measure BRWC in protein-protein interaction network (PIN) [10,29]. The used PIN has 1867 vertices and $P(k)$ satisfies the power-law, $P(k) \sim k^{-\gamma}$, with $\gamma \simeq 2.4$. The average degree is $\langle k \rangle \approx 2.4$. From the measurement of $P(b_v)$, we obtain $\delta_v \simeq 2.2$ for large k which is consistent with the previous studies (Fig. 3a) [15]. To compare the obtained results of $P(n_v)$ with $P(b_v)$ in PIN, we measure $P(n_v)$ for various α. We find that $\sigma_v = \delta_v(\simeq 2.2)$ at $\alpha \simeq 0.5$ (see Fig. 3a). This value of α deviates from the value of α calculated from equation (5) with $\delta_v(\simeq 2.2)$ and $\gamma(\simeq 2.4)$. Figure 3b shows $n_v(k)$ against $b_v(k)$ for various α(=0, 0.5, and 0.75). From the best fit of equation (4) to the data, we obtain $\beta_v = 0.8$, 1.0, and 1.2. As shown in Figure 3a, we find that $\sigma_v \simeq 2.2$ for $\alpha = 0.5$ and the relation $\sigma_v = \delta_v$ is satisfied within the estimated errors. The inset of Figure 3b shows $n_v(k)$ of a vertex of degree k. From the best fit of equation (2) to the data, we obtain $\nu_v = 1.0$, 1.3, and 1.45 for $\alpha = 0$, 0.5, and 0.75, respectively. These values of ν_v slight deviate from analytic derivation, $\alpha + 1 = \nu_v$ (Eq. (2)). We expect that the discrepancies between the analytic results and measurements come from the detailed structure of the underlaying network such as degree-degree correlation.

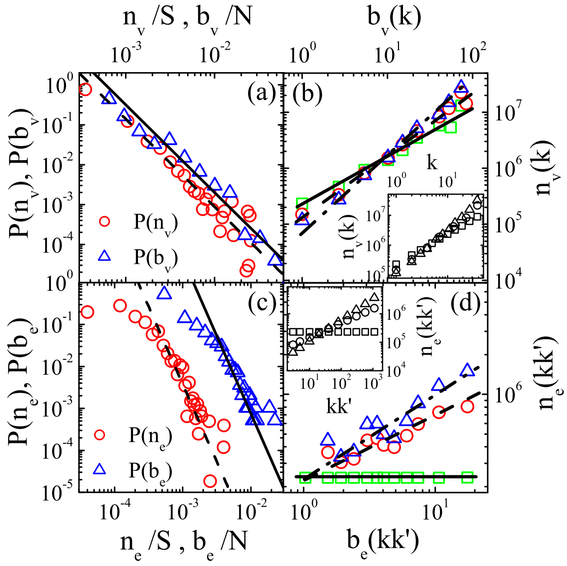

Fig. 3. (Color online) (a) Plots of $P(n_v)$ and $P(b_v)$ in PIN. S is the total number of steps of BRW. We use $\alpha = 0.5$. The dashed line and solid line represent the power-law $P(n_v) \sim {n_v}^{-2.2}$ and $P(b_v) \sim {b_v}^{-2.2}$, respectively. (b) Plot of $n_v(k)$ against $b_v(k)$ with $\alpha = 0$ (□), 0.5 (◯), and 0.75 (△). The lines indicate $\beta_v = 0.8$ (solid line), 1.0 (dashed line), and 1.2 (dashed-dotted line). The inset shows the plot of $n_v(k)$ against k. (c) Plot of $P(n_e)$ and $P(b_e)$ against n_e and b_e. We use $\alpha = 0.75$. The obtained exponents are $\sigma_e = \delta_e \simeq 3.7$ (dashed and solid lines). (d) Plot of $n_e(kk')$ against $b_e(kk')$ with $\alpha = 0$ (□), 0.5 (◯), and 0.75 (△). The lines indicate $\beta_e = 0$ (solid line), 0.65 (dashed line), and 1.0 (dashed-dotted line). In the inset we verified the relation $n_e(kk') \sim (kk')^{\nu_e}$.

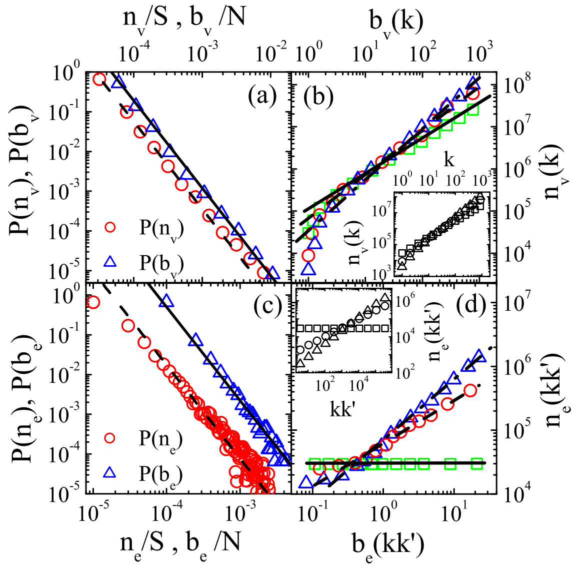

Fig. 4. (Color online) (a) Plots of $P(n_v)$ and $P(b_v)$ in IAS. We use $\alpha = 0.5$. The dashed line and solid line represent the power-law $P(n_v) \sim {n_v}^{-2.3}$ and $P(b_v) \sim {b_v}^{-2.3}$, respectively. (b) Plot of $n_v(k)$ against $b_v(k)$ with $\alpha = 0$ (□), 0.5 (◯), and 0.75 (△). The lines indicate $\beta_v = 0.85$ (solid line), 1.0 (dashed line), and 1.2 (dashed-dotted line). The inset shows the plot of $n_v(k)$ against k. (c) Plot of $P(n_e)$ and $P(b_e)$. The obtained exponents are $\sigma_e = \delta_e \simeq 2.3$ (dashed and solid lines) with $\alpha = 0.75$. (d) Plot of $n_e(kk')$ against $b_e(kk')$ when $\alpha = 0$ (□), 0.5 (◯), and 0.75 (△). The lines indicate $\beta_e = 0$ (solid line), 0.65 (dashed line), and 1.0 (dashed-dotted line). In the inset we show that the relation $n_e(kk') \sim (kk')^{\nu_e}$ is valid in IAS.

In Figure 3c, we display the $P(n_e)$ and $P(b_e)$ in PIN and find that both $P(b_e)$ and $P(n_e)$ satisfy the power-law $P(n_e) \sim {n_e}^{-\sigma_e}$ and $P(b_e) \sim {b_e}^{-\delta_e}$, respectively, for large n_e and b_e. From the data in Figure 3c, we obtain $\sigma_e = \delta_e \simeq 3.7$ for $\alpha \simeq 0.75$. This value of α deviates from the estimated α using equation (11). Using the scaling relation (7), we obtain $\eta_e = 0.7(1)$ (which is not shown). Figure 3d shows $n_e(kk')$ against $b_e(kk')$ with $\alpha = 0, 0.5$, and 0.75. From the data, we obtain $\beta_e = 0$, 0.65, and 1.0. As shown in Figure 3c, we find that $\sigma_e \simeq 3.7$ when $\alpha = 0.75$ and the relation $\sigma_v = \delta_v$ is satisfied within the estimated errors. As shown in the inset of Figure 3d, $n_e(kk')$ satisfies equation (9), even though ν_e slight deviates from equation (11) due to the detailed structure of the underlying network.

We also find that the five largest vertices of BRWC in PIN for $\alpha \geq 0$ include the high ranked essential proteins such as YIL061C [9]. Among the five largest vertices, three proteins which are responsible for important biological functions such as pre-autophagosomal structure organization (YLR423C) and ATPase (YDL100C) [30] are included even though they are non-essential proteins in knock out experiment. This result indicates that the BRWC can find not only the topologically important vertex or edge related to the knock out experiment, but also the dynamically important vertex or edge. We also measure the BRWC in the Internet autonomous systems (IAS) and find the similar results with PIN (Fig. 4).

In summary, we propose a novel centrality measure based on BRW. The relation between BRWC and other centralities is shown by analytic and numerical methods for relatively large k. We also show that the distribution of BC (or LC) for an edge satisfies a power-law in SFNs for the first time. We expect that the results provide fundamental theoretical bases to understand various dynamical phenomena such as jamming transition and information transmission on SFNs.

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