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Centrally extended BMS4 Lie algebroid

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Abstract: We explicitly show how the field dependent 2-cocycle that arises in the current algebra of 4 dimensional asymptotically flat spacetimes can be used as a central extension to turn the BMS4 Lie algebra, or more precisely, the BMS4 action Lie algebroid, into a genuine Lie algebroid with field dependent structure functions. Both a BRST formulation, where the extension appears as a ghost number 2 cocyle, and a formulation in terms of vertex operator algebras are introduced. The mapping of the celestial sphere to the cylinder then implies zero mode shifts of the asymptotic part of the shear and of the news tensor.

Keywords: Anomalies in Field and String Theories, Classical Theories of Gravity, Conformal and W Symmetry, Gauge-gravity correspondence

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1 Introduction

The holographic properties of three-dimensional anti de-Sitter gravity are very special: the asymptotic symmetry algebra represents the two dimensional conformal algebra with a definite prediction for the classical central charge in the Dirac bracket algebra of the canonical generators [1], the solution space includes rotating black holes [2], and the dual theory on the classical level is Liouville theory [3]. Furthermore, using the classical central extension in Cardy's formula allows one to reproduce the Bekenstein-Hawking entropy of the BTZ black holes [4] (see also [5]).

Whereas all these results have natural analogues in asymptotically flat gravity at null infinity in three dimensions [6–12], the four-dimensional case [13–15] is more involved. One reason is the non-integrability of the charges in the presence of gravitational radiation [16]. Another source of complication is due to the desire to include superrotations in the BMS group [17–19] or the associated algebra [20–23], which goes together with allowing for suitable singularities in the solution space on which superrotations act. Besides formal reasons that drive one to include these additional symmetries, they have been shown to lead to new physical applications such as the subleading soft graviton theorem [24, 25] or the spin memory effect [26, 27]. A naive treatment of the charges associated to the extended algebra, that is to say integrating regular solutions on the celestial sphere with generators that have poles, generically leads to divergences [28, 29]. One way to avoid dealing directly with those is simply not to integrate over the celestial sphere and to work with the local current algebra instead [30]. For some problems this is however not good enough as they do require a well-defined integration, or more precisely, a suitable moment map from solution space into the dual of the symmetry algebra. One such problem concerns the central charge.

The aim of the present paper is to clarify formal aspects of the field dependent central extension that appears in the modified Dirac bracket algebra of charges and currents [28, 30], by explicitly constructing the centrally extended Lie algebroid that comes with the $\mathfrak{bms}_4$ algebra and its action on the free data at null infinity.

More precisely, we first reformulate the field dependent central extension as a local BRST 2-cocycle, in much the same way the Adler-Bardeen non-abelian gauge anomaly can be reformulated as the BRST 1-cocycle that appears in the transgression from the characteristic class $\mathrm{Tr}\, F^3$ to the primitive element $\mathrm{Tr}\, C^5$ (see e.g. [31–34]). Despite the BRST-type formulation, we do not imply that we are dealing with gauge symmetries. Rather, as in the three-dimensional case where the asymptotic symmetries of the gravitational/Chern-Simons theory become the global symmetries of the dual Wess-Zumino-Witten or Liouville-type theory, we consider BMS_4 as the global symmetry group of a suitable dual theory. The BRST formulation here is just a convenient way to encode Lie algebra or algebroid cohomology.

For the above considerations, the associated local functionals are formal in the sense that they are given by equivalence classes of top forms up to exact ones, which means that one disregards all boundary terms that come from integrations by parts. When looking for concrete realizations, one is led towards formulations in terms of vertex operator algebras of conformal field theories where spatial integrals correspond to taking residues (see e.g. [35] or [36, 37] for elementary introductions, [38] where related contour integrals have appeared in the current context and [39, 40] for related constructions applied to BMS_3).

Alternatively, as proposed in [18], one may map $\mathscr{I}^+$ to a cylinder times a line and explicitly realize the centrally extended $\mathfrak{bms}_4$ algebroid using Fourier analysis. The effect of mapping a gravitational solution from the 2-punctured Riemann sphere to the cylinder is a shift of the zero mode of the asymptotic part of the shear, and thus also of the subleading part of the angular metric, that is linear in retarded time. As a consequence, this implies a constant shift of the zero mode of the news, in direct analogy with the standard shift of the zero mode of the energy momentum tensor in a conformal field theory.

Since it might not be widely known in the physics literature, we start by briefly recalling the general framework for central extensions in a Lie algebroid/Lie-Rinehart pair (see e.g. [41, 42]) before applying the construction to the case of interest.

2 Local description of a Lie algebroid

Consider an algebra of functions A in variables ϕ^i with elements denoted by $f(\phi)$ and a vector space $\mathfrak{g}$ generated over A by a set e_α, with elements denoted by $\xi = \xi^\alpha(\phi) e_\alpha$. The vector space $\mathfrak{g}$ is turned into a Lie algebra by defining

$$[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta}(\phi) e_\gamma, \tag{2.1}$$

$$[e_\alpha, f(\phi)] = R^i_\alpha(\phi)\partial_i f, \tag{2.2}$$

where $\partial_i = \frac{\partial}{\partial \phi^i}$, by extending the bracket using skew-symmetry and the Leibniz rule, and by requiring that

$$2R^i_{[\alpha}\partial_i R^j_{\beta]} = f^\gamma_{\alpha\beta} R^j_\gamma, \tag{2.3}$$

$$R^i_{[\gamma}\partial_i f^\epsilon_{\alpha\beta]} = f^\epsilon_{\delta[\gamma} f^\delta_{\alpha\beta]}, \tag{2.4}$$

where square brackets denote skew-symmetrization of the included indices of the same type, divided by the factorial of the number of these indices. In other words, all Jacobi identities hold when using the rules and conditions (2.3) and (2.4). Alternatively, instead of (2.2), one can define $\delta_\xi f = \xi^\alpha R^i_\alpha \partial_i f$ and

$$[\xi_1, \xi_2] = (\xi_1^\alpha \xi_2^\beta f^\gamma_{\alpha\beta} + \delta_{\xi_1}\xi_2^\gamma - \delta_{\xi_2}\xi_1^\gamma) e_\gamma, \tag{2.5}$$

with

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1,\xi_2]}. \tag{2.6}$$

Note that here and below, we will systematically use the notation

$$\delta_\xi f = [\xi, f]. \tag{2.7}$$

Introducing Grassmann odd variables C^α and $\partial_\alpha = \frac{\partial}{\partial C^\alpha}$, the graded space of polynomials in these variables taking values in A is denoted by Ω^*. Its elements are denoted by

$$\omega = \sum_{p=0} \frac{1}{p!} \omega_{\alpha_1 \dots \alpha_p}(\phi) C^{\alpha_1} \dots C^{\alpha_p}, \tag{2.8}$$

where $\omega_{\alpha_1\dots\alpha_p} = \omega_{[\alpha_1\dots\alpha_p]}$. Equations (2.3) and (2.4) are then equivalent to the requirement that

$$\gamma = C^\alpha R^i_\alpha \partial_i - \frac{1}{2} C^\alpha C^\beta f^\gamma_{\alpha\beta} \partial_\gamma, \tag{2.9}$$

is a differential on Ω^*,

$$\gamma^2 = 0. \tag{2.10}$$

The particular case where the $f^\gamma_{\alpha\beta}$ are constants and do not depend explicitly on the fields is referred to as an action algebroid.

Note that in the case of interest below, fields and their derivatives are relevant, so that these formulas have to be suitably interpreted in, respectively extended to, the context of jet-bundles, see e.g. [43–46].

3 Central extensions

A trivial central extension is constructed by adding a generator Z to $\mathfrak{g}$, with $\widehat{\mathfrak{g}}$ consisting of elements $\hat{\xi} = \xi^\alpha(\phi) e_\alpha + \xi^Z(\phi) Z$, keeping (2.1) unchanged, while

$$[Z, e_\alpha] = 0 = [Z, f(\phi)], \tag{3.1}$$

and the bracket again extended by skew-symetry and Leibniz rule.

Consider a 2-cocycle,

$$\gamma\omega^2 = 0 \iff R^i_{[\gamma}\partial_i\omega_{\alpha\beta]} = \omega_{\delta[\gamma}f^\delta_{\alpha\beta]}. \tag{3.2}$$

This condition is equivalent to saying that $\widehat{\mathfrak{g}}$, where (2.1) is changed to

$$[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta}(\phi)e_\gamma + \omega_{\alpha\beta}(\phi)Z, \tag{3.3}$$

and all other relations are kept unchanged, is still a Lie algebroid. In the case where ω^2 is a coboundary,

$$\omega^2 = \gamma\eta^1 \iff \omega_{\alpha\beta} = 2R^i_{[\alpha}\partial_i\eta_{\beta]} - f^\gamma_{\alpha\beta}\eta_\gamma, \tag{3.4}$$

this extended Lie algebroid is equivalent to the trivially extended Lie algebroid by the change of generators

$$e'_\alpha = e_\alpha - \eta_\alpha Z, \quad Z' = Z. \tag{3.5}$$

The differential of the extended Lie algebroid is

$$\hat{\gamma} = \gamma - \frac{1}{2}C^\alpha C^\beta\omega_{\alpha\beta}\frac{\partial}{\partial C^Z} \tag{3.6}$$

in the space of polynomials in C^α, C^Z with values in functions of ϕ^i. By construction, the 2-cocycle ω^2, becomes trivial in the extended complex, $\omega^2 = -\hat{\gamma}C^Z$.

For later use, we note that, if $K_{\xi_1,\xi_2} = \omega_{\alpha\beta}\xi_1^\alpha\xi_2^\beta$, $\eta_\xi = \eta_\alpha\xi^\alpha$, the cocycle condition (3.2) and the coboundary condition (3.4) can also be written as

$$K_{[\xi_1,\xi_2],\xi_3} - [\xi_3, K_{\xi_1,\xi_2}] + \text{cyclic}(1,2,3) = 0, \tag{3.7}$$
$$K_{\xi_1,\xi_2} = [\xi_1, \eta_{\xi_2}] - [\xi_2, \eta_{\xi_1}] - \eta_{[\xi_1,\xi_2]}, \tag{3.8}$$

and that the extension defined by (3.3) is equivalent to

$$[\hat{\xi}_1, \hat{\xi}_2] = [\xi_1, \xi_2] + K_{\xi_1,\xi_2}Z. \tag{3.9}$$

4 BMS4 action algebroid

We describe here relevant elements of the symmetry structure of four-dimensional asymptotically flat spacetimes at null infinity. We will adopt the point of view developed in [18, 21–23, 28, 30, 47], to which we refer for more details and assume at the outset to be in the simplest case for our purpose here, with future null infinity $\mathscr{I}^+$ taken as the 2-punctured Riemann sphere times a line.

The independent variables are $u, \zeta, \bar{\zeta}$. The variable u is real, $\bar{u} = u$. In this context, the $\mathfrak{bms}_4$ algebra is parametrized by $T(\zeta,\bar{\zeta}) = \bar{T}$, $Y(\zeta), \bar{Y}(\bar{\zeta})$. It is the Lie algebra of vector fields

$$\xi = f\partial_u + Y\partial + \bar{Y}\bar{\partial}, \tag{4.1}$$

where $\partial = \partial_\zeta$, $\bar{\partial} = \partial_{\bar{\zeta}}$,

$$f = T + \frac{1}{2}u\psi, \quad \psi = \partial Y + \bar{\partial}\bar{Y}. \tag{4.2}$$

Writing $[\xi_{T_1,Y_1,\bar{Y}_1}, \xi_{T_2,Y_2,\bar{Y}_2}] = \xi_{\hat{T},\hat{Y},\hat{\bar{Y}}}$, this gives

$$\widehat{T} = Y_1 \partial T_2 - \frac{1}{2}\partial Y_1 T_2 - (1 \leftrightarrow 2) + \text{c.c.}, \tag{4.3}$$

$$\widehat{Y} = Y_1 \partial Y_2 - (1 \leftrightarrow 2), \tag{4.4}$$

where c.c. denotes complex conjugation and $\widehat{\bar{Y}} = \bar{\widehat{Y}}$.

The relevant fields are $\sigma(u,\zeta,\bar{\zeta})$, its complex conjugate $\bar{\sigma}$ and their derivatives. They correspond to the asymptotic part of the complex shear, but for notational simplicity, we have dropped the standard superscript 0. On-shell, they encode the subleading components of the angular part of the BMS metric. They transform as

$$-[\xi,\sigma] = \left[f\partial_u + Y\partial + \bar{Y}\bar{\partial} + \frac{3}{2}\bar{\partial}\bar{Y} - \frac{1}{2}\partial Y\right]\sigma - \bar{\partial}^2 f, \tag{4.5}$$

with $\delta_\xi \bar{\sigma} = \overline{\delta_\xi \sigma}$. Furthermore, the transformation of the derivative of a field corresponds to the derivative of the transformation of the field,

$$[\xi, \partial^k \sigma] = \partial^k([\xi,\sigma]), \quad [\xi, \bar{\partial}^k \sigma] = \bar{\partial}^k([\xi,\sigma]), \quad [\xi, \partial_u^k \sigma] = \partial_u^k([\xi,\sigma]), \tag{4.6}$$

together with the complex conjugates of these relations. Note that this implies in particular the following transformation law for the news tensor $\dot{\sigma} = \partial_u \sigma$,

$$-[\xi,\dot{\sigma}] = [f\partial_u + Y\partial + \bar{Y}\bar{\partial} + 2\bar{\partial}\bar{Y}]\dot{\sigma} - \frac{1}{2}\bar{\partial}^3 \bar{Y}. \tag{4.7}$$

It also follows that $[\xi_1,[\xi_2,\sigma]] - [\xi_2,[\xi_1,\sigma]] = [[\xi_1,\xi_2],\sigma]$, as required by (2.3). There are other fields on which $\mathfrak{bms}_4$ acts non trivially, but they are passive in the sense that they do not modify the commutators below.

It follows from the computations in [28, 30, 47] that the expression

$$K_{\xi_1,\xi_2} = \int d\zeta \int d\bar{\zeta} \left[\left(\sigma f_1 \partial^3 Y_2 - (1 \leftrightarrow 2)\right) + \text{c.c.}\right], \tag{4.8}$$

satisfies the cocycle condition (3.7) provided that the integral annihilates ∂ and $\bar{\partial}$ derivatives.

5 BRST formulation

Introducing the Grassmann odd fields $\eta(\zeta)$, $\bar{\eta}(\bar{\zeta})$, $C(\zeta,\bar{\zeta})$ with C real, associated to $Y, \bar{Y}, T$, and the combination $\chi = C + \frac{u}{2}(\partial\eta + \bar{\partial}\bar{\eta})$, the BRST differential of the BMS4 action algebroid is defined through

$$\begin{aligned} \gamma\eta &= -\eta\partial\eta, \quad \gamma C = -\eta\partial C + \frac{1}{2}\partial\eta C + \text{c.c.}, \\ \gamma\sigma &= -\left(\chi\partial_u + \eta\partial + \bar{\eta}\bar{\partial} + \frac{3}{2}\bar{\partial}\bar{\eta} - \frac{1}{2}\partial\eta\right)\sigma + \bar{\partial}^2\chi. \end{aligned} \tag{5.1}$$

When changing variables and using $\chi(u,\zeta,\bar\zeta)$ instead of C, we have

$$\dot\chi = \frac{1}{2}(\partial\eta + \bar\partial\bar\eta), \quad \gamma\chi = -\left(\eta\partial + \bar\eta\bar\partial - \frac{1}{2}\partial\eta - \frac{1}{2}\bar\partial\bar\eta\right)\chi. \tag{5.2}$$

The differential is extended so as to commute with complex conjugation and all derivatives. The expression corresponding to the integrand of (4.8) and the associated spatial components computed in [30] is given by

$$\omega^{2,2} = d\zeta d\bar\zeta K^u - dud\bar\zeta K + dud\zeta\bar K, \tag{5.3}$$

where

$$K^u = \chi(Q+\bar Q), \quad K = \eta(Q+\bar Q) + \bar\partial^3\bar\eta\partial\chi, \quad Q = \partial^3\eta\sigma. \tag{5.4}$$

Introducing in addition

$$N = \chi K, \quad \bar O = \eta\bar\eta(Q+\bar Q) + \eta\partial^3\eta\bar\partial\chi - \bar\eta\bar\partial^3\bar\eta\partial\chi, \tag{5.5}$$

and

$$\begin{aligned} \omega^{3,1} &= -(du\bar O - d\zeta\bar N + d\bar\zeta N), \\ \omega^{4,0} &= \eta\bar\eta\chi(Q+\bar Q) + \eta\partial^3\eta\chi\bar\partial\chi - \bar\eta\bar\partial^3\bar\eta\chi\partial\chi \\ &= \eta\partial^3\eta(\bar\eta\chi\sigma + \chi\bar\partial\chi) - \bar\eta\bar\partial^3\bar\eta(\eta\chi\bar\sigma + \chi\partial\chi)\,, \end{aligned} \tag{5.6}$$

the relations

$$\begin{aligned} &\gamma Q = -\partial(\eta Q) - \partial_u(\chi Q) - \bar\partial(\bar\eta Q + \partial^3\eta\bar\partial\chi), \qquad \gamma K^u = \partial N + \bar\partial\bar N, \\ &\gamma K = -\partial_u N + \bar\partial\bar O, \qquad \gamma\bar O = -\partial_u\omega^{4,0}, \qquad \gamma N = -\bar\partial\omega^{4,0}, \end{aligned} \tag{5.7}$$

allow one to easily derive the descent equations

$$\begin{aligned} \gamma\omega^{2,2} + d_H\omega^{3,1} &= 0, \\ \gamma\omega^{3,1} + d_H\omega^{4,0} &= 0, \\ \gamma\omega^{4,0} &= 0, \end{aligned} \tag{5.8}$$

where $d_H = du\partial_u + d\zeta\partial + d\bar\zeta\bar\partial$. It follows that $\omega^{2,2}$ is a BRST cocycle modulo d_H in ghost number 2 and form degree 2. One way to show that this cocycle is non-trivial, $\omega^{2,2} \neq \gamma\eta^{1,2} + d_H\eta^{2,1}$ is to show that $\omega^{4,0}$ is non trivial, $\omega^{4,0} \neq \gamma\eta^{3,0}$. This analysis will be completed elsewhere.

6 Centrally extended BMS4 Lie algebroid

6.1 Realization on the two-punctured Riemann sphere

Provided the integral annihilates spatial boundary terms, the centrally extended Lie algebroid $\widehat{\mathfrak{bms}}_4$ is defined by the commutators given in (3.9). More explicitly, parametrizing $\widehat{\mathfrak{bms}}_4$ through $(T(\zeta,\bar\zeta), Y(\zeta), \bar Y(\bar\zeta), V)$, with the understanding that the elements in each slot can be multiplied by functions of $\sigma, \bar{\sigma}$ and their derivatives, the commutation relations (4.3) and (4.4) are completed by

$$\hat{V} = K_{\xi_1,\xi_2}. \tag{6.1}$$

A concrete framework where multiplication is well defined and spatial boundary terms can indeed be neglected is provided by vertex operator algebras where one considers either polynomials with formal power series or Laurent series and the integral is defined to select the residue separately in ζ and $\bar{\zeta}$. For instance, a set-up that accommodates singular solutions with delta function singularities is to take for $Y, \bar{Y}, T$ Laurent polynomials, while $\partial_u^n \sigma$ are formal power series.

In terms of the following generators for $Y, \bar{Y}, T$,

$$l_m = -\zeta^{m+1}\partial, \quad \bar{l}_m = -\bar{\zeta}^{m+1}\bar{\partial}, \quad P_{k,l} = \zeta^{k+\frac{1}{2}}\bar{\zeta}^{l+\frac{1}{2}}, \tag{6.2}$$

the algebra reads

$$\begin{aligned}
[l_m, l_n] &= (m-n)l_{m+n}, & [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n},\\
[l_m, P_{k,l}] &= \left(\frac{1}{2}m - k\right) P_{m+k,l}, & [\bar{l}_m, P_{k,l}] &= \left(\frac{1}{2}m - l\right) P_{k,m+l},\\
[l_m, \bar{l}_n] &= 0 = [P_{k,l}, P_{o,p}]. & &
\end{aligned} \tag{6.3}$$

It follows from (4.5), (4.7) that the conformal weights of $\sigma, \partial_u\sigma$ are $(-\frac{1}{2}, \frac{3}{2})$ and $(0,2)$ respectively, so that u is of conformal weights $(-\frac{1}{2}, -\frac{1}{2})$. This leads to the expansions

$$\begin{aligned}
\partial_u^n \sigma(u,\zeta,\bar{\zeta}) &= \sum_{k,l} (\partial_u^n\sigma)_{k,l}(u)\zeta^{-k-\frac{n-1}{2}}\bar{\zeta}^{-l-\frac{n+3}{2}},\\
\partial_u^n \bar{\sigma}(u,\zeta,\bar{\zeta}) &= \sum_{k,l} (\partial_u^n\bar{\sigma})_{k,l}(u)\zeta^{-k-\frac{n+3}{2}}\bar{\zeta}^{-l-\frac{n-1}{2}}.
\end{aligned} \tag{6.4}$$

Equation (4.5), (4.7) and their higher order time derivatives become

$$\begin{aligned}
[l_m, (\partial_u^n\sigma)_{k,l}] &= \left(\frac{n-3}{2}m - k\right)(\partial_u^n\sigma)_{m+k,l} + \frac{m+1}{2}u(\partial_u^{n+1}\sigma)_{m+k-\frac{1}{2},l-\frac{1}{2}},\\
[\bar{l}_m, (\partial_u^n\sigma)_{k,l}] &= \left(\frac{n+1}{2}m - k\right)(\partial_u^n\sigma)_{k,m+l} + \frac{m+1}{2}u(\partial_u^{n+1}\sigma)_{k-\frac{1}{2},m+l-\frac{1}{2}}\\
&\quad - \frac{1}{2}m(m^2-1)\left(u\delta_n^0\delta^0_{k-\frac{1}{2}}\delta^0_{m+l-\frac{1}{2}} + \delta_n^1\delta_k^0\delta^0_{m+l}\right),\\
[P_{k,l}, (\partial_u^n\sigma)_{o,p}] &= -(\partial_u^{n+1}\sigma)_{k+o,l+p} + \delta_n^0\left(l^2 - \frac{1}{4}\right)\delta^0_{k+o}\delta^0_{l+p}.
\end{aligned} \tag{6.5}$$

In these expansions, $m, n \in \mathbb{Z}$. One consistent choice, called (NS) below, which makes all the inhomogeneous terms above non-vanishing, is $k, l, o, p \in \frac{1}{2} + \mathbb{Z}$ when carried by P or by an even number of time derivatives of σ and $k, l, o, p \in \mathbb{Z}$ when carried by an odd number of time derivatives of σ. This means that fields with odd conformal weights satisfy Neveu-Schwarz boundary conditions and will be anti-periodic on the cylinder, while fields with even conformal weights will be periodic on the cylider. Another possibility (R) is to take $k, l, o, p \in \mathbb{Z}$ in all cases. Other possibilities should of course be systematically explored.

Equation (4.8) now implies the following explicit expression for the central extension in terms of generators,

$$
\begin{aligned}
K_{l_m,l_n} &= \frac{1}{2}u(m+1)(n+1)\sigma_{m+n-\frac{1}{2},-\frac{1}{2}}[n(n-1)-m(m-1)],\\
K_{l_m,\bar{l}_n} &= -\frac{1}{2}u(m+1)(n+1)\left[\sigma_{m-\frac{1}{2},n-\frac{1}{2}}m(m-1)-\bar{\sigma}_{m-\frac{1}{2},n-\frac{1}{2}}n(n-1)\right],\\
K_{l_m,P_{k,l}} &= \sigma_{m+k,l}m(m^2-1),\\
K_{\bar{l}_m,\bar{l}_n} &= \frac{1}{2}u(m+1)(n+1)\bar{\sigma}_{-\frac{1}{2},m+n-\frac{1}{2}}[n(n-1)-m(m-1)],\\
K_{\bar{l}_m,P_{k,l}} &= \bar{\sigma}_{k,m+l}m(m^2-1),\\
K_{P_{k,l},P_{o,p}} &= 0.
\end{aligned}
\tag{6.6}
$$

In case (NS), all these terms may be non-vanishing, while in case (R) only $K_{l_m,P_{k,l}}$ and $K_{\bar{l}_m,P_{k,l}}$ may be.

Even though it is not directly necessary for the construction of the classical $\widehat{\mathfrak{bms}}_4$ Lie algebroid, we note in case (NS) for instance, the $\mathfrak{bms}_4$ Lie algebra itself may be encoded through the series

$$
\begin{aligned}
J(\zeta) &= \sum_{m\in\mathbb{Z}}\zeta^{-m-2}l_m, \qquad \bar{J}(\bar{\zeta}) = \sum_{m\in\mathbb{Z}}\bar{\zeta}^{-m-2}\bar{l}_m,\\
P(\zeta,\bar{\zeta}) &= \sum_{k,l\in\frac{1}{2}+\mathbb{Z}}\zeta^{-k-\frac{3}{2}}\bar{\zeta}^{-l-\frac{3}{2}}P_{k,l},
\end{aligned}
\tag{6.7}
$$

the commutation relations (6.3) being equivalent to

$$
\begin{aligned}
[J(\zeta),J(\omega)] &= (\delta(\zeta-\omega)D+2D\delta(\zeta-\omega))J(\omega),\\
[\bar{J}(\bar{\zeta}),\bar{J}(\bar{\omega})] &= (\delta(\bar{\zeta}-\bar{\omega})\bar{D}+2\bar{D}\delta(\bar{\zeta}-\bar{\omega}))\bar{J}(\bar{\omega}),\\
[J(\zeta),P(\omega,\bar{\omega})] &= \left(\delta(\zeta-\omega)D+\frac{3}{2}D\delta(\zeta-\omega)\right)P(\omega,\bar{\omega}),\\
[\bar{J}(\bar{\zeta}),P(\omega,\bar{\omega})] &= \left(\delta(\bar{\zeta}-\bar{\omega})\bar{D}+\frac{3}{2}\bar{D}\delta(\bar{\zeta}-\bar{\omega})\right)P(\omega,\bar{\omega}),\\
[J(\zeta),\bar{J}(\bar{\omega})] &= 0 = [J(\zeta,\bar{\zeta}),P(\omega,\bar{\omega})]\,,
\end{aligned}
\tag{6.8}
$$

with $D^k = \frac{1}{k!}\partial_\omega^k$ and

$$
D^k\delta(\zeta-\omega) = \sum_{n\in\mathbb{Z}}\binom{n}{k}\zeta^{-n-1}\omega^{n-k}. \tag{6.9}
$$

As usual, in the space of formal distributions with values in the universal enveloping algebra of $\mathfrak{bms}_4$, one can write the singular parts as

$$
\begin{aligned}
J(\zeta)J(\omega) &\sim \frac{DJ(\omega)}{\zeta-\omega}+\frac{2J(\omega)}{(\zeta-\omega)^2},\\
\bar{J}(\bar{\zeta})\bar{J}(\bar{\omega}) &\sim \frac{\bar{D}\bar{J}(\bar{\omega})}{\bar{\zeta}-\bar{\omega}}+\frac{2\bar{J}(\bar{\omega})}{(\bar{\zeta}-\bar{\omega})^2},\\
J(\zeta)P(\omega,\bar{\omega}) &\sim \frac{DP(\omega,\bar{\omega})}{\zeta-\omega}+\frac{3P(\omega,\bar{\omega})}{2(\zeta-\omega)^2},\\
\bar{J}(\bar{\zeta})P(\omega,\bar{\omega}) &\sim \frac{\bar{D}P(\omega,\bar{\omega})}{\bar{\zeta}-\bar{\omega}}+\frac{3P(\omega,\bar{\omega})}{2(\bar{\zeta}-\bar{\omega})^2},\\
J(\zeta)\bar{J}(\overline{\omega}) &\sim 0 \sim P(\zeta,\bar{\zeta})P(\omega,\bar{\omega})\,,
\end{aligned}
\tag{6.10}
$$

while

$$
\begin{aligned}
[J(\zeta),\partial_u^n\sigma(u,\omega,\bar{\omega})] &= \left(\delta(\zeta-\omega)D+D\delta(\zeta-\omega)\left[\frac{n-1}{2}+\frac{u}{2}\partial_u\right]\right)\partial_u^n\sigma(u,\omega,\bar{\omega}),\\
[\bar{J}(\bar{\zeta}),\partial_u^n\sigma(u,\omega,\bar{\omega})] &= \left(\delta(\bar{\zeta}-\bar{\omega})\bar{D}+\bar{D}\delta(\bar{\zeta}-\bar{\omega})\left[\frac{n+3}{2}+\frac{u}{2}\partial_u\right]\right)\partial_u^n\sigma(u,\omega,\bar{\omega})\\
&\quad -3(u\delta_n^0+\delta_n^1)\bar{D}^3\delta(\bar{\zeta}-\bar{\omega}),\\
[P(\zeta,\bar{\zeta}),\partial_u^n\sigma(u,\omega,\bar{\omega})] &= -\delta(\zeta-\omega)\delta(\bar{\zeta}-\bar{\omega})\partial_u^{n+1}\sigma(u,\omega,\bar{\omega})\\
&\quad +2\delta_n^0\delta(\zeta-\omega)\bar{D}^2\delta(\bar{\zeta}-\bar{\omega}).
\end{aligned}
\tag{6.11}
$$

In the case of a suitable (free-field) representation of $\mathfrak{bms}_4$ with locality conditions so that the various series can be multiplied, one would write

$$
\begin{aligned}
J(\zeta)\partial_u^n\sigma(u,\omega,\bar{\omega}) &\sim \left(\frac{1}{\zeta-\omega}D+\frac{1}{(\zeta-\omega)^2}\left[\frac{n-1}{2}+\frac{u}{2}\partial_u\right]\right)\partial_u^n\sigma(u,\omega,\bar{\omega}),\\
\bar{J}(\bar{\zeta})\partial_u^n\sigma(u,\omega,\bar{\omega}) &\sim \left(\frac{1}{\bar{\zeta}-\bar{\omega}}\bar{D}+\frac{1}{(\bar{\zeta}-\bar{\omega})^2}\left[\frac{n+3}{2}+\frac{u}{2}\partial_u\right]\right)\partial_u^n\sigma(u,\omega,\bar{\omega})\\
&\quad -3(u\delta_n^0+\delta_n^1)\frac{1}{(\bar{\zeta}-\bar{\omega})^4},\\
P(\zeta,\bar{\zeta})\partial_u^n\sigma(u,\omega,\bar{\omega}) &\sim -\frac{1}{\zeta-\omega}\frac{1}{\bar{\zeta}-\bar{\omega}}\partial_u^{n+1}\sigma(u,\omega,\bar{\omega})\\
&\quad +2\delta_n^0\frac{1}{\zeta-\omega}\frac{1}{(\bar{\zeta}-\bar{\omega})^3}.
\end{aligned}
\tag{6.12}
$$

6.2 Realization on the cylinder

Alternatively, one may map $\mathscr{I}^+$ to a cylinder times a line and consider Fourier series that can simply be multiplied under standard assumptions.

As defined in [22, 28], the transformation laws of the $\mathfrak{bms}_4$ algebra under finite superrotations are

$$
\begin{aligned}
Y'(\zeta') &= Y(\zeta(\zeta'))\frac{\partial\zeta'}{\partial\zeta}, & \bar{Y}'(\bar{\zeta}') &= \bar{Y}(\bar{\zeta}(\bar{\zeta}'))\frac{\partial\bar{\zeta}'}{\partial\bar{\zeta}},\\
T'(\zeta',\bar{\zeta}') &= J^{-\frac{1}{2}}T(\zeta(\zeta'),\bar{\zeta}(\bar{\zeta}')), & J &= \frac{\partial\zeta}{\partial\zeta'}\frac{\partial\bar{\zeta}}{\partial\bar{\zeta}'},
\end{aligned}
\tag{6.13}
$$

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while for the asymptotic part of the shear and its time derivatives, equation (6.104) of [18] implies that

$$\partial^n_{u'}\sigma'(u',\zeta',\bar{\zeta}') = \left(\frac{\partial\zeta}{\partial\zeta'}\right)^{\frac{n-1}{2}}\left(\frac{\partial\bar{\zeta}}{\partial\bar{\zeta}'}\right)^{\frac{3+n}{2}}\left[\partial^n_u\sigma + \frac{1}{2}(u\delta^0_n + \delta^1_n)\{\bar{\zeta}',\bar{\zeta}\}\right], \quad u' = J^{-\frac{1}{2}}u. \tag{6.14}$$

The standard mapping from the cylinder to the 2-punctured Riemann sphere is described by $\zeta = e^{\frac{2\pi}{L}\omega}$ with $\omega = x_1 + ix_2$, $x_2 \sim x_2 + L$, coordinates on the cylinder. Taking $\zeta' = \omega, \bar{\zeta}' = \bar{\omega}$ in the above then gives $l_m = -\frac{L}{2\pi}e^{\frac{2\pi}{L}m\omega}\partial_\omega$, $\bar{l}_m = -\frac{L}{2\pi}e^{\frac{2\pi}{L}m\bar{\omega}}\partial_{\bar{\omega}}$, $u' = \frac{L}{2\pi}(\zeta\bar{\zeta})^{-\frac{1}{2}}u$, $P_{kl} = \frac{L}{2\pi}e^{\frac{2\pi}{L}k\omega}e^{\frac{2\pi}{L}l\bar{\omega}}$, and the mode expansion

$$\partial^n_{u'}\sigma'(u',\omega,\bar{\omega}) = \left(\frac{2\pi}{L}\right)^{n+1}\left[(\partial^n_u\sigma)_{k,l}(u)e^{-\frac{2\pi}{L}kw}e^{-\frac{2\pi}{L}l\bar{w}}\right] + \left(\frac{2\pi}{L}\right)^2\frac{1}{4}(\delta^0_n u' + \delta^1_n). \tag{6.15}$$

7 Conclusion

In this work, we have explicitly constructed a centrally extended Lie algebroid associated to $\mathfrak{bms}_4$ on the two-punctured Riemann sphere and the cylinder by suitably adapting the integration rules and allowing for formal distributions.

Note that one could also have worked with appropriate distributions directly on the celestial sphere (see e.g. [48–53]). The point of view taken here consists in first using transformation rules and invariance properties of various quantities such as the Bondi mass aspect under conformal rescalings [54, 55] to transpose everything to the Riemann sphere before considering distributions.

Working out the details when starting from the celestial sphere provides one with the normalizations for mass and angular momentum. In this context, note that we have put the coefficient of the central charge in (4.8) to one. One should keep in mind however that the correct normalization coming from the Einstein-Hilbert action is $(16\pi G)^{-1}$ when integrated over the celestial sphere. For instance, for asymptotically anti-de Sitter spacetimes in three dimensions, it is this normalization that determines the precise values $c^\pm = 3l/2G$ [1]. The correct normalization is thus liable to play an important role in applications such as Cardyology [4, 5, 10, 11, 56–58] at null infinity where iu becomes a coordinate on the thermal circle, and so is the shift in (6.15) since the asymptotic part of the shear for the Kerr black hole vanishes on the celestial sphere.

Apart from the concrete application considered in this work, the current set-up paves the way for analyzing what happens to gravitational solutions when replacing the celestial sphere by a generic Riemann surface.

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