3-2003

Centralized and Competitive Inventory Models With Demand Substitution

Serguei Netessine
University of Pennsylvania

Nils Rudi

Recommended Citation
Netessine, S., & Rudi, N. (2003). Centralized and Competitive Inventory Models With Demand Substitution. *Operations Research, 51* (2), 329-335. http://dx.doi.org/10.1287/opre.51.2.329.12788

Centralized and Competitive Inventory Models With Demand Substitution

Abstract

A standard problem in operations literature is optimal stocking of substitutable products. We consider a consumer-driven substitution problem with an arbitrary number of products under both centralized inventory management and competition. Substitution is modeled by letting the unsatisfied demand for a product flow to other products in deterministic proportions. We obtain analytically tractable solutions that facilitate comparisons between centralized and competitive inventory management under substitution. For the centralized problem we show that, when demand is multivariate normal, the total profit is decreasing in demand correlation.

Keywords

Inventory/production, stochastic, multiproduct: substitution, Games/group decisions, noncooperative: Nash equilibrium

Disciplines

Business Administration, Management, and Operations | Organization Development

Centralized and competitive inventory models with demand substitution*

Forthcoming in *Operations Research*

Serguei Netessine
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104

Nils Rudi
The Simon School
University of Rochester
Rochester, NY 14627

July 2000, Revised July 2001 and June 2002

Abstract

A standard problem in operations literature is optimal stocking of substitutable products. We consider a consumer-driven substitution problem with an arbitrary number of products under both centralized inventory management and competition. Substitution is modeled by letting the unsatisfied demand for a product flow to other products in deterministic proportions. We obtain analytically tractable solutions that facilitate comparisons between centralized and competitive inventory management under substitution. For the centralized problem we show that, when demand is multivariate Normal, the total profit is decreasing in demand correlation.

1 Introduction

This paper examines the optimal inventory stocking policies for a given product line under the notion that consumers who do not find their first-choice product in the current inventory might substitute a similar product for it (consumer-driven substitution). Namely, there is an arbitrary number of products and each consumer has a first-choice product. If this product is out of stock, the consumer might choose one of the other products as a substitute. In modeling the substitution process we employ an abstraction that is frequently used in the literature (see McGillivray and Silver [10], Parlar and Goyal [14], Noonan [13], Parlar [15], Wang and Parlar [20], Rajaram and Tang [16], Ernst and Kouvelis [3]). Specifically, for each product there is a random demand that is exogenously specified. In the case where demand realization exceeds the stocking quantity of

*We would like to thank two anonymous referees for valuable feedback, including a more compact way to derive the result of Proposition 1.

a product, then (a part of) excess demand is re-allocated to the other products in deterministic proportions. If re-allocated demand cannot be satisfied the sale is lost. We use a single-period formulation with demand for products following an arbitrary continuous multivariate distribution.

We consider two fundamentally distinct scenarios: centralized inventory management, where all products are managed by a central decision-maker whose objective is to maximize the expected aggregate profit, and de-centralized inventory management, where each product is managed by an independent decision-maker maximizing the expected profit generated by this specific product while interacting strategically with the other decision-makers. For the model that we employ, the contributions of this paper are as follows: (*i*) we obtain necessary optimality conditions (which may not be sufficient due to the potential existence of multiple local maxima) for the non-competitive case with n products thus extending the work of Parlar and Goyal [14] (2 products), Ernst and Kouvelis [3] (3 products with partial substitution) and Noonan [13] (n products but no analytical expression for the optimality conditions); (*ii*) we show that concavity of the objective function in the non-competitive setting established by Parlar and Goyal [14] (2 products) and Ernst and Kouvelis [3] (3 products with partial substitution) does not extend to n products with full substitution structure; (*iii*) we establish uniqueness of the equilibrium for the competitive n-product case, thus extending the work of Parlar [15] (2 products) and Wang and Parlar [20] (3 products but no proof of uniqueness); (*iv*) we obtain optimality conditions for the competitive n-product case, thus extending the work of Parlar [15] (2 products) and Wang and Parlar [20] (3 products); (*v*) we provide comparison between non-competitive and competitive solutions and (*vi*) we characterize the impact of demand correlation on profits under demand substitution for the centralized case, thus analytically confirming the numerical results of Rajaram and Tang [16] and Ernst and Kouvelis [3]. A non-intuitive result is the finding that competition might lead to understocking some of the substitutable products as compared to the centralized solution.

A number of papers employ probabilistic models of demand substitution that are different from the deterministic model in this paper. For the centralized setting, Smith and Agrawal [17] and Agrawal and Smith [1] study the problem of jointly deciding the stocking levels and assortment under probabilistic substitution. Mahajan and van Ryzin [8] and Mahajan and van Ryzin [9] analyze, correspondingly, a centralized model and a de-centralized model where customers dynamically and probabilistically substitute among products within a retail assortment in the case of a stock-out. Their consumer choice model is based on utility maximization. Lippman and McCardle [6] study a de-centralized model where the aggregate demand for all firms is a random variable, and demand for each firm results from an initial allocation and very general forms of re-allocation of excess demand. Anupindi et al. [2] propose a model to estimate substitution probabilities and apply it to vending machines. For a more comprehensive survey of research on demand substitution, the reader is referred to Mahajan and van Ryzin [7].

2 Analysis

We consider the problem with n products indexed by $i = 1, \ldots, n$ offered for sale in a single period. At the beginning of the period, Q_i units of product i are stocked at a unit cost c_i and sold at unit price r_i. We assume that leftover inventory is salvaged at the end of the period at unit salvage value s_i. We also make the following assumption that is standard for newsvendor-type problems: $r_i > c_i > s_i > 0$. There is an initial demand for product i denoted by D_i (random variable) where the demand vector $\underline{D} = (D_1, \ldots, D_n)$ follows a known continuous multivariate demand distribution with positive support. The deterministic fraction $a_{ij} \in [0, 1]$, where $\sum_{j=1}^{n} a_{ij} < 1$, of the excess demand (i.e., demand that cannot be satisfied by Q_i) for product i will buy product j (if available) as a substitute (where $a_{ii} = 0$ for all i). We assume that at this point unmet demand is lost, i.e., there is no second substitution attempt. We will consider two alternative models. In the centralized model, all n products are managed by a single company. In the decentralized (competitive) model, each product is managed by a separate company. Let π denote the company's profit under centralization and π_i denote the company i's profit under competition. Define $x^+ = \max(0, x)$.

2.1 Centralized inventory management

The expression for the expected profit consists of three parts: revenue, acquisition cost and salvage value for each product:

$$\pi = E \sum_i \left[r_i \min\left(D_i + \sum_{j \neq i} a_{ji} \left(D_j - Q_j\right)^+, Q_i \right) - c_i Q_i + s_i \left(Q_i - \left(D_i + \sum_{j \neq i} a_{ji} \left(D_j - Q_j\right)^+ \right) \right)^+ \right].$$

Define $D_i^s = D_i + \sum_{j \neq i} a_{ji} \left(D_j - Q_j\right)^+$, where the superscript s indicates that the effect from substitution has been accounted for. In words, D_i^s is the sum of the first-choice demand and demand from substitution. It is conventional to define $u_i = r_i - c_i$, the unit underage cost; and $o_i = c_i - s_i$, the unit overage cost. By algebraic manipulation and collecting similar terms, we get:

$$\pi = E \sum_i \left[u_i D_i^s - u_i \left(D_i^s - Q_i\right)^+ - o_i \left(Q_i - D_i^s\right)^+ \right]. \tag{1}$$

Notice that D_i^s depends on the stocking quantities of the other products (i.e., Q_j for $j \neq i$). While for the simple newsvendor problem minimizing the expected opportunity cost is equivalent to maximizing expected profit, it follows that this is not the case under substitution because the expected profit under "perfect information" (i.e., the first term of (1)) depends on the decision variables. For certain problem parameters, Parlar and Goyal [14] have demonstrated that the objective function in the case of two products is jointly concave. Ernst and Kouvelis [3] have shown that the problem with three partially substitutable products is jointly concave as well. However, we will verify analytically and through numerical experiments that the objective function with more than two products and full substitution structure might not be concave and not even quasiconcave, which parallels a finding of Mahajan and van Ryzin [8] in a different setting. To see this, consider the deterministic analog of the problem. We will show that the deterministic objective function, $\hat{\pi}$, may not be concave in at least one of the decision variables. The objective function can be re-written as follows:

$$\begin{aligned}
\hat{\pi} &= \sum_i \left[(u_i + o_i)\min\left(Q_i, D_i^s\right) - o_i Q_i\right] \\
&= \sum_i \left[(u_i + o_i)\min\left(Q_i, D_i + \sum_{j\neq i} a_{ji} D_j - \sum_{j\neq i} a_{ji}\min(Q_j, D_j)\right) - o_i Q_i\right] \\
&= \sum_i \left[(u_i + o_i)\min\left(Q_i + \sum_{j\neq i} a_{ji}\min(Q_j, D_j), D_i + \sum_{j\neq i} a_{ji} D_j\right) - o_i Q_i - \sum_{j\neq i} a_{ji}\min(Q_j, D_j)\right] \\
&= (u_i + o_i)\min\left(Q_i + \sum_{j\neq i} a_{ji}\min(Q_j, D_j), D_i + \sum_{j\neq i} a_{ji} D_j\right) && (2) \\
&\quad + \sum_{k\neq i}(u_k + o_k)\min\left(Q_k + \sum_{j\neq k,i} a_{jk}\min(Q_k, D_k) + a_{ik}\min(Q_i, D_i), D_k + \sum_{j\neq i} a_{jk} D_j\right) && (3) \\
&\quad - \min(Q_i, D_i)\sum_{j\neq i} a_{ij}(u_j + o_j) && (4) \\
&\quad - o_i Q_i && (5) \\
&\quad - \sum_{k\neq i}\min(Q_k, D_k)\sum_{j\neq i} a_{kj}(u_j + o_j). && (6)
\end{aligned}$$

Now hold Q_j, $j \neq i$ fixed and consider the slope of the objective function for different values of Q_i.

1. $Q_i < D_i$. Term (2) rises at a rate $(u_i + o_i)$. Term (3) rises at a rate between 0 and $\sum_{k\neq i} a_{ik}(u_k + o_k)$. Terms (4) and (5) decrease at rates $\sum_{j\neq i} a_{ij}(u_j + o_j)$ and o_i, correspondingly. Hence, on this interval the slope of the objective function is between $u_i - \sum_{j\neq i} a_{ij}(u_j + o_j)$ and u_i.

2. $D_i < Q_i < D_i^s$. Term (2) rises at a rate $(u_i + o_i)$ and term (5) decreases at a rate o_i. Hence, on this interval the slope of the objective function is u_i.

3. $Q_i > D_i^s$. Term (5) decreases at a rate o_i. Hence, on this interval the slope of the objective function is $-o_i$.

We can now plot the objective function. If $u_i > \sum_{j\neq i} a_{ij}(u_j + o_j)$, then the objective function is quasiconcave (Figure 1 left). Otherwise, the objective function might actually be bi-modal (Figure 1 right).

This deterministic counter-example of non-concavity also works when demand distribution has very low variability. We verified that this is the case through numerical experiments (see Figure 2, left, for an example of the objective function. We use Normal distribution with mean 100 and a standard deviation of 1). Although it appears through our derivation that as long as $u_i > \sum_{j\neq i} a_{ij}(u_j + o_j)$ the objective function should be unimodal in each of the decision variables (but not necessarily jointly),

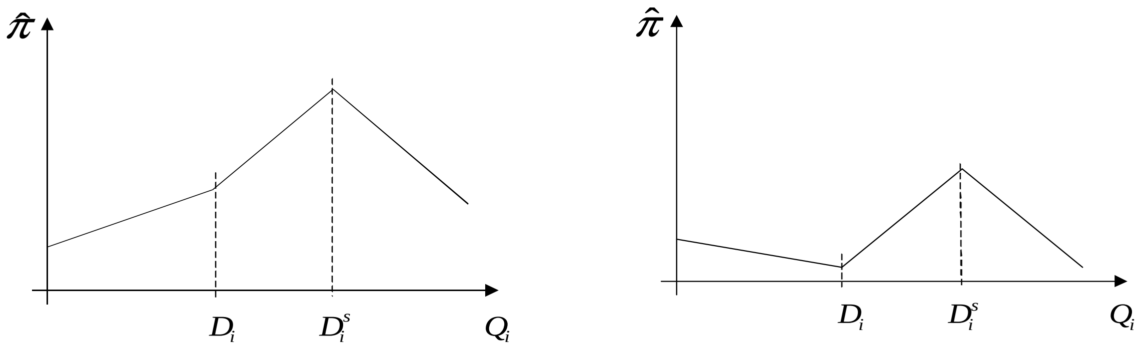

Figure 1: Example of quasiconcave (left) and non-quasiconcave (right) objective functions.

we were unable to verify this result analytically. Moreover, numerical experiments indicate that for any reasonable demand variability (coefficient of variation more than 0.1) the objective function is, in fact, concave in each variable (see Figure 2, right, for the same objective function as on the left but with the standard deviation of 10).

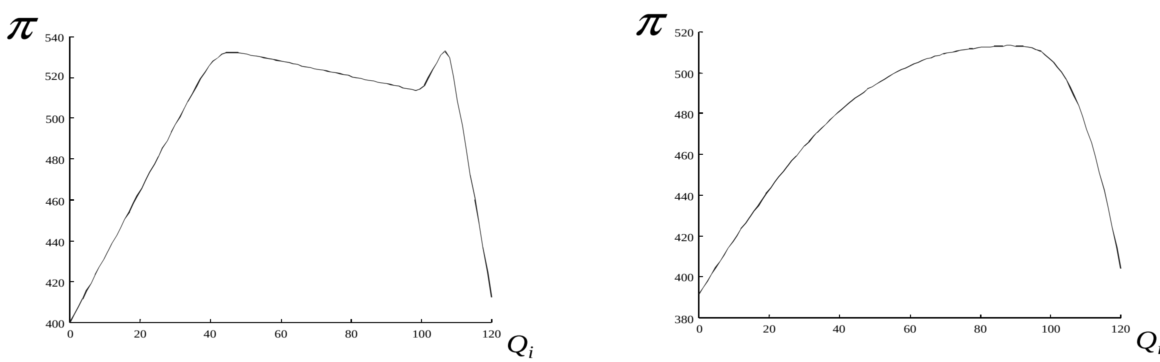

Figure 2: Example of objective function under low demand variability (left) and high demand variability (right)

In a variety of numerical experiments with Normal, uniform, and bi-modal demand distributions with reasonably high coefficients of variation we were unable to find any examples with multiple solutions. Note, however, that all our subsequent results do not rely on concavity in any way: they hold for any solution (in case there are multiple local maxima) since all of the solutions will satisfy the first-order conditions.

Since the objective function might not be concave, satisfying the first-order optimality conditions does not guarantee the global optimum. We will now obtain the first-order necessary optimality conditions. Taking derivatives using Leibnitz's formula is difficult for this problem due to the necessity of dealing with nested integrals of high dimensionality over regions formed by intersections of a large number of hyperplanes. This problem has been acknowledged by Noonan [13]. Also, solutions obtained with Leibnitz's formula are quite cumbersome with little analytical structure. We, however, utilize an alternative technique that is based on the interchange of the derivative and expectation.

Proposition 1 *The first-order necessary optimality conditions of the centralized problem are given by*

$$\Pr\left(D_i < Q_i^c\right) - P(D_i < Q_i^c < D_i^s) + \sum_{j \neq i} \frac{u_j + o_j}{u_i + o_i} a_{ij} \Pr\left(D_j^s < Q_j^c, D_i > Q_i^c\right) = \frac{u_i}{u_i + o_i}, \qquad (7)$$

$i = 1, ..., n$, where Q_i^c denotes the optimal order quantity for product i.

Proof:

We will obtain a derivative for one term in the objective function: $E \sum_j u_j (D_j^s - Q_j)^+$. The other terms can be analyzed in a similar way. The derivative of this term can be expressed as:

$$\frac{\partial E \sum_j u_j (D_j^s - Q_j)^+}{\partial Q_i} = u_i \frac{\partial E (D_i^s - Q_i)^+}{\partial Q_i} + \sum_{j \neq i} u_j \frac{\partial E (D_j^s - Q_j)^+}{\partial Q_i}.$$

The derivative of the first term can easily be taken using Leibnitz's formula so we will demonstrate how to find the derivative of the second term. Since the function under the expectation is integrable and has a bounded derivative, it satisfies the Lipschits condition of order one and hence the expectation and the derivative can be interchanged (see Glasserman [4])

$$\frac{\partial E (D_j^s - Q_j)^+}{\partial Q_i} = E\left(\frac{\partial (D_j^s - Q_j)^+}{\partial Q_i}\right) = E\left(\frac{\partial (D_j^s - Q_j)^+}{\partial D_j^s} \frac{\partial D_j^s}{\partial Q_i}\right).$$

Let $1_{\{\omega\}}$ be the indicator function of the event ω, i.e., $1_{\{\omega\}} = 1$ if ω is true and $1_{\{\omega\}} = 0$. Using the indicator function we get

$$\frac{\partial E (D_j^s - Q_j)^+}{\partial Q_i} = E\left(1_{\{D_j^s > Q_j\}} \left(-a_{ij} 1_{\{D_i > Q_i\}}\right)\right) = -a_{ij} \Pr\left(D_j^s > Q_j, D_i > Q_i\right).$$

Applying the technique to all terms results in the following expression of the derivative of (1) with respect to Q_i:

$$\begin{aligned} \frac{\partial \pi}{\partial Q_i} &= -\sum_{j \neq i} \left[u_j a_{ij} \Pr\left(D_i > Q_i\right)\right] + \left[u_i \Pr\left(D_i^s > Q_i\right) - o_i \Pr\left(D_i^s < Q_i\right)\right] \\ &\quad + \sum_{j \neq i} \left[u_j a_{ij} \Pr\left(D_j^s > Q_j, D_i > Q_i\right) - o_j a_{ij} \Pr\left(D_j^s < Q_j, D_i > Q_i\right)\right]. \end{aligned} \qquad (8)$$

Equating to zero and rearranging gives the desired result. This completes the proof. □

Expression (7) has an intuitive interpretation that in part parallels Noonan's [13] intuition. Without the second and third terms on the left-hand side, the expression becomes the solution to the simple newsvendor problem without substitution. Further, the optimal order quantity of product i is adjusted up (the second term on the left-hand side) for the extra demand due to substitution. On the other hand, the optimal order quantity is also adjusted down (the third term on the left-hand side) due to the possibility that a stock-out of product i might not necessarily result in a lost sale but might instead result in substitution. We see that the simple newsvendor solution is generally not optimal under substitution. From the second term on the left-hand side of (7) it follows that, the higher the degree of substitution *to* product i (i.e., large a_{ji}), the *more* one would expect to order of product i (i.e., Q_i). Further, from the third term on the left-hand side of (7) it follows that, the higher the degree of substitution *from* product i (i.e., large a_{ij}), the *less* one would expect to order of product i (i.e., Q_i).

In practice, it is important to recognize environments where demand substitution is more/less beneficial. One useful characteristic of such environments is the dependence structure of the multivariate demand distribution. This issue has been addressed in other substitution papers through numerical experiments (see Ernst and Kouvelis [3] and Rajaram and Tang [16]). We will now demonstrate analytically that, if demand is Normally distributed, then expected profit is a decreasing function of demand correlation when stocking quantities are either held fixed or changed optimally as correlation changes. To show this result, we use the fact that the multivariate Normal random variables can be ordered in the sense of supermodular stochastic order as long as their covariance matrices are ordered.

Proposition 2 *Suppose* $\underline{D} \sim N(\underline{\mu}, \Sigma)$. *Then the retailer's profit is decreasing in any coefficient of correlation* ρ_{ij} *if stocking quantities* Q_i*s are either held fixed or adjusted optimally as correlation changes.*

Proof: Muller and Scarsini [12] define supermodular stochastic order as follows: a random vector $\underline{D}^1$ is said to be smaller than the random vector $\underline{D}^2$ in the supermodular order, written $\underline{D}^1 \leq_{sm} \underline{D}^2$, if $Ef(\underline{D}^1) \leq Ef\left(\underline{D}^2\right)$ for all supermodular functions f such that the expectation exists. Using this definition, Muller and Scarsini obtain the following result: let $\underline{D}^1$ and $\underline{D}^2$ be multivariate Normal random vectors with parameters $\underline{D}^1 \sim N(\underline{\mu}, \Sigma^1)$ and $\underline{D}^2 \sim N(\underline{\mu}, \Sigma^2)$ where Σ^1, Σ^2 are covariance matrices such that $\sigma^1_{ii} = \sigma^2_{ii}, \sigma^1_{ij} \leq \sigma^2_{ij}$. Then $\underline{D}^1 \leq_{sm} \underline{D}^2$. To use this result, it must be shown that the retailer's objective function is supermodular in $\underline{D}$ for *each demand realization*. Instead, we will show that the objective function is submodular for each demand realization and later use the fact that if $f(x)$ is submodular then $-f(x)$ is supermodular. To prove this, we need to show that the following function is submodular in $\underline{D}$

$$\pi(\underline{D}) = \sum_i \left[(u_i + o_i) \min \left(D_i + \sum_{j \neq i} a_{ji}(D_j - Q_j)^+, Q_i \right) - o_i Q_i \right].$$

The last term does not depend on the D_is and hence can be ignored. Further, a sum of submodular functions is a submodular function, and hence it suffices to prove submodularity for each term under the summation sign. Notice that the following function

$$h(\underline{D}) = D_i + \sum_{j \neq i} a_{ji}(D_j - Q_j)^+$$

is a valuation (see page 43 in Topkis [19]), i.e., it is both submodular and supermodular. This is a consequence of the fact that this function is separable in the D_i's. Table 1 in Topkis [18] shows that if $f(x)$ is a concave increasing function and $h(\underline{D})$ is submodular, then $f(h(\underline{D}))$ is also submodular. Clearly, $f(x) = \min(x, Q_i)$ is concave and increasing in x so that $f(\underline{D}) = \min(h(\underline{D}), Q_i)$ is submodular in $\underline{D}$ and hence $\pi(\underline{D})$ is submodular. We are now ready to prove the final result. We are interested in the full differential of the objective function w.r.t. correlation, i.e.,

$$\frac{d\pi}{d\rho_{ij}} = \frac{\partial \pi}{\partial \rho_{ij}} + \sum_k \frac{\partial \pi}{\partial Q_k}\frac{\partial Q_k}{\partial \rho_{ij}}.$$

If we hold the Qs fixed then $\partial Q_k/\partial \rho = 0$. Similarly, if we change the Qs optimally, then $\partial \pi/\partial Q_k = 0$. Hence, under our assumptions $d\pi/d\rho_{ij} = \partial \pi/\partial \rho_{ij}$. Finally, by Muller and Scarsini [12], if $\rho_{ij}^1 \leq \rho_{ij}^2$, $\underline{D}^1 \sim N(\underline{\mu}, \Sigma^1)$ and $\underline{D}^2 \sim N(\underline{\mu}, \Sigma^2)$ then $\underline{D}^1 \leq_{sm} \underline{D}^2$ and hence $E\left[-\pi(\underline{D}^1)\right] \leq E\left[-\pi(\underline{D}^2)\right]$ since $-\pi(\underline{D})$ is supermodular in $\underline{D}$. This completes the proof. □

This result is in line with the intuition that substitution is more beneficial when demand for products is less correlated. To gain an underlying insight, consider the following casual interpretation of the impact of demand correlation in the two-product case. When demands are highly positively correlated, demand realizations tend to be high together and low together. In the former case, there will not be a sufficient inventory to satisfy the substituting excess demand and in the latter case there will not be excess demand to substitute. On the contrary, when demands are highly negatively correlated, if one product has a high demand realization the other product tends to have a low demand realization. Hence, a majority of substituting excess demand for the former product can be satisfied by the available supply of the latter product, making substitution more profitable.

2.2 De-centralized inventory management

We will next consider the alternative model where a separate company controls the stocking policy of each product. Firm i's optimal decision will then depend on the vector of inventory levels of the other firms denoted by Q_{-i}. Let the expected profit of firm i given the other firms' decisions be denoted by $\pi_i = \pi_i(Q_i|Q_{-i})$. We have

$$\pi_i = E\left[u_i D_i^s - u_i\left(D_i^s - Q_i\right)^+ - o_i\left(Q_i - D_i^s\right)^+\right], i = 1, ..., n. \tag{9}$$

Note that the demand distribution is a function of the order quantities of the other firms. Hence, a game theoretic situation arises where firm i will employ the best response, defined as follows: the strategy Q_i^d (where d implies decentralized) is player i's best response to Q_{-i} if

$$\pi_i(Q_i^d, Q_{-i}) = \max_{Q_i} \pi_i(Q_i, Q_{-i}).$$

This best response will be denoted by $Q_i^d \equiv Br_i(Q_{-i})$. Given Q_{-i} it is easy to verify that π_i is concave in Q_i. To obtain the best response, we take the derivative of π_i with respect to Q_i:

$$\frac{\partial \pi_i}{\partial Q_i} = u_i - (u_i + o_i)\Pr\left(D_i^s < Q_i\right).$$

The best response is then characterized by the familiar expression:

Proposition 3 *Any Nash equilibrium is characterized by the following optimality conditions:*

$$\Pr\left(D_i < Q_i^d\right) - \Pr(D_i < Q_i^d < D_i^s) = \frac{u_i}{u_i + o_i},\ i = 1, ..., n. \tag{10}$$

The explanation is as follows: the simple newsvendor quantity is adjusted up by the second term on the left-hand side in order to account for extra demand from substitution. Notice that the difference between the non-competitive solution (7) and the competitive solution (10) is that the order quantity is not adjusted down since excess demand for product i is lost for company i. We see again that the simple newsvendor solution is suboptimal. Moreover, the competitive solution is suboptimal in terms of the system profit, since it does not account for the advantage in the case that a stock-out might lead to additional sales of a substitute product (i.e., the third term of the optimality condition for the centralized case (7)). Due to this effect, we expect to stock more under competition than in the centralized case. From the second term on the left-hand side of (10) it follows that, the higher the degree of substitution *to* a product i (i.e., large a_{ji}), the *more* one would order of product i (i.e., Q_i). Contrary to the centralized case, however, a higher degree of substitution *from* a product i (i.e., large a_{ij}) does not have a direct effect on the order quantity of product i (i.e., Q_i). With respect to the existence and uniqueness of the Nash equilibrium, we establish the following result:

Proposition 4 *A Nash equilibrium exists in the competitive case and can be found from the first order conditions (10). Further, if either* $\sum_{i=1}^n a_{ij} < 1$ *for all* j*, or* $\sum_{j=1}^n a_{ij} < 1$ *for all* i*, the Nash equilibrium is unique and globally stable.*

Proof: Existence of the equilibrium has been demonstrated by Lippman and McCardle [6]. We then proceed to prove the uniqueness of the Nash equilibrium using the contraction mapping principle. It is straightforward to verify that the second derivative of i's objective function is negative, and hence each objective function is concave in the order quantity. It follows that each best response function is single-valued and, given the order quantities of all other players, the unique best response of each player can then be found from the first order conditions. Consider the following system of equations: $Q_i^d = Br_i(Q_{-i})$, $i = 1, ..., n$. This system is a $R^n \to R^n$ mapping. In order to conclude that it has a unique solution, it is sufficient to prove that this mapping is a contraction (see Moulin [11]). To show that the mapping is a contraction, it is sufficient to demonstrate that the spectral radius (denoted by ρ) of the Jacobian of the mapping is bounded by one, i.e., $\rho(J_{Br}) < 1$. Using Theorem 5.6.9 from Horn and Johnson [5], the spectral radius of the matrix is bounded by any of the matrix norms, i.e., $\rho(J_{Br}) \leq \|J_{Br}\|$. By using maximum column sum (one-norm) and maximum row sum (infinity-norm) matrix norms, we can show that the proposition holds if either $\sum_{i=1}^n a_{ij} < 1$ for all j or $\sum_{j=1}^n a_{ij} < 1$ for all i. We only consider the case of the maximum column sum norm here, since the case of the maximum row sum norm follows by transposing the Jacobian. The maximum column sum norm is defined as:

$$\|J_{Br}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |J_{ij}| .$$

Each element for the Jacobian $J_{ij} = \partial Br_i / \partial Q_j$, $i \neq j$, $J_{ii} = 0$, $i, j = 1, ..., n$ represents the slope of a best response function. In order to obtain entries of the Jacobian we will employ implicit differentiation. The slope of the best response function is:

$$J_{ij} = \frac{\partial Br_i}{\partial Q_j} = -\frac{\frac{\partial \Theta}{\partial Q_j}}{\frac{\partial \Theta}{\partial Q_i}} = -\frac{a_{ji} f_{D_i^s | D_j > Q_j}(Q_i^d) \Pr(D_j > Q_j)}{f_{D_i^s}(Q_i^d)},$$

where f_X is the density function of the random variable X. It is easy to see that the best response function $Br_i(Q_{-i})$ is monotonic and the slope is between 0 and a_{ji} in absolute value. The spectral radius of the Jacobian is bounded above as follows:

$$\rho(J_{Br}) \leq \|J_{Br}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |J_{ij}| \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} a_{ji} < 1.$$

Hence, the best response mapping is a contraction, and correspondingly it has a unique, globally stable fixed point that is the Nash equilibrium of the game. □

In the case of competition and a multivariate Normal demand distribution we are not able to prove that each players' equilibrium profit decreases as correlation rises. However, it can be shown that, when all stocking quantities are kept fixed and correlation rises, the expected profit for each retailer will decrease (the proof follows along the lines of the proof of Proposition 2).

2.3 Comparison of optimal stocking levels

The literature has noted that retailers behave suboptimally under competition, i.e., their stocking policies deviate from the centralized solution. The following question arises: will retailers stock more or less under competition? An intuitive answer is that competition drives inventories up for all retailers. In settings different from ours and with symmetric products this result was demonstrated by Mahajan and van Ryzin [9] and Lippman and McCardle [6]. Seemingly, in our model (both in symmetric and asymmetric cases) a similar conclusion follows from the fact that, in the centralized solution, the newsvendor inventory level for product i is adjusted up to account for demand switching to product i, and adjusted down to account for demand switching from product i. Meanwhile, in the de-centralized solution, the inventory level is only adjusted up. While we confirm that in the symmetric case this intuition holds, we also provide a counter-example illustrating that in the asymmetric case there are situations in which the inventory level for at least one product will be higher in the non-competitive solution than in the competitive.

Proposition 5 *The following relationships between Q_i^c and Q_i^d hold:*

(i) *There exist situations when $Q_i^c \geq Q_i^d$ for some i.*

(ii) *It is always true that $Q_i^c \leq Q_i^d$ for at least one i.*

(iii) *Suppose that all the costs and revenues are symmetric among firms, demands are independent and identically distributed, and consumers are equally likely to switch to any of the $(N-1)$ products, i.e., $a_{ij} = a < \frac{1}{N-1}$ for all i, j. Then $Q_i^c \leq Q_i^d$ for all i.*

Proof:

(i) The proof is by counter-example. First, consider the centralized solution. Suppose for some product i, $D_i = 0$ and $a_{ij} = 0$, for all j. Hence, the stocking policy for product i depends solely on the demand from substitution to product i from other products and $D_i^s = \sum_j a_{ji}(D_j - Q_j^c)$. Note that in this situation (7) looks like (10). Next, we turn to the same scenario but in the de-centralized setting. Assume that $Q_j^c \le Q_j^d$ for all $j \neq i$ (otherwise the counter-example is complete). Hence, demand for product i, D_i^s is stochastically smaller, resulting in a decrease in the stocking quantity, $Q_i^c \ge Q_i^d$.

(ii) The proof is by contradiction. Assume that $Q_j^c \ge Q_j^d$ for all j. We will use notation D_i^{sc} and D_i^{sd} to denote demand with the effect of substitution in the centralized and de-centralized problems, respectively. Consider an arbitrary product i. For the centralized problem, the first order condition is

$$\Pr\left(D_i^{sc} < Q_i^c\right) + \sum_{j \neq i} \frac{u_j + o_j}{u_i + o_i} a_{ij} \Pr\left(D_j^{sc} < Q_j^c, D_i > Q_i^c\right) = \frac{u_i}{u_i + o_i}.$$

For the de-centralized case, the condition for the Nash equilibrium is

$$\Pr\left(D_i^{sd} < Q_i^d\right) = \frac{u_i}{u_i + o_i}.$$

Since the right-hand sides of these two equations are equal, the left-hand sides must be equal too. Further, it is easy to see that

$$\Pr\left(D_i^{sc} < Q_i^c\right) \le \Pr\left(D_i^{sd} < Q_i^d\right),$$

or, similarly, after expanding,

$$\Pr\left(D_i + \sum_j a_{ji}\left(D_j - Q_j^c\right)^+ < Q_i^c\right) \le \Pr\left(D_i + \sum_j a_{ji}\left(D_j - Q_j^d\right)^+ < Q_i^d\right).$$

Observe that $D_i + \sum_j a_{ji}\left(D_j - Q_j^c\right)^+ \le D_i + \sum_j a_{ji}\left(D_j - Q_j^d\right)^+$ due to the assumption that $Q_j^c \ge Q_j^d$ for all j. This, however, implies that $Q_i^c \le Q_i^d$ for an inequality to hold. This is a contradiction.

(iii) Under the symmetry assumption, $Q_i^c = Q_j^c$ and $Q_i^d = Q_j^d$ for all i, j. In the previous proposition we demonstrated that at least for one i it is always true that $Q_i^c \le Q_i^d$. Hence, $Q_i^c \le Q_i^d$ for all i. □

The counter-intuitive situation (i) that the stocking quantity under centralization might exceed the corresponding stocking quantity under competition occurs when all or most of the effective demand for a product stems from second-choice demand (demand from substitution). One plausible situation that would lead to such a result is if the set of products in consideration consists of a subset of products whose demand primarily arises from first-choice demand but rarely serves as a substitute, while products from another set primarily serve as substitutes.

References

[1] Agrawal, N. and S. Smith. 1998. Optimal retail assortments for substitutable items purchased in sets. Working paper, Santa Clara University.

[2] Anupindi, R., M. Dada and S. Gupta. 1998. Estimation of consumer demand with stock-out based substitution: an application to vending machine products. *Marketing Science*, Vol.17, No.4, 406-423.

[3] Ernst, R. and P. Kouvelis. 1999. The effects of selling packaged goods on inventory decisions. *Management Science*, Vol.45, No.8, 1142-1155.

[4] Glasserman, P. 1994. Perturbation analysis of production networks. D. Yao, Ed. Stochastic modeling and analysis of manufacturing systems. Springer-Verlag, New-York.

[5] Horn, A. R. and C. A. Johnson. Matrix analysis. Cambridge University Press.

[6] Lippman, S. A. and K. F. McCardle. 1997. The competitive newsboy. *Operations Research*, Vol.45, No.1, 54-65.

[7] Mahajan, S and G. van Ryzin. 1999. Retail inventories and consumer choice. In Quantitative models for supply chain management, S. Tayur, R. Ganeshan and M. Magazine (eds.), Kluwer Academic Publishers.

[8] Mahajan, S. and G. van Ryzin. 2001. Stocking retail assortments under dynamic consumer substitution. *Operations Research*, Vol.49, No.3, 334-351.

[9] Mahajan, S and G. van Ryzin. 2001. Inventory competition under dynamic consumer choice. *Operations Research*, Vol.49, No.5, 646-657.

[10] McGillivray, A. R. and E. A. Silver. 1978. Some concepts for inventory control under substitutable demand. *INFOR*, Vol.16, No.1, 47-63.

[11] Moulin, H. 1986. Game theory for the social sciences. Second edition. New York University Press.

[12] Muller, A. and M. Scarsini. 2000. Some remarks on the supermodular order. *Journal of multivariate analysis*, Vol.23, 107-119.

[13] Noonan, P. S. 1995. When customers choose: a multi-product, multi-location newsboy model with substitution. Working Paper, Emory University.

[14] Parlar, M. and S. K. Goyal. 1984. Optimal ordering decisions for two substitutable products with stochastic demand. *OPSEARCH*, Vol.21, No.1, 1-15.

[15] Parlar, M. 1988. Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics*, Vol.35, 397-409.

[16] Rajaram, K. and C. S. Tang. 2001. The impact of product substitution on retail merchandising. *European Journal of Operational Research*, Vol. 135, 582-601.

[17] Smith, S. A. and N. Agrawal. 2000. Management of multi-item retail inventory system with demand substitution. *Operations Research*, Vol.48, No.1, 50-64.

[18] Topkis, D. 1978. Minimizing a submodular function on a lattice. *Operations Research*, Vol.26, No.2, 305-321.

[19] Topkis, D. 1998. Supermodularity and complementarity. Princeton University Press.

[20] Wang, Q. and M. Parlar. 1994. A three-person game theory model arising in stochastic inventory control theory. *European Journal of Operations Research*, Vol.76, 83-97.