Tohoku Math. J.
50 (1998), 261-290

CERTAIN ALGEBRAIC SURFACES OF GENERAL TYPE WITH IRREGULARITY ONE AND THEIR CANONICAL MAPPINGS

TOMOKUNI TAKAHASHI

(Received October 8, 1996, revised May 12, 1997)

Abstract. In this paper, we show the existence of certain algebraic surfaces of general type with irregularity one, and investigate the canonical mappings of these
surfaces. Such a surface has a pencil of non-hyperelliptic curves of genus 3 over an
elliptic curve, and is obtained as the minimal resolution of a relative quartic hypersurface
with at most rational double points as singularities, of the projective plane bundle over
an elliptic curve. We use some results on locally free sheaves over elliptic curves by
Atiyah and Oda to prove the existence.

1. Introduction. Let S be a minimal nonsingular projective surface defined over
C. S is said to be canonical if the rational mapping ƒ³|Ks| defined by the canonical linear system |Ks| is birational.
In this paper, we show for all values of •¬ the existence of minimal algebraic
surfaces of general type with Ks2=3pg(S) and q(S)=1, and study their canonical
mappings. Note that the case pg(S)=1 was studied by Catanese and Ciliberto [7].

(I) (Castelnuovo-Horikawa's inequality, cf. [5, Theoreme 5.5], [12, Lemma 1.1]).
If S is a canonical surface, then

•¬

(II) Castelnuovo obtained canonical surfaces with Ks2=3pg(S)-7 (cf. [6]). Such a surface S satisfies q(S)=0, and with a few exceptions S is birational to a relative quartic hypersurface of a P2-bundle over P1 which has at most rational double points as singularities.
In general, a nonsingular relative quartic hypersurface in a P2-bundle over a nonsingular curve C of genus b satisfies

•¬

We may ask whether a canonical surface S satisfying these equalities is obtained as the minimal resolution of a relative quartic hypersurface with at most rational double points, of a P2-bundle over a nonsingular curve C of genus b. Konno [15, Lemma 3.1, Theorem 3.2] proved that it is the case if b=1. Namely, if S is a canonical surface satisfying Ks2=3pg(S) and q(S)=1, then S is the minimal resolution of a relative quartic

*1991 Mathematics Subject Classification. Primary 14J10.

262 T. TAKAHASHI

hypersurface in a P2-boundle over an elliptic curve.
More precisely, S has a pencil f:S•¨C=Alb(S) whose general fiber is a
non-hyperelliptic curve of genus 3. Hence, the direct image •¬ of the relative dualiz
ing sheaf •¬ is a locally free sheaf of rank 3 over C. If we let ƒÎ:
W:=P(f*ƒÖs/c)•¨C to be the P2-bundle associated to f*ƒÖs/c, T•¸Pic(W) a tautologi
cal divisor with •¬, and D•¸Pic(C) a divisor with •¬,
then there exists a member S'•¸|4T-ƒÎ*D| which has at most rational double points as
singularities, and S is the minimal resolution of S'(cf. [15]).

Not all the irreducible relative quartic hypersurfaces in the P2-bundles over elliptic curves which have at most rational double points as singularities have canonical surfaces as the minimal resolutions of singularities. For example, we have the possibilities pg(S)=1, 2, 3, and S is not canonical in these cases.
In this paper, we study whether a complete linear system of •¬
has members which have at most rational double points as singularities for every locally
free sheaf E of rank three over an elliptic curve C, where ƒÎ:W:=P(E)•¨C is the
P2-bundle associated to E and T is a tautological divisor with •¬. In particular,
we check for all values of •¬ the existence of minimal algebraic surfaces of general
type satisfying Ks2=3pg(S) and q(S)=1. We then study their canonical mappings ƒ³|Ks|
including the cases •¬.

We obtain the following results on the existence of minimal algebraic surfaces with

Ks=3pg(S) and q(S)=1, using the results about vector bundles over an elliptic curve C by Atiyah [4] and Oda [19].

(1) The case where f*ƒÖs/c is isomorphic to the direct sum of three invertible
sheaves over •¬ is necessary, and conversely, for every integer •¬,
there exists minimal algebraic surfaces of general type with pg(S)=N, Ks2=3pg(S) and
q(S)=1. (See Theorem 3.1.)
(2) The case where f*ƒÖs/c is isomorphic to the direct sum of an invertible sheaf
and an indecomposable locally free sheaf of rank 2 over •¬ is necessary,
and conversely, for every integer •¬, there exist minimal algebraic surfaces of general
type with pg(S)=N, Ks2=3pg(S) and q(S)=1. (See Theorems 3.9 and 3.10.)
(3) The case where f*ƒÖs/c is indecomposable •¬ is necessary, and
conversely, for every integer •¬, there exist minimal algebraic surfaces of general

type with pg(S)=N, Ks2=3pg(S) and q(S)=1. (See Theorem 3.19.)
As for the canonical mappings of the above surfaces, we obtain the following results:
(1) In the case where f*ƒÖs/c is the direct sum of three invertible sheaves, if •¬
holds, then ƒ³|Ks| is always birational onto its image with the exception of only one
case •¬ where L0 is an invertible sheaf of degree 2 over C.

If pg(S)=5 and if f*ƒÖs/c is not some special locally free sheaf, then ƒ³|K s| is always
birational onto its image, too.

If pg(S)=5 and f*ƒÖs/c is some special locally free sheaf, or if pg(S)=4, then ƒ³|K s|
is birational onto its image in most cases. Although there is a possibility of the existence

ALGEBRAIC SURFACES OF GENERAL TYPE 263

of a surface whose canonical mapping is not birational onto its image, we have not
obtained an example of such a surface.
If pg(S)=3, then ƒ³|Ks| is a generically finite mapping onto the projective plane
whose degree varies according to the isomorphism class of f*ƒÖs/c. In most cases, the
degree of the canonical mapping is 6, 8 or 9.
(2) In the case where f*ƒÖs/c is the direct sum of an invertible sheaf and an
indecomposable locally free sheaf of rank 2, if •¬ holds, then ƒ³|Ks|, is always
birational onto its image.
If pg(S)=4, then ƒ³|Ks| is birational onto its image in most cases. Although there
is a possibility of the existence of a surface whose canonical mapping is not birational
onto its image, we have not obtained an example of such a surface.
If pg(S)=3, then ƒ³|Ks| is a generically finite mapping onto the projective plane
whose degree varies according to the isomorphism class of f*ƒÖs/c. In most cases, the
degree of the canonical mapping is 4, 8 or 9.

If pg(S)=2, then |Ks| is a linear pencil and the genus of a general member is 7.

(3) In the case where f*ƒÖs/c is mdecomposable, it •¬ holds, tnen ƒ³|Ks| is
always holomorphic and birational onto its image.
If pg(S)=4, then ƒ³|Ks| is birational onto its image in most cases. Although there
is a possibility of the existence of a surface whose canonical mapping is not birational
onto its image, we have not obtained an example of such a surface.
If pg(S)=3, then ƒ³|Ks| is a generically finite mapping of degree 8 onto the projective
plane in most cases.

If pg(S)=2, then |Ks| is a linear pencil and the genus of a general member is 7. We obtain some examples of canonical surfaces whose canonical mappings are not holomorphic. Such surfaces do not appear in the cases treated by Ashikaga [2] and Konno [16].
This paper is a revised version of the author's thesis [22].

ACKNOWLEDGEMENT. The author would like to thank Professor Tadao Oda for
constant encouragement and advice. Thanks are also due to Professors Tadashi Ashikaga
and Kazuhiro Konno who suggested the problem to the author and provided valuable information over the years. The author is grateful to the refree for valuable comments and suggestions.

2. Preliminaries. Let us mention some results which we need later.

THEOREM 2.1 (cf. Konno [15, Corollary 6.4]). If S is a canonical surface with
q(S)=l and •¬, then a general fiber of the Albanese mapping f:
S•¨C:=Alb(S) is a nonsingular curve of genus 3.

THEOREM 2.2 (cf. Konno [15, Lemma 3.1, and Theorem 3.2]). Let f:S•¨C be
a relatively minimal non-hyperelliptic fibration of genus 3, where S is a nonsingular surface,

264 T. TAKAHASHI

and C is a nonsingular curve of genus b. Then

(*) •¬
Let ƒÎ:W:=P(f*ƒÖs/c)•¨C be the P2-bundle over C defined by the locally free sheaf
f*ƒÖs/c of rank 3, T a tautological divisor with •¬, and •¬ the
rational mapping induced by the natural sheaf homomorphism •¬ . If the
equality holds in (*), then •¬ has at most rational double points as singularities

,

and we have

•¬

where •¬ is the •¬c-module dual to f*ƒÖs/c.

REMARK. The inequality stated in the first half of Theorem 2.2 was proved by Horikawa [13], [14, Proposition 2.1] and Reid [20] in a different way. Konno [16, Theorem 2.1] himself also gave another proof.

PROPOSITION 2.3. Let C be a nonsingular curve of genus b, and E a locally free
sheaf of rank 3 over C. Let ƒÎ:W:=P(E)•¨C be the P2-bundle over C associated to E
,

T a tautological divisor with •¬, and D•¸Div(C) a divisor on C such that
•¬. If |4T-ƒÎ*D| has an irreducible member S' with at most rational double
points as singularities, then the following equalities hold for the minimal resolution v:S•¨S'
of singularities.

•¬

Furthermore, if we denote •¬ , then we have •¬

PROOF. We have •¬ and q(S')=q(S) by the hypothesis that S'
has at most rational double points as singularities. Since •¬

by the adjunction formula, and since T3-(degE)T2F=0 , we have •¬

.

By considering the cohomology long exact sequence induced by the exact sequence

•¬

we obtain the equality for pg(S) and q(S).

Since S' has at most rational double points as singularities
, •¬

, hold. Since •¬ by the adjunction formula
, we have

•¬

Since we have the long exact sequence

•¬

265 ALGEBRAIC SURFACES OF GENERAL TYPE

and since •¬ det E=0 for j=0, 1, we
obtain •¬, and hence •¬. q.e.d.

REMARK. By the last assertion of Proposition 2.3, we see that two different P2-bundles do not contain the same surfaces.

THEOREM 2.4 (cf. Atiyah [4, Theorem 5, Theorem 7 and Corollary, Theorem 9],
Oda [19, Theorem 1.2]). Let C be an elliptic curve and •¬(r, d) (r, d•¸Z) the set of
isomorphism classes of indecomposable locally free sheaves of rank r and degree d over C.
(1) If(r, d)=1, and if we fix any isogeny •¬ of degree r, we have a bijection

•¬

Denote G=ker •¬, and let TƒÐ, be the translation by ƒÐ•¸G on C. Then we get

(2) For any r•¸N, there exists a unique •¬ such that H0(C, Fr)•‚0. Fr is a
successive extension of •¬c, and •¬ holds. Furthermore, dimHi(C, Fr)=1
(i=0, 1). We have the following bijective mapping for m•¸Z:

REMARK. Although not necessary in this paper, we have the following in general:
If (r, d)=h, then •¬ is a bijective mapping.

We use the following lemma in •˜3.2 and •˜3.3:

LEMMA 2.5., Let C be an elliptic curve, ƒÊ:Y=P(F2)•¨C the ruled surface associated
to F2, and C'•¼Y the unique section of y with •¬. For any point p•¸C and for
any positive integer i, we have •¬, where ƒ¡p:=ƒÊ-1(p) and y0:=C'•¿ƒ¡p.
Furthermore, general members of|iC'+ƒ¡p| are nonsingular at y0, and all the members
which are nonsingular have the same tangent at y0. If i and j are positive integers with
i•‚j, then a nonsingular member of|iC'+ƒ¡p| and a nonsingular member of |jC'+ƒ¡p| have
different tangents at y0.

PROOF. We have Bs |iC'+ƒ¡p|•¼C'•¾ƒ¡p. Since dim Im•¬
, and since dim •¬, there exists at most one base point
on Fp. On the other hand, since dim •¬
, C' is not a fixed component of |iC'+ƒ¡p|. Furthermore, since (iC'+ƒ¡p)C'=1
and •¬, only •¬ is the base point of |iC'+ƒ¡P| lying on C'. Hence, we
obtain Bs•¬. Since (C'+ƒ¡p)C'=1, general members of |iC'+ƒ¡p| are
nonsingular at y0.
Let M•¸|iC'+ƒ¡p| be a nonsingular member. If we consider the cohomology long
exact sequence induced by the exact sequence of sheaves

266 T. TAKAHASHI

•¬

we get dim •¬, and dim Im•¬ . The
subsystem of the complete linear system of M|M corresponding to the image of
•¬ may be regarded as the complete linear system of
M|M-y0, and its dimension is i-1. On the other hand, since deg •¬
, the genus of M is equal to i. Since •¬ by the adjunc

tion formula, the complete linear system of M|M-2y0 is also (i-1)-dimensional . Hence,
yo is the base point of the complete linear system of M|M-Y0, and the intersection
multiplicity of any nonsingular member M'•¸|iC'+ƒ¡p| with M at yo is at least two
,

i.e., M and M' have the same tangent.

The last assertion can be proved in the same way as above. q.e.d.

The following lemma is trivial:

LEMMA 2.6. Let X be a complete variety, and D an effective divisor on X. Assume

dim •¬, and dim •¬. If D1•¿D2•‚Bs|D| for any distinct members D1, D2•¸|D|,
then |D| is not composite with a pencil. In particular, |D| is not composite with a pencil
if one of the following holds:
(i) Bs •¬ and Dn>0,
(ii) Bs •¬ and the dimension of any component of Bs |D| is less than dim X-2.

3. Existence and birationality. By Theorems 2.1 and 2.2, to classify canonical
surfaces with Ks2=3pg(S) and q(S)=1, we need to have a necessary and sufficient

condition for the complete linear system |4T-ƒÎ*D| , on the P2-bundle W=P(E)
associated to a locally free sheaf E of rank 3 over an elliptic curve C, to have irreducible
members with at most rational double points as singularities , where T is a tautological
divisor on W with •¬, and D•¸Div(C) satisfies •¬ . We should then

choose those members whose minimal resolutions are canonical

.

Locally free sheaves of rank 3 over an elliptic curve C are expressed uniquely up
to order as direct sums of indecomposable locally free sheaves (cf. [4]). Hence we should consider the following three cases:
(1) E is the direct sum of three invertible sheaves.
(2) E is the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2.

(3) E is indecomposable.

DEFINITION. Let ƒÎ:W•¨C be the P2-bundle over an elliptic curve C associated

to a locally free sheaf E of rank 3, T a tautological divisor with •¬, and
D•¸Div(C) a divisor with •¬ . We say that E satisfies the condition (A) if
| 4T-ƒÎ*D| has a member S' satisfying the following conditions:

(i) S' has at most rational double points as singularities,
(ii) The minimal resolution S of S' is of general type,

ALGEBRAIC SURFACES OF GENERAL TYPE 267

(iii) S satisfies Ks2=3pg(S) and q(S)=1.

REMARK. We only have to consider the locally free sheaves E with •¬
by Proposition 2.3. If E satisfies the condition (A), then X(•¬s)=deg E>0. Furthermore,
by Fujita [8, (1.2) Proposition], we only have to consider locally free sheaves such that
any quotient locally free sheaf has nonnegative degree.
3.1. The case where E is a direct sum of three invertible sheaves. Let L0, L1,
L2 be invertible sheaves over an elliptic curve C such that •¬, and
denote di:=deg Li(i=0, 1, 2). Furthermore, let ƒÎ:W•¨C be the p2-bundle associated
to E, and T a tautological divisor with •¬. In •˜3.1, we prove the existence of
a surface S of general type with Ks2=3pg(S), q(S)=1 and pg(S)=N for any integer •¬
by obtaining such a locally free sheaf E of rank three satisfying the condition (A)
(Theorem 3.1). We then study the canonical mapping of the surfaces thus obtained.
The results about the canonical mappings are stated in Corollaries 3.3 and 3.4, and
Propositions 3.5, 3.7 and 3.8.
3.1.1. Existences. We may assume •¬. We only have to consider the
case •¬, •¬ and d2>0 by the remark immediately before •˜3.1.

THEOREM 3.1. Let ƒÎ:W=P(E)•¨C be the P2-bundle over an elliptic curve C
associated to •¬, T a tautological divisor with •¬, and D•¸Div(C)
satisfies •¬det E. Denote di:=deg Li (i=0, 1, 2), and suppose •¬ and
d2>0. Then the locally free sheaf E satisfies the condition (A) if and only if the following
(1), (2) and (3) hold.
(1) One of the following (i), (ii) and (iii) holds:
(i) •¬,
(ii) •¬,
(iii) d0=d1 and at least one of•¬, •¬, •¬ and •¬ is isomorphic
to L2.
(2) One of the following (i), (ii) and (iii) holds:
(i) d1<2d0,
(ii) •¬,
(iii) •¬ and •¬.
(3) If d0=d1=d2=1 holds, then one of L0, L1, L2 is not isomorphic to the others.

PROOF. We can choose •¬ (i=0, 1, 2) which give ho
mogeneous coordinates on each fiber of it. Then any •¬
can be written as

•¬

In the same way as in the proof of Claim III in [3], we can show that E does not satisfy the condition (A) when one of (1) and (2) does not hold. If (3) does not hold,

268 T. TAKAHASHI

then Bs |4T-ƒÎ*D| consists of a fiber of ƒÎ.
From now on, we assume that (1), (2) and (3) hold.

(I) Let us look at the case where •¬. (If, moreover,
d0=d1 and •¬ hold, we may assume •¬
Clearly, |4T-ƒÎ*D| has no base point if and only if •¬. If
•¬ and •¬, then Bs •¬ consists of one point. Assume
•¬ and •¬. If •¬
, then Bs •¬ consists of two points. If •¬
, then Bs•¬ is a line contained in some fiber of ƒÎ. Next
assume •¬, i.e., •¬. If L0,
L1 and L2 are pairwise different, Bs •¬ consists of three points. If two of L0
, L1 and L2 are isomorphic, then Bs •¬ consists of a point and a line contained
in some fiber of ƒÎ. We can show that a general member of •¬ is nonsingular
at any point of Bs •¬ in any case above by considering the local equation.
Clearly, •¬ is not composite with a pencil in any case above by Lemma 2.6.
Hence a general member of •¬ is irreducible and nonsingular by Bertini's
theorem.

(II) Let us look at the case where •¬ or (•¬ and •¬
). We have Z0•¼Bs •¬, where Z0 is a curve defined by X1=X2=0.
If •¬, or •¬, then we have Bs•¬. If
•¬ and •¬, then Bs•¬ consists of Z0 and a
point. If •¬ and •¬
, then Bs•¬ consists of Z0 and two points. If •¬
and •¬, then Bs•¬ consists of Z0

and a line contained in some fiber of ƒÎ. If •¬ and •¬, then
we must have •¬, and Bs•¬, where Z1, is the curve
defined by X0=X2=0.
We can show that a general member of •¬ is nonsingular at the base
points which are not contained in Z0 when •¬ holds by considering
the local equations. Hence it is sufficient to look at the multiplicity of a general member
of •¬ at Z0 when •¬, or at Z0•¾Z1 when •¬.
Let us look at the case where 2d0>d2 or (•¬ and d0<d1). (When •¬,
if we assume d1=d2 and •¬, further, interchange L1 and L2 and regard this case
as the case •¬. Hence, we may assume •¬ when •¬ and d1=d2.) In
this case, we have •¬. Since we have •¬ and
•¬, a general member of •¬ is nonsingular at Z0 except in
the case where •¬ and •¬ hold. In this case, we can show that
any general member of •¬ has a rational double point of type A1 on Z0 by
considering the local equation.
Let us look at the case where 2d0<d2 or (2d0=d2, d0<d, and •¬). In this
case, the coefficients of X04 and X03X1 are 0, and •¬ holds. If 2d0=d1

ALGEBRAIC SURFACES OF GENERAL TYPE 269

then we have •¬, by the assumption of the theorem , and the coefficient of •¬
is a constant. Hence a general member is nonsingular at Z0. Assume d1<2d0 holds,
and that •¬ is general . Let p•¸C be one of the points
with ,•¬ for •¬, and t a local coordinate around p.
Furthermore, let q•¸W be a point with t=X1=X2=0, and denote x1:=X1/X0 and
X2:=X2/X0. Then •¬ can be written as

•¬

around q. The equation •¬ gives a rational double point of type A1 at q except in
the case

•¬, and d0+d1-d2=2d0-d1=1.

In this case, •¬ holds around q for some constant c'•¸C, and the equation •¬
gives a rational double point of type A2 at q. If 2d0=2d1=d2, we can show that a
general member of |4T-ƒÎ*D| has at most rational double points of tyep A, on Z0•¾Z1
in the same way as above.
Let S1 and S2 be general members of |4T-ƒÎ*D|, and F a general fiber of ƒÎ

.

Furthermore, denote qi:=Zi•¿F for i=0, 1. We can show that the intersection
multiplicity of S1|F and S2|F at q0 in F is at most two. When Z1•¼Bs|4T-ƒÎ*D|, we
can also show that the intersection multiplicity of S1|F and S2|F at q1 in F is at most
two. Since S1|F and S2|F are quartic curves, they have other intersection points. Therefore,
we see that |4T-ƒÎ*D| is not composite with a pencil by Lemma 2.6. Hence a general
member of |4T-ƒÎ*D| is irreducible and nonsingular by Bertini's theorem. q.e.d.

3.1.2. The canonical mappings. In this section, we consider the canonical
mappings of those surfaces whose existences were shown in •˜3.1.1.

LEMMA 3.2. Let L0, L1, L2 be invertible sheaves over an elliptic curve C, and
denote di:=degLi, (i=0, 1, 2). Assume that L0, L1, L2 satisfy the conditions of Theorem

3.1. If ƒÎ:W:=P(E)•¨C is the P2-bundle over C associated to •¬ , and
T is a tautological divisor such that •¬, then ƒ³|T| is birational onto its image
when one of the following holds.
(i) •¬.
(ii) (d0, d1, d2)=(1, 2, 3).
(iii) (d0, d1, d2)=(2, 2, 2) and one of L0, L1, L2 is not isomorphic to the others.
(iv) (d0, d1, d2)=(1, 2, 2) and •¬.

PROOF. If F is a general fiber of ƒÎ, we have •¬ . Hence the
restriction mapping •¬ is surjective, and the restriction of

ƒ³|T| to F gives an isomorphism of F onto its image.
Let •¬(i= 0, 1, 2) be as in the proof of Theorem 3.1. Any

•¬ can be written as

270 T. TAKAHASHI

•¬

We can easily prove that there exists a Zariski open subset of W such that the restric
tion of ƒ³|T| on it gives an isomorphism onto the image under the assumption of the
lemma. q.e.d.

COROLLARY 3.3. The canonical mapping of any surface S, whose existence is guaranteed by Theorem 3.1 and the condition (A), is a birational morphism if one of the following holds.
(i) •¬ and •¬,
(ii) (d0, d1, d2)=(2, 2, 2), and one of L0, L1, L2 is not isomorphic to the others.

PROOF. Let the notation be as in Proposition 2.3. Since •¬, and since
Hi(W, ƒÖw)=0, for i=0, 1, we have •¬. Since S' has at most
rational double points as singularities, we have •¬, where •¬
is a minimal resolution. Clearly, S' has nonempty intersection with the Zariski open
subset of W appearing in the proof of Lemma 3.2, and hence the birationality follows
from Lemma 3.2. Since •¬, we have Bs•¬, and hence Bs•¬ holds.
q.e.d.

COROLLARY 3.4. The canonical mapping of any surface S, whose existence is guaranteed by Theorem 3.1 and the condition (A), is birational onto its image but is not a morphism, and its image is non-normal, if one of the following holds:

(i) (d0, d1, d2)=(1, 2, 5),
(ii) (d0, d1, d2)=(1, 2, 4),
(iii) (d0, d1, d2)=(1, 2, 3),
(iv) (d0, d1, d2)=(1, 2, 2) and •¬.

PROOF. We can show the birationality of ƒ³|Ks| as in the proof of Corollary 3.3.
In the rest of the proof, we use our notation in Theorem 3.1. By considering Bs|T|
and Bs•¬, we see that |Ks| has only one base point q0. The restriction of |T|
to the fiber F0 containing q0 may be regarded as a subsystem of the complete linear
system of •¬p2(1) consisting of all lines going through q0. Each line of this system
intersects the fiber •¬ of S at four points, one of which is q0. Hence we have
deg(•¬), and the canonical image of S is non-normal by Zariski's main
theorem. q.e.d.

PROPOSITION 3.5. In the notation of Lemma 3.2, assume di=2(i=0, 1, 2) and •¬

. Then the canonical mapping of a general member of |4T-ƒÎ*D| is a mor
phism of degree 2 onto the image, where D•¸Div(C) satisfies •¬det E.

PROOF. If we denote •¬, we have •¬, and hence •¬
. Therefore, if we denote •¬, we have the
following commutative diagram:

271 ALGEBRAIC SURFACES OF GENERAL TYPE

•¬

Let T0 be a tautological divisor with •¬. We have •¬, and both
•¬ and •¬ are 6-dimensional. Hence, we get •¬
Since •¬ is an embedding we have deg ƒ³|T|=2.
Since dim |4T-ƒÎ*D|=dim•¬, and since Bs•¬, the canonical
mapping of a general member of •¬ is a morphism of degree 2 onto the
image. q.e.d.

REMARK. W0 in the proof of Proposition 3.5 is isomorphic to P2•~P1. Let
•¬ be a nonsingular member. If S0 is the image of S in W0, and if r:W0•¨P2
is a natural projection, we see that •¬ holds, and that r|s0:S0•¨P2 is a birational
morphism by easy caluculations. Hence, r|s0 is the blowing-up at sixteen points of
P2, and maps each fiber of S0•¨P1 onto a plane quartic curve birationally. Therefore,
the surfaces in Proposition 3.5 are obtained in another way as follows:
Let B1, B2, B3, B4•¼P2 be nonsingular quartic curves intersecting each other at
sixteen points A1,..., A16 transversally. Let ƒÌ:X•¨P2 be the blowing-up at A1,...,
A16, and Bj the proper transform of Bj(j=1, 2, 3, 4), and denote •¬. We have
•¬ where H•¼P2 is a line. Let h:S•¨X be a double cover
ing branched along •¬. Then •¬ holds, and we have •¬
. On the other hand, we have •¬
. Since •¬ is equal to the dimension of the subsystem of
| 5H| which consists of all the quintic curves going through A1,...,A16, we obtain
dim•¬ by using the Caylay-Bacharach theorem (cf.,
e.g., [10]), and hence pg(S)=6. Since

•¬

we have q(S)=1.

LEMMA 3.6. Let L0, L1, L2 be invertible sheaves over an elliptic curve C, ƒÎ
:W:=P(E)•¨C be P2-bundle associated to the locally free sheaf •¬,
and T a tautological divisor with •¬, and denote di:=degLi (i=0, 1, 2). If L0,
L1 and L2 satisfy one of the following (i) and (ii), then deg ƒ³|T|=2 holds:
(i) (d0,d1,d2)=(1,2,2) and •¬.
(ii) (d0, d1, d2)=(1, 1, 2).

PROOF. In the case (i), let p•¸C be a point with •¬. There exists a point
q•¸ƒÎ -1(
p) with Bs|T|={q}. Denote •¬ and •¬. We

272 T. TAKAHASHI

have the following commutative diagram as a special case of Maruyama [17, Chapter 1]:

•¬

where ƒÎ':W':=P(E')•¨C is the P2-boundle associated to •¬ is the
blowing-up at q •¬ is the blowing-up along P(F'), and
ƒÎ0:W0•¨P1 is the P2-bundle associated to •¬. Let T' be a
tautological divisor of W' with •¬, and T the proper transform of T by •¬.
We have •¬. If T0 is a tautological divisor of W0 satisfying •¬, then
we have ƒ³*T0•`T', and dim |T'|=dim|T0|=4. Hence we have •¬. We
can show that ƒ³|T0| is a birational morphism onto the image in a way similar to Lemma
3.2. Therefore we have deg ƒ³|T|=degƒ³|T'|=degƒ³=2.
In the case (ii), if we assume •¬, then the statement can be proved in a way
similar to that in the case (i).
Assume •¬, in the case (ii). If p•¸C is the point with •¬, then there
exists a line Z•¼ƒÎ-1(p) with Bs|T|=Z. We obtain the same commutative diagram as
above, and in this case, •¬ is the blowing-up along Z. We can show that
deg ƒ³|T|=2 by the same argument as in the case (i). q.e.d

PROPOSITION 3.7. Let the notation and the assumption be as in Lemma 3.6. Then
the minimal resolution of a general member S•¸|4T-ƒÎ*D| is canonical.

PROOF. First, we consider the case •¬.

There is nothing to prove if ƒ³|Ks| is birational onto its image. Thus suppose ƒ³|K s| is
not birational. Hence ƒ³|Ks| gives an unramified two-to-one covering

•¬

where Fi:=ƒÎ-1(pi) with pi•¸C (i=0, 1, 2, 3) the ramification points of •¬
and S0 is the image. Let C0•¼W be a curve which is the base locus of •¬
Fix a point •¬ and let q•¸W be the other point which is mapped to
ƒ³|T|(q) by the two-to-one map ƒ³|T|.
Since dim •¬, we obtain •¬
such that X0 vanishes at q and q' and that the divisor (X0) is irreducible. Similarly,
since dim •¬, we obtain •¬ such
that X1 vanishes at q and q', and that X2 does not vanish at q and q'. Furthermore,
the divisors (X1) and (X2) are irreducible. A global section •¬
defining S satisfies •¬ by our choice of X0, X1, X2, where

ALGEBRAIC SURFACES OF GENERAL TYPE 273

•¬04 is as in the previous section. Hence q' is not contained in S if and only if •¬
holds. Since S is general, we are done.
Next, we consider the case •¬.
Let •¬ be a general member. Except when the following (i) and (ii) are
satisfied, if we assume deg ƒ³|Ks|=2, then a fiber of S'•¨C which has a multiple
component and another fiber which has no multiple component are mapped onto the
same curve isomorphically, which is absurd.
(i) •¬
(ii) •¬
In the cases (i) and (ii), we can show that S is a canonical surface in the same way
as in the case •¬. q.e.d

REMARK. In the situation of Proposition 3.7, we have a possibility that there exist
special members, with at most rational double points as singularities, of •¬
whose canonical mapping is of degree 2.

PROPOSITION 3.8. Let L0, L1 and L2 be invertible sheaves over an elliptic curve C
satisfying deg Li=1 (i=0, 1, 2) and the condition (3) of Theorem 3.1, ƒÎ:W:=P(E)•¨C
the P2-bundle associated to •¬, T a tautological divisor with •¬,
D•¸Div(C) a divisor with •¬, and S•¸|4T-ƒÎ*D| a general member. We have
the following about ƒ³|Ks|:
(i) If •¬ and •¬, then ƒ³|Ks| gives a
covering of degree 9 onto P2.
(ii) If only one of •¬ and •¬ holds, then
| Ks| has one isolated base point, and ƒ³|Ks| gives a covering of degree 8 over P2.
(iii) If all of •¬ hold, then |Ks|
has three isolated base points, and ƒ³|Ks| gives a covering of degree 6 over P2.

PROOF. First we assume that L0, L1, L2 are pairwise non-isomorphic. Let the
notation be as in Theorem 3.1. If qi•¸W (i=0, 1, 2) is the point defined by •¬
, where •¬, and ƒÑ is the cyclic permutation (012),
then we have Bs|T|={q0, q1, q2}.
In the case (i), we have Bs|T|•¿Bs•¬. Hence ƒ³|Ks| is a surjective
morphism onto P2. Since Ks2=9 and the degree of P2 is 1, we are done in the case (i).
Next, we consider the case (ii). We only have to consider the case •¬
by renumbering L0, L1 and L2 if necessary. In this case, all the members of |4T-ƒÎ*D|
go through q0. Since S•¸|4T-ƒÎ*D| is general, it does not contain q1 and q2. Hence
we obtain Bs|Ks|={q0}. Let •¬ be the blowing-up at q0, and T a proper
transform of T by ƒÓ. It is easy to see that Bs•¬ holds, and hence q0 is the simple
base point of |Ks|. Denote •¬, where S is a proper transform of S by ƒÓ, and
•¬. If |V| is the variable part of •¬, then we have •¬.
Therefore, we obtain

274 T. TAKAHASHI

degƒ³|Ks|=V2=(ƒÌ*Ks-E)2=8

Similarly, since qi(i=0, 1, 2) is the simple base point of |Ks| in the case (iii), we
have deg•¬.
The proof is essentially the same when •¬
. q.e.d.

3.2. E is the direct sum of an invertible sheaf and an indecomposable locally free
sheaf of rank 2. We denote •¬, where E0 is an indecomposable locally free
sheaf of rank 2 with deg E0=:e, and L is an invertible sheaf over an elliptic curve C
with deg L=:d. We only have to consider the case •¬, •¬ and (e, d)•‚(0, 0) by the
remark immediately before •˜3.1.
We prove the existence of a surface S with •¬, q(S)=1 and pg(S)=N for
any integer •¬ in •˜3.2.1 (Theorem 3.9) when e is even, and in •˜3.2.2 (Theorem 3.10)
when e is odd. (When e is even, however, the case pg(S)=2 does not occur.) In •˜3.2.3,
we study the canonical mapping of the surfaces obtained in •˜3.2.1 and •˜3.2.2. The
results about the canonical mappings are stated in Corollary 3.13, and Propositions
3.14, 3.17, 3.18 and 3.30.

Let ƒÎ:W:=P2(E)•¨C be the P2-bundle associated to E, and T a tautological
divisor with •¬ is the blowing-up along •¬, then
X is a P1-bundle ƒÐ:X•¨Y:=P(E0). Let ƒÊ:Y•¨C be the ruling, and denote •¬
and •¬. If C0•¸Div(Y) is a tautological divisor with •¬, then we
have •¬, and •¬. Let Y0•¸Div(X) be a divisor with •¬
, and let •¬ be global
sections with (Z0)=Y0 and (Z•‡)=Y•‡. Then Z0 and Z•‡ give homogeneous coordinates
of each fiber of the P1-bundle ƒÐ.
We study the complete linear system of the invertible sheaf •¬
over X.
Any •¬can
be written as

•¬

3.2.1. Existence in the case where e is even. Denote e=2e0. There exist •¬
, and •¬, with •¬, hence we have

•¬.

THEOREM 3.9. Let the conditions and notation be as above. Then the locally free sheaf E satisfies the condition (A) if and only if one of the following (1), (2) and (3) holds:

(1) •¬ and •¬,
(2) d<e<4d,
(3) •¬ and •¬.

ALGEBRAIC SURFACES OF GENERAL TYPE 275

PROOF. We deal with different cases. Let D•¸Div(C) satisfy •¬det E.
(i) The case where (e=d and •¬), or (e<d). Since we have

•¬

by Theorem 2.4, •¬ is divisible by Z0. Hence, the image of (•¬) in W is reducible.
(ii) The case where (•¬), (d<e<3d), or (•¬).
Let us look at the complete linear system of •¬. Since we have
deg•¬, it does not have base points when 3d-e•‚1. If 3d-e=
1 holds, we have •¬, where q•¸C satisfies •¬
. We easily check that ƒ¡ is a (-1)-curve on S":=(•¬).
Let us look at •¬. Clearly, Bs•¬ holds when
deg•¬. When e-d=1, a general member of |4C0-ƒÊ*D| is non-
singular by Lemma 2.5. Thus a general member of |4Y1-ƒÐ*ƒÊ*D| is nonsingular
at the base point Y0•¿ƒÐ-1(y0), where y0 is the base point of |4C0-ƒÊ*D|. If e-d=0,
since we assume •¬, we have •¬. Thus we have

•¬, where C' is a section of ƒÊ with •¬. The
divisor (•¬3) on Y defined by general •¬3 intersects C' at e0 points transversally. Let
p be one of these intersection points, (t, u) a local coordinate around p so that t=0
and u=0 are the local equations for C' and (•¬3) around p, respectively, and denote
z0:=Z0/Z•‡, •¬ can be written as

•¬

near p0:=ƒÐ-1(p)•¿Y0. This is an equation defining a rational double point of type A3.
We have to consider the case •¬ with •¬(1, 1). (This is the case
where 3d-e=1 and e-d=1 above hold at the same time.) In this case, •¬i is contained
in •¬. We have Bs|iC'+ƒ¡0|={y0} by Lemma 2.5, and hence
•¬. We only have to prove that it is
nonsingular along ƒÐ-1(y0). Since all the nonsingular members of |4C'+ƒ¡0| have the
same tangent at y0 by Lemma 2.5, we can choose a local coordinate (t, u) around y0
so that t=0 is the local equation of ƒ¡0 and that u=0 gives the tangent line of nonsingular
members of |4C'+ƒ¡0| at y0. If we denote z:=Z0/Z•‡, then •¬ can be written as

•¬

near •¬, where ai, bj•¸C, (i=0, 1, 2, 3, j=1, 2, 3, 4), and lj(t, u), (j=1, 2, 3, 4)
is the sum of all the monomials with respect to t and u with degree at least two. Since
•¬ is general, we may assume a0•‚0 and b4•‚0. If we fix a1, a2 and a3, then b1, b2 and
b3 are uniquely determined. On the other hand, a0 and b4 can be chosen independently
of them. Hence the two equations •¬ and •¬ do not have the same
solutions, and (•¬) is nonsingular along ƒÐ-1(y0).
Clearly, •¬ is not composite with a pencil when e>d by Lemma 2.6.

276 T. TAKAHASHI

If e=d holds, let S1 and S2 be two distinct general members of •¬ , and F
a general fiber of •¬, and denote •¬. We can easily check that the
intersection multiplicity of S1|F and S2|F at q is four. Hence S1|F and S2|F have other
intersection points, and we see that •¬ is not composite with a pencil even
in the case e=d by Lemma 2.6. Therefore, a general member of •¬ is
irreducible.

It is easily seen that •¬ consists of 3d-e pieces of (-1)-curves , where S":=(•¬)
Hence the image of •¬ in W is a finite set of nonsingular points of •¬

.

(iii) The case where (e=3d and •¬), or (•¬).

•¬ is divisible by •¬, i.e., the image of (•¬) in W contains C1. In this case, we have
to consider the complete linear system of •¬ . Any
•¬ can be written as

•¬

Let us look at •¬ , then base points do not
exist. If 4d-e=2, then there exists a unique isolated base point on C. If 4d-e=0,
then we have |C'|={C'}. In each case, we easily see that a general (•¬) is nonsingular
over the base points of •¬.

Let us look at •¬ . Since deg•¬

hold, we have Bs•¬.
In the same way as in the proof of (ii), we can show that •¬ is
not composite with a pencil

.

By what we have seen so far, a general member of •¬ is irreducible
and nonsingular. It is easy to see that the intersection of the member with •¬ is
irreducible, and hence its image by •¬ is nonsingular

.

(iv) The case where (e=4d and •¬ or (4d<e). (•¬) has •¬ as a
component, and the image of (•¬) in W contains C1 as a singular curve

.

3.2.2. Existence in the case where e is odd. Denote •¬. If we fix

any F2,•¬, then there exist L0•¸•¬ and •¬ with •¬
and •¬. Let •¬ (k=1,2,3) be the invertible sheaves with •¬
, and

•¬.

THEOREM 3.10. Let the conditions and notation be as above. Then the locally free sheaf E satisfies the condition (A) if and only if one of the following (1) and (2) holds:

(1) e=d>0 and det •¬ is isomorphic to one of •¬ and •¬k(k=1, 2, 3)

.

(2) d<e<4d.

We use the following result on symmetric products by Ashikaga [1] to prove this theorem. Since [1] is unpublished, we give the proof for the readers' convenience

.

ALGEBRAIC SURFACES OF GENERAL TYPE 277

LEMMA 3.11 (cf. [1]). If •¬k(k=1, 2, 3) are the three nontrivial line bundles satisfying

,•¬, and if F2,1 is an indecomposable locally free sheaf of rank 2 and degree 1 on
an elliptic curve C, then the following hold for any nonnegative integer m:

(1) •¬

(2) •¬

PROOF. First, we show the statement for S2 (F2,1). We have

•¬

by the Clebsch-Gordan formula [4, p. 438], and Atiyah's result [4, Theorem 14]. Hence we obtain an isomorphism

To complete the proof, it is sufficient to show the following (i), (ii) and (iii):

(i) •¬

(ii) (1) of the lemma is true under the assumption that the lemma is true for all the even integers less than or equal to 4m-2.
(iii) (2) of the lemma is true under the assumption that the lemma is true for all the even integers less than or equal to 4m.
We show only (iii) here. (i) and (ii) can be shown in the same way.
We have

•¬

for m>0 by the Clebsch-Gordan formula. On the other hand, we have

by the induction assumption, and furthermore, we have

Hence we have

by the Krull-Schmidt theorem. q.e.d.

278 T. TAKAHASHI

PROOF OF THEOREM 3.10. Let •¬ and D be as in the previous section.
(i) The case where (e<d), or (e=d and det •¬ is isomorphic to
none of •¬c and •¬k(k=1, 2, 3)). Using Lemma 3.11, we can show that (•¬) has Y0 as
a component in the same way as in the proof of (i) of Theorem 3.9.
(ii) The case where (det •¬ is isomorphic to one of •¬c and •¬k
(k=1, 2, 3)), (d<e<3d), or (•¬).
We can show that E satisfies the condition (A) when •¬ holds in the same
way as in the proof of (ii) of Theorem 3.9. We only have to prove that a general
member of |4Y1-ƒÐ*ƒÊ*D| is nonsingular at the base points dominating the base points
of •¬ when e-d=0, 1.
For that purpose, we need to study the structure of Y more precisely. We can show
that dim •¬ by using Lemma 3.11, and that this linear
pencil has no base point. Let ƒÄ:Y•¨P1 be the corresponding fibration. The in
vertible sheaves •¬(k=1, 2, 3) satisfy •¬
, and •¬ by Lemma 3.11, hence ƒÄ has three multiple fibers •¬
(k=1, 2, 3) with •¬k satisfying •¬.
Next, we study the complete linear system of •¬. We
obtain •¬ by Lemma 3.11. Since

•¬

where ƒÑ is a cyclic permutation, each of these three complete linear systems consists
only of •¬.
If e-d=1, there exists a point p•¸C with det•¬, and we
have Bs•¬. Let pk•¸C be a point with •¬,
and denote ƒ¡k:=ƒÊ-1(pk)(k=1,2,3). Then we have •¬
(k=1, 2, 3). Since p, p1, p2, p3 are pairwise distinct, and since •¬1, •¬2 and •¬3 intersect
T at distinct points, we obtain Bs•¬.
If e-d=0, then det •¬ is isomorphic to one of •¬c and •¬
by assumption. If det •¬, then Bs•¬ holds. If
det•¬, we have •¬. We can easily show
that a general member of |4Y1-ƒÐ*ƒÊ*D| is nonsingular along •¬.
In the same way as in the proof of Theorem 3.9, we can show that |4Y1-ƒÐ*ƒÊ*D|
is not composite with a pencil in each case above, and hence a general member of
| 4Y1-ƒÐ*ƒÊ*D| is irreducible and nonsingular.
As in the case of (ii) with e even, we can show that •¬ is a disjoint union of
3d-e pieces of (-1)-curves.
(iii) The case where (e=3d>0 and •¬), or (3d<e<4d). Using
Lemma 3.11, we can show that a general member of |3Y1+ƒÐ*(C0-ƒÊ*D)| is irreducible
and nonsingular in the same way as in the proof of (iii) of Theorem 3.9.
(iv) The case where 4d<e. The images of all the members of |4Y1-ƒÐ*ƒÊ*D| in

ALGEBRAIC SURFACES OF GENERAL TYPE 279

W have non-isolated singularity for the same reason as in the case (iv) with e even. q.e.d.

REMARK. (1) We can prove Theorem 3.10 above by using an isogeny •¬

with deg•¬ of elliptic curves as in •˜3.3.4 .below where we use an isogeny of degree 3.
(2) The existence of the linear pencil ƒÄ:Y•¨P1 and the multiple fibers •¬,
(k=1, 2, 3) above was proved by Suwa [21, •˜4]. What we mentioned in the proof of
Theorem 3.10 is a re-interpretation of Suwa's result by means of Lemma 3.11 due to
Ashikaga.

3.2.3. The canonical mapping. In this section, we study the canonical mappings
of those surfaces whose existence was shown in •˜•˜3.9-3.10. Let E0 and L be as above
satisfying the conditions of Theorem 3.9 when e is even, and Theorem 3.10 when e is odd.

LEMMA 3.12. If ƒÊ:Y:=P(E0)•¨C is the ruled surface associated to •¬,
and C0 is a section of ƒÊ with •¬, then ƒ³|C0| is birational onto its image.

PROOF. Let ƒÂ•¸Div(C) be a divisor satisfying •¬, and C' a section of ƒÊ
with •¬, where L0 is an invertible sheaf with •¬.

Let q1, q2•¸Y•_C' be any pair of points contained in different fibers of ƒÊ, and ƒ¡1,
the fiber of ƒÊ containing q1. Since Bs|C0-ƒ¡1| consists of one point on C' by Lemma
2.5, there exists a member C'0 of |C0-ƒ¡1| with •¬. Then C'0+ƒ¡1, contains q1, but
not q2. Hence |C0| separates q1 and q2, and ƒ³|C0| co is birational onto its image. q.e.d.

PROPOSITION 3.13. Any surface whose existence is guaranteed by Theorems 3.9
and 3.10 and the condition (A) is canonical if •¬. If (e, d)•‚(4, 1), then the canonical
mapping is a morphism. If (e, d)=(4, 1), then |Ks| has a unique isolated base point, and
its canonical image is non-normal.

PROOF. We use the same notation as in •˜•˜3.2.1-3.2.2. We only have to prove
that ƒ³|T| is birational onto the image to show the birationality of the canonical mapping.
We can show that the restriction of ƒ³|T| to a general fiber F of it gives an iso
morphism of F onto its image as in the proof of Lemma 3.2.
If ƒ³|Y1| is birational onto its image, then ƒ³|T| is also birational onto its image.
Therefore, we consider ƒ³|Y1|.
Any •¬ can be written as

•¬

Hence it is easy to see that ƒ³|Y1| is birational onto its image if one of •¬, •¬
and (e, d)=(5, 2) holds. Similarly, in the cases (e, d)=(4, 2), (4, 1), ƒ³|Y1| is birational
by Lemma 3.12. If (e, d)=(3, 2), then we have Bs•¬. Since •¬ and the degree
of the image of X cannot be 1, ƒ³|Y1| i is a birational morphism.
If •¬ and •¬, then Bs•¬ by the equation above. The statement in the
case (e, d)=(4, 1) is proved in the same way as in the proof of Corollary 3.4. q.e.d.

280 T. TAKAHASHI

PROPOSITION 3.14. Let ƒÎ:W:=P(E)•¨C be the P2-bundle associated to a locally
free sheaf •¬, T a tautological divisor with
•¬, and •¬ the invertible sheaves satisfying •¬
. We have the following for a general S•¸|4T-ƒÎ*D|:
(1) deg•¬.
(ii) deg•¬ and •¬.
(iii) deg•¬.

PROOF. Let p•¸C be a point with •¬, and q•¸Y a point with Bs|C0|={q} .
(See Lemma 2.5.) Bs•¬ is proved in the same
way as in the proof of Corollary 3.13.
Let C' be a section of ƒÊ with •¬ a point with
•¬ the intersection point of ƒÊ-1(p0) with C', and p'•¸C a
point with •¬. We have already seen that Bs
•¬ holds in Theorem 3.9.
q coincides with q' if and only if •¬, hence •¬. p coincides with
•¬ if and only if •¬. This is equivalent to •¬.
Hence, if •¬ holds, the complete linear system of the canonical divisor of
a general member of |4Y1-ƒÐ*ƒÊ*D| has no base point, and we obtain deg ƒ³|K
s|=9.
if •¬ and •¬ holds, then we have q•‚q' and p•‚p'. Hence the canonical
system of a general member of |4T-ƒÎ*D| has one isolated base point. We have the
following elementary transformation (cf. [17]):

•¬

where ƒÎ':W'-C is the P2-bundle associated to a locally free sheaf •¬ of

rank 3 over C, •¬ is the blowing-up at the isolated base point of |4T-ƒÎ*D| , and •¬ is
the blowing-up along the line •¬ . Let T' be a tautological divisor with •¬
. The complete linear system |T| on W is mapped to the complete linear

system |T'| by this elementary transformation. Furthermore , if S is the proper transform
of a general member S by •¬, and if we denote •¬ , then we have S'•`4T' by the
assumption •¬ Since Bs•¬, the complete linear system of •¬
on S' has no base point. Since ƒ³|Ks| factors as

•¬

we have deg •¬
Finally, we consider the case (iii), i.e., the case •¬ . We see that
Bs|T|=Bs|4T-ƒÎ*D| holds, and it is a line contained in a fiber ƒÎ-1(p)•¼W .

We have the same elementary transformation as in the case (ii). (We use the same notation as above.) In this case, Bs|T'| consists of one point contained in

ALGEBRAIC SURFACES OF GENERAL TYPE 281

ƒÎ'-1(p)
, and the image S' in W' of the proper transform of a general member
S•¸|4T-ƒÎ*D| goes through this point. Let T'0 be the image of P(E/L)•¼W in W'.
Regarding C0, C' and ƒÊ-1(p) as divisors on P(E/L) or T'0 in view of •¬,
we have •¬. Since the restriction of S to P(E/L) is
linearly equivalent to 4C'+ƒÊ-1(p), the restriction of S' to T'0 is the sum of a divisor G
which is linearly equivalent to 4C'+ƒÊ-1(p) and 3ƒÊ-1(p). G goes through q=ƒÊ-1(p)•¿C',
and since S is generic, G is nonsingular at q. C0 also goes through q and is nonsingular
at q, and C0 and G have different tangents by Lemma 2.5.
Let v:W•¨W' be the blowing-up at q, let T and S be the proper transforms of
T' and S', respectively, and denote •¬. Since •¬, we can prove
•¬ by the above result. Although |T| has one isolated base point, S does
not go through the point. Hence we have

•¬

q.e.d.

We treat the case (e, d)=(1, 1) in Proposition 3.30 in the next section.
In the case (e, d)=(3, 1), we have the following:

LEMMA 3.15. Let ƒÎ:W:=P(E)•¨C be the P2-bundle associated to a locally free
sheaf •¬, and T a tautological divisor with •¬.
Then ƒ³|T| is a triple covering of W over P3.

PROOF. Let •¬ be the ruled surface associated to •¬,

and C0 a section of ƒÊ with •¬. Then the restriction mapping
•¬ is surjective for any fiber ƒ¡ of ƒÊ, and we have Bs•¬.

Consider the pull-back |Y1| of |T| to X. Since Bs•¬, we have •¬
, where p•¸C is the point with •¬. This curve is contracted
to a point q by •¬, and we have Bs|T|={q}.
Let ƒÎ:W':=P(E')•¨C be the P2-bundle associated to the locally free sheaf •¬
, and T' a tautological divisor with •¬. We obtain an elementary
transformation

•¬

as before, where ƒÓ is the blowing-up at q. The image under ƒÓ' of the proper trans
form of T by ƒÓ is linearly equivalent to T'. Clearly Bs•¬, and hence,
deg ƒ³|T|=degƒ³|T|=(T')3=3 holds.

282 T. TAKAHASHI

LEMMA 3.16. Let ƒÊ:Y•¨C be the ruled surface associated to a locally free sheaf
•¬(2, 3), and C0 a section of ƒÊ with •¬. Then •¬ is a birational
morphism onto its image for any divisor D•¸Div(C) of degree 4.

PROOF. It is known that Y is isomorphic to the symmetric product of C of degree
2 (cf. [7]). Let ƒÅ:C•~C•¨Y be the quotient morphism. C•~{p} and {p}•~C are
mapped by ƒÅ to the same curve Cp on Yfor any point p•¸C. Since (C•~{p}+{p}•~C)2=2
and deg ƒÅ=2, this curve Cp is a section of ƒÅ with self-intersection number 1.

| 4C0-ƒÊ*D| contains a member of the form •¬,(pi•¸C , i=1, 2, 3, 4). Since

•¬, there exist points pi•¸C such that
•¬ does not contain q for an point q•¸Y. Hence we have Bs•¬
Let q, q*•¸Y be distinct points which are not contained in the image under ƒÅ of
the diagonal of C•~C, and denote q=(p, p') for p, p'•¸C in view of the above. Then
Cp and Cp'. are two distinct sections of ƒÊ. Since CpCp'=1, at least one of C
p and Cp'
does not go through q*. We may assume that Cp does not go through q*.
•¬ contains a member of the form •¬, and there

exists points pi•¸C (i=1, 2, 3) such that •¬ , does not go through q'. Hence the
complete linear system •¬ separates q and q'. q.e.d.

Proposition 3.17. In the notation of Lemma 3.15, if D•¸Div(C) is a divisor with
•¬ det E, then a general member •¬ is a canonical surface.

PROOF. By the proof of Theorem 3.10, we know that Bs•¬ holds
when •¬det E2,1, and that Bs•¬ holds when •¬det F2.1.
Since deg ƒ³|T|=3 by Lemma 3.15, we have deg ƒ³|Ks|=1, 2 or 3.
First, we consider the case L•¬det F2,1.
Assume deg ƒ³|Ks|=2. Bs|T| consists of one point q•¸W, and S does not contain
q. Let ƒ³:W•¨W be the blowing-up at q and T•¼W the proper transform of T by ƒÓ,
and denote •¬. The proper transform S of S is linearly equivalent to
•¬ is as in the proof of Lemma 3.15, then we have •¬

. We may identify as S=S, and ƒ³|Ks|, is factored as

•¬

Since •¬, there exists a divisor •¬ with •¬

.

Since deg ƒ³|T'|=3 and deg ƒ³|Ks|=2 hold, Q is birationally equivalent to ƒ³|K
s|(S). On
the other hand, Q is birationally equivalent to a ruled surface over C. Thus S' is
birationally equivalent to a double covering of a ruled surface over C, which is absurd.
Assume deg ƒ³|Ks|=3. Let q0•¸S'•_C1 be a point such that •¬ consists
of three distinct points q0, q1, q2. Since the restriction of ƒ³|T'| to any fiber of ƒÎ is an
isomorphism onto its image, q0, q1, q2 are contained in distinct fibers of ƒÎ'. Let the
notation be as in Theorem 3.10. Since dim •¬ , we may choose
•¬ in such a way that Z0(q0)=0 and that the divisor (Z0)
is irreducible. Let •¬ be the global section defining the

ALGEBRAIC SURFACES OF GENERAL TYPE 283

proper transform S of S by ƒÏ. Since |T| does not separate q0, q1, and q2, we have
Z0(q1)=Z0(q2)=0. Hence we have •¬ where •¬4 is as in the previous

section. On the other hand, since •¬ , we have •¬
for a general S by Lemma 3.16, a contradiction.

Next, we consider the case •¬ F2.1
Since Bs•¬, if we let the notation to be as above, we have •¬

, and •¬, where •¬, and p'•¸C is the
point with •¬. Hence, the invertible sheaf •¬ cannot be the pull-back

of any invertible sheaf over P3, and we have deg ƒ³|K s|•‚3.
We can prove that ƒ³|Ks| does not give rise to a double covering onto its image as
in the case •¬. Therefore, S is canonical in this case, too. q.e.d.

In the case (e, d)=(2, 2), we have the following:

PROPOSITION 3.18. Let ƒÎ:W•¨C be the P2-bundle associated to E with •¬
and •¬, T a tautological divisor with •¬, and D•¸Div(C) the divisor
with •¬det E. Then the minimal resolution S of a general member S'•¸|4T-ƒÎ*D|

is canonical.

PROOF. Let the notation be as above, and y0•¸Y as in Lemma 2.5. We have

Bs|Y1|={q0}, where •¬. If we identify q0 with ƒÏ(q0) so that q0•¸W, then
we have Bs|T|={q0}. Again by Lemma 2.5, general members of |C0| have the same

tangent y0. Hence if we let ƒÄ1:W•¨W to be the blowing-up at q0, and T' the proper
transform of T, then the complete linear system |T'| has one base point q'0. If we let
ƒÄ2:W2•¨W1, to be the blowing-up at q'0, and T" the proper transform of T', then we
have Bs•¬,dim|T"|=dim|T|=3 and |T"|3=T3-2=2, and •¬ is
the double covering. Hence, the canonical mapping of S has degree one or two.
Since Bs|4T-ƒÎ*D|=C' and since S' has a rational double point of type A3 at
q0•¸C' by the proof of Theorem 3.9, we have •¬, where •¬

and S'1, is the proper transform of S' by ƒÄ1. S1 has a rational double point of type A1.
On the other hand, if we regard Y0 as a divisor of W, since the support of the intersec-
tion of S' with Y0 is C', this rational double point does not coincide with q'0. Hence
the proper transform S'2 of S'1 by ƒÄ2 satisfies •¬, where •¬2 is
the exceptional divisor of ƒÄ2, and •¬1 is the proper transform of •¬1 by ƒÄ2. Since •¬
, we see that S'2 cannot be the pull-back of any effective divisor of
P3 by •¬ Therefore, S is canonical. q.e.d.

3.3. E is indecomposable. Let E be an indecomposable locally free sheaf of rank
3 over an elliptic curve C. Denote d:=deg E. We prove the following theorem in
•˜•˜ 3.3.1-3.3.4. We consider the case d•¬0 (mod 3) and d•‚1, 2 in •˜3.3.1, the case d•ß0
(mod3) and d•‚3 in •˜3.3.2, the case d=3 in •˜3.3.3, and the case d=2 in •˜3.3.4. We
omit the case d=1 because it was investigated by Catanese and Ciliberto [7]. In •˜3.3.5,
we study the canonical mappings of the surfaces obtained in •˜•˜3.3.1-3.3.4. The results

284 T. TAKAHASHI

about the canonical mappings are stated in Propositions 3.28, 3.29 and 3.30.
We only have to consider the case d>0 by the remark immediately before •˜3.1.

THEOREM 3.19. Let ƒÎ:W:=P(E)•¨C be the P2-bundle associated to •¬(3, d),
and T a tautological divisor with •¬. Then the locally free sheaf E satisfies the
condition (A) if and only if d•¬1.

REMARK. In this case, |4T-ƒÎ*D| turns out not to have base points except in the
case d=3, where D•¸Div(C) satisfies •¬. Hence its general members are irreducible
and nonsingular by Bertini's theorem and Lemma 2.6. In particular, it suffices to show
the following lemma when d•¬4 (•˜•˜3.3.1-3.3.2).

LEMMA 3.20. Let the notation be as in Theorem 3.19. Then the restriction mapping
•¬ is surjective for any fiber F of ƒÎ when d•¬4.

3.3.1. The proof when deg E•¬4 is not divisible by 3. Suppose d:=deg E•¬0
(mod 3). By Theorem 2.4, if we choose and fix any isogeny ƒÓ:C•¨C of degree 3, there
exists an invertible sheaf L0 of degree d over C such that •¬. Furthermore, if we
denote •¬ and L:=T*iƒÐL0(i=1, 2) where TiƒÐ is the translation by
iƒÐ•¸G, then we have •¬.
Let ƒÎ:W:=P(E)•¨C and ƒÎ:W:=P(E)•¨C be the P2-bundles associated to E
and E, respectively. Let T and T be tautological divisors on W and W, respectively,
such that •¬. Consider the following diagram:

•¬

If we denote ƒ³:W•¨W, then T•`ƒ³*T.
Denote •¬, where F is any fiber of ƒÎ. It suffices to show

that •¬=0 holds.
The proof of the following is immediate:

LEMMA 3.21. If {•¬} is the kernel ker ƒÓ* of the isogeny •¬
corresponding to ƒÓ:C•¨C, then we have •¬

By Lemma 3.21, we get •¬. Since the action
of G on W is fixed point free, we have •¬, [11, p. 202,
Corollaire]). On the other hand, if we denote F0+F1+F2:=ƒ³*F, and •¬
(i=0, 1, 2), then we have •¬ and

•¬

ALGEBRAIC SURFACES OF GENERAL TYPE 285

Since d•¬4, this cohomology group vanishes , and hence H1(W,•¬)=0. Therefore,
Lemma 3.20 in the case d•¬0 (mod 3) is proved . q.e.d.

3.3.2. The proof when deg E•‚3 is divisible by 3. If we denote d0=d/3 , there
exists an invertible sheaf L of degree d0 such that •¬ .

Denote p:=ƒÎ(F) for any fiber F of ƒÎ. Since •¬ by

[4, Theorem 9] and [9, p. 156], we have an isomorphism

•¬

which vanishes if d0•¬2. q.e.d.

3.3.3. The proof when deg E=3 holds. There exists an invertible sheaf •¬

with •¬. Let p0•¸C be a point satisfying •¬ . Let ƒÎ:W•¨C
be the P2-bundle associated to E and T a tautological divisor with •¬ . Denote •¬

LEMMA 3.22. Let the notation be as above, T0 a relative hyperplane with T0•`T-F0
,

and C0-T0 the section of ƒÊ:=ƒÎ|T0:T0•¨C with •¬ . Then we have
Bs|4T-3F0|={q0} where q0:=C0•¿F0.

PROOF. In the same way as in the proof of •˜3.3.2, we can show that there is no

base point of |4T-3F0| on any fiber except F0. Furthermore , the base points of
| 4T-3F0| exist only on the line •¬ , since •¬.

Since •¬ (cf. [4, Theorem 9], [9, p. 156]), the restriction mapping

•¬

is surjective, where ƒ¡0:=F0•¿T0, and the statement follows from Lemma 2.5. q.e.d.

The restriction of a general member S of |4T-3F0| to T0 is nonsingular by Lemmas
2.5 and 3.22. Hence S is irreducible and nonsingular by Lemma 2.6. q.e.d.

3.3.4. The proof when degE=2 holds. Let the notation be an in •˜3.3.1, and

denote •¬.
G acts on •¬. Let •¬ be the
subspace which consists of all the members which are invariant under this action

.

LEMMA 3.23. In the above notation, we have

•¬

PROOF. Since we have •¬ by the base change theorem (cf., e.g.,
Mumford [18]), if we denote •¬ , then we obtain isomorphisms

•¬

286 T. TAKAHASHI

The eigenspace •¬ of T*ƒÐ for the eigenvalue 1 corresponds to •¬, and is the image of the injection •¬. q.e.d.

We describe the action of G on •¬ to study Bs •¬.
We choose and fix •¬(i=O, 1,2) so that X1=T*ƒÐX0
and •¬ hold. Then any •¬ can be written as

•¬

Since we have

•¬

we get •¬ if and only if •¬

.

Let •¬ be defined by •¬ for y•¸C, where Ty is the
translation by y on C. Then it is a group homomorphism by the theorem of square
(cf., e.g., [18]). Since •¬ and •¬, we have iso
morphism.

•¬ where ƒÑ is the cyclic permutation (123). Hence we have

•¬

Let •¬ be a basis, and denote •¬
. We can choose a basis of •¬ consisting of
the following for j=1, 2:

LEMMA 3.24. We can choose the basis {s1, s2} of H0(C, L0) so that sj(p)tj(p)uj(p)•‚0
holds for any p•¸C and for at least one of j=1, 2. Furthermore, we have sj(p)sj(p')sj(p")•‚0,
where p':=TƒÐ(p) and p":=T2ƒÐ(p).

PROOF. To avoid confusion in this proof, we denote by (q) the divisor on C
determined by q•¸C. There exist distinct points p1, •¬
by Abel's theorem. If we denote •¬, we have

ALGEBRAIC SURFACES OF GENERAL TYPE 287

•¬

Let s1, s2•¸H0(C, L0) be the global sections defining the divisors 2(p1), 2(p2)
respectively, and denote tj:=T*ƒÐSj, and uj:=T*2ƒÐSj(j=1, 2). Then one of s1 and s2
satisfies the condition of the lemma for any point. q.e.d.

We choose a basis •¬ as in Lemma 3.24, fix any point p•¸C, and
denote p':=TƒÐ(p) and p":=T2ƒÐ(p). We assume that j•¸{1, 2} satisfies s j(p)sj(p')sj(p")•‚0.
Let us restrict •¬, and study if they have common
solutions on it. Note •¬.

The following lemma is trivial.

LEMMA 3.25. If we fix j•¸{1, 2} satisfying sj(p)sj(p')sj(p")•‚0, then Xi=0, •¬
and •¬ do not have common solutions for any i=0, 1, 2.

In view of Lemma 3.25, we consider only the solutions satisfying X0X1X2•‚0
in the rest of our argument. Denote •¬.

LEMMA 3.26. If we fix j•¸{1,2} with sj(p)sj(p')sj(p")•‚0, then (p,(1:a:b)) is a
common solution of •¬ if any only if a, b are cube roots of 1, and
Sj(p)+atj(p)+buj(p)=0.

PROOF. Since we have •¬, we may exclude •¬
from our consideration. We have

•¬

If •¬ holds, since we have •¬ and since sj(p)•‚0 and
X0•‚0, we obtain •¬. Similarly, if we assume •¬, we have •¬. Hence,
the common solutions are of the form (p,(1:a:b)) where a and b are one of 1, ƒÖ, and

ƒÖ2, and ƒÖ is a cube root of 1. If (p, (1:1:1)) is a common solution, we obtain
sj(p)+tj(p)+uj(p)=0 by substituting (p, (1:1:1)) into •¬ (i=0, 1, 3, 4, 5). We can
obtain the same result in the other cases. q.e.d.

PROPOSITION 3.27. •¬ has no base point. Hence a general member of •¬ is irreducible
and nonsingular by Bertini's theorem and Lemma 2.6.

PROOF. Assume (p, (1:a:b))•¬. Let g:C•¨P1 be defined by the complete
linear system of L0. Denote •¬.
First assume that g(p)=g(p') holds. If s'•¸H0(C, L0) is a global section defining
the divisor p +p', then s'(p)+as'(p')+bs'(p")•‚0. This contradicts Lemma 3.26. We can
obtain the same results when g(p')=g(p") or g(p")=g(p) holds.
Next, assume that g(p), g(p') and g(p") are pairwise distinct. Let ƒÌ0, ƒÌ•‡ be global
sections of •¬ defining g(p), g(p'), respectively. Then any •¬ is written
as •¬, and clearly there exist A, B•¸C such that •¬
satisfies s(p)+as(p')+bx(p")•‚0. This contradicts Lemma 3.26. q.e.d.

288 T. TAKAHASHI

Let •¬ be irreducible and nonsingular. We have S:=ƒ³(S)•¸|4T-ƒÎ*D| , and

•¬ is the restriction of the action of G on W Since this action has no fixed
point, S is irreducible and nonsingular as well. q.e.d.

REMARK. Instead of our argument in •˜3.3.1, we can use the above argument also
in the case d•¬4 and d•¬0 (mod 3).

3.3.5. The canonical mapping. In this section, we study the canonical mappings
of those surfaces whose existence was shown in •˜•˜3.3.1-3.3.4.

PROPOSITION 3.28. Let ƒÎ:W:=P(E)•¨C be the P2-boundle associated to •¬
, T a tautological divisor with •¬, and D•¸Div(C) satisfy
•¬ det E. Then a genral member of |4T-ƒÎ*D| is canonical when •¬.

PROOF. Since dim •¬, for any fiber F of ƒÎ, the restriction
mapping •¬ is surjective when •¬.
We first show that ƒ³|T| is birational onto the image when •¬, hence a general
member of |4T-ƒÎ*D| is canonical. We can prove this in the same way as in the proof
of Lemma 3.12 when •¬ in view of what we just saw above, and when d=6 in view
of Lemma 3.22. If d=5, then since 5=T3=degƒ³|T| deg ƒ³|T|(W) and deg •¬,
we see that ƒ³|T| is birational onto the image.
In the case where d=4, since Bs•¬ , and since T3=4>0, we see that ƒ³|T|
gives a 4-fold covering of W onto P3. Hence, ƒ³|Ks| is a morphism, and the degree of
ƒ³|Ks| is 1, 2, 3 or 4 for general •¬.
Since •¬ holds, if deg ƒ³|Ks|=4, then S":=ƒ³|Ks|(S)•¼P3 is a cubic
surface. Hence, we have •¬ which is absurd since S•`4T-ƒÎ*D.
If deg ƒ³|Ks|=3, we have •¬ as above. Therefore, there exist fibers F1, F2,
F3, F4 of ƒÎ satisfying ƒ³|T|*S"=F1+F2+F3+F4. ƒ³|T| is a birational morphism of
F1•¾ F2•¾F3•¾F4 onto its image, since deg ƒ³|T|=4 and deg ƒ³|Ks|=deg •¬. This
means that the image is not irreducible, a contradiction.
Finally, we show that the case deg ƒ³|Ks|=2 does not occur. Let p, p'•¸C be two
distinct general points. Furthermore, denote Fp:=ƒÎ-1(p) and Fp':=ƒÎ-1(p'), and let Tp
and Tp' be the relative hyperplanes of W satisfying T•`Tp+Fp•`Tp'+Fp'. Since p, p'
and S are generic, S•¿Tp•¿Fp'. S•¿Tp'•¿FP and S•¿Tp•¿Tp' all consist of four distinct
points set-theoretically. Since any fiber of it is mapped onto its image in P3 by ƒ³|T|, if
deg ƒ³|Ks|=2, then some point of S•¿TP•¿Fp', and some point of S•¿Tp'•¿Fp are mapped
to the same point by ƒ³|T|. Hence if we fix any point q•¸S•¿Tp•¿Fp' and any point
•¬, we only have to find a member of |T| containing q but not q'.
It is well-known that W is isomorphic to the symmetric product of C of degree 3
(cf. e.g., [7]). We can show that the image of (C•~C•~{p})•¾(C•~{p}•~C)•¾({p}•~C•~
C)•¼C•~C•~C in W is a relative hyperplane with self-intersection number one in the
same way as in the proof of Lemma 3.16. Therefore, for a general point of W, there

289 ALGEBRAIC SURFACES OF GENERAL TYPE

exist three distinct relative hyperplanes with self-intersection number one containing
the point.
Since p, p' and S are general, there exist two distinct relative hyperplanes T'p and
T"p distinct from Tp and containing q. If F'p and F"p are fibers of ƒÎ satisfying

•¬ , respectively, then one of T'p+F'p and T"p+F"p does not contain q'.
Hence ƒ³|Ks| is a birational morphism onto its image. q.e.d.

Next, we investigate the canonical mapping in the case pg(S)=d=3. We use the
notation of •˜3.3.3.

PROPOSITION 3.29. Let the notation be as in •˜3.3.3. Then the canonical mapping of
a nonsingular member S•¸|4T-3F0| has degree 8.

PROOF. In the same way as in the proof of Lemma 3.22, we can show that
Bs•¬. Hence, the canonical system of S has one base point. If
v:W•¨W is the blowing-up at q0, the complete linear system of the proper transform
T of T by v has one base point by Lemma 2.5. On the other hand, the proper transform
S of S by v does not go through the base point of |T| by Lemma 2.5. Hence, if we
denote •¬, we have deg•¬ q.e.d.

Finally, we study the canonical mapping in the case pg(S)=2. In •˜3.2.2, we proved
the existence of a surface S with •¬, but did not study
the canonical mapping ƒ³|Ks| in the case •¬. On the
other hand, we showed the existence of a surface S with the same invariants in the case •¬
. We obtain the following result in these two cases:

PROPOSITION 3.30. Let E be one of the following:

(i) •¬
(ii) •¬
In the same notation as in Proposition 3.28, the canonical mapping of the minimal
resolution of a general member •¬ gives a linear pencil whose general fibers
are irreducible nonsingular curves of genus 7.

PROOF. Since H0(S, ƒÖs) is 2-dimensional, |Ks| is a linear pencil. Furthermore, we
have •¬ for a general member of Z of |Ks|. q.e.d.

REFERENCES

[1] T. ASHIKAGA, A remark on lower semi-continuity of Kodaira dimension, Master's thesis, Tohoku
Univ., 1978 (in Japanese).
[2] T. ASHIKAGA, A remark on the geography of surfaces with birational canonical morphism, Math.
Ann. 290 (1991), 63-76.
[3] T. ASHIKAGA AND K. KONNO, Algebraic surfaces of general type with •¬, Tohoku Math. J.
42 (1990), 517-536.
[4] M. F. ATIYAH, Vector bundles over an elliptic curve, Proc. London Math. Soc. (3) 7 (1957), 414-452.

290 T. TAKAHASHI

[5] A. BEAUVILLE, L'application canonique pour les surfaces de type general, Invent. Math. 55 (1979),
121-140.

[6] G. CASTELNUOVO, Osservazioni intorno alla geometria sopra una superficie, Nota II. Rendiconti del
R. Instituto Lombardo, s. II, vol. 24, 1891.
[7] F. CATANESE AND C. CILIBERTO, Symmetric products of elliptic curves and surfaces of general type
with pg=q=1, J. Algebraic Geometry 2 (1993), 389-411.
[8] T. FUJITA, On Kahler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), 779-794.
[9] W. FULTON AND J. HARRIS, Representation Theory, Springer-Verlag, New York-Heidelberg-London-
Paris-Tokyo-Hong Kong-Barcelona-Budapest, 1991.
[10] P. GRIFFITHS AND J. HARRIS, Principles of Algebraic Geometry, John Willey and Sons, 1978.
[11] A. GROTHENDIECK, Sur quelques points d'algebre homologique, Tohoku Math. J. 9 (1957), 119-221.
[12] E. HORIKAWA, Algebraic surfaces of general type with small •¬, II, Invent. Math. 37 (1976), 121-155.
[13] E. HORIKAWA, Notes on canonical surfaces, Tohoku Math. J. 43 (1991), 141-148.
[14] E. HORIKAWA, Certain degenerate fibers in pencils of non-hyperelliptic curves of genus three, preprint.
[15] K. KONNO, Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces, Ann.
Scuola Norm. Sup. Pisa Sci. Fis. Mat. (4) 20 (1993), 575-595.
[16] K. KONNO, A note on surfaces with pencils of non-hyperelliptic curves of genus 3, Osaka J. Math. 28
(1991), 737-745.
[17] M. MARUYAMA, On a family of algebraic vector bundles, Number Theory, Algebraic Geometry and
Commutative Algebra in honor of Y. Akizuki (Y. Kusunoki et al., eds.), Kinokuniya, Tokyo, (1973),
95-146.
[18] D. MUMFORD, Abelian Varieties, Tata Inst. Studies in Math., Oxford Univ. Press, 1970.
[19] T. ODA, Vector bundles on an elliptic curve, Nagoya Math. J. 43 (1971), 41-72.
[20] M. REID, Problems on pencils of small genus, preprint.
[21] T. SUWA, On ruled surfaces of genus 1, J. Math. Soc. Japan 21 (1969), 291-311.
[22] T. TAKAHASHI, Certain algebraic surfaces of general type with irregularity one and their canonical
mappings, Tohoku Mathematical Publications 2 (1996), 1-60.

FACULTY OF GENERAL EDUCATION
ICHINOSEKI NATIONAL COLLEGE OF TECHNOLOGY
ICHINOSEKI 021-0902
JAPAN

E-mail address: tomokuni@ichinoseki.ac.jp