CENTROID TRIANGULATIONS FROM k-SETS

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ABSTRACT

Given a set V of n points in the plane, no three of them being collinear, a convex inclusion chain of V is an ordering of the points of V such that no point belongs to the convex hull of the points preceding it in the ordering. We call k-set of the convex inclusion chain, every k-set of an initial subsequence of at least k points of the ordering. We show that the number of such k-sets (without multiplicity) is an invariant of V, that is, it does not depend on the choice of the convex inclusion chain. Moreover, this number is equal to the number of regions of the order-k Voronoi diagram of V (when no four points are cocircular).

The dual of the order-k Voronoi diagram belongs to the set of so-called centroid triangulations that have been originally introduced to generate bivariate simplex spline spaces. We show that the centroids of the k-sets of a convex inclusion chain are the vertices of such a centroid triangulation. This leads to the currently most efficient algorithm to construct particular centroid triangulations of any given point set; it runs in $O(n \log n + k(n-k) \log k)$ worst case time.

Keywords: k-set enumeration; convex inclusion chains; order-k Voronoi diagrams; centroid triangulations.

1. Introduction

Given a finite set V of n points in the plane, a k-set of V is a subset of k points of V that can be separated from the remaining points by a straight line. Since the 1970s, finding matching upper and lower bounds for the maximum number of k-sets of a set of n points in the plane has been an important problem in combinatorial geometry (see, e.g., Chapter 11 in Matoušek's textbook[1], or Wagner's survey[2]). The first (published) result is due to Lovász[3] who proved an upper bound of $O(n^{\frac{3}{2}})$ for $k = n/2$. Erdös, Lovász, Simmons and Straus[4] extended the result to general k, getting an upper bound of $O(nk^{\frac{1}{2}})$ and a lower bound of $\Omega(n \log k)$. The currently best bounds have been given by Dey[5] and by Tóth[6]. Dey showed that no point set has more than $O(nk^{\frac{1}{3}})$ k-sets and Tóth constructed sets with $n2^{\Omega(\sqrt{\log k})}$ k-sets. More precise results have been obtained by adding up the numbers of k-sets over different values of k. Calling every i-set of V $(i \leq k)$ an $(\leq k)$-set of V, Peck[7] and, independently, Alon and Györi[8], showed that the number of $(\leq k)$-sets of any set of n points is bounded by kn; the bound is achieved for points in convex position.

In this paper we obtain a new result on another summation of numbers of k-sets. In opposition to previous works, we fix the value of k and consider the k-sets of different subsets of V (without multiplicity). The subsets are obtained in the following way: Let $\mathcal{V} = (v_1, v_2, ..., v_n)$ be an ordering of the points of V such that, for every $i \in \{2, ..., n\}$, v_i does not belong to the convex hull of $\{v_1, ..., v_{i-1}\}$. $\mathcal{V}$ is called a convex inclusion chain of V, and we call k-set of $\mathcal{V}$ any k-set of $\{v_1, ..., v_i\}$, for all $i \in \{k, ..., n\}$. We show that, when no three points of V are collinear, the number of distinct k-sets of a convex inclusion chain of V is an invariant of V, that is, it does not depend on the chosen convex inclusion chain. More precisely, it is equal to $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} a_j(V)$, with $a_j(V)$ the number of j-sets of V and with $\sum_1^0 = 0$.

Surprisingly, this number is equal to the number of regions of the order-k Voronoi diagram of V (when no four points of V are cocircular)[9]. This diagram is a partition of the plane whose every region is the set of points of the plane having the same k nearest neighbors in V. In order to establish another connection between convex inclusion chains and order-k Voronoi diagrams, we first recall how these diagrams occur in spline theory.

An (univariate) degree k B-spline is a piecewise polynomial function, every piece of which being a degree k polynomial[10]. This function is defined through a set of $k+2$ reals, called knots. Given a set K of n knots, the B-splines defined by all subsets of $k+2$ consecutive knots of K are linearly independent and form the basis of a degree k B-spline space. Spline functions are then obtained by linear combinations of B-splines. These functions have many properties that make them attractive, namely for curve modeling. In order to model surfaces, a multivariate generalization of B-splines is needed. In 1976, de Boor[11] introduced the notion of simplex spline: a degree k simplex spline is a piecewise polynomial function defined through a set of $k+d+1$ knots, which are points in $\mathbb{R}^d$. Given a set K of n knots, different methods have then been proposed to select subsets of $k+d+1$ knots of K (also called configurations) to form the basis of a simplex spline space; for an overview, see the survey of Neamtu[12]. In 2007, Neamtu[13] proposed the first solution that really reduces to B-splines when $d = 1$. The selected configurations are all $(k+d+1)$-subsets of K for which there exists a sphere passing through $d+1$ of these knots, having the k other knots inside, and the rest of K outside. The centers of these spheres form a subset of the vertices of the order-$(k+1)$ Voronoi diagram of K. An application of Neamtu's configurations to surface reconstruction can be found in [14]. However, Liu and Snoeyink[15] pointed out that Neamtu's solution is too restrictive on the types of splines that can be generated. They showed that there exist more general families of configurations that generalize B-splines. The simplex splines defined with these more general configurations are interesting notably to model sharp features[16].

In order to generate their configurations in the case $d = 2$, Liu and Snoeyink generalized the order-k Voronoi diagram in the following way. The order-k Voronoi diagram of a set V of points in the plane admits a straight line dual graph whose vertices are the centroids of the k-point subsets defining the order-k Voronoi regions[17,18]. When no four points of V are cocircular, this dual graph is a triangulation called the order-k centroid Delaunay triangulation of V. Such a triangulation can be constructed by an iterative algorithm that deduces the order-k centroid Delaunay triangulation from the order-$(k-1)$ centroid Delaunay triangulation[9,17]. Liu and Snoeyink extended this algorithm to construct more general triangulations, called order-k centroid triangulations. The extended algorithm is applied to any given order-$(k-1)$ centroid triangulation. When it succeeds in constructing a new triangulation (and not overlapping triangles) then the generated triangulation is an order-k centroid triangulation. An order-1 centroid triangulation is an arbitrary (classical) triangulation of V. The vertices of an order-k centroid triangulation are centroids of k-point subsets of V, and its triangles are of two types. The triangles of the first type are defined with subsets of $k+2$ points of V, and these subsets are configurations defining a bivariate degree $k-1$ simplex spline space. The triangles of the second type are defined with subsets of $k+1$ points that define a bivariate degree $k-2$ simplex spline space.

However, Liu and Snoeyink could prove that their algorithm really generates triangulations only for the cases $k = 2$ and $k = 3$. Even though experimental results indicate that it also works for higher values of k, the family of centroid Delaunay triangulations is still the only family of centroid triangulations for which it has been proved that it can be generated for all k. In this paper, we prove the existence of a new family of centroid triangulations, which are related to convex inclusion chains. In fact we show that, for all k, the centroids of the k-sets of a convex inclusion chain of V are the vertices of an order-k centroid triangulation of V.

Up to now, the algorithm of Liu and Snoeyink is the only algorithm that allows to generate every existing centroid triangulation. Its time complexity depends on the generated triangulation, but it cannot be less than $\Omega(n \log n + k^2(n-k))$. A particular centroid triangulation, the order-k centroid Delaunay triangulation, can be generated using algorithms that construct the order-k Voronoi diagram. The algorithm of Agarwal and Matoušek[19], for example, allows to construct this diagram in $O(n^{1+\epsilon}k)$ time, where $\epsilon > 0$ is an arbitrarily small constant. The currently best randomized algorithm is the one given by Agarwal, de Berg, Matoušek, and Schwarzkopf[20], improved by Chan[21], which runs in $O(n \log n + k(n-k) \log k)$ expected time.

We give here a deterministic algorithm that constructs different order-k centroid triangulations of any set of n points in the plane in $O(n \log n + k(n-k) \log k)$ worst case time. Our algorithm is a generalization of the simple and fast Beneath-Beyond algorithm used to construct a triangulation of a point set, after a presort in some direction[22]. The triangulations generated by our algorithm can serve as input for an algorithm (for example a flip algorithm) that has to build a centroid triangulation optimizing various criteria.

2. k-sets of a convex inclusion chain

In the whole paper V denotes a finite set of n points in the plane, no three of them being collinear.

For any subset E of the plane, we denote by $\overline{E}$ the closure of E, by $\mathring{E}$ the relative interior of E, by $\delta(E) = \overline{E} \setminus \mathring{E}$ the boundary of E, and by $conv(E)$ the convex hull of E.

For every oriented straight line Δ, let Δ^+ (resp. Δ^-) be the closed half plane on the left (resp. on the right) of Δ.

Given to points s and t, we denote by st the closed line segment with endpoints s and t oriented from s to t, and by (st) the oriented straight line generated by st.

Given a non-negative integer k, a k-point subset T of V is called a k-set of V if their exists an oriented straight line Δ such that $\mathring{\Delta}^- \cap V = T$. An oriented segment st with endpoints in V is called a k-edge of V if $|(\mathring{st})^- \cap V| = k$.

A convex inclusion chain of V is an ordering $(v_1, v_2, ..., v_n)$ of the points of V such that, for every $i \in \{2, ..., n\}$, $v_i \notin conv(v_1, ..., v_{i-1})$. Every k-set of $\{v_1, ..., v_i\}$, $i \in \{1, ..., n\}$, is called a k-set of the convex inclusion chain $(v_1, v_2, ..., v_n)$.

Theorem 1. *For every $k \in \{1, \ldots, n\}$, any convex inclusion chain of V admits $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} a_j(V)$ distinct k-sets, where $a_j(V)$ is the number of j-sets of V and $\sum_1^0 = 0$.*

Proof. (i) Let $\mathcal{V} = (v_1, \ldots, v_n)$ be a convex inclusion chain of V. Consider first, for any $k \in \{0, ..., n-2\}$ and for any $i \in \{1, ..., n\}$, the set of k-edges of $V_i = \{v_1, ..., v_i\}$. Every k-edge st of V_i, if some exists (it is the case if and only if $i \geq k+2$), falls into one of the following three categories:

(a) $v_i \in (\mathring{st})^+$; in this case, $i \geq k+3$ and st is also a k-edge of V_{i-1},

(b) $v_i = s$ or $v_i = t$; st is clearly not a k-edge of V_{i-1},

(c) $v_i \in (\mathring{st})^-$; in this case, $k \geq 1$, $i \geq k+2$, and st is a $(k-1)$-edge of V_{i-1}.

Since v_i is an extreme point of V_i by assumption, if $i \geq k+2$ there is precisely one point s in V_i such that $|(\mathring{sv_i})^- \cap V_i| = k$. It follows that V_i admits precisely one k-edge of type (b) with $v_i = t$. In the same way, V_i admits precisely one k-edge of type (b) with $v_i = s$.

Furthermore, the k-edges of type (c) are all the $(k-1)$-edges of V_{i-1} that are not $(k-1)$-edges of V_i.

For every $i \in \{2, ..., n\}$, let us denote by $c_k(i)$ the number of k-edges of V_i that are not k-edges of V_{i-1} (these are “created” when inserting v_i), and by $d_k(i)$ the number of k-edges of V_{i-1} that are not k-edges of V_i (these are “deleted” when inserting v_i). Then, from the three categories above, $c_0(i) = 2$ and, for every $k \in \{1, ..., n-2\}$,

- $c_k(i) = 2 + d_{k-1}(i)$ for $i \in \{k+2, ..., n\}$,
- $c_k(i) = 0$ for $i \in \{2, ..., k+1\}$,
- $d_k(i) = 0$ for $i \in \{2, ..., k+2\}$.

Furthermore, the total number of k-edges of V is equal to $e_k(V) = \sum_{i=2}^{n} c_k(i) - \sum_{i=2}^{n} d_k(i)$. Thus, for every $k \in \{1, ..., n-2\}$,

$$\sum_{i=2}^{n} c_k(i) = \sum_{i=k+2}^{n} 2 + \sum_{i=2}^{n} d_{k-1}(i) = 2(n-k-1) + \sum_{i=2}^{n} c_{k-1}(i) - e_{k-1}(V).$$

Since $\sum_{i=2}^{n} c_0(i) = 2(n-1)$, we get by induction, for every $k \in \{0, ..., n-2\}$,

$$\sum_{i=2}^{n} c_k(i) = (2n-k-2)(k+1) - \sum_{j=0}^{k-1} e_j(V).$$

(ii) Consider now, for any $k \in \{1, ..., n\}$, the set of k-sets of all the sets V_i, when i ranges over $\{1, ..., n\}$. Let $b_k(i)$ be the number of k-sets created when inserting v_i. Clearly, $b_k(i) = 0$ when $i \in \{1, ..., k-1\}$, and $b_k(k) = 1$.

When $i \geq k+1$, there is a bijection between the k-sets and the $(k-1)$-edges of V_i. Indeed, if st is a $(k-1)$-edge of V_i then $T = ((\mathring{st})^- \cap V_i) \cup \{t\}$ is a k-set of V_i, which can be associated to the $(k-1)$-edge st (a separating line for T can be obtained by rotating st slightly counter clockwise about its midpoint). Conversely, for every k-set T of V_i, T and $V_i \setminus T$ admit exactly one common internal tangent (st) such that $s \in V \setminus T$, $t \in T$, and $T \subset (st)^-$. The segment st is then a $(k-1)$-edge and T is its associated k-set. Furthermore, if st is of type (c), or of type (b) with $v_i = t$, then $v_i \in T$ and T is not a k-set of V_{i-1}. In the other cases, $v_i \notin T$ and T is a k-set of V_{i-1}. It follows that $b_k(i) = c_{k-1}(i) - 1$ when $i \in \{k+1, ..., n\}$, and that the total number of k-sets of the convex inclusion chain $\mathcal{V}$ is

$$\sum_{i=1}^{n} b_k(i) = 1 + \sum_{i=k+1}^{n} c_{k-1}(i) - (n-k).$$

From (i), since $\sum_{i=2}^{k} c_{k-1}(i) = 0$, this number equals

$$1 + (2n-k-1)k - \sum_{j=0}^{k-2} e_j(V) - (n-k) = 2kn - n - k^2 + 1 - \sum_{j=0}^{k-2} e_j(V).$$

The statement of the theorem follows from the fact that, by the bijection, $e_j(V)$ is equal to the number $a_{j+1}(V)$ of $(j+1)$-sets of V. □

This theorem shows that the number of k-sets of a convex inclusion chain of a planar point set is an invariant of the point set, that is, it does not depend on the chosen convex inclusion chain. It notably means that the total number of k-sets that have to be generated when determining the k-sets of a point set by adding the points one by one does not depend on the order in which the points are treated, provided that every new inserted point does not belong to the convex hull of the previously inserted ones. Since $\sum_{j=1}^{k-1} a_j(V)$ is the number of $(\leq (k-1))$-sets of V, which is known to be bounded by $(k-1)n$ (see [7,8]), it follows that the worst case complexity of the incremental determination of the k-sets of n points is $\Omega(k(n-k))$.

A more intriguing consequence of the theorem arises from its connection with order-k Voronoi diagrams. Recall that the order-k Voronoi diagram of the point set V is a partition of the plane whose every region is associated with a k-point subset of V. More precisely, the order-k Voronoi region associated with a subset T of V includes all points of the plane closer to each element of T than to any element of $V \setminus T$. Lee[9] has shown that, if no four points of V are cocircular, the order-k Voronoi diagram of V admits $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} a_j(V)$ regions; the same number as the one found in Theorem 1. Since a k-point subset T is associated with an order-k Voronoi region if and only if T can be separated from the remaining points by a circle, it follows that:

Corollary 1. *Given a set V of points in the plane, no three of them being collinear and no four of them being cocircular, the number of k-sets of a convex inclusion chain of V is equal to the number of k-point subsets of V that can be separated from the remaining by a circle.*

Before we go further, we have to wonder if the subsets of k points of V separable from the others by a circle are the k-sets of a particular convex inclusion chain of V. The following example shows that it is not the case for every set V.

Example 1. Let V be a set of six points, five of them being the vertices of a regular pentagon $\mathcal{P}$ and the sixth being placed at the center of the circle circumscribed to $\mathcal{P}$. We can slightly perturb the vertices of $\mathcal{P}$ so that no four points are cocircular. By definition, the last element of any convex inclusion chain $\mathcal{V}$ of V is a vertex of $conv(V)$, that is, a vertex s of $\mathcal{P}$. The two neighbors r and t of s on $\mathcal{P}$ form an edge of $conv(V \setminus \{s\})$ and, therefore, a 2-set of $V \setminus \{s\}$. $\{r, t\}$ is then a 2-set of $\mathcal{V}$, but it can not be separated from V by a circle. It results that this point set V has no convex inclusion chain whose every 2-set can be separated from the other points by a circle.

3. Convex inclusion chains and k-neighbor triangulations

In the remainder of the paper, we will try to better understand the result of Corollary 1 by establishing other relations between k-sets of convex inclusion chains and order-k Voronoi diagrams.

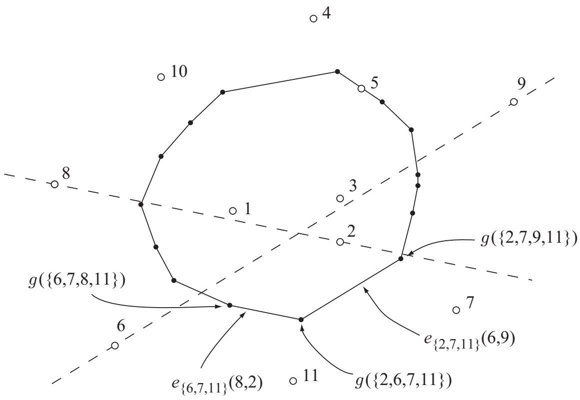

Fig. 1. Edges and vertices of a 4-set polygon of 11 points.

3.1. *Convex inclusion chains and* k*-set polygons*

From now on we will suppose that the integer k belongs to $\{1, ..., n\}$.

A practical way to handle a k-point subset T of V consists in representing it by its centroid $g(T)$. The convex hull of the centroids of all k-point subsets of V is called the k-set polygon of V. It has been introduced by Edelsbrunner, Valtr, and Welzl[23], and is denoted by $\mathcal{Q}_k(V)$. Notice that $\mathcal{Q}_1(V)$ is the convex hull of V and that $\mathcal{Q}_n(V)$ is a unique point, the centroid of V.

The characterization of the vertices and of the edges of $\mathcal{Q}_k(V)$ has been given by Andrzejak and Fukuda[24], and by Andrzejak and Welzl[25] (see Fig. 1):

Proposition 1. *(i) The centroid $g(T)$ of T is a vertex of $\mathcal{Q}_k(V)$ if and only if T is a k-set of V. Distinct k-sets have distinct centroids.*

(ii) The line segment $g(T)g(T')$ is an edge of $\mathcal{Q}_k(V)$ if and only if there exists a $(k-1)$-edge st of V such that, if $P = (\mathring{st})^- \cap V$, then $T = P \cup \{s\}$ and $T' = P \cup \{t\}$.

Such an oriented edge $g(P \cup \{s\})g(P \cup \{t\})$ will be denoted by $e_P(s,t)$. Obviously, $e_P(s,t)$ is parallel to (st).

Given an oriented straight line Δ, we say that a set T is Δ-separable from V if T is a subset of V such that $\mathring{\Delta}^- \cap V = T$. T is said to be $//_\Delta$-separable from V if there exists a straight line Δ', parallel to Δ and with the same orientation as Δ, such that T is Δ'-separable from V. For short, we say also that a vertex of a convex polygon $\mathcal{P}$ is Δ-separable (resp. $//_\Delta$-separable) from $\mathcal{P}$ if it is Δ-separable (resp. $//_\Delta$-separable) from the set of vertices of $\mathcal{P}$.

Lemma 1. *Given a k-point subset T of V and an oriented straight line Δ, the following assertions are equivalent:*

- *T is $//_\Delta$-separable from V,*
- *$g(T)$ is $//_\Delta$-separable from the set of centroids of all k-point subsets of V.*

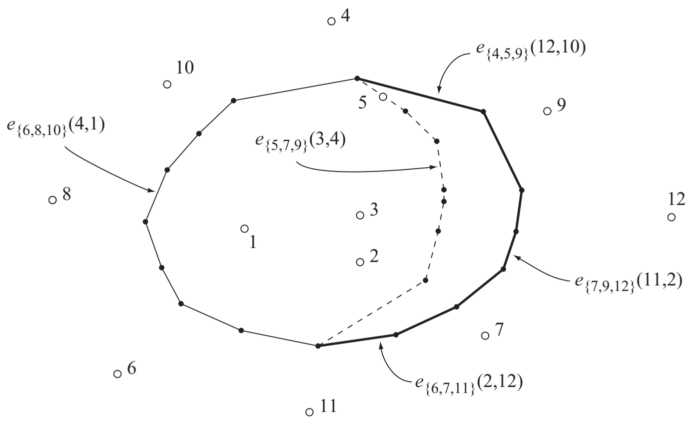

Fig. 2. Updating the 4-set polygon of $V_{i-1} = \{1, ..., 11\}$ when $v_i = \{12\}$ is added. The edges to delete are in dashed lines and the edges to create in bold lines.

Proof. (i) If $g(T)$ is $//_\Delta$-separable from $\mathcal{Q}_k(V)$, there exists a straight line Δ' parallel to Δ, with same orientation as Δ, that passes through $g(T)$, and such that all other points of $\mathcal{Q}_k(V)$ are in $\mathring{\Delta}'^+$. Let Δ'' be a straight line parallel to Δ, with same orientation as Δ, and such that $|\mathring{\Delta}''^- \cap V| < k$ and $|\Delta''^- \cap V| \geq k$. At least one point a of T belongs to Δ''^+. There is no point b of $V \setminus T$ in Δ''^-; otherwise the point $g((T \setminus \{a\}) \cup \{b\})$ of $\mathcal{Q}_k(V)$ would belong to Δ'^- since the segments $g(T)g((T \setminus \{a\}) \cup \{b\})$ and ab are parallel and have same orientation. Thus $\Delta''^- \cap V = T$ and T is $//_\Delta$-separable from V.

(ii) Conversely, if T is $//_\Delta$-separable from V, let Δ'' be a straight line parallel to Δ, with same orientation as Δ, and such that T is Δ''-separable from V. Let Δ' be the straight line parallel to Δ, with same orientation as Δ, and that passes through $g(T)$. For every k-point subset T' of V distinct from T, $A = T \setminus T' \subset \mathring{\Delta}''^-$ and $B = T' \setminus T \subset \Delta''^+$. Thus, $g(A)g(B)$ is oriented from $\mathring{\Delta}''^-$ to Δ''^+ and, since $T' = (T \setminus A) \cup B$, $g(T)g(T')$ is parallel to $g(A)g(B)$ and has the same orientation. Hence, $g(T')$ belongs to $\mathring{\Delta}'^+$, and $g(T)$ is $//_\Delta$-separable from $\mathcal{Q}_k(V)$. □

Let now $\mathcal{V} = (v_1, v_2, ..., v_n)$ be a convex inclusion chain of V and let $V_i = \{v_1, ..., v_i\}$, for all $i \in \{1, ..., n\}$.

Using the arguments of the proof of Theorem 1, we can characterize the k-set polygon edges to create and those to delete when constructing $\mathcal{Q}_k(V_i)$ from $\mathcal{Q}_k(V_{i-1})$ (see Figure 2).

Proposition 2. *(i) If $k < i \leq n$, the edges of $\mathcal{Q}_k(V_i)$ that are not edges of $\mathcal{Q}_k(V_{i-1})$ form a (connected) polygonal line of at least two edges. These edges are of the form:*

- *$e_P(s, v_i)$ for the first edge (in counter clockwise direction),*
- *$e_P(v_i, t)$ for the last edge,*
- *$e_P(s, t)$ with $v_i \in P$ for the other edges, if some exist.*

(ii) If $k < i \leq n$, the common edges of $\mathcal{Q}_k(V_i)$ and of $\mathcal{Q}_k(V_{i-1})$, if some exist, form a (connected) polygonal line, and are of the form $e_P(s,t)$ with $v_i \in (\overset{\circ}{st})^+$.

(iii) If $k+1 < i \leq n$, the edges of $\mathcal{Q}_k(V_{i-1})$ that are not edges of $\mathcal{Q}_k(V_i)$ form a (connected) polygonal line of at least one edge, and are of the form $e_P(s,t)$ with $v_i \in (\overset{\circ}{st})^-$.

Proof. (i) From proof of Theorem 1, V_i admits exactly one $(k-1)$-edge of the form sv_i. Thus, from Proposition 1, $\mathcal{Q}_k(V_i)$ admits exactly one edge of the form $e_P(s,v_i) = g(P \cup \{s\})g(P \cup \{v_i\})$, and this edge is the unique edge of $\mathcal{Q}_k(V_i)$ that starts at a vertex of $\delta(\mathcal{Q}_k(V_i)) \cap \delta(\mathcal{Q}_k(V_{i-1}))$ and that ends at a vertex of $\delta(\mathcal{Q}_k(V_i)) \setminus \delta(\mathcal{Q}_k(V_{i-1}))$. Similarly, $\mathcal{Q}_k(V_i)$ admits exactly one edge of the form $e_{P'}(v_i,t')$ that starts at a vertex of $\delta(\mathcal{Q}_k(V_i)) \setminus \delta(\mathcal{Q}_k(V_{i-1}))$ and ends at a vertex of $\delta(\mathcal{Q}_k(V_i)) \cap \delta(\mathcal{Q}_k(V_{i-1}))$. It follows that no edge of $\mathcal{Q}_k(V_i)$ between $e_P(s,v_i)$ and $e_{P'}(v_i,t')$ in counter clockwise direction is an edge of $\mathcal{Q}_k(V_{i-1})$. From proof of Theorem 1 and from Proposition 1, these edges are of the form $e_{P''}(s'',t'')$ with $v_i \in (\overset{\circ}{s''t''})^- \cap V_i = P''$.

(ii) From (i), the edges of $\mathcal{Q}_k(V_i)$ that are also edges of $\mathcal{Q}_k(V_{i-1})$ form a polygonal line. From proof of Theorem 1, they are defined by $(k-1)$-edges st such that $v_i \in (\overset{\circ}{st})^+$.

(iii) If $i > k+1$, V_{i-1} admits at least two k-sets and thus $\mathcal{Q}_k(V_{i-1})$ admits at least two (oriented) edges. From (i), the edges of $\mathcal{Q}_k(V_{i-1})$ that are not edges of $\mathcal{Q}_k(V_i)$ form a polygonal line of at least one edge. From proof of Theorem 1, these edges are defined by $(k-1)$-edges st with $v_i \in (\overset{\circ}{st})^-$. □

3.2. k-*neighbor triangulations*

The order-k Voronoi diagram admits a straight line dual graph whose vertices are the centroids of the k-point subsets associated with the order-k Voronoi regions[17,18]. When no four points of V are cocircular, this dual graph induces a triangulation of the k-set polygon of V (with additional inner points), called the order-k centroid Delaunay triangulation of V (see[26] and Fig. 3). Every edge of this triangulation connects the centroids of two k-point subsets that differ from each other by only one point (the same holds for the subsets associated with two order-k Voronoi regions sharing an edge[9]). Recall that it is also the case with the edges of the k-set polygon (Proposition 1).

More generally, we call k-neighbor triangulation of V any triangulation $\mathcal{T}$ of $\mathcal{Q}_k(V)$ (with possible additional inner points) such that

- there exists a set $\mathcal{R}$ of k-point subsets of V such that every vertex of $\mathcal{T}$ is the centroid of a unique element of $\mathcal{R}$,
- every edge of $\mathcal{T}$ is of the form $g(T)g(T')$ with $\{T,T'\} \subseteq \mathcal{R}$ and $|T \cap T'| = k-1$.

From this definition, if V admits different k-point subsets with same centroid, then at most one of these subsets is in $\mathcal{R}$. In the following, when we will say that $g(T)$ is a vertex of the k-neighbor triangulation $\mathcal{T}$, this will imply that T is an

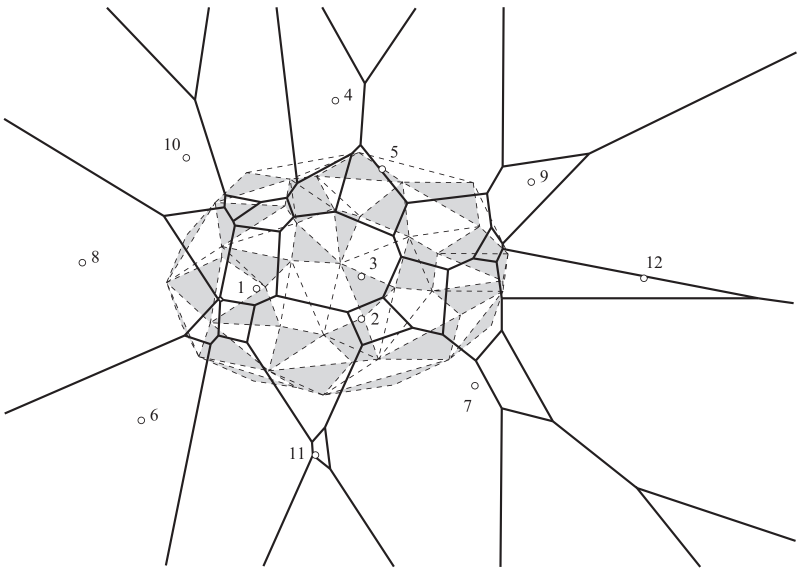

Fig. 3. Order-4 Voronoi diagram (full lines) and order-4 centroid Delaunay triangulation (dashed lines). White triangles are type-1 and gray triangles are type-2.

element of $\mathcal{R}$. When, moreover, the centroid of every element of $\mathcal{R}$ is a vertex of $\mathcal{T}$, $\mathcal{R}$ is said to determine the vertices of $\mathcal{T}$.

Now it is easy to see that $\mathcal{T}$ has only two types of triangles:

– so called type-1 triangles of the form $g(P \cup \{r\})g(P \cup \{s\})g(P \cup \{t\})$, with P a $(k-1)$-point subset of V and r, s, t three distinct points of $V \setminus P$,
– so called type-2 triangles of the form $g(P \cup \{r, s\})g(P \cup \{s, t\})g(P \cup \{r, t\})$, with P a $(k-2)$-point subset of V and r, s, t three distinct points of $V \setminus P$.

Remark 1. It is important to note that this property would be wrong if $\mathcal{R}$ could contain two elements with same centroid (see Fig. 4).

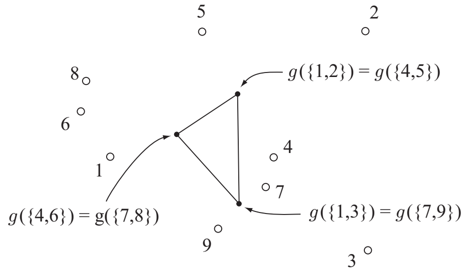

Fig. 4. The three edges of this triangle are of the form $g(T)g(T')$ with $|T \cap T'| = k - 1$; but the triangle is neither type-1 nor type-2.

Notice also that a 1-neighbor triangulation has only type-1 triangles, for its vertices are points of V, whereas an $(n-1)$-neighbor triangulation has only type-2 triangles, for its vertices are centroids of all but one point of V.

The aim of this section is to show that the set of k-sets of the convex inclusion chain $\mathcal{V} = (v_1, \ldots, v_n)$ of V determines the vertices of a k-neighbor triangulation of V.

As in proof of Proposition 2, let $e_P(s, v_i)$ and $e_{P'}(v_i, t')$ be the first and the last edge of the polygonal line formed by the edges of $\mathcal{Q}_k(V_i)$ that are not edges of $\mathcal{Q}_k(V_{i-1})$, for every $k \in \{1, ..., n-1\}$ and for every $i \in \{k+1, ..., n\}$. Let $\mathcal{C}_k(i)$ be the part of the line without these two edges ($\mathcal{C}_k(i)$ is possibly reduced to a point).

Furthermore, when $k \in \{1, ..., n-2\}$ and $i \in \{k+2, ..., n\}$, we denote by $\mathcal{D}_k(i)$ the polygonal line of edges of $\mathcal{Q}_k(V_{i-1})$ that are not edges of $\mathcal{Q}_k(V_i)$.

Since $\mathcal{Q}_k(V_k)$ is reduced to the unique point $g(V_k)$, we set $\mathcal{C}_k(k) = g(V_k)$, for all $k \in \{1, \ldots, n\}$, and $\mathcal{D}_k(k+1) = g(V_k)$, for all $k \in \{1, \ldots, n-1\}$.

Suppose now $i \in \{k+1, \ldots, n\}$. Since the edges $e_P(s, v_i)$ and $e_{P'}(v_i, t')$ are parallel to the straight lines (s, v_i) and (v_i, t') respectively, the vertices of $\mathcal{C}_k(i)$ are the vertices $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ with the oriented straight lines Δ verifying $0 < \measuredangle((sv_i), \Delta) < \measuredangle((sv_i), (v_i t'))$. Now, the vertices $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$ with such oriented straight lines Δ are the vertices of $\mathcal{D}_k(i)$. So let $(g(T_{i,1}), \ldots, g(T_{i,m_i}))$ be the sequence of vertices of $\mathcal{D}_k(i)$ ordered in counter clockwise direction, and, for every vertex $g(T_{i,j})$, let $\mathcal{C}_k(i, j)$ be the set

- of vertices $g(T)$ of $\mathcal{C}_k(i)$ such that $g(T)$ and $g(T_{i,j})$ are respectively $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ and from $\mathcal{Q}_k(V_{i-1})$, with a same straight line Δ,
- and of edges of $\mathcal{C}_k(i)$ that connect these vertices.

Clearly, the set of straight lines Δ for which $g(T_{i,j})$ is $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$ defines a (connected) interval of angles $\measuredangle((sv_i), \Delta)$. Hence, $\mathcal{C}_k(i, j)$ is a connected subset of $\mathcal{C}_k(i)$, possibly reduced to a point. Moreover, if $h > j$, $\mathcal{C}_k(i, h)$ is after $\mathcal{C}_k(i, j)$ on $\mathcal{C}_k(i)$, and these two polygonal lines do not overlap, except possibly at their endpoints. Furthermore:

Lemma 2. *(i) $g(P \cup \{v_i\})$ is the first endpoint of $\mathcal{C}_k(i, 1)$ and $g(P' \cup \{v_i\})$ is the last endpoint of $\mathcal{C}_k(i, m_i)$.*

(ii) For all $j \in \{2, ..., m_i\}$, if $e_{P_j}(s_j, t_j)$ is the edge of $\mathcal{D}_k(i)$ connecting $g(T_{i,j-1})$ to $g(T_{i,j})$, then $\mathcal{C}_k(i, j-1)$ and $\mathcal{C}_k(i, j)$ admit $g(P_j \cup \{v_i\})$ as common endpoint.

Proof. (i) $g(P \cup \{v_i\})$ is the second endpoint of $e_P(s, v_i)$ and is thus the first endpoint of $\mathcal{C}_k(i)$. $g(P \cup \{v_i\})$ is also $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ with oriented straight lines Δ such that $0 < \measuredangle((sv_i), \Delta) < \epsilon$ (for some $\epsilon > 0$). Moreover, since the straight line spanned by $e_P(s, v_i)$ is tangent to $\mathcal{Q}_k(V_{i-1})$ at $g(T_{i,1})$, $g(T_{i,1})$ is also $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$ with such straight lines Δ (and ϵ sufficiently small). By the definition of $\mathcal{C}_k(i, 1)$, it follows that $g(P \cup \{v_i\})$ belongs to $\mathcal{C}_k(i, 1)$ and, since it is the first endpoint of $\mathcal{C}_k(i)$, it is also the first endpoint of $\mathcal{C}_k(i, 1)$.

Similarly, $g(P' \cup \{v_i\})$ is the last endpoint of $\mathcal{C}_k(i, m_i)$.

(ii) Since $P_j = (s_j\overset{\circ}{}t_j)^- \cap V_{i-1}$ and since, from Proposition 2, $v_i \in (s_j\overset{\circ}{}t_j)^-$, we have $P_j \cup \{v_i\} = (s_j\overset{\circ}{}t_j)^- \cap V_i$. Therefore, from Lemma 1, $g(P_j \cup \{v_i\})$ is the vertex $//_{(s_j t_j)}$-separable from $\mathcal{Q}_k(V_i)$. Now, $g(P_j \cup \{v_i\})$ is also $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ with oriented straight lines Δ such that $\measuredangle((sv_i), \Delta)$ is slightly smaller or slightly greater than $\measuredangle((sv_i), (s_j t_j))$. In the first case $g(T_{i,j-1})$ is $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$, and in the second case $g(T_{i,j})$ is $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$. It follows that $g(P_j \cup \{v_i\})$ is a vertex of both $\mathcal{C}_k(i, j-1)$ and $\mathcal{C}_k(i, j)$. □

It then results that $\mathcal{C}_k(i)$ is the sequence of polygonal lines $\mathcal{C}_k(i, 1), \ldots, \mathcal{C}_k(i, m_i)$, which do not overlap except at their endpoints (see Fig. 5).

Lemma 3. *(i) For every vertex $g(T_{i,j})$ of $\mathcal{D}_k(i)$ and for every vertex $g(T)$ of $\mathcal{C}_k(i, j)$, there exists $q \in T_{i,j}$ such that $T = (T_{i,j} \setminus \{q\}) \cup \{v_i\}$.*

(ii) The segment $g(T_{i,j})g(T)$ is included in $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$.

Proof. Let Δ be a straight line passing through $g(T_{i,j})$ such that $g(T_{i,j})$ and $g(T)$ are respectively $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ and from $\mathcal{Q}_k(V_{i-1})$. Obviously, $\mathcal{Q}_k(V_{i-1}) \subset \Delta^+$.

From Lemma 1, there exist two straight lines Δ' and Δ'', which are parallel to and oriented in the same direction as Δ, which avoid the points of V_i, and such that $\mathring{\Delta}'^- \cap V_{i-1} = T_{i,j}$ and $\mathring{\Delta}''^- \cap V_i = T$. Thus, since $v_i \in T$, there is exactly one point q of V_{i-1} between Δ and Δ'. It follows that $T = (T_{i,j} \setminus \{q\}) \cup \{v_i\}$.

Furthermore, $g(T) \in \mathring{\Delta}'^-$ since $v_i \in \mathring{\Delta}''^-$, $q \in \mathring{\Delta}''^+$, and $g(T_{i,j})g(T)$ is parallel to and has same orientation as qv_i. Thus, $g(T_{i,j})g(T)$ intersects $\overline{\mathcal{Q}_k(V_{i-1})}$ only in $g(T_{i,j})$. Since $g(T_{i,j})$ and $g(T)$ belong to $\mathcal{Q}_k(V_i)$, $g(T_{i,j})g(T) \subset \overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$. □

From now on, we denote by $\mathcal{E}_k(i)$ the set of segments $g(T_{i,j})g(T)$ determined by the previous lemma when j runs over $\{1, ..., m_i\}$ (see Fig. 5).

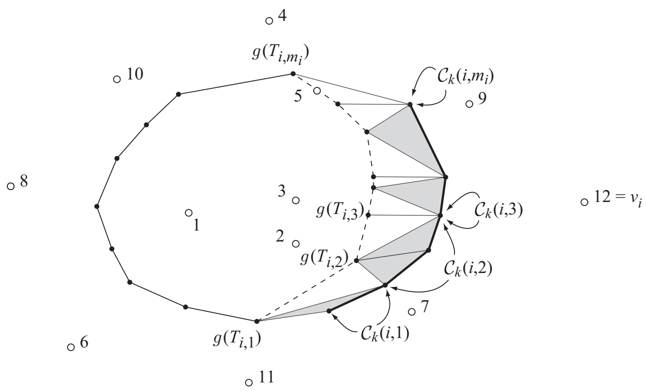

Fig. 5. The edges of $\mathcal{D}_k(i)$ are in dashed lines, the edges of $\mathcal{C}_k(i)$ in bold lines, and the edges of $\mathcal{E}_k(i)$ connect the vertices of $\mathcal{D}_k(i)$ with the vertices of $\mathcal{C}_k(i)$. The triangles generated by these edges are either of type-1 (white) or of type-2 (gray).

Lemma 4. *(i) The segments of $\mathcal{E}_k(i)$ induce a triangulation of $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$.*

(ii) The triangles of this triangulation are

- *the type-2 triangles $g(T_{i,j})g(T)g(T')$, where $g(T_{i,j})$ is a vertex of $\mathcal{D}_k(i)$ and $g(T)g(T')$ is an edge of $\mathcal{C}_k(i,j)$,*
- *the type-1 triangles $g(T_{i,j})g(T_{i,j+1})g(T)$, where $g(T_{i,j})g(T_{i,j+1})$ is an edge of $\mathcal{D}_k(i)$ and $g(T)$ is the common vertex of $\mathcal{C}_k(i,j)$ and $\mathcal{C}_k(i,j+1)$.*

Proof. (i) $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$ is reduced to a line segment when $\mathcal{D}_k(i)$ is reduced to the point $g(T_{i,1})$ and when $\mathcal{C}_k(i) = \mathcal{C}_k(i,1)$ is reduced to a unique point $g(T)$. $\mathcal{E}_k(i)$ is then reduced to the segment $g(T_{i,1})g(T)$ and forms a degenerate triangulation of $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$. (Actually this only occurs when $k = 1$ and $i = 2$.)

We deal now with the case where $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$ is not reduced to a segment.

$g(T_{i,j})$ precedes $g(T_{i,h})$ on $\mathcal{D}_k(i)$ if and only if $\mathcal{C}_k(i,j)$ precedes $\mathcal{C}_k(i,h)$ on $\mathcal{C}_k(i)$; therefore, the segments of $\mathcal{E}_k(i)$ are pairwise disjoint (up to their endpoints).

The boundary of $\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})$ is composed of the edges of $\mathcal{D}_k(i)$, of the edges of $\mathcal{C}_k(i)$, and of two edges of $\mathcal{E}_k(i)$: The one linking $g(T_{i,1})$ to the first vertex of $\mathcal{C}_k(i,1)$ and the one linking $g(T_{i,m_i})$ to the last vertex of $\mathcal{C}_k(i,m_i)$. Since every segment of $\mathcal{E}_k(i)$ connects a point of $\mathcal{D}_k(i)$ to a point of $\mathcal{C}_k(i)$, the boundary Γ of every connected component of $(\mathring{\mathcal{Q}_k}(V_i) \setminus \mathcal{Q}_k(V_{i-1})) \setminus \mathcal{E}_k(i)$ is also composed of edges of $\mathcal{D}_k(i)$, of edges of $\mathcal{C}_k(i)$, and of two edges of $\mathcal{E}_k(i)$.

If Γ contains an edge $g(T)g(T')$ of $\mathcal{C}_k(i)$, $g(T)g(T')$ is an edge of exactly one line $\mathcal{C}_k(i,j)$. By definition, the segments $g(T)g(T_{i,j})$ and $g(T')g(T_{i,j})$ belong to $\mathcal{E}_k(i)$, and Γ is the triangle $g(T)g(T')g(T_{i,j})$.

If Γ contains an edge $g(T_{i,j})g(T_{i,j+1})$ of $\mathcal{D}_k(i)$, from Lemma 2, $\mathcal{C}_k(i,j)$ and $\mathcal{C}_k(i,j+1)$ have a common vertex $g(T)$. By definition, $g(T_{i,j})g(T)$ and $g(T_{i,j+1})g(T)$ are then also segments of $\mathcal{E}_k(i)$, and Γ is the triangle $g(T_{i,j})g(T_{i,j+1})g(T)$.

It follows that every connected component of $(\mathring{\mathcal{Q}_k}(V_i) \setminus \mathcal{Q}_k(V_{i-1})) \setminus \mathcal{E}_k(i)$ is a triangle. Hence, $\mathcal{E}_k(i)$ induces a triangulation of $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$.

(ii) From (i), the triangulation induced by $\mathcal{E}_k(i)$ has two types of triangles. The triangles $g(T)g(T')g(T_{i,j})$ are type-2 triangles since, from Lemma 3, there exist two distinct points q and q' of $T_{i,j}$ such that $T = (T_{i,j} \setminus \{q\}) \cup \{v_i\}$ and $T' = (T_{i,j} \setminus \{q'\}) \cup \{v_i\}$. The triangles $g(T_{i,j})g(T_{i,j+1})g(T)$ are type-1 triangles since, from Lemma 3, there exist $q \in T_{i,j}$ and $q' \in T_{i,j+1}$ such that $T_{i,j} = (T \setminus \{v_i\}) \cup \{q\}$ and $T_{i,j+1} = (T \setminus \{v_i\}) \cup \{q'\}$, where $q \neq q'$. □

Theorem 2. *The edges of the k-set polygons $\mathcal{Q}_k(V_i)$ and of the sets $\mathcal{E}_k(i)$, for all integers i of $\{k+1, ..., n\}$, form a k-neighbor triangulation of V whose vertices are determined by the k-sets of the convex inclusion chain $\mathcal{V}$.*

Proof. The k-set polygon $\mathcal{Q}_k(V_k)$ is reduced to a unique point. From Lemma 4, if $i \in \{k+1, ..., n\}$, $\mathcal{E}_k(i)$ induces a triangulation of $\overline{\mathcal{Q}_k(V_i) \setminus \mathcal{Q}_k(V_{i-1})}$. It follows that the set of edges of all k-set polygons $\mathcal{Q}_k(V_i)$ and of all sets $\mathcal{E}_k(i)$, $i \in \{k+1, ..., n\}$, forms a triangulation $\mathcal{T}$ of $\mathcal{Q}_k(V_n) = \mathcal{Q}_k(V)$.

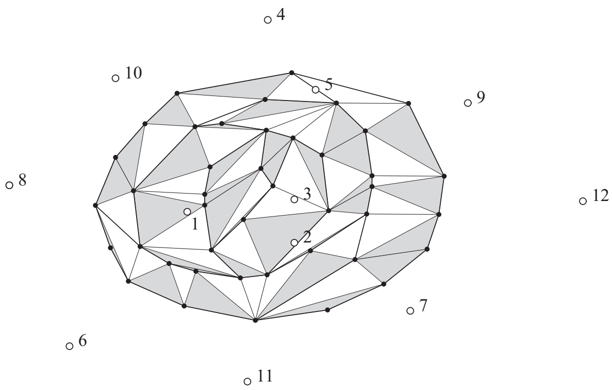

Fig. 6. The 4-neighbor triangulation $\mathcal{T}_4(1, 2, \ldots, 12)$. The white triangles are type-1 and the gray triangles are type-2.

Moreover, from Proposition 1, every edge of $\mathcal{Q}_k(V_i)$ is of the form $g(T)g(T')$ with $|T \cap T'| = k - 1$. From Lemma 3, it is the same with the edges of $\mathcal{E}_k(i)$.

The vertices of $\mathcal{T}$ are the vertices of the k-set polygons $\mathcal{Q}_k(V_i)$, $\forall i \in \{k, ..., n\}$, that is, the centroids of the k-sets of the convex inclusion chain $\mathcal{V}$. Since distinct k-sets have distinct centroids, it follows that $\mathcal{T}$ is a k-neighbor triangulation of V whose vertices are determined by the k-sets of the convex inclusion chain $\mathcal{V}$. □

For every convex inclusion chain $\mathcal{V}$ of V, the triangulation defined by this theorem is said to be associated to $\mathcal{V}$ and is denoted by $\mathcal{T}_k(\mathcal{V})$ (see Fig. 6). In the particular case $k = n$, we set $\mathcal{T}_n(\mathcal{V}) = \mathcal{Q}_n(V) = g(V)$.

4. Convex inclusion chains and centroid triangulations

If $g(P \cup \{r\})g(P \cup \{s\})g(P \cup \{t\})$ is a type-1 triangle whose vertices are centroids of $(k-1)$-point subsets of V then, by definition, $g(P \cup \{r, s\})g(P \cup \{r, t\})g(P \cup \{s, t\})$ is a type-2 triangle whose vertices are centroids of k-point subsets of V. Using this property, Liu and Snoeyink[15] suggested to apply Algorithm 1 below to $(k-1)$-neighbor triangulations. By the term "constrained triangulation of $\overline{\mathcal{Q}_k(V) \setminus \tau}$" in step (2) of the algorithm is meant a partition of $\overline{\mathcal{Q}_k(V) \setminus \tau}$ in triangles such that the vertices of $\overline{\mathcal{Q}_k(V) \setminus \tau}$ are the vertices of the partition and every edge of $\overline{\mathcal{Q}_k(V) \setminus \tau}$ is an edge of the partition.

Algorithm 1: To apply to a $(k-1)$-neighbor triangulation $\mathcal{T}$ of V
(1) **for** *every type*-1 *triangle* $g(P \cup \{r\})g(P \cup \{s\})g(P \cup \{t\})$ *of* $\mathcal{T}$ **do** compute the type-2 triangle $g(P \cup \{r, s\})g(P \cup \{r, t\})g(P \cup \{s, t\})$ (2) Construct a constrained triangulation of $\overline{\mathcal{Q}_k(V) \setminus \tau}$, where τ is the set of triangles computed in loop (1)

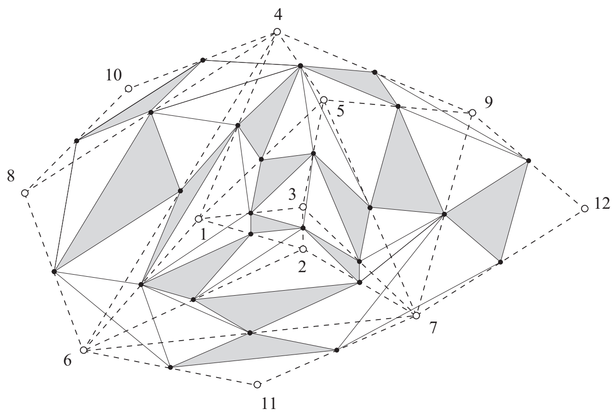

Fig. 7. A 2-neighbor triangulation (full lines) obtained from a 1-neighbor triangulation (dashed lines) by Algorithm 1. The triangles computed in loop (1) are gray.

Actually, Algorithm 1 is a generalization of an algorithm that constructs the order-k centroid Delaunay triangulation from the order-$(k-1)$ centroid Delaunay triangulation[9,17]. The difference is that that algorithm constructs a constrained Delaunay triangulation in step (2) instead of an arbitrary constrained triangulation.

Unfortunately, Algorithm 1 does not work with any given $(k-1)$-neighbor triangulation, in the sense that the type-2 triangles computed in loop (1) may overlap. However, it is easy to see that it works when applied to a 1-neighbor triangulation, *i.e.*, to a (classical) triangulation of the point set (see Fig. 7). In this case, it generates a 2-neighbor triangulation. Liu and Snoeyink proved that, if the algorithm is applied to such a 2-neighbor triangulation, then it also generates a 3-neighbor triangulation. Experimental results indicate that the algorithm works as long as it is applied to a $(k-1)$-neighbor triangulation that has been iteratively generated by the algorithm itself. A triangulation generated in this way is called an order-k centroid triangulation.

Currently, the order-k centroid Delaunay triangulations form the only family of k-neighbor triangulations for which it has been proved that the algorithm works for $k > 3$. The aim of this section is to show that it is also the case for the triangulations associated to convex inclusion chains.

For every set V of n points, we call iterative centroid triangulation sequence of V, every sequence $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of centroid triangulations of V such that $\mathcal{A}_1$ is a (classical) triangulation of V and, for every integer $i \in \{2, \ldots, n\}$, $\mathcal{A}_i$ is obtained from $\mathcal{A}_{i-1}$ using Algorithm 1.

Theorem 3. *For every convex inclusion chain* $\mathcal{V} = (v_1, \ldots, v_n)$ *of* V, $(\mathcal{T}_1(\mathcal{V}), \ldots, \mathcal{T}_n(\mathcal{V}))$ *is an iterative centroid triangulation sequence of* V.

Proof. (i) The set $V_1 = \{v_1\}$ admits a unique convex inclusion chain $\mathcal{V}_1 = (v_1)$.

The triangulation $\mathcal{T}_1(\mathcal{V}_1)$ (which is reduced to the point v_1) is then the unique element of the unique iterative centroid triangulation sequence of V_1, $(\mathcal{T}_1(\mathcal{V}_1))$.

(ii) Now, assume $n > 1$ and let i be an integer of $\{2, \ldots, n\}$. Let $\mathcal{V}_{i-1}$ be the convex inclusion chain $(v_1, \ldots, v_{i-1})$ of $V_{i-1} = \{v_1, \ldots, v_{i-1}\}$. Assume the following induction hypothesis: $(\mathcal{T}_1(\mathcal{V}_{i-1}), \ldots, \mathcal{T}_{i-1}(\mathcal{V}_{i-1}))$ is an iterative centroid triangulation sequence of V_{i-1}. We will furthermore assume by induction that, for every $l \in \{2, \ldots, i-2\}$, all type-2 triangles of $\mathcal{T}_l(\mathcal{V}_{i-1})$ are obtained from the type-1 triangles of $\mathcal{T}_{l-1}(\mathcal{V}_{i-1})$ (*i.e.*, they are constructed by loop (1) of Algorithm 1) and that every vertex of $\mathcal{T}_l(\mathcal{V}_{i-1})$ is a vertex of a type-2 triangle of $\mathcal{T}_l(\mathcal{V}_{i-1})$. In the remainder of the proof we will refer to this condition as the triangle and vertex criterion.

We show now that, if $\mathcal{V}_i$ is the convex inclusion chain $(v_1, \ldots, v_i)$ of $V_i = \{v_1, \ldots, v_i\}$, then $(\mathcal{T}_1(\mathcal{V}_i), \ldots, \mathcal{T}_i(\mathcal{V}_i))$ is still an iterative centroid triangulation sequence of V_i verifying the triangle and vertex criterion. From Theorem 2, $\mathcal{T}_1(\mathcal{V}_i)$ is a triangulation of V_i and is then the first element of an iterative centroid triangulation sequence of V_i. Suppose that the following second induction hypothesis holds: For an integer $h \in \{2, \ldots, i\}$, $(\mathcal{T}_1(\mathcal{V}_i), \ldots, \mathcal{T}_{h-1}(\mathcal{V}_i))$ is an initial subsequence of an iterative centroid triangulation sequence of V_i, verifying the triangle and vertex criterion.

(ii.1) If $h = i$, on the one hand, $\mathcal{T}_h(\mathcal{V}_h)$ is reduced to the unique point $g(V_h)$. On the other hand, $\mathcal{T}_{h-1}(\mathcal{V}_h)$ is an $(h-1)$-neighbor triangulation of h points; thus, all its triangles are type-2. When Algorithm 1 is applied to this triangulation, loop (1) does not construct any triangle. Since $\mathcal{Q}_h(V_h)$ is reduced to the point $g(V_h)$, step (2) generates a degenerate triangulation which is also reduced to this point. Therefore, from the second induction hypothesis, $(\mathcal{T}_1(\mathcal{V}_h), \ldots, \mathcal{T}_h(\mathcal{V}_h))$ is an iterative centroid triangulation sequence of V_h, verifying the triangle and vertex criterion.

(ii.2) Suppose now that $h \in \{2, \ldots, i-1\}$.

- By construction, the set of type-2 triangles of $\mathcal{T}_h(\mathcal{V}_i)$ is the union of the type-2 triangles of $\mathcal{T}_h(\mathcal{V}_{i-1})$ and of those of $\overline{\mathcal{T}_h(\mathcal{V}_i) \setminus \mathcal{T}_h(\mathcal{V}_{i-1})}$. In the same way, the set of type-1 triangles of $\mathcal{T}_{h-1}(\mathcal{V}_i)$ is the union of the type-1 triangles of $\mathcal{T}_{h-1}(\mathcal{V}_{i-1})$ and of those of $\overline{\mathcal{T}_{h-1}(\mathcal{V}_i) \setminus \mathcal{T}_{h-1}(\mathcal{V}_{i-1})}$.

  Now, by the first induction hypothesis, the type-2 triangles of $\mathcal{T}_h(\mathcal{V}_{i-1})$ are obtained from the type-1 triangles of $\mathcal{T}_{h-1}(\mathcal{V}_{i-1})$ by loop (1) of Algorithm 1.

  From Lemma 4, every type-1 triangle of $\overline{\mathcal{T}_{h-1}(\mathcal{V}_i) \setminus \mathcal{T}_{h-1}(\mathcal{V}_{i-1})}$ is of the form $g(P \cup \{q\})g(P \cup \{q'\})g(P \cup \{v_i\})$, where $g(P \cup \{q\})g(P \cup \{q'\})$ is the edge $e_P(q, q')$ of $\mathcal{D}_{h-1}(i)$. Now, from Proposition 2, $v_i \in (\mathring{qq'})^-$ and $e_{P\cup\{v_i\}}(q, q') = g(P \cup \{v_i, q\})g(P \cup \{v_i, q'\})$ is an edge of $\mathcal{C}_h(i)$. Then, from Lemma 4, the triangle $g(P \cup \{v_i, q\})g(P \cup \{v_i, q'\})g(P \cup \{q, q'\})$ is a type-2 triangle of $\overline{\mathcal{T}_h(\mathcal{V}_i) \setminus \mathcal{T}_h(\mathcal{V}_{i-1})}$. Conversely, every edge of $\mathcal{C}_h(i)$ is an edge of such a type-2 triangle (see Fig. 8). (Notice that $\mathcal{C}_h(i)$ admits at least one edge since, from Proposition 2, $\mathcal{D}_{h-1}(i)$ admits at least one edge.)

  It follows that all type-2 triangles of $\mathcal{T}_h(\mathcal{V}_i)$ are obtained by applying loop (1) of Algorithm 1 to $\mathcal{T}_{h-1}(\mathcal{V}_i)$.
- The set of vertices of $\mathcal{T}_h(\mathcal{V}_i)$ is the union of the vertices of $\mathcal{T}_h(\mathcal{V}_{i-1})$ and of the

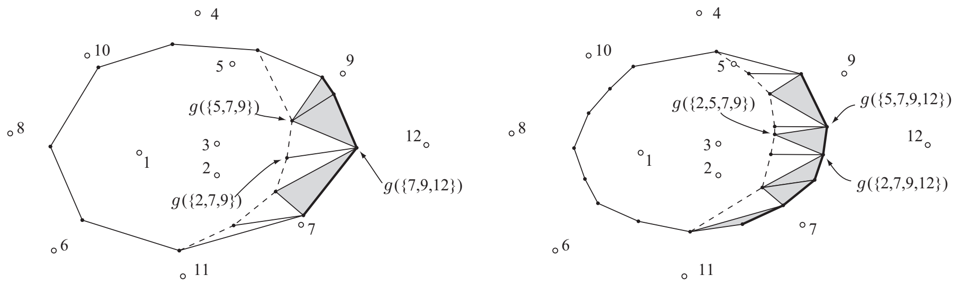

Fig. 8. The type-2 triangles of $\overline{\mathcal{T}_4(1,...,12) \setminus \mathcal{T}_4(1,...,11)}$ (gray on the figure on the right) can be obtained from the type-1 triangles of $\overline{\mathcal{T}_3(1,...,12) \setminus \mathcal{T}_3(1,...,11)}$ (white on the figure on the left) by loop (1) of Algorithm 1.

vertices of $\mathcal{C}_h(i)$.

From the previous item, all edges of $\mathcal{C}_h(i)$ are edges of type-2 triangles of $\mathcal{T}_h(\mathcal{V}_i)$, and $\mathcal{C}_h(i)$ admits at least one edge. Hence, every vertex of $\mathcal{C}_h(i)$ is also a vertex of a type-2 triangle.

From the first induction hypothesis, the vertices of $\mathcal{T}_h(\mathcal{V}_{i-1})$ are also vertices of type-2 triangles, when $h < i-1$. When $h = i-1$, $\mathcal{T}_h(\mathcal{V}_{i-1})$ is reduced to the unique vertex $\mathcal{Q}_h(V_h) = \mathcal{D}_h(h+1)$ and, from Lemma 4, it is a vertex of all type-2 triangles having an edge on $\mathcal{C}_h(h+1)$.

Thus, all vertices of $\mathcal{T}_h(\mathcal{V}_i)$ are vertices of type-2 triangles of $\mathcal{T}_h(\mathcal{V}_i)$.

– Since $\mathcal{T}_h(\mathcal{V}_i)$ is a triangulation of $\mathcal{Q}_h(V_i)$, it follows from the preceding item that the type-1 triangles of $\mathcal{T}_h(\mathcal{V}_i)$ form a constrained triangulation of $\overline{\mathcal{Q}_h(V_i) \setminus \tau}$, where τ is the set of type-2 triangles of $\mathcal{T}_h(\mathcal{V}_i)$. Hence, from the first item, the type-1 triangles of $\mathcal{T}_h(\mathcal{V}_i)$ can be constructed by step (2) of Algorithm 1.

Thus, $\mathcal{T}_h(\mathcal{V}_i)$ can be obtained from $\mathcal{T}_{h-1}(\mathcal{V}_i)$ using Algorithm 1 and it verifies the triangle and vertex criterion. Therefore, from the second induction hypothesis, $(\mathcal{T}_1(\mathcal{V}_i), \ldots, \mathcal{T}_h(\mathcal{V}_i))$ is an initial subsequence of an iterative centroid triangulation sequence of V_i verifying the triangle and vertex criterion, for all $h \in \{2, \ldots, i\}$.

Finally, from the first induction hypothesis, $(\mathcal{T}_1(\mathcal{V}_i), \ldots, \mathcal{T}_i(\mathcal{V}_i))$ is an iterative centroid triangulation sequence of V_i verifying the triangle and vertex criterion, for every $i \in \{2, \ldots, n\}$. □

An immediate consequence of this theorem is that,

Corollary 2. *The k-neighbor triangulation associated to any convex inclusion chain of V is an order-k centroid triangulation of V.*

Since the order-k centroid Delaunay triangulation of V is also an order-k centroid triangulation, this result is a first step toward the understanding why the number of k-sets of a convex inclusion chain of V is equal to the number of regions of the order-k Voronoi diagram of V. For the explanation to be complete, it should be proved that all order-k centroid triangulations of V have the same number of vertices, that is, $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} a_j(V)$ vertices. This would also corroborate a conjecture of Liu and Snoeyink[15], which claims that the size of every order-k centroid triangulation of n points is in $\Theta(kn)$ or, more precisely, in $\Theta(k(n-k))$.

5. Construction of particular centroid triangulations

If the conjecture on the size of a centroid triangulation holds, it implies that the construction of any order-k centroid triangulation of n points with the help of Algorithm 1 takes $\Omega(n \log n + k^2(n-k))$ time: $\Omega(n \log n)$ for an (order-1 centroid) triangulation and $\Omega(i(n-i))$ for each of the next $k-1$ order-i centroid triangulations.

The aim of this last section is to show that order-k centroid triangulations associated to some special convex inclusion chains can be constructed in $O(n \log n + k(n-k) \log k)$ worst case time. Moreover, every point set admits such convex inclusion chains.

Let $\mathcal{V} = (v_1, \dots, v_n)$ be a convex inclusion chain of V and, as in proof of Theorem 3, let $\mathcal{V}_i = (v_1, \dots, v_i)$, for all $i \in \{1, \dots, n\}$. Assume that the order-k centroid triangulation $\mathcal{T}_k(\mathcal{V}_{i-1})$ associated to $\mathcal{V}_{i-1}$ is given, for an $i > k$. To construct $\mathcal{T}_k(\mathcal{V}_i)$, we need then to determine the vertices of $\mathcal{C}_k(i)$, to connect them to each others with the edges of $\mathcal{C}_k(i)$, and to connect them to $\mathcal{T}_k(\mathcal{V}_{i-1})$ with the edges of $\mathcal{E}_k(i)$.

As already pointed out in section 3, $\mathcal{C}_k(i)$ can be decomposed into a sequence of lines $\mathcal{C}_k(i,j)$, each of them being associated to a vertex $g(T_{i,j})$ of $\mathcal{D}_k(i)$. More precisely, the vertices of $\mathcal{C}_k(i,j)$ are the vertices of $\mathcal{C}_k(i)$ that are $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ with the oriented straight lines Δ for which $g(T_{i,j})$ is $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$. For such a line Δ we then have:

Lemma 5. *If $g(T)$ is the vertex $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$, then it is also the vertex $//_\Delta$-separable from $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$.*

Proof. Since, from Lemma 1, $T_{i,j}$ is $//_\Delta$-separable from V_{i-1}, the k-set T $//_\Delta$-separable from $V_i = V_{i-1} \cup \{v_i\}$ is either $T_{i,j}$ itself, or a subset of $T_{i,j} \cup \{v_i\}$ containing v_i. Hence, the vertex $g(T)$ $//_\Delta$-separable from $\mathcal{Q}_k(V_i)$ is also $//_\Delta$-separable from $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$. □

It follows that determining the vertices of $\mathcal{C}_k(i,j)$ comes to compute the k-set polygon $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$ and to extract the vertices that are $//_\Delta$-separable from $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$ with the oriented straight lines Δ for which $g(T_{i,j})$ is $//_\Delta$-separable from $\mathcal{Q}_k(V_{i-1})$. Actually, since we only want vertices of $\mathcal{C}_k(i)$, we have only to consider the lines Δ such that $0 < \measuredangle((sv_i), \Delta) < \measuredangle((sv_i), (v_i t'))$, where $e_P(s, v_i)$ and $e_{P'}(v_i, t')$ are the edges respectively preceding and following $\mathcal{C}_k(i)$ on $\mathcal{Q}_k(V_i)$. The set of these lines Δ determines a set of consecutive vertices of $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$. We show now who to find the first and the last vertex of the line $\mathcal{C}_k(i,j)$ on $\delta(\mathcal{Q}_k(T_{i,j} \cup \{v_i\}))$.

Lemma 6. *(i) The first vertex of the line $\mathcal{C}_k(i,1)$ is the successor of $g(T_{i,1})$ on $\delta(\mathcal{Q}_k(T_{i,1} \cup \{v_i\}))$, and the last vertex of $\mathcal{C}_k(i,m_i)$ is the predecessor of $g(T_{i,m_i})$ on $\delta(\mathcal{Q}_k(T_{i,m_i} \cup \{v_i\}))$.*

(ii) If $j \in \{2, \ldots, m_i\}$ and if $e_{P_j}(s_j, t_j)$ is the edge of $\mathcal{D}_k(i)$ linking $g(T_{i,j-1})$ to $g(T_{i,j})$, then $g((T_{i,j-1} \cup \{v_i\}) \setminus \{s_j\}) = g((T_{i,j} \cup \{v_i\}) \setminus \{t_j\})$ is both the last vertex of $\mathcal{C}_k(i, j-1)$ and the first vertex of $\mathcal{C}_k(i, j)$.

Proof. (i) The edge that links $g(T_{i,1})$ to the first vertex of $\mathcal{C}_k(i,1)$ on $\delta(\mathcal{Q}_k(V_i))$ is of the form $e_P(s, v_i)$, with $P \cup \{s\} = T_{i,1}$ and $g(P \cup \{v_i\})$ the first vertex of $\mathcal{C}_k(i,1)$. Thus, $P \cup \{s, v_i\} \subseteq T_{i,1} \cup \{v_i\}$ and $e_P(s, v_i)$ is also an edge of $\mathcal{Q}_k(T_{i,1} \cup \{v_i\})$. Hence, $g(P \cup \{v_i\})$ is also the successor of $g(T_{i,1})$ on $\delta(\mathcal{Q}_k(T_{i,1} \cup \{v_i\}))$. Similarly, an edge of $\mathcal{Q}_k(V_i)$ of the form $e_{P'}(v_i, t')$ links the last vertex of $\mathcal{C}_k(i, m_i)$ to $g(T_{i,m_i})$ on $\delta(\mathcal{Q}_k(T_{i,m_i} \cup \{v_i\}))$.

(ii) For all $j \in \{2, \ldots, m_i\}$, if $e_{P_j}(s_j, t_j)$ links $g(T_{i,j-1})$ to $g(T_{i,j})$ then, from Lemma 2, $g(P_j \cup \{v_i\})$ is the last vertex of $\mathcal{C}_k(i, j-1)$ and the first vertex of $\mathcal{C}_k(i,j)$. Since $T_{i,j-1} = P_j \cup \{s_j\}$ and $T_{i,j} = P_j \cup \{t_j\}$, we have $g(P_j \cup \{v_i\}) = g((T_{i,j-1} \setminus \{s_j\}) \cup \{v_i\}) = g((T_{i,j} \setminus \{t_j\}) \cup \{v_i\})$. □

The last thing to see to compute $\mathcal{C}_k(i,j)$ efficiently is that, for all $j \in \{1, \ldots, m_i\}$, the k-set polygon $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$ is the image of $conv(T_{i,j} \cup \{v_i\})$ by the homothety with center $g(T_{i,j} \cup \{v_i\})$ and ratio $-1/k$. Indeed, q is a vertex of $conv(T_{i,j} \cup \{v_i\})$ if and only if it can be separated from $(T_{i,j} \cup \{v_i\}) \setminus \{q\}$ by a straight line, that is, if and only if $g((T_{i,j} \cup \{v_i\}) \setminus \{q\})$ is a vertex of $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$. Thus, every counter clockwise oriented edge qq' of $conv(T_{i,j} \cup \{v_i\})$ corresponds to an edge $g((T_{i,j} \cup \{v_i\}) \setminus \{q\}) g((T_{i,j} \cup \{v_i\}) \setminus \{q'\}) = e_{(T_{i,j} \cup \{v_i\}) \setminus \{q,q'\}}(q', q)$ of $\mathcal{Q}_k(T_{i,j} \cup \{v_i\})$ (see Fig. 9).

The edges of $\mathcal{E}_k(i)$ can also be easily built while constructing the lines $\mathcal{C}_k(i,j)$. Indeed, from Lemma 3, for every vertex $g((T_{i,j} \cup \{v_i\}) \setminus \{q\})$ of a line $\mathcal{C}_k(i,j)$, $g((T_{i,j} \cup \{v_i\}) \setminus \{q\}) g(T_{i,j})$ is an edge of $\mathcal{E}_k(i)$; and every edge of $\mathcal{E}_k(i)$ is of this form.

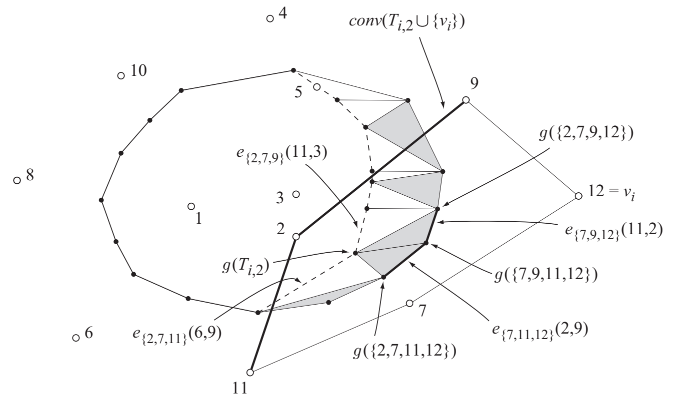

Fig. 9. The line $\mathcal{C}_4(i, 2) = (g(\{2, 7, 11, 12\}), g(\{7, 9, 11, 12\}), g(\{2, 7, 9, 12\}))$ is the image of the line $(9, 2, 11)$ by the homothety with center $g(T_{i,2} \cup \{v_i\}) = g(\{2, 7, 9, 11, 12\})$ and ratio $-1/4$.

$\mathcal{T}_k(\mathcal{V}_i)$ can then be constructed from $\mathcal{T}_k(\mathcal{V}_{i-1})$ by Algorithm 2. When implementing the algorithm, attention has to be paid to the case where $\mathcal{T}_k(\mathcal{V}_{i-1})$ is reduced to the point $g(T_{i,1}) = g(T_{i,m_i})$ (as already seen, this only occurs when $i - 1 = k$). In this case, the vertices to traverse on $\delta(conv(T_{i,1} \cup \{v_i\}))$ go from the successor of v_i to the predecessor of v_i, in counter clockwise direction.

Algorithm 2: Construction of $\mathcal{T}_k(\mathcal{V}_i)$ from $\mathcal{T}_k(\mathcal{V}_{i-1})$

```
// construction of C_k(i,1) and of the edges of E_k(i) incident in g(T_{i,1})
let e_{P_2}(s_2, t_2) be the edge leaving g(T_{i,1}) on δ(Q_k(V_{i-1})) = δ(T_k(V_{i-1}))
q ← successor of v_i on δ(conv(T_{i,1} ∪ {v_i}))
create the first vertex g((T_{i,1} ∪ {v_i}) \ {q}) of C_k(i,1)
create the edge g((T_{i,1} ∪ {v_i}) \ {q}) g(T_{i,1}) of E_k(i)
(1) while q ≠ s_2 do
    q' ← successor of q on δ(conv(T_{i,1} ∪ {v_i}))
    create the next vertex g((T_{i,1} ∪ {v_i}) \ {q'}) of C_k(i,1)
    create the edge g((T_{i,1} ∪ {v_i}) \ {q'}) g((T_{i,1} ∪ {v_i}) \ {q}) of C_k(i,1)
    create the edge g((T_{i,1} ∪ {v_i}) \ {q'}) g(T_{i,1}) of E_k(i)
    q ← q'

// treatment of C_k(i,2), ..., C_k(i, m_i − 1)
for all j ∈ {2, ..., m_i − 1} do
    let e_{P_j}(s_j, t_j) be the edge entering g(T_{i,j}) on δ(T_k(V_{i-1}))
    let e_{P_{j+1}}(s_{j+1}, t_{j+1}) be the edge leaving g(T_{i,j}) on δ(T_k(V_{i-1}))
    create the edge g((T_{i,j} ∪ {v_i}) \ {t_j}) g(T_{i,j}) of E_k(i)
    // g((T_{i,j} ∪ {v_i}) \ {t_j}) = g((T_{i,j-1} ∪ {v_i}) \ {s_j}) is the last vertex of
    // the just created line C_k(i, j − 1) and the first vertex of C_k(i, j)
    q ← t_j
    (2) while q ≠ s_{j+1} do
        q' ← successor of q on δ(conv(T_{i,j} ∪ {v_i}))
        create the next vertex g((T_{i,j} ∪ {v_i}) \ {q'}) of C_k(i,j)
        create the edge g((T_{i,j} ∪ {v_i}) \ {q'}) g((T_{i,j} ∪ {v_i}) \ {q}) of C_k(i,j)
        create the edge g((T_{i,j} ∪ {v_i}) \ {q'}) g(T_{i,j}) of E_k(i)
        q ← q'

// treatment of C_k(i, m_i)
let e_{P_{m_i}}(s_{m_i}, t_{m_i}) be the edge entering g(T_{i,m_i}) on δ(T_k(V_{i-1}))
create the edge g((T_{i,m_i} ∪ {v_i}) \ {t_{m_i}}) g(T_{i,m_i}) of E_k(i)
q ← t_{m_i}
(3) while q ≠ predecessor of v_i on δ(conv(T_{i,m_i} ∪ {v_i})) do
    q' ← successor of q on δ(conv(T_{i,m_i} ∪ {v_i}))
    create the next vertex g((T_{i,m_i} ∪ {v_i}) \ {q'}) of C_k(i, m_i)
    create the edge g((T_{i,m_i} ∪ {v_i}) \ {q'}) g((T_{i,m_i} ∪ {v_i}) \ {q}) of C_k(i, m_i)
    create the edge g((T_{i,m_i} ∪ {v_i}) \ {q'}) g(T_{i,m_i}) of E_k(i)
    q ← q'
```

We store the constructed centroid triangulation in a combinatorial map. Every edge of the triangulation is of the form $g(P \cup \{s\})g(P \cup \{t\})$. So we associate the points s and t to the edge connecting $g(P \cup \{s\})$ to $g(P \cup \{t\})$ in the map, but we do not store explicitly the set P. In fact, we only need to store the set T for a unique vertex $g(T)$ of the centroid triangulation. Starting from $g(T)$, the sets defining the other vertices can then be computed while traversing the map, by alternatively removing and adding the points associated to the edges. Hence, an order-k centroid triangulation with e edges can be stored with $O(e + k)$ space.

For the need of our algorithm, $g(T)$ is a particular vertex on the boundary of the triangulation, which will be made precise later on. As seen above, we will not only need the set T, but also its convex hull. Moreover, while traversing the triangulation, we need to insert and remove points in this convex hull. So we store the convex hull of T in a dynamic convex hull data structure, as described by Brodal and Jacob[27]. Recall that the convex hull of h points can be stored in such a structure CH with $O(h)$ space. After inserting or deleting a point, CH can be updated in amortized $O(\log h)$ time. The data structure also supports queries for the neighbor points on the convex hull in $O(\log h)$ time. It means that any sequence of $O(Q)$ operations in CH, mixing insertions, deletions, and neighbor queries, takes $O(Q \log h)$ worst case time, provided that CH is empty before the first operation, and that it never contains more than $O(h)$ points.

Denoting by $|\mathcal{L}|$ the number of edges of any polygonal line $\mathcal{L}$, we then have:

Proposition 3. *If the vertex $g(T_{i,1})$ is given and if $conv(T_{i,1})$ is stored in the data structure CH, then the complexity of Algorithm 2 depends only on the complexity of the $O(|\mathcal{C}_k(i)|+|\mathcal{D}_k(i)|)$ operations performed in CH. Moreover, CH always contains the convex hull of $O(k)$ points.*

Proof. The total number of passes in loops (1), (2), and (3) is equal (within a margin of one) to the number of vertices created by the algorithm, and is thus in $O(|\mathcal{C}_k(i)|)$. At each pass in such a loop, the successor q' of one vertex q is searched in CH and two edges are created: An edge of $\mathcal{C}_k(i)$ to which are associated the points q' and q, and an edge of $\mathcal{E}_k(i)$ to which are associated the points q' and v_i. Each edge can be created in constant time, and the overall complexity of loops (1), (2), and (3) only depends on the number $O(|\mathcal{C}_k(i)|)$ of successor queries in CH.

The other instructions in the algorithm consist in traversing the vertices $g(T_{i,j})$ of $\mathcal{D}_k(i)$ and constructing the convex hulls $conv(T_{i,j} \cup \{v_i\})$. Traversing $\mathcal{D}_k(i)$ consists in traversing the edges $e_P(s,t)$ of $\delta(\mathcal{T}_k(\mathcal{V}_{i-1}))$ such that $v_i \in (\mathring{st})^-$. Since s and t are stored in the map and since the first vertex $g(T_{i,1})$ of $\mathcal{D}_k(i)$ is given, $\mathcal{D}_k(i)$ can be traversed in $O(|\mathcal{D}_k(i)|)$ time.

As the convex hull $conv(T_{i,1})$ is supposed to be stored in CH, $conv(T_{i,1} \cup \{v_i\})$ is obtained with only one insertion in CH. For every $j \in \{2, \ldots, m_i\}$, if $e_{P_j}(s_j, t_j)$ is the edge connecting $g(T_{i,j-1})$ and $g(T_{i,j})$, $conv(T_{i,j} \cup \{v_i\})$ is obtained from $conv(T_{i,j-1} \cup \{v_i\})$ by removing s_j from CH and inserting t_j. Thus, the complexity of all convex hull constructions only depends on the number $O(|\mathcal{D}_k(i)|)$ of insertions and removals in CH.

Finally, the overall complexity of the algorithm only depends on the complexity of the $O(|\mathcal{C}_k(i)| + |\mathcal{D}_k(i)|)$ operations performed in CH. Moreover, CH always contains the convex hull of less than $k+2$ points. □

The aim is now to use Algorithm 2, with i ranging over $\{k+1, \ldots, n\}$, for the incremental construction of $\mathcal{T}_k(\mathcal{V})$. Since $\mathcal{T}_k(\mathcal{V}_k)$ is reduced to the point $g(V_k)$, the map is initialized with this unique vertex, and the convex hull of V_k is stored in CH with $O(k)$ insertions (CH is initially empty). The problem left is to find the first vertex $g(T_{i,1})$ of $\mathcal{D}_k(i)$, for all $i \in \{k+1, \ldots, n\}$. It is straightforward when $i = k+1$, since $\mathcal{T}_k(\mathcal{V}_k)$ is reduced to the point $g(T_{i,1})$. When $i > k+1$, it comes to find the unique vertex of $\delta(\mathcal{T}_k(\mathcal{V}_{i-1}))$ such that, if $e_{P_1}(s_1, t_1)$ and $e_{P_2}(s_2, t_2)$ are the edges of $\delta(\mathcal{T}_k(\mathcal{V}_{i-1}))$ respectively entering and leaving this vertex, then $v_i \in (s_1 \overset{\circ}{} t_1)^+$ and $v_i \in (s_2 \overset{\circ}{} t_2)^-$. In the general case, we may have to traverse the whole boundary of $\mathcal{T}_k(\mathcal{V}_{i-1})$ to find this vertex, leading to a non-efficient centroid triangulation construction. So let us concentrate on the particular case where the convex inclusion chain $\mathcal{V} = (v_1, \ldots, v_n)$ forms a simple polygonal line.

Lemma 7. *If the convex inclusion chain $\mathcal{V} = (v_1, \ldots, v_n)$ forms a simple polygonal line, at least one vertex of $\mathcal{C}_k(i-1)$ is also a vertex of $\mathcal{D}_k(i)$, for all $i \in \{k+2, \ldots, n\}$.*

Proof. (i) By definition of a convex inclusion chain, v_{i-1} is a vertex of $conv(V_{i-1})$ and v_i does not belong to $conv(V_{i-1})$. Assume that the segment $v_{i-1} \overset{\circ}{} v_i$ intersects $conv(V_{i-1})$. It then divides $conv(V_{i-1})$ in two and it cuts an edge $v_a \overset{\circ}{} v_b$ of $conv(V_{i-1})$. The part of the polygonal line $(v_1, \ldots, v_i)$ connecting v_a and v_b is contained in $conv(V_{i-1})$ and is therefore also intersected by $v_{i-1} \overset{\circ}{} v_i$. But this is impossible since $(v_1, \ldots, v_i)$ is a simple polygonal line. It follows that $v_{i-1} \overset{\circ}{} v_i$ does not intersect $conv(V_{i-1})$.

(ii) Hence, there exists a straight line Δ that passes through exactly one point of V_{i-1} and such that $v_{i-1} \in \Delta^-$, $v_i \in \mathring{\Delta}^-$, and $|\Delta^- \cap V_{i-1}| = k$. Setting $T = \Delta^- \cap V_{i-1}$, $g(T)$ is a vertex of $\mathcal{Q}_k(V_{i-1})$ and, since $v_{i-1} \in T$, $g(T)$ is a vertex of $\mathcal{C}_k(i-1)$.

The two edges of $\mathcal{Q}_k(V_{i-1})$ incident in $g(T)$, are determined by the two $(k-1)$-edges st and $s't'$ of V_{i-1} such that (st) and $(s't')$ are the common internal tangents of T and $V_{i-1} \setminus T$, with $T \subset (st)^- \cap (s't')^-$. We then have $\mathring{\Delta}^- \subset (s\overset{\circ}{}t)^- \cup (s'\overset{\circ}{}t')^-$ and, since $v_i \in \mathring{\Delta}^-$, at least one of st and $s't'$ is not a $(k-1)$-edge of V_i. Thus, at least one of the edges incident in $g(T)$ belongs to $\mathcal{D}_k(i)$ and it is the same with $g(T)$. □

It suffices now to see that, for all $i \in \{k+1, \ldots, n-1\}$, Algorithm 2 stops on the vertex $g(T_{i,m_i})$ of $\mathcal{D}_k(i)$ and with the convex hull of $T_{i,m_i} \cup \{v_i\}$ stored in CH. Since $g(T_{i,m_i})$ is the successor of $\mathcal{C}_k(i)$ on $\delta(\mathcal{Q}_k(V_i))$, it results from Lemma 7, that $g(T_{i+1,1})$ can be found by traversing at most the edges of $\mathcal{C}_k(i)$ and of $\mathcal{D}_k(i+1)$. For each edge $e_P(s,t)$ traversed, it is checked on which side of (st) the point v_i lies, s is remove from CH, and t is inserted in CH. Thus, the complexity of the search for $g(T_{i+1,1})$ depends only on the number $O(|\mathcal{C}_k(i)| + |\mathcal{D}_k(i+1)|)$ of insertions and removals performed in CH. Moreover, CH contains always the convex hull of $O(k)$ points.

Theorem 4. *Any set of n points in the plane admits (usually several) order-k centroid triangulations that can be constructed in $O(n \log n + k(n-k) \log k)$ worst case time.*

Proof. For any set of n points, it is possible to find several convex inclusion chains $\mathcal{V}$ that also form simple polygonal lines, in $O(n \log n)$ time; for example, by sorting the points in various directions. In general, the sets of k-sets of these convex inclusion chains are distinct. From the discussion above, the complexity of the incremental construction of $\mathcal{T}_k(\mathcal{V}_k), \ldots, \mathcal{T}_k(\mathcal{V}_n)$ only depends on the complexity of the operations performed in CH. CH is initially empty. When constructing $\mathcal{T}_k(\mathcal{V}_k)$, k points are inserted in CH. Then, for every $i \in \{k+1, \ldots, n\}$, $\mathcal{T}_k(\mathcal{V}_i)$ is constructed from $\mathcal{T}_k(\mathcal{V}_{i-1})$ with $O(|\mathcal{C}_k(i-1)| + |\mathcal{C}_k(i)| + |\mathcal{D}_k(i)|)$ operations in CH (Proposition 3 and discussion above). Since CH contains always the convex hull of $O(k)$ points, the total worst case complexity of the construction of $\mathcal{T}_k(\mathcal{V})$ is

$$O(n \log n + k \log k + \sum_{i=k+1}^{n} (|\mathcal{C}_k(i-1)| + |\mathcal{C}_k(i)| + |\mathcal{D}_k(i)|) \log k).$$

Each of the three sums is bounded by the total number of distinct edges of the k-set polygons $\mathcal{Q}_k(V_k), \ldots, \mathcal{Q}_k(V_n)$, that is, from Proposition 1, by the number of $(k-1)$-edges of the sets $V_1, \ldots, V_n$ (without multiplicity). It then follows from proof of Theorem 1 that the total complexity of the algorithm is $O(n \log n + k(n-k) \log k)$. □

6. Conclusion

In this paper, we have shown that the number of k-sets of a convex inclusion chain of a set V of points in the plane is an invariant of V. Furthermore, it is equal to the number of regions of the order-k Voronoi diagram of V. We get that way a completely new method to compute the size of the order-k Voronoi diagram in the plane. We hope that the study of convex inclusion chains brings new insights into the important open problem of the size of order-k Voronoi diagrams in higher dimensions.

We have also shown that the centroids of the k-sets of a convex inclusion chain are the vertices of a triangulation that belongs to the set of centroid triangulations. This set also contains the order-k centroid Delaunay triangulation. Fully explaining the reason why the number of k-sets of a convex inclusion chain is equal to the number of order-k Voronoi regions comes then to showing that all centroid triangulations of a given point set have the same number of vertices. A sufficient condition for the result to hold has been given in [28]: If, in any centroid triangulation, every set of type-2 triangles defined with the same $k+1$ points triangulates a convex polygon, then all centroid triangulations have the same number of vertices.

The difficulty of proving results on centroid triangulations comes from their recursive definition. The existence of an order-k centroid triangulation depends on the existence of a sequence of order-i centroid triangulations, for all $i < k$. We deem it necessary to find a direct geometric characterization of order-k centroid triangulations, not depending on lower order triangulations. We are currently proving that every vertex of an order-k centroid triangulation is the centroid of a k-point subset that can be separated from the remaining points by a convex curve. More precisely, the separating curves of the k-point subsets that define the vertices of such a triangulation form a set of convex pseudo-circles[29].

Finally, we have given an algorithm that allows to construct several order-k centroid triangulations of any given point set in $O(n \log n + k(n-k) \log k)$ time. This is nearly optimal since the size of an order-k centroid triangulation is expected to be $\Theta(k(n-k))$ and since the construction of an order-1 centroid triangulation is in $\Omega(n \log n)$. However, the generated centroid triangulations should have some regularity properties to be of practical interest, namely for B-spline construction. Therefore, it is now important to investigate how they can be improved by local transformations, such as flips.

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References

1. J. Matoušek. *Lectures on Discrete Geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.
2. U. Wagner. k-sets and k-facets. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, Contemporary Mathematics, pages 443–514. American Mathematical Society, 2008.
3. L. Lovász. On the number of halving lines. *Ann. Univ. Sci. Budapest, Eötvös, Sect. Math.*, 14:107–108, 1971.
4. P. Erdös, L. Lovász, A. Simmons, and E. G. Straus. Dissection graphs of planar point sets. In *A Survey of Combinatorial Theory*, pages 139–149. North-Holland, 1973.
5. T. K. Dey. Improved bounds on planar k-sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.
6. G. Tóth. Point sets with many k-sets. *Discrete Comput. Geom.*, 26(2):187–194, 2001.
7. G. W. Peck. On k-sets in the plane. *Discrete Math.*, 56:73–74, 1985.
8. N. Alon and E. Györi. The number of small semispaces of a finite set of points in the plane. *J. Combinatorial Theory, Ser. A*, 41:154–157, 1986.
9. D. T. Lee. On k-nearest neighbor Voronoi diagrams in the plane. *IEEE Trans. Comput.*, C-31:478–487, 1982.
10. H. B. Curry and I. J. Schoenberg. On pólya frequency functions IV: The fundamental spline functions and their limits. *J. Analyse Math.*, 17:71–107, 1966.
11. C. de Boor. Splines as linear combinations of B-splines: A survey. *Approximation Theory*, II:1–47, 1976.
12. M. Neamtu. What is the natural generalization of univariate splines to higher dimen-

sions? In *Mathematical Methods for Curves and Surfaces: Oslo 2000*, pages 355–392, 2001.

13. M. Neamtu. Delaunay configurations and multivariate splines: A generalization of a result of B. N. Delaunay. *Trans. Amer. Soc.*, 359:2993–3004, 2007.
14. J. Cao, X. Li, G. Wang, and H. Qin. Surface reconstruction using bivariate simplex splines on Delaunay configurations. *Computers & Graphics*, 33(3):341–350, 2009.
15. Y. Liu and J. Snoeyink. Quadratic and cubic B-splines by generalizing higher-order Voronoi diagrams. In *Symposium on Computational Geometry*, pages 150–157, 2007.
16. Y. Liu and J. Snoeyink. Bivariate B-splines from centroid triangulations. In *Computational and Conformal Geometry*, `http://www.math.sunysb.edu/~ccg2007/snoeyink.ppt`, 2007.
17. D. Schmitt and J.-C. Spehner. On Delaunay and Voronoi diagrams of order k in the plane. In *Proc. 3rd Canad. Conf. Comput. Geom.*, pages 29–32, 1991.
18. F. Aurenhammer and O. Schwarzkopf. A simple on-line randomized incremental algorithm for computing higher order Voronoi diagrams. *Internat. J. Comput. Geom. Appl.*, 2:363–381, 1992.
19. P. K. Agarwal and J. Matoušek. Dynamic half-space range reporting and its applications. *Algorithmica*, 13(4):325–345, 1995.
20. P. K. Agarwal, M. de Berg, J. Matousek, and O. Schwarzkopf. Constructing levels in arrangements and higher order Voronoi diagrams. *SIAM J. Comput.*, 27(3):654–667, 1998.
21. T. M. Chan. Random sampling, halfspace range reporting, and construction of ($\leq k$)-levels in three dimensions. *SIAM J. Comput.*, 30(2):561–575, 2000.
22. J.-D. Boissonnat and M. Yvinec. *Algorithmic geometry.* Cambridge University Press, 1998.
23. H. Edelsbrunner, P. Valtr, and Emo Welzl. Cutting dense point sets in half. *Discrete Comput. Geom.*, 17:243–255, 1997.
24. A. Andrzejak and K. Fukuda. Optimization over k-set polytopes and efficient k-set enumeration. In *Proc. 6th Workshop Algorithms Data Struct.*, volume 1663 of *Lecture Notes Comput. Sci.*, pages 1–12. Springer-Verlag, 1999.
25. A. Andrzejak and E. Welzl. In between k-sets, j-facets, and i-faces: (i, j)-partitions. *Discrete Comput. Geom.*, 29:105–131, 2003.
26. D. Schmitt and J.-C. Spehner. k-set polytopes and order-k Delaunay diagrams. In *Proceedings of the 3rd International Symposium on Voronoi Diagrams in Science and Engineering*, pages 173–185, 2006.
27. G. S. Brodal and R. Jacob. Dynamic planar convex hull. In *Proc. 43rd IEEE Sympos. Found. Comput. Sci*, pages 617–626, 2002.
28. W. El Oraiby. *k-set polygons and centroid triangulations.* Ph.D. thesis, Université de Haute–Alsace, Mulhouse, France, 2009.
29. D. Schmitt and J.-C. Spehner. Generation of k-point subsets separable by convex pseudo-circles. In preparation.