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# CENTRALIZERS OF $C^1$-DIFFEOMORPHISMS

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ABSTRACT. In this paper we prove that $Z(f) = \{f^k | k \in Z\}$ for generic Axiom A diffeomorphisms. We also prove that generic diffeomorphisms have no $k$-roots.

**Introduction.** Let $M$ be a compact connected $C^\infty$-manifold without boundary. A $C^r$-dynamical system on $M$, $0 \leqslant r \leqslant \infty$, is the triple $(M, \mathfrak{C}, f)$, where $\mathfrak{C}$ is the $C^r$-structure of $M$ and $C$ is a $C^r$-diffeomorphism of $M$. We simply let $f$ refer to it. A $C^r$-dynamical system naturally has the structure of $C^s$-dynamical system for any $0 \leqslant s \leqslant r$. Then a $C^s$-diffeomorphism $g$ commuting with $f$, i.e., $f \circ g = g \circ f$, is a *$C^s$-symmetry of $f$* in the sense that $g$ preserves the $C^s$-structure of the dynamical system $f$. Throughout we consider $C^1$-symmetries of $C^1$-dynamical systems. Let Diff$(M)$ be the set of $C^1$-diffeomorphisms of $M$ with uniform $C^1$-topology. The centralizer of $f$, $Z(f)$, is the set of all symmetries of $f$. Clearly, $f^k \in Z(f)$, for any $k \in Z$ ($Z$ denotes the set of integers). Then $g \in Z(f)$ is said to be *trivial* if $g = f^k$ for some $k \in Z$. Given a periodic point $p$ of $f$, we say that $g \in Z(f)$ is *$W^s$-trivial at $p$* if $g|W^s(p) = f^k|W^s(p)$ for some $k \in Z$, and *$W^s$-trivial* if $g$ is $W^s$-trivial at every periodic point of $f$.

Using the above notations, we can state our results as follows:

THEOREM 1. *Generic diffeomorphisms have only $W^s$-trivial symmetries. More precisely, there exists a generic subset $K^*$ of* Diff$(M)$ *such that $Z(f)$ consists only of $W^s$-trivial symmetries for any $f \in K^*$.*

We say $g \in$ Diff$(M)$ is *a $k$-root of $f$* $\in$ Diff$(M)$ if $f = g^k$.

COROLLARY. *Generic diffeomorphisms have no $k$-root for any $k \in Z$, $k \neq \pm 1$.*

Let $A$ be the set of Axiom A diffeomorphisms.

THEOREM 2. *There exists a generic subset $A^*$ of $A$ such that*

$$Z(f) = \{f^k | k \in Z\}$$

*for any $f \in A^*$.*

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For $C^\infty$-centralizers of $C^\infty$-diffeomorphisms, B. Anderson in [1] has proved that having discrete centralizer is $C^3$-open $C^\infty$-dense property in $MS$. See also [2], [6], and [7, p. 809].

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**1. Local symmetries.** In this section, we consider local symmetries of embeddings. Let $E(D_r) = \mathrm{emb}(D_r^n, R^n) \subset C^1(D_r^n, R^n)$ be the set of embeddings with $C^1$-topology, where $D_r = D_r^n = \{x \in R^n \mid \|x\| \leqslant r\}$. If $f \in E(D_r)$ has a fixed point $p$ in int $D_r$, *a local symmetry at* $p$ of $f$ means an embedding $g$ from some neighbourhood $U_p$ of $p$ into $R^n$ such that

(i) $g(p) = p$,

(ii) $g \circ f$ and $f \circ g$ are germ equivalent at $p$, i.e., there is a neighbourhood $V_p$ of $p$ such that $f \circ g$ and $g \circ f$ are defined and coincide on it.

Then $g$ is said to be *trivial* if $g$ and $f^k$ are germ equivalent at $p$ for some $k \in Z$, and $W^s$-*trivial* if $g|W^s(p)$ and $f^k|W^s(p)$ are germ equivalent at $p$.

Let $CE(D_r)$ be the set of contractions, i.e., the set of $f$'s in $E(D_r)$ such that

(i) $f(D_r) \subset \mathrm{int}\, D_r$,

(ii) $\bigcap_{k=1}^{k=\infty} f^k(D_r)$ is just one point, and we let $p_f$ denote it,

(iii) $Df(p_f)$ is a linear contraction.

Then $CE(D_r)$ is equipped with the topology as a subspace of $E(D_r)$.

PROPOSITION (1.1) *There exists a $C^1$-generic subset $CE^*(D_r)$ of $CE(D_r)$ such that any $f \in CE^*(D_r)$ has only trivial local symmetries at $p_f$.*

The proof of this proposition is similar to the one of the theorem in our previous paper [8], and we omit it.

Suppose that $f_0 \in E(D^n)$; fix $0 \in R^n$, and let $L = Df_0(0)$ be hyperbolic with skewness $\tau$, where $D^n$ denotes the unit disc. Let $R^n = R^s \oplus R^u$ be the splitting of $R^n$ to the contracting and expanding subspace of $L$. Choose any $0 < \varepsilon < \frac{1}{2}(1-\tau)(1+\tau)^{-1}$, then there exists $r > 0$ such that $\|Df_0(x) - L\| < \varepsilon/2$ for any $x \in D_r^s \times D_r^u$. Choose any $0 < \delta < \min(\varepsilon^2 r\tau, \varepsilon/2)$. Let $Q(f_0)$ be the $\delta$-neighbourhood of $f_0$ in $E(D^n)$, i.e.,

$$Q(f_0) = \{f \in E(D^n) \mid \|f - f_0\|_1 < \delta\},$$

where $\|\ \|_1$ denotes the $C^1$-norm. Notice that if $f \in Q(f_0)$, then $\|f(0)\| < \delta$ and $\mathrm{Lip}(f - L) < \varepsilon$ in $D_r^s \times D_r^u$.

Now we can apply the Stable Manifold Theorem [4] and get the following:

LEMMA (1.1) *If $f \in Q(f_0)$, then $f$ has unique fixed point $p_f$ in $D_r$ and there exists a continuous map*

$$\mathfrak{g}\colon Q(f_0) \to C^1(D_r^s, D_r^u), \quad f \mapsto \mathfrak{g}_r,$$

*such that the graph of $\mathfrak{g}_f$ gives the stable manifold for $f$.*

Consider the map

$$\psi\colon Q(f_0) \to CE(D_{r/2}^s), \quad f \mapsto p_s \circ f \circ (I_s, \mathfrak{g}_f)|D_{r/2},$$

where $p_s: R^n \to R^s$ denotes the projection and $I_s$ denotes the identity map of $D_r^s$.

LEMMA (1.2) *$\psi$ is an open continuous map.*

PROOF. Since the composition map $\circ$ is continuous [5], so is $\psi$. Let $\varepsilon > 0$ and $f \in Q(f_0)$ be given. We show that if $g \in CE(D_{r/2}^s)$ is sufficiently close to $\psi(f)$, then there is a map $\tilde{f} \in Q(f_0)$ such that $\psi(\tilde{f}) = g$ and $\|\tilde{f} - f\|_1 < \varepsilon$. This implies that $\psi$ is an open map. Let $W_f^s = (I_s, \mathfrak{g}_f)(D_r^s)$. Let $U$ be a tubular neighbourhood of $W_f^s$ and $\pi: U \to W_f^s$ be the projection of this bundle. Let $\alpha$ be a bump function on $U$ with $\alpha | W_f^s = 1$. First we extend $g$ to $D_r^s$ so that $g$ coincides with $p_s \circ f \circ (I_s, \mathfrak{g}_f)$ out of $D_{2r/3}^s$. Using the diffeomorphism $p_s | W_f^s$, we lift this extension of $g$ to $W_f^s$ and get a map $\tilde{g}: W_f^s \to W_f^s$. Then the required map $\tilde{f}$ is defined by

$$\tilde{f}\,|D - U = f|D - U, \quad \tilde{f}\,|U = f|U + ((\tilde{g} - f) \circ \pi)\alpha.$$

It is clear that $f \in \psi^{-1}(CE^*(D_{r/2}^s))$ has only $W^s$-trivial symmetries. Since $\psi$ is an open continuous map, we can conclude that $\psi^{-1}(CE^*(D_{r/2}^s))$ is generic in $Q(f_0)$, because the inverse image of a generic subset by an open continuous map is also generic. Hence we get:

PROPOSITION (1.2) *If $f_0 \in E(D^n)$ has $0$ as a hyperbolic fixed point, then there exist an $r$-disc $D_r$, a neighbourhood $Q(f_0)$ of $f_0$ in $E(D^n)$, and its generic subset $Q^*(f_0)$ such that any $f \in Q^*(f_0)$ has only $W^s$-trivial local symmetries at $p_f$, which is the unique fixed point of $f$ in $D_r$.*

**2. Proof of Theorem 1**. Let per$(f, m)$ be the set of the periodic points of $f$ of period $m \in N$ ($N$ denotes the set of the natural numbers). Let $K_m$ be the set of $f$'s $\in$ Diff$(M)$ such that

(i) each $p \in$ per$(f, m)$ is hyperbolic,

(ii) if $p, q \in$ per$(f, m)$ have different orbits, then they have different eigenvalues.

Property (ii) implies that $g(O_f(p)) = O_f(p)$ for any $p \in$ per$(f, m)$ and any $g \in Z(f)$, where $O_f(p)$ denotes the orbit of $p$ under $f$. Notice that each per$(f, m)$ is finite, each $K_m$ is open dense, and $K = \bigcap_{m=1}^{m=\infty} K_m$ is generic in Diff$(M)$. Let $K_m^*$ be the set of $f$'s $\in K_m$ such that any $g \in Z(f)$ is $W^s$-trivial at any $p \in$ per$(f, m)$.

To prove Theorem 1, we have only to show that each $K_m^*$ is generic in $K_m$, because this implies that $K^* = \bigcap_{m=1}^{m=\infty} K_m^*$ is generic in $K$ and any $f \in K^*$ has only $W^s$-trivial symmetries. Further, because Diff$(M)$ is second countable, it is sufficient to prove that any $f_0 \in K_m^*$ has a neighbourhood $U(f_0)$ in $K_m$ such that $U(f_0) \cap K_m^*$ is generic in $U(f_0)$.

Choose a neighbourhood $V(p_i)$ for each $p_i \in$ per$(f_0, m)$ such that $V(p_i) \cap$ per$(f_0, m) = \{p_i\}$. Then we can take a neighbourhood $U_1(f_0)$ of $f_0$ in $K_m$ such that if $f \in U_1(f_0)$,

per$(f, m) \cap V(p_i) =$ one point $p_{f,i}$, for each $p_i \in$ per$(f, m)$, and

$$\operatorname{per}(f, m) \subset \bigcup V(p_i), \quad p_i \in \operatorname{per}(f_0, m).$$

LEMMA (2.1) *Using the above notations, there exist a neighbourhood* $U(f_0, p_i) \subset U_1(f_0)$ *and its generic subset* $U^*(f_0, p_i)$ *for each* $p_i$ *such that any* $f \in U^*(f_0, p_i)$ *has only* $W^s$*-trivial symmetries at* $p_{f,i}$.

Let $U(f_0) = \bigcap U(f_0, p_i)$, $p_i \in \operatorname{per}(f_0, m)$. Then $U(f_0) \cap K_m^*$ contains a generic set $\bigcap U^*(f_0, p_i)$ of $U(f_0)$, hence is generic itself. So we get Theorem 1 from the above lemma.

We now prove Lemma (2.1). We can choose a neighbourhood $V_1(p_i)$ such that $f_0^j(\operatorname{Cl}(V_1(p_i))) \subset V(f_0^j(p_i))$, $0 \leqslant j \leqslant m$.

Without loss of any generality, we can assume that there exists a chart $\{\varphi_i, U(p_i)\}$ such that $\varphi_i(p_i) = 0$ and $\varphi_i(\operatorname{Cl}(V_1(p_i)))$ is the unit disc of $R^n$, $n = \dim(M)$. We choose a neighbourhood $U_2(f_0, p_i) \subset U_1(f_0)$ in $K_m$ such that for any $f \in U_1(f_0)$

(i) $p_{f,i} \in V_1(p_i)$,

(ii) $f^j(\operatorname{Cl}(V_1(p_i))) \subset V(f^j(p_i))$, $0 \leqslant j \leqslant m$.

We define a map

$$\Lambda_i \colon U_2(f_0, p_i) \to E(D^n)$$

by

$$\Lambda_i(f) = \varphi_i \circ f^m \circ \varphi_i^{-1}|D.$$

Then $\Lambda_i$ is open continuous. Suppose that $f \in U_2(f_0, p_i)$ and $g \in Z(f)$ is not $W^s$-trivial at $p_{f,i}$. Because $f^j \circ g(p_{f,i}) = p_{f,i}$ for some $0 \leqslant j < m$, and $f^j \circ g \in Z(f)$ is not $W^s$-trivial at $p_{f,i}$, we can suppose $g(p_{f,i}) = p_{f,i}$, without loss of any generality. Then the germ of $\varphi_i \circ g \circ \varphi_i^{-1}$ at $\varphi(p_{f,i})$ gives a germ of a local symmetry of $\Lambda_i(f)$ at $\varphi_i(p_{f,i})$ which is not $W^s$-trivial at $\varphi_i(p_{f,i})$. Hence $\Lambda_i^{-1}(Q(\Lambda(f_0)))$ and its generic subset $\Lambda_i^{-1}(Q^*(\Lambda(f_0)))$ give the required $U(f_0, p_i)$ and $U^*(f_0, p_i)$ of Lemma (2.1) respectively, proving Theorem 1.

**3. Proof of Theorem 2.** We shall prove Theorem 2. Let $K^{*-1} = \{f^{-1} | f \in K^*\}$. Then $K^{*-1}$ is generic in $\operatorname{Diff}(M)$, and any $f \in K^{*-1}$ has only $W^u$*-trivial symmetries*. We shall prove that any $f \in K^* \cap K^{*-1} \cap A$ has only trivial symmetries. Let $f \in K^* \cap K^{*-1} \cap A$ and $g \in Z(f)$. Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ be the spectral decomposition. First we show that there is an integer $k(i)$ for each $\Omega_i$ such that $g|W^s(p) = f^{k(i)}|W^s(p)$ for any $p \in \Omega_i \cap \operatorname{per} f$. This is trivial if $\Omega_i$ is an orbit of a periodic point. So we assume that $\Omega_i$ is infinite. Let $p \in \Omega_i \cap \operatorname{per} f$, and let

$$g|W^s(p) = f^k|W^s(p), \qquad g|W^u(p) = f^j|W^u(p).$$

Since $\Omega_i$ is topological transitive and periodic points are dense in $\Omega_i$, $p$ is not isolated in $\Omega_i \cap \operatorname{per} f$. Since the family of stable manifolds is smooth, then $W^s(p) \cap W^u(q) \neq \varnothing$, $W^u(p) \cap W^s(q) \neq \varnothing$ for sufficiently near $q \in \Omega_i \cap \operatorname{per} f$. Using the $\lambda$-lemma, we can conclude that $W^s(p) \cap W^u(p) - \operatorname{per} f \neq \varnothing$, and this implies that $k = j$. Suppose that $p_1, p_2 \in \Omega_i \cap \operatorname{per} f$, and let

$$g|W^s(p_1) = f^k|W^s(p_1), \qquad g|W^s(p_2) = f^j|W^s(p_2).$$

Since $W^s(p_1)$ is dense in $\Omega_i$ [7, p. 783], then $W^s(p_1) \cap W^u(p_2) \neq \emptyset$. This implies that $k = j$. Hence there is an integer $k(i)$ such that $g|W^s(p) = f^{k(i)}|W^s(p)$ for any $p \in \Omega_i \cap \operatorname{per} f$.

Since the continuity of the family of the stable manifolds on $\Omega_i$ implies that

$$\mathrm{Cl}\Big( \bigcup W^s(p), p \in \operatorname{per} f \cap \Omega_i \Big) \supset \bigcup W^s(x), \qquad x \in \Omega_i,$$

and since $W^s(\Omega_i) = \bigcup W^s(x)$, $x \in \Omega_i$ [3], then

$$g|\mathrm{Cl}(W^s(\Omega_i)) = f^{k(i)}|\mathrm{Cl}(W^s(\Omega_i)).$$

Since $M = \bigcup W^s(\Omega_i)$, connectedness of $M$ implies that $k(1) = \cdots = k(m)$, hence $g$ is trivial, proving the Theorem 2.

## References


1. B. Anderson, *Diffeomorphisms with discrete centralizer*, Topology **15** (1976), 143–147.

2. ______, *Centralizer of C-diffeomorphisms*, Trans. Amer. Math. Soc. **222** (1976), 97–106.

3. M. Hirsh, J. Palis, C. Pugh, and M. Shub, *Neighbourhood of hyperbolic sets*, Invent. Math. **9** (1970), 121–134.

4. M. Hirsh and C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. of A. M. S. Summer Institute on Global Analysis at Berkeley, 1968, pp. 133–165.

5. M. Irvin, *On the smoothness of the composition map*, Quart. J. Math. Oxford Ser. (2) **23** (1972), 112–133.

6. J. Palis, *Vector fields generate few diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 503–505.

7. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

8. Y. Togawa, *Generic Morse-Smale diffeomorphisms have only trivial symmetries*, Proc. Amer. Math. Soc. (to appear).


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