COLLOQUIUM MATHEMATICUM

VOL. LXIII 1992 FASC. 1

CERTAIN CURVATURE CHARACTERIZATIONS OF AFFINE HYPERSURFACES

BY

RYSZARD DESZCZ (WROCŁAW)

DEDICATED TO THE MEMORY OF MY FRIEND DR. WIESŁAW GRYCAK

1. Introduction. Let A^{n+1} be an $(n+1)$-dimensional, $n \geq 2$, affine space considered as a homogeneous space under the action of the unimodular affine group $\mathrm{ASL}(n+1,\mathbb{R})$. We denote by $(\widetilde{\nabla}, \widetilde{\Theta})$ the natural equiaffine structure on A^{n+1}, i.e. the standard torsion-free connection $\widetilde{\nabla}$ and the volume element $\widetilde{\Theta}$ given by the determinant which is parallel with respect to this connection.

Suppose that M is a non-degenerate hypersurface in A^{n+1} with the affine normal ξ and with induced equiaffine structure (∇, Θ) (we refer to [23] and [31] for the construction of ξ and (∇, Θ)). Thus we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi\,, \tag{1.1}$$

$$\widetilde{\nabla}_X \xi = -SX \tag{1.2}$$

for all vector fields X and Y tangent to M, where h is a non-degenerate symmetric bilinear form and S a (1, 1)-tensor field on M. The tensor field S is called the *affine shape operator*. The fundamental equations of M in A^{n+1} (i.e. the equations of Gauss, Codazzi and Ricci) are (see [23], [24], [22]):

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY\,, \tag{1.3}$$

$$C(X,Y,Z) = C(Y,X,Z)\,, \tag{1.4}$$

$$(\nabla_X S)Y = (\nabla_Y S)X\,, \tag{1.5}$$

$$h(X,SY) = h(SX,Y)\,, \tag{1.6}$$

where R is the curvature tensor of ∇ defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and the $(0,3)$-tensor C is given by

$$C(X,Y,Z) = (\nabla h)(Y,Z;X) = Xh(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\,,$$

X, Y, Z begin vector fields tangent to M. The tensor field C is called the *cubic form* of M.

A non-degenerate hypersurface M in A^{n+1} is said to be an *affine hypersphere* if the equality

$$S = \lambda I, \qquad \lambda \in \mathbb{R}\,,$$

holds on M, where I denotes the identity $(1,1)$-tensor field on M. If $\lambda \neq 0$ (resp. $\lambda = 0$), then a hypersurface M is called a *proper affine hypersphere* (resp. an *improper affine hypersphere*).

The basic definitions and formulas are given in Section 2. In Section 3 we obtain some results on non-degenerate hypersurfaces M in A^{n+1} satisfying certain curvature conditions imposed on its cubic form. In Section 4 we consider a curvature condition imposed on the tensor R of M. These subjects are a continuation of the investigations presented in [1] and [31], respectively. In these sections we also consider curvature conditions imposed on the generalized curvature tensor R^* defined in [27]. Moreover, in Section 4 we consider affine-quasi-umbilical hypersurfaces M in A^{n+1}. This class of hypersurfaces was introduced in [26]. We prove (see Theorem 4.8) that such hypersurfaces in A^{n+1}, $n \geq 4$, are characterized by the vanishing on M of the Weyl curvature tensor $W(R^*)$ corresponding to the tensor R^*.

This paper was prepared during my stay at the Katholieke Universiteit Leuven in 1989, sponsored by a Research Fellowship and by Research Project KUL OT/89/11 of that University.

I would like to express my hearty thanks to Professors K. Nomizu and L. Verstraelen for the invitation to the subject of affine differential geometry. Thanks are also due to Drs. B. Opozda, F. Dillen, Z. Olszak, P. Verheyen and L. Vrancken for the comments and discussions during the preparation of this paper.

2. Pseudosymmetry curvature conditions on affine hypersurfaces. Let M be a connected n-dimensional, $n \geq 2$, smooth Riemannian manifold with a not necessarily definite metric h. We denote by ∇, R, $\overline{R}$, $\mathrm{Ricc}(R)$ and $W(R)$ the Levi-Cività connection, the curvature tensor, the Riemann–Christoffel curvature tensor, the Ricci tensor and the Weyl conformal curvature tensor of (M,h), respectively. Denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M.

For a symmetric (0, 2)-tensor field D on M define the endomorphism $X \wedge_D Y$ of $\mathfrak{X}(M)$ by

$$(X \wedge_D Y)Z = D(Y,Z)X - D(X,Z)Y\,, \tag{2.1}$$

where $X, Y, Z \in \mathfrak{X}(M)$. The endomorphism $X \wedge_h Y$ will be denoted simply by $X \wedge Y$. Further, for a (1, 3)-tensor field B on M satisfying

$$B(X, Y, Z) = -B(Y, X, Z)\,, \tag{2.2}$$

we denote by $B(X, Y)$ the endomorphism of $\mathfrak{X}(M)$ defined by

$$B(X, Y)Z = B(X, Y, Z)\,.$$

We extend the endomorphisms $X \wedge_D Y$ and $B(X, Y)$ to derivations $(X \wedge_D Y)\cdot$ and $B(X, Y)\cdot$ of the algebra of tensor fields on M, assuming that they commute with contractions and

$$(X \wedge_D Y) \cdot f = 0\,, \qquad B(X, Y) \cdot f = 0\,,$$

for any function f on M. Now, for any (l, k)-tensor field T on M we define the $(l, k + 2)$-tensors $B \cdot T$ and $Q(D, T)$ by

$$\begin{aligned}
(B \cdot T)(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k; X, Y)&\\
&= (B(X, Y) \cdot T)(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k)\\
&= -\sum_{i=1}^{l} T(\omega^1, \ldots, B(X, Y) \cdot \omega^i, \ldots, \omega^l, X_1, \ldots, X_k)\\
&\quad -\sum_{j=1}^{k} T(\omega^1, \ldots, \omega^l, X_1, \ldots, B(X, Y) \cdot X_j, \ldots, X_k)\,,
\end{aligned}$$

$$\begin{aligned}
Q(D, T)(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k; X, Y)&\\
&= -((X \wedge_D Y) \cdot T)(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k)\\
&= \sum_{i=1}^{l} T(\omega^1, \ldots, (X \wedge_D Y) \cdot \omega^i, \ldots, \omega^l, X_1, \ldots, X_k)\\
&\quad +\sum_{j=1}^{k} T(\omega^1, \ldots, \omega^l, X_1, \ldots, (X \wedge_D Y) \cdot X_j, \ldots, X_k)
\end{aligned}$$

respectively, where $X, Y, X_1, \ldots, X_k \in \mathfrak{X}(M)$ and $\omega^1, \ldots, \omega^l$ are real-valued 1-forms on M.

Two (l, k)-tensor fields T_1 and T_2 on M are *pseudosymmetric related* with respect to a $(1, 3)$-tensor B satisfying (2.2) and a symmetric $(0, 2)$-tensor D if

$(*)$ $\quad B \cdot T_1$ and $Q(D, T_2)$ are linearly dependent at every point of M.

In the special case when $T_1 = T_2 = T$, we say that the tensor field T is *pseudosymmetric* with respect to B and D. A tensor field T on M will be called *semisymmetric* with respect to a $(1, 3)$-tensor B satisfying (2.2) if $B \cdot T$ vanishes on M.

A tensor field B of type $(1,3)$ on M is said to be a *generalized curvature tensor* [21] if

$$B(X_1,X_2,X_3)+B(X_2,X_3,X_1)+B(X_3,X_1,X_2)=0\,,$$

$B(X_1,X_2,X_3)=-B(X_2,X_1,X_3), \overline{B}(X_1,X_2,X_3,X_4)=\overline{B}(X_3,X_4,X_1,X_2)$, where $\overline{B}(X_1,X_2,X_3,X_4)=h(B(X_1,X_2,X_3),X_4)$ and $X_1,\ldots,X_4\in\mathfrak{X}(M)$. The *Ricci tensor* $\mathrm{Ricc}(B)$ of B is the trace of the linear mapping $X_1\longmapsto B(X_1,X_2,X_3)$. If $n=\dim M\geq 3$ then we can define the *Weyl curvature tensor* $W(B)$ of B by

$$W(B)(X_1,\ldots,X_4)=\overline{B}(X_1,\ldots,X_4)+\frac{K(B)}{(n-1)(n-2)}h((X_1\wedge X_2)X_3,X_4)$$
$$-\frac{1}{n-2}(h((X_1\wedge_{\mathrm{Ricc}(B)}X_2)X_3,X_4)-h((X_1\wedge_{\mathrm{Ricc}(B)}X_2)X_4,X_3))\,,$$

where $K(B)$ is the scalar curvature of B.

For a generalized curvature tensor B we define the tensor $Z(B)$ by

$$Z(B)(X_1,\ldots,X_4)=\overline{B}(X_1,\ldots,X_4)-\frac{K(B)}{n(n-1)}h((X_1\wedge X_2)X_3,X_4)\,.$$

The following proposition gives examples of pseudosymmetric related tensors.

PROPOSITION 2.1. *If B is a $(1,3)$-tensor field on a manifold (M,h) of the form*

$$B(X,Y,Z)=\overline{A}(Y,Z)AX-\overline{A}(X,Z)AY\,, \tag{2.3}$$

where A is a $(1,1)$-tensor field on M and $\overline{A}$ is the symmetric $(0,2)$-tensor field defined by $\overline{A}(X,Z)=h(X,AZ)$, then

$$B\cdot\overline{B}=Q(\mathrm{Ricc}(B),\overline{B}) \tag{2.4}$$

on M.

The above assertion is an immediate consequence of (2.3) and the definitions of $\overline{B}$, $B\cdot\overline{B}$ and $Q(\mathrm{Ricc}(B),\overline{B})$.

Manifolds satisfying (2.4) were considered in [14]. For instance, it was proved (see [14], Theorem 1) that if B is a generalized curvature tensor field on M, then (2.4) is satisfied at each point of M at which there exists a non-zero covector w satisfying

$$w(X_1)B(X_2,X_3,X_4)+w(X_2)B(X_3,X_1,X_4)+w(X_3)B(X_1,X_2,X_4)=0\,, \tag{2.5}$$

where $X_1,\ldots,X_4$ are tangent vectors at x. Examples of manifolds fulfilling (2.4) for $B=R$ or $B=W(R)$ are given in [14] and [2]. Furthermore, in [8] it was proved that any conformally flat manifold M of dimension ≥ 4

satisfying (2.4) with $B = R$ is pseudosymmetric. A Riemannian manifold M is said to be *pseudosymmetric* if the Riemann–Christoffel curvature tensor $\overline{R}$ is pseudosymmetric with respect to the curvature tensor R and the metric tensor h [12]. Recently, pseudosymmetric manifolds were studied in [3]–[5], [10]–[12], [15]–[16], and [18]–[20].

Ricci-pseudosymmetric manifolds and Weyl-pseudosymmetric manifolds can be defined in a similar manner. Such manifolds were investigated in [6], [7], [9], [13], [17] and [25]. Certain properties of pseudosymmetric (with respect to R and h) generalized curvature tensors were also obtained in [11], [12] and [16].

A $(0,k)$-tensor field T on M is said to be *totally symmetric* (cf. [1]) if

$$T(X_1,\dots,X_k) = T(X_{\sigma(1)},\dots,X_{\sigma(k)})$$

for any permutation σ of $\{1,\dots,k\}$ and $X_1,\dots,X_k \in \mathfrak{X}(M)$.

Let now M be a non-degenerate hypersurface of the affine space A^{n+1}. The following proposition states that on M there exist tensor fields which are pseudosymmetric related with respect to the tensors R and h.

Proposition 2.2. *On any non-degenerate hypersurface M in A^{n+1} we have*

$$R \cdot h = Q(h, \overline{S}), \tag{2.6}$$

$$R \cdot \overline{S} = Q(h, \overline{S}^2), \tag{2.7}$$

$$R \cdot C = Q(h, CS), \tag{2.8}$$

where CS is the $(0,3)$-tensor fields defined by

$$CS(X,Y,Z) = C(SX,Y,Z) \tag{2.9}$$

and $\overline{S}$ and $\overline{S}^2$ are the $(0,2)$-tensor fields defined by

$$\overline{S}(X,Y) = h(SX,Y) \quad \text{and} \quad \overline{S}^2(X,Y) = \overline{S}(SX,Y),$$

respectively.

The above proposition is an immediate consequence of the following proposition:

Proposition 2.3. *Let T be a totally symmetric $(0,k)$-tensor field, $k \geq 1$, on a non-degenerate hypersurface M in A^{n+1}. Then*

$$R \cdot T = Q(h, TS)$$

on M, where TS is the $(0,k)$-tensor field defined by

$$TS(X_1,\dots,X_k) = T(SX_1, X_2,\dots,X_k).$$

Proof. This is a consequence of the Gauss equation (1.3) and the definition of pseudosymmetric related tensors.

Moreover, it can be easily noted that the following proposition is also true.

PROPOSITION 2.4. *Any tensor field on an affine hypersphere M in A^{n+1} is pseudosymmetric with respect to R and h.*

We give another example of pseudosymmetric related tensors. On a non-degenerate hypersurface M in A^{n+1} consider the generalized curvature tensor R defined by ([27])

$$R^*(X,Y,Z) = R(X,Y)SZ = \overline{S}(Y,Z)SX - \overline{S}(X,Z)SY\,. \tag{2.10}$$

The equality (2.10) implies

$$\begin{aligned} \overline{R}^*(X_1,\ldots,X_4) &= h(R^*(X_1,X_2,X_3),X_4) \\ &= \overline{S}(X_1,X_4)\overline{S}(X_2,X_3) - \overline{S}(X_1,X_3)\overline{S}(X_2,X_4)\,. \end{aligned} \tag{2.11}$$

From Proposition 2.1 we easily obtain the following corollary.

COROLLARY 2.5. *Let M be a non-degenerate hypersurface in A^{n+1}. Then*

$$R^*\cdot\overline{R}^* = Q(\mathrm{Ricc}(R^*),\overline{R}^*)$$

on M.

A non-degenerate hypersurface M in A^{n+1} is said to be an *affine Einstein hypersurface* ([27]) if $\mathrm{Ricc}(R^*)$ is proportional to h. Thus Corollary 2.5 yields

COROLLARY 2.6. *The curvature tensor $\overline{R}^*$ of a non-degenerate affine Einstein hypersurface in A^{n+1} is pseudosymmetric with respect to R^* and h.*

To end this section, we prove some lemmas.

LEMMA 2.7 ([8], Theorem 3.5). *Let B be a generalized curvature tensor on a Riemannian manifold (M,h), $n\ge 4$, with a not necessarily definite metric h. Moreover, suppose the Weyl curvature tensor $W(B)$ vanishes on M. Then*

$$B\cdot\overline{B} = Q(\mathrm{Ricc}(B),\overline{B})$$

on M if and only if at each point of M the Ricci tensor $\mathrm{Ricc}(B)$ *has the form*

$$\mathrm{Ricc}(B) = \alpha h + \beta a\otimes a\,,\quad \alpha,\beta\in\mathbb{R}\,,$$

where a is a covector.

LEMMA 2.8. *Let A and B be two symmetric $(0,2)$-tensors at a point x of a Riemannian manifold (M,h) with a not necessarily definite metric h.*

(i) *If $Q(A,B)=0$ at x then A and B are linearly dependent.*
(ii) *If $A\circ B = B\circ A$ at x and*

$$\alpha Q(h,A) + \gamma Q(A,B) + \beta Q(h,B) = 0\,,\qquad \alpha,\beta,\gamma\in\mathbb{R}\,,\ \gamma\ne 0\,, \tag{2.12}$$

at x then the tensors

$$A - \frac{1}{n}\operatorname{tr}(A)h \quad \textit{and} \quad B - \frac{1}{n}\operatorname{tr}(B)h$$

are linearly dependent, where $A \circ B$ is the $(0,2)$-tensor with the local components $(A \circ B)_{rs} = h^{pq}A_{rp}B_{qs}$.

Proof. (i) The proof was given in [8] (see the proof of Lemma 3.4).

(ii) Contracting the equality

$$\alpha Q(h,A)_{rstu} + \gamma Q(A,B)_{rstu} + \beta Q(h,B)_{rstu} = 0$$

with h^{ru} we obtain

$$\alpha\Big(A - \frac{1}{n}\operatorname{tr}(A)h\Big) + \gamma\Big(\frac{1}{n}\operatorname{tr}(A)B - \frac{1}{n}\operatorname{tr}(B)A\Big) + \beta\Big(B - \frac{1}{n}\operatorname{tr}(B)h\Big) = 0\,,$$

which yields

$$\alpha Q(h,A) + \gamma Q\Big(\frac{1}{n}\operatorname{tr}(A)h, B\Big) - \gamma Q\Big(\frac{1}{n}\operatorname{tr}(B)h, A\Big) + \beta Q(h,B) = 0\,.$$

Next, subtracting the above equality from (2.12) we get

$$\gamma Q\Big(A - \frac{1}{n}\operatorname{tr}(A)h, B - \frac{1}{n}\operatorname{tr}(B)h\Big) = 0\,.$$

Now (i) completes the proof.

LEMMA 2.9. *Let (M,h), $\dim M \ge 4$, be a Riemannian manifold with a not necessarily definite metric h. If a symmetric $(0,2)$-tensor A satisfies at $x \in M$ the condition*

$$\begin{aligned} A((X_1 \wedge_A X_2)X_3, X_4) &= \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)h((X_1 \wedge X_2)X_3, X_4) \\ &\quad + \frac{1}{n-2}(h((X_1 \wedge_{a\otimes a} X_2)X_3, X_4) \\ &\quad - h((X_1 \wedge_{a\otimes a} X_2)X_4, X_3))\,, \quad \tau, \varrho \in \mathbb{R}\,, \end{aligned} \tag{2.13}$$

then $A = \alpha h + \beta b \otimes b$, $\alpha, \beta \in \mathbb{R}$, at x, where $\|a\|^2$ is the square of the norm of the covector a.

Proof. (2.13) can be written in the form

$$\begin{aligned} A_{ru}A_{st} - A_{rt}A_{su} &= \frac{1}{(n-1)(n-2)}((n-2)\tau - \|a\|^2\varrho)(h_{ru}h_{st} - h_{rt}h_{su}) \\ &\quad + \frac{1}{n-2}\varrho(h_{ru}a_s a_t + h_{st}a_r a_u - h_{rt}a_s a_u - h_{su}a_r a_t)\,. \end{aligned} \tag{2.14}$$

Contracting this with h^{ru} we obtain

$$\operatorname{tr}(A)A_{st} - A^2{}_{st} = \tau h_{st} + \varrho a_s a_t\,, \tag{2.15}$$

where A_{ru}, $A^2{}_{ru}$, h_{ru}, and a_r are the local components of the tensors A, $A^2 = A \circ A$, h and a, respectively.

Next, transvecting (2.15) with $A^r{}_p = A_{qp}h^{rq}$ we obtain

$$\begin{aligned} A^2{}_{ru}A_{st} - A^2{}_{rt}A_{su} &= \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)(A_{ru}h_{st} - A_{rt}h_{su}) \\ &\quad + \frac{1}{n-2}\varrho(A_{ru}a_s a_t - A_{rt}a_s a_u + h_{st}P_r a_u - h_{su}P_r a_t)\,, \end{aligned} \tag{2.16}$$

where $P_r = a_p A^p{}_r$. Further, contracting the above relation with h^{st}, we get

$$\operatorname{tr}(A)A^2{}_{ru} - A^3{}_{ru} = \tau A_{ru} + \varrho P_r a_u\,, \qquad A^3{}_{ru} = A^2{}_{pr}h^{pq}A_{qu}\,.$$

From this it follows immediately that

$$\varrho P_r = \varrho\lambda a_r\,, \qquad \lambda \in \mathbb{R}\,.$$

Using the above equality and (2.15) we can write (2.16) in the form

$$\begin{aligned} &\operatorname{tr}(A)(A_{ru}A_{st} - A_{rt}A_{su}) - \tau(h_{ru}A_{st} - h_{rt}A_{su}) - \beta(a_r a_u A_{st} - a_r a_t A_{su}) \\ &\quad = \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)(A_{ru}a_s a_t - A_{rt}a_s a_u) \\ &\qquad + \frac{1}{n-2}\varrho(a_s a_t A_{ru} - a_s a_u A_{rt} + \lambda(a_r a_u h_{st} - a_r a_t h_{su}))\,. \end{aligned}$$

This, by symmetrization in r and s, yields

$$\begin{aligned} \left(\frac{n-2}{n-1}\tau + \frac{1}{n-2}\varrho\|a\|^2\right)Q(h,A)_{rstu} + \frac{n-3}{n-2}\varrho Q(a\otimes a, A)_{rstu} \\ + \frac{1}{n-2}\lambda\varrho Q(a\otimes a, h)_{rstu} = 0\,. \end{aligned} \tag{2.17}$$

If $\varrho \neq 0$ then from Lemma 2.8(ii) it follows that the tensors $A - \frac{1}{n}\operatorname{tr}(A)h$ and $a\otimes a - \frac{1}{n}\|a\|^2 h$ are linearly dependent. So A has the required form. If $\varrho = 0$ then (2.17) yields $\tau Q(h,A) = 0$, whence we get $A - \frac{1}{n}\operatorname{tr}(A)h = 0$ or $\tau = 0$. In the second case, i.e. when $\tau = 0$, (2.14) implies that $\operatorname{rank}(A) = 1$. The last remark completes the proof.

3. Curvature conditions imposed on the cubic form. Let M be non-degenerate hypersurface in the affine space A^{n+1}, $n \geq 2$. Let A be the $(0,4)$-tensor field on M defined by

$$\begin{aligned} &A(X_1, X_2, X_3, X_4) \\ &\quad = (\nabla C)(X_1, X_2, X_3; X_4) - (\nabla C)(X_1, X_2, X_4; X_3)\,, \end{aligned} \tag{3.1}$$

where $X_1, \ldots, X_4$ are vector fields tangent to M. Note that in virtue of (1.3), A satisfies the condition

$$A = R \cdot h\,. \tag{3.2}$$

From this and (2.6) it follows that A vanishes if and only if the tensor $\overline{S}$ is proportional to h (i.e. ∇C is totally symmetric). Non-degenerate hypersurface in A^{n+1} with ∇C and $\nabla^2 C$ totally symmetric were considered in [1]. ∇C and $\nabla^2 C$ are both totally symmetric if and only if $C = 0$ or $S = 0$ ([1], Theorem 1). In the special case when $\nabla C = 0$ and $C \neq 0$, then $S = 0$, ∇ is flat, the Pick invariant of M vanishes and h is a hyperbolic metric with zero Ricci tensor ([1], Corollary). Of course, if $\nabla^2 C$ is totally symmetric, then $R \cdot C = 0$ (i.e. C is semisymmetric with respect to R). As a generalization of the semisymmetry of C with respect to R, we can consider the pseudosymmerty of C with respect to R and h. Note that $Q(h, C)$ vanishes at $x \in M$ if and only if C vanishes at x. Thus

$$R \cdot C = L_C Q(h, C) \tag{3.3}$$

on the set U_C of all points of M at which C is non-zero, where L_C is a function defined on U_C.

PROPOSITION 3.1. *Let M be a non-degenerate hypersurface in A^{n+1}. Then C satisfies* (3.3) *on U_C if and only if*

$$C((S - L_C I)X, Y, Z) = 0 \tag{3.4}$$

on U_C, where X, Y, Z are vector fields tangent to U_C.

Proof. If (3.4) holds on U_C then (3.3) is also satisfied. This is an immediate consequence of (2.8). Assume now that (3.3) holds on U_C. Let $U \subset U_C$ be a coordinate neighbourhood. We can write (3.3) in the form

$$\begin{aligned} &- C_{pst} R^p{}_{rvw} - C_{rpt} R^p{}_{svw} - C_{rsp} R^p{}_{tvw} \\ &\quad = L_C (h_{rw} C_{vst} + h_{sw} C_{rvt} + h_{tw} C_{rsv} - h_{rv} C_{wst} - h_{sv} C_{rwt} - h_{tv} C_{rsw})\,, \end{aligned}$$

where $R^p{}_{rvw}$, C_{rst} and h_{rw} are the local components of R, C and h, respectively. Applying (1.3) to the above equality, we get

$$\begin{aligned} &h_{rw} V_v{}^p C_{pst} + h_{sw} V_v{}^p C_{prt} + h_{tw} V_v{}^p C_{prs} \\ &\qquad - h_{rv} V_w{}^p C_{pst} - h_{sv} V_w{}^p C_{prt} - h_{tv} V_w{}^p C_{prs} = 0\,, \end{aligned}$$

where $V_v{}^p = S_v{}^p - L_C \delta^p_v$. The above relation, by contraction with h^{rw}, yields

$$(n+1) V_v{}^p C_{pst} = h_{sv} V^{pq} C_{pqt} + h_{tv} V^{pq} C_{pqs}\,, \tag{3.5}$$

whence, by contraction with h^{st} and making use of the apolarity condition $h^{pq} C_{pqs} = 0$, we obtain

$$V^{pq} C_{pqt} = 0\,, \tag{3.6}$$

where $V^{pq} = h^{ps}V_s{}^q$ and h^{ps} are the local components of h^{-1}. Substituting (3.6) into (3.5) we obtain (3.4) on U, which completes the proof.

Now we will consider non-degenerate hypersurfaces M in A^{n+1} which have a tensor field A pseudosymmetric with respect to R and h.

LEMMA 3.2. *Let M be a non-degenerate hypersurface in A^{n+1}. Then the tensor $Q(h, A)$ vanishes at $x \in M$ if and only if $\overline{S}$ is proportional to h at x.*

Proof. Of course, if $\overline{S}$ is proportional to h at x then both A and $Q(h, A)$ vanish at x. Assume now that $Q(h, A)$ vanishes at x. Let $U \subset M$ be a coordinate neighbourhood of x. Thus, at x,

$$\begin{aligned} h_{rw}A_{vstu} + h_{sw}A_{rvtu} + h_{tw}A_{rsvu} + h_{uw}A_{rstv} \\ -h_{rv}A_{wstu} - h_{sv}A_{rwtu} - h_{tv}A_{rswu} - h_{uv}A_{rstw} = 0\,. \end{aligned} \tag{3.7}$$

Contracting this with h^{tw} and h^{uv} we obtain

$$(n-2)A_{rsvu} = A_{vsur} + A_{vrus} - h_{sv}h^{pq}A_{pruq} - h_{rv}h^{pq}A_{psuq}\,, \tag{3.8}$$

$$h^{pq}A_{psrq} = -h^{pq}A_{prsq}\,. \tag{3.9}$$

Similarly, contracting (3.7) with h^{rw} and h^{uv} and using (3.9) we find

$$2h^{pq}A_{pstq} = h^{pq}A_{pqts}\,. \tag{3.10}$$

On the other hand, from (2.6) and (3.2) it follows that

$$h^{pq}A_{pqts} = 0\,. \tag{3.11}$$

Now (3.8), by (3.11) and (3.10), takes the form

$$(n-1)(h_{ru}\overline{S}_{vs} - h_{rv}\overline{S}_{us} + h_{us}\overline{S}_{rv} - h_{sv}\overline{S}_{ru}) = 2(h_{rs}\overline{S}_{uv} - h_{uv}\overline{S}_{rs})\,,$$

whence, by contraction with h^{rs}, we obtain our assertion.

From the last lemma it follows that if a tensor field A is pseudosymmetric with respect to R and h then

$$R \cdot A = L_A Q(h, A) \tag{3.12}$$

on the set U_A of all points of M at which A is non-zero, where L_A is a function defined on U_A.

THEOREM 3.3. *Let M be a non-degenerate hypersurface in A^{n+1} with a tensor field A pseudosymmetric with respect to R and h.*

(i) *If $\dim M \geq 3$ then M is an affine hypersphere.*
(ii) *If $\dim M = 2$ and h is positive definite then M is an affine sphere.*
(iii) *If $\dim M = 2$ and h is indefinite then M is an affine Einstein surface.*

Proof. Let U_A be the set defined above. We note that (3.12) can be written in the following form (on a coordinate neighbourhood $U \subset U_A$):

$$h_{rw}V_v{}^pA_{pstu} + h_{sw}V_v{}^pA_{prtu} - h_{rv}V_w{}^pA_{pstu} - h_{sv}V_w{}^pA_{prtu} \tag{3.13}$$

$$-h_{tw}V_v{}^pA_{rsup} + h_{uw}V_v{}^pA_{rstp} + h_{tv}V_w{}^pA_{rsup} - h_{uv}V_w{}^pA_{rstp} = 0\,,$$

where $V_w{}^p = S_w{}^p - L_A\delta^p_w$. Contracting (3.13) with h^{tw} we obtain

$$(n-2)V_v{}^pA_{rsup} = V_v{}^pA_{psru} + V_v{}^pA_{prsu} + h_{rw}E_{su} + h_{sv}E_{ru}\,, \tag{3.14}$$

where $E_{rs} = V^{pq}A_{prsq}$ and $V^{pq} = V_r{}^qh^{rp}$. Similarly, contracting (3.13) with h^{rw} and h^{uv}, we find

$$nV_v{}^pA_{pstu} = V_v{}^pA_{tsup} - V_v{}^pA_{ustp} + h_{sv}V^{pq}A_{pqtu} - h_{tv}E_{su} + h_{uv}E_{st}\,, \tag{3.15}$$

$$V^{pq}A_{pqts} = 2E_{st}\,. \tag{3.16}$$

On the other hand, from (2.6) and (3.2) it follows that

$$V^{pq}A_{pqts} = 0\,. \tag{3.17}$$

Applying this and (3.16) in (3.14) and (3.15) we find

$$(n-2)V_v{}^pA_{rsup} = V_v{}^pA_{psru} + V_v{}^pA_{prsu}\,, \tag{3.18}$$

$$nV_v{}^pA_{pstu} = V_v{}^pA_{tsup} - V_v{}^pA_{ustp}\,, \tag{3.19}$$

respectively. Moreover, using (2.6) and (3.2), we can express A_{pstu} in the form

$$A_{pstu} = h_{pu}V_{ts} + h_{su}V_{pt} - h_{pt}V_{us} - h_{st}V_{pu}\,, \tag{3.20}$$

where $V_{up} = h_{pq}V_u{}^q$. Now (3.18) and (3.19) take the forms

$$\begin{aligned}(n-1)(V_{vr}V_{us} + V_{vs}V_{ur} - h_{ur}V^2{}_{sv} - h_{us}V^2{}_{rv})&\\ = 2(V_{uv}V_{rs} - h_{rs}V^2{}_{ur})&\,,\end{aligned} \tag{3.21}$$

$$(n+1)(V_{uv}V_{ts} - V_{tv}V_{us} + h_{us}V^2{}_{tv} - h_{ts}V^2{}_{uv}) = 0\,, \tag{3.22}$$

respectively. From the last equality, by contraction with h^{us}, we obtain

$$V^2{}_{tv} = \frac{1}{n}\lambda V_{tv}\,, \qquad \lambda = h^{pq}V_{pq}\,. \tag{3.23}$$

Further, contracting (3.21) with h^{us}, we get

$$\lambda\left(V_{rs} - \frac{1}{n}\lambda h_{rs}\right) = 0\,. \tag{3.24}$$

From (3.20) and the definition of the set U_A it follows that $V_{rs} - \frac{1}{n}\lambda h_{rs}$ is non-zero at every point of U_A. Thus (3.24) yields $\lambda = 0$ and, by (3.23), we also have $V^2{}_{rs} = 0$. Now (3.21) and (3.22) turn into

$$(n-1)(V_{vr}V_{us} + V_{vs}V_{ur}) = 2V_{uv}V_{rs}\,,$$

$$V_{uv}V_{ts} = V_{us}V_{tv}\,, \tag{3.25}$$

respectively. But from the last two equalities and the assumption $n \geq 3$ it follows immediately that V_{us} must vanish on U_A, i.e. the set U_A is empty. This completes the proof of (i).

(ii) From (3.25) it follows that

$$V_{rs} = \lambda\psi_r\psi_s\,, \qquad \lambda \in \mathbb{R} - \{0\}\,,$$

at every $x \in U$. Together with (3.20), this yields

$$\begin{aligned}\psi^r A_{rstu} &= \lambda\psi^r\psi_r(\psi_t h_{us} - \psi_u h_{ts})\,,\\ \psi^r A_{ustr} &= \psi_u V_{ts} + \psi_s V_{tu} - \lambda\psi^r\psi_r(\psi_s h_{ut} + \psi_u h_{st})\,.\end{aligned}$$

Now (3.19) turns into

$$\lambda\psi^r\psi_r(\psi_t h_{us} - \psi_u h_{ts}) = 0\,.$$

Thus, we see that U_A must be empty, which completes the proof of (ii).

(iii) The relation (3.25) can be written in the form

$$\begin{aligned}\overline{S}_{ur}\overline{S}_{ts} - \overline{S}_{us}\overline{S}_{tr} &= L_A(h_{ur}\overline{S}_{ts} + h_{ts}\overline{S}_{ur} - h_{us}\overline{S}_{tr} - h_{tr}\overline{S}_{us})\\ &\quad - {L_A}^2(h_{ur}h_{ts} - h_{us}h_{tr})\,,\end{aligned}$$

which, by the identity (cf. [9], Lemma 2(iii))

$$h_{ur}\overline{S}_{ts} + h_{ts}\overline{S}_{ur} - h_{us}\overline{S}_{tr} - h_{tr}\overline{S}_{us} = \operatorname{tr}(S)(h_{ur}h_{ts} - h_{us}h_{tr})\,,$$

turns into

$$\overline{S}_{ur}\overline{S}_{ts} - \overline{S}_{us}\overline{S}_{tr} = L_A(\operatorname{tr}(S) - L_A)(h_{ur}h_{ts} - h_{us}h_{tr})\,.$$

Thus, the tensor $\operatorname{Ricc}(R^*)$ is proportional to h on U_A. But this completes the proof of (iii).

THEOREM 3.4. *Let M be an affine Einstein surface in the affine space A^3. If the set U_A is non-empty and if $(\operatorname{tr}(S))^2 = 2\operatorname{tr}(S^2)$ on U_A then A is pseudosymmetric with respect to R and h and L_A is defined by $2L_A = \operatorname{tr}(S)$.*

Proof. Since M is an Einstein affine surface we have, on a coordinate neighbourhood $U \subset U_A$,

$$\operatorname{tr}(S)\overline{S}_{ts} - \overline{S}^2{}_{ts} = \tfrac{1}{2}((\operatorname{tr}(S))^2 - \operatorname{tr}(S^2))h_{ts}\,.$$

We put $V_{rs} = \overline{S}_{rs} - L_A h_{rs}$, $L_A = \frac{1}{2}\operatorname{tr}(S)$. Now we can easily verify that

$$h^{pq}V_{pq} = 0\,,\ V^2{}_{ur} = 0\,,\ V_{ur}V_{ts} - V_{us}V_{tr} = 0\,,\ V_{vr}V_{us} + V_{vs}V_{ur} = 2V_{uv}V_{rs}\,.$$

Further, using the above equalities, we can express the tensor field $R \cdot A - L_A Q(h, A)$ in the form

$$(R \cdot A - L_A Q(h, A))_{rstuvw} = 2V_{rs}(h_{uw}V_{vt} + h_{tv}V_{uw} - h_{tw}V_{uv} - h_{uv}V_{tw})\,,$$

which implies $(R \cdot A - L_A Q(h, A))_{rstuvw} = 0$, completing the proof.

4. Affine hypersurfaces with pseudosymmetric curvature tensors. Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 3$. In [31]

(Proposition 2) it was proved that the curvature tensor R of M is semisymmetric (more precisely: $R \cdot R = 0$ on M) if and only if M is an affine hypersphere. This fact will be used in the proof of the following theorem.

THEOREM 4.1. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 3$. If the curvature tensor R is pseudosymmetric with respect to R and h, then M is an affine hypersphere.*

Proof. We remark that $Q(h,R)$ vanishes at $x \in M$ if and only if the shape operator S is proportional to the identity transformation at x. Denote by U_R the set of all points of M at which $Q(h,R)$ is non-zero. Thus, on U_R,

$$R \cdot R = L_R Q(h,R)\,, \tag{4.1}$$

where L_R is a function defined on U_R. Let $U \subset U_R$ be a coordinate neighbourhood. We write (4.1) in the form

$$\begin{aligned}
&V_{rv}(h_{st}V_{uw} - h_{us}V_{tw}) - V_{rw}(h_{st}V_{uv} - h_{us}V_{tv})\\
&+ V_{rt}(h_{sw}V_{uv} - h_{sv}V_{uw} + h_{uw}V_{sv} - h_{uv}V_{sw})\\
&- V_{ru}(h_{sw}V_{tv} - h_{sv}V_{tw} + h_{tw}V_{sv} - h_{tv}V_{sw})\\
&+ V^2{}_{rv}(h_{us}h_{tw} - h_{ts}h_{uw}) - V^2{}_{rw}(h_{us}h_{tv} - h_{ts}h_{uv})\\
&+ L_R h_{tr}(h_{sw}V_{uv} - h_{sv}V_{uw} + h_{uw}V_{sv} - h_{uv}V_{sw})\\
&- L_R h_{ur}(h_{sw}V_{tv} - h_{sv}V_{tw} + h_{tw}V_{sv} - h_{tv}V_{sw}) = 0\,,
\end{aligned}$$

where $V_{rt} = h_{rp}V_t{}^p$, $V_t{}^p = S_t{}^p - L_R\delta_t^p$ and $V^2{}_{rt} = h_{rq}V_r{}^pV_p{}^q$. This, by contractions with h^{st}, h^{uv} and h^{rw}, gives

$$\begin{aligned}
&(n-2)(V_{uw}V_{rv} - V_{uv}V_{rw} + h_{uv}V^2{}_{rw} - h_{uw}V^2{}_{rv})\\
&\qquad + L_R(h_{rw}V_{uv} - h_{rv}V_{uw} + h_{uw}V_{rv} - h_{uv}V_{rw}) = 0\,,
\end{aligned} \tag{4.2}$$

$$n(n-2)V^2{}_{rw} - (nL_R + (n-2)\operatorname{tr}(V))V_{rw} + L_R \operatorname{tr}(V)h_{rw} = 0\,, \tag{4.3}$$

$$n\operatorname{tr}(V^2) = (\operatorname{tr}(V))^2\,, \tag{4.4}$$

where $\operatorname{tr}(V) = h^{pq}V_{pq}$ and $\operatorname{tr}(V^2) = h^{pq}V^2{}_{pq}$. From (4.2), by antisymmetrization and symmetrization with respect to u and r, respectively, we obtain

$$V_{rw}V_{uv} - V_{rv}V_{uw} = h_{uv}V^2{}_{rw} - h_{uw}V^2{}_{rv} + h_{rw}V^2{}_{uv} - h_{rv}V^2{}_{uw}\,, \tag{4.5}$$

$$\begin{aligned}
&2L_R(h_{uw}V_{rv} - h_{uv}V_{rw} + h_{rw}V_{uv} - h_{rv}V_{uw})\\
&\qquad + (n-2)(h_{rv}V^2{}_{uw} - h_{rw}V^2{}_{uv} + h_{uv}V^2{}_{rw} - h_{uw}V^2{}_{rv}) = 0\,,
\end{aligned} \tag{4.6}$$

respectively. Contracting (4.5) and (4.6) with h^{rw} and h^{uw} we get

$$V^2{}_{st} = \frac{1}{n-1}(\operatorname{tr}(V)V_{st} - \operatorname{tr}(V^2)h_{st})\,, \tag{4.7}$$

$$V^2{}_{st} = \frac{2}{n-2} L_R V_{st} + \frac{1}{n}\left(\operatorname{tr}(V^2) - \frac{2}{n-2} L_R \operatorname{tr}(V)\right) h_{st}\,, \tag{4.8}$$

respectively. Next comparing the right sides of (4.3) and (4.7) and using (4.4) we obtain

$$\left(\frac{1}{n-2} L_R - \frac{1}{n(n-1)} \operatorname{tr}(V)\right)\left(V_{rs} - \frac{1}{n} \operatorname{tr}(V) h_{rs}\right) = 0\,.$$

Of course, the tensor field $V - \frac{1}{n}\operatorname{tr}(V)h$ is non-zero at every point of U_R. By the last equality, we have on U

$$\frac{1}{n-2} L_R = \frac{1}{n(n-1)} \operatorname{tr}(V)\,.$$

Now, applying this and (4.4) in (4.8), we obtain

$$V^2{}_{st} = \frac{1}{n(n-1)} \operatorname{tr}(V)\left(2V_{st} + \frac{n-3}{n} \operatorname{tr}(V) h_{st}\right),$$

which, together with (4.7), yields $\operatorname{tr}(V) = 0$ and, in consequence, $L_R = 0$. Thus, (4.1) turns into $R\cdot R = 0$. On the other hand, in view of Proposition 2 of [31], $R\cdot R = 0$ implies that S is proportional to the identity transformation. Thus we see that the set U_R must be empty. This completes the proof.

Let B be a generalized curvature tensor field on a non-degenerate hypersurface M in A^{n+1}, $n \geq 2$. The tensor $Q(h, \overline{B})$ vanishes at $x \in M$ if and only if $Z(\overline{B}) = 0$ at x (cf. [4], Lemma 1.1(iii)). Denote by $U_{\bar{B}}$ the set of all points of M at which $Z(\overline{B}) = 0$. If $\overline{B}$ is pseudosymmetric with respect to R and h then

$$R \cdot \overline{B} = L_{\bar{B}} Q(h, \overline{B}) \tag{4.9}$$

on $U_{\bar{B}}$, where $L_{\bar{B}}$ is a function defined on $U_{\bar{B}}$. We can easily prove the following property of generalized curvature tensors which are pseudosymmetric with respect to R and h.

LEMMA 4.2. *Let B be a generalized curvature tensor field on non-degenerate hypersurface M in A^{n+1}, $n \geq 2$. Then (4.9) holds on $U_{\bar{B}}$ if and only if*

$$\overline{B}((S - L_{\bar{B}} I)X_1, X_2, X_3, X_4) = \frac{1}{n-1} h((X_4 \wedge_D X_3)X_2, X_1) \tag{4.10}$$

on $U_{\bar{B}}$, where D is a $(0,2)$-tensor field on $U_{\bar{B}}$ defined by

$$D(X_1, X_2) = \sum_{i=1}^{n} \varepsilon_i \varepsilon_j (\overline{S} - L_{\bar{B}} h)(E_i, E_j) \overline{B}(E_i, X_1, X_2, E_j) \tag{4.11}$$

for any local orthonormal basis $\{E_1, \ldots, E_n\}$.

Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 2$. We define on M the following tensors:

$$V(X_1, X_2) = \overline{S}(X_1, X_2) - Lh(X_1, X_2)\,, \tag{4.12}$$

$$V^2(X_1, X_2) = \sum_{i=1}^{n} \varepsilon_i V(X_1, E_i) V(X_2, E_i)\,, \tag{4.13}$$

$$V^3(X_1, X_2) = \sum_{i=1}^{n} \varepsilon_i V^2(X_1, E_i) V(X_2, E_i)\,, \tag{4.14}$$

$$\begin{aligned} F(X_1, X_2, X_3, X_4) &= -\frac{1}{n-1} h((X_1 \wedge_E X_2)X_3, X_4) \\ &+ (V^2 + LV)((X_1 \wedge_V X_2)X_3, X_4) + L(V^2 + LV)((X_1 \wedge X_2)X_3, X_4)\,, \end{aligned} \tag{4.15}$$

for any orthonormal basis $\{E_1, \dots, E_n\}$, where L is a function on M and

$$E = -V^3 - 2LV^2 + (\operatorname{tr}(V^2) + L \operatorname{tr}(V) - L^2)V + L(\operatorname{tr}(V^2) + L \operatorname{tr}(V))h\,.$$

Now using (4.12) and (4.13) we can write the curvature tensor $\overline{R}^*$, the Ricci tensor $\operatorname{Ricc}(R^*)$ and the scalar curvature $K(R^*)$ in the following form:

$$\begin{aligned} &\overline{R}^*(X_1, X_2, X_3, X_4) \\ &\quad = V((X_1 \wedge_V X_2)X_3, X_4) + L^2 h((X_1 \wedge X_2)X_3, X_4) \\ &\quad\quad + L(h((X_1 \wedge_V X_2)X_3, X_4) - h((X_1 \wedge_V X_2)X_4, X_3))\,, \end{aligned} \tag{4.16}$$

$$\operatorname{Ricc}(R^*) = -V^2 + (\operatorname{tr}(V) + (n-2)L)V + L(\operatorname{tr}(V) + (n-1)L)h\,, \tag{4.17}$$

$$K(R^*) = -\operatorname{tr}(V^2) + (\operatorname{tr}(V))^2 + 2(n-1)L \operatorname{tr}(V) + n(n-1)L^2\,, \tag{4.18}$$

respectively. We note that the tensor D corresponding to the tensor $\overline{R}^*$ satisfies the equation

$$D = E\,. \tag{4.19}$$

If $n = \dim M \geq 3$ then we can define the Weyl curvature tensor $W(R^*)$ of R^* ([26]) by

$$\begin{aligned} &W(R^*)(X_1, X_2, X_3, X_4) \\ &\quad = \overline{R}^*(X_1, X_2, X_3, X_4) + \frac{1}{(n-1)(n-2)} K(R^*) h((X_1 \wedge X_2)X_3, X_4) \\ &\quad - \frac{1}{n-2}(h((X_1 \wedge_{\operatorname{Ricc}(R^*)} X_2)X_3, X_4) - h((X_1 \wedge_{\operatorname{Ricc}(R^*)} X_2)X_4, X_3))\,. \end{aligned}$$

This, by making use of (4.16)–(4.18), turns into

$$W(R^*)(X_1, X_2, X_3, X_4) = -\frac{1}{n-2} \operatorname{tr}(V)(h((X_1 \wedge_V X_2)X_3, X_4) \tag{4.20}$$

$$- h((X_1 \wedge_V X_2)X_4, X_3)) + V((X_1 \wedge_V X_2)X_3, X_4)$$
$$+ \frac{1}{(n-1)(n-2)}((\mathrm{tr}(V))^2 - \mathrm{tr}(V^2))h((X_1 \wedge X_2)X_3, X_4)$$
$$+ \frac{1}{n-2}(h((X_1 \wedge_{V^2} X_2)X_3, X_4) - h((X_1 \wedge_{V^2} X_2)X_4, X_3)) .$$

The Weyl curvature tensor corresponding to the curvature tensor R of a hypersurface in A^{n+1} was constructed in [30]. Of course, the two tensors are different in general.

We denote by $U_{\bar{R}^*}$ the set all points of M at which $Z(R^*) \neq 0$. If $\overline{R}^*$ is pseudosymmetric with respect to R and h then

$$R \cdot \overline{R}^* = L_{\bar{R}^*} Q(h, \overline{R}^*) \tag{4.21}$$

on $U_{\bar{R}^*}$, where $L_{\bar{R}^*}$ is a function on $U_{\bar{R}^*}$.

The following lemma is an immediate consequence of Lemma 4.2.

LEMMA 4.3. *Let the tensor field $\overline{R}^*$ of a non-degenerate hypersurface M in A^{n+1}, $n \geq 2$, be pseudosymmetric with respect to R and h. Then on $U_{\bar{R}^*}$ the equalities* (4.21) *and $F = 0$ (with $L = L_{\bar{R}^*}$) are equivalent.*

LEMMA 4.4. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 2$. If the shape operator S of M satisfies*

$$S^2 + \mu S = \varrho I , \qquad \mu, \varrho \in \mathbb{R} , \tag{4.22}$$

at $x \in M$, then $R \cdot \overline{R}^* = -\mu Q(h, \overline{R}^*)$ at x.

Proof. We note that (4.22) can be written at x in the form

$$V^2 - \frac{1}{n}\,\mathrm{tr}(V^2)h = \mu\left(V - \frac{1}{n}\,\mathrm{tr}(V)h\right) , \tag{4.23}$$

where V^2 and V (with $L = -\mu$) are defined by (4.13) and (4.12), respectively. Next, applying (4.23) and (4.14) in (4.15) we see that F vanishes at x. Now Lemma 4.3 completes the proof.

LEMMA 4.5. *Suppose the curvature tensor $\overline{R}^*$ of a non-degenerate hypersurface M in A^{n+1}, $n \geq 2$, satisfies*

$$R \cdot \overline{R}^* = -\mu Q(h, \overline{R}^*) , \qquad \mu \in \mathbb{R} ,$$

at $x \in M$. Then (4.22) *holds at x.*

Proof. From Lemma 4.3 it follows that $F(X_1, X_2, X_3, X_4) = 0$ at x (with $L = -\mu$). From this, by symmetrization with respect to X_3 and X_4, we obtain

$$\begin{aligned} &(n-1)(Q(V, V^2) + LQ(h, V^2) + L^2 Q(h, V)) \\ &\quad -Q(h, V^3) - 2LQ(h, V^2) + (\mathrm{tr}(V^2) + L\,\mathrm{tr}(V) - L^2)Q(h, V) = 0 , \end{aligned} \tag{4.24}$$

whence we get

$$V^3 = \frac{1}{n}\operatorname{tr}(V^3)h + (n-3)L\left(V^2 - \frac{1}{n}\operatorname{tr}(V)h\right) \tag{4.25}$$

$$+(\operatorname{tr}(V^2)+L\operatorname{tr}(V)+(n-2)L^2)\left(V-\frac{1}{n}\operatorname{tr}(V)h\right)+\frac{n-1}{n}(\operatorname{tr}(V)V^2-\operatorname{tr}(V^2)V)\,.$$

Substituting this in (4.24) we find

$$Q(V,V^2) = \frac{1}{n}\operatorname{tr}(V)Q(h,V^2) - \frac{1}{n}\operatorname{tr}(V^2)Q(h,V)\,,$$

whence, in view of Lemma 2.8(ii), it follows that the tensors $V^2 - \frac{1}{n}\operatorname{tr}(V^2)h$ and $V - \frac{1}{n}\operatorname{tr}(V)h$ are linearly dependent. Evidently, our assertion is true when $V - \frac{1}{n}\operatorname{tr}(V)h = 0$. Otherwise

$$V^2 - \frac{1}{n}\operatorname{tr}(V^2)h = \lambda\left(V - \frac{1}{n}\operatorname{tr}(V)h\right), \qquad \lambda \in \mathbb{R}\,,$$

at x. The last formula can be written in the form

$$S^2 + (2\mu - \lambda)S = \frac{1}{n}(\operatorname{tr}(S^2) - (\lambda - 2\mu)\operatorname{tr}(S))I\,.$$

From Lemma 4.4 it follows that $2\mu - \lambda = \mu$. Thus the above relation yields (4.22), which completes the proof.

Combining the last two lemmas we obtain the following

THEOREM 4.6. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 2$. Then on $U_{\bar{R}^*}$ the equations $R \cdot \overline{R}^* = L_{\bar{R}^*}Q(h,\overline{R}^*)$ and $S^2 - \frac{1}{n}\operatorname{tr}(S^2)I = L_{\bar{R}^*}(S - \frac{1}{n}\operatorname{tr}(S)I)$ are equivalent.*

Using this theorem we can obtain a curvature characterization of affine Einstein hypersurfaces.

COROLLARY 4.7. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 2$. Then M is an affine Einstein hypersurface if and only if $R \cdot \overline{R}^* = \operatorname{tr}(S)Q(h,\overline{R}^*)$ on $U_{\bar{R}^*}$.*

A non-degenerate hypersurface M in A^{n+1}, $n \geq 2$, is said to be *affine-quasi-umbilical* ([26]) if

$$\overline{S} = \alpha h + \beta a \otimes a\,, \qquad \alpha, \beta \in \mathbb{R}\,, \tag{4.26}$$

at every $x \in M$, where a is a covector at x.

THEOREM 4.8. *Let M be a non-degenerate hypersurface in A^{n+1}, $n \geq 4$. Then M is affine-quasi-umbilical if and only if the Weyl conformal curvature tensor $W(\overline{R}^*)$ vanishes on M.*

Proof. Assume that (4.26) is fulfilled at $x \in M$. We put $V = \overline{S} - Lh$, $L = \alpha$. Thus we have

$$V^2 = \operatorname{tr}(V)V \quad \text{and} \quad \operatorname{tr}(V^2) = (\operatorname{tr}(V))^2 .$$

Applying these formulas in (4.20) we easily obtain $W(R^*) = 0$.

Assume now that $W(R^*)$ vanishes at x. From Corollary 2.5 and Lemma 2.7 it follows that

$$\operatorname{Ricc}(R^*) = \tau h + \varrho a \otimes a , \qquad \tau, \varrho \in \mathbb{R} ,$$

at x, where a is a covector. Now, using the above formula and the definitions of $\overline{R}^*$ and $\operatorname{Ricc}(R^*)$, we can rewrite the equality $W(R^*) = 0$ in the form

$$\overline{S}((X_1 \wedge_{\bar{S}} X_2)X_3, X_4) = \frac{1}{(n-1)(n-2)}((n-2)\tau - \varrho\|a\|^2)h((X_1 \wedge X_2)X_3, X_4)$$
$$+ \frac{1}{n-2}\varrho(h((X_1 \wedge_{a\otimes a} X_2)X_3, X_4) - h((X_1 \wedge_{a\otimes a} X_2)X_4, X_3)) .$$

Now Lemma 2.9 completes the proof.

REFERENCES

[1] N. Bokan, K. Nomizu and U. Simon, *Affine hypersurfaces with parallel cubic forms*, Tôhoku Math. J. 42 (1990), 101–108.

[2] F. Defever and R. Deszcz, *On warped product manifolds satisfying a certain curvature condition*, Atti Acad. Peloritana Cl. Sci. Fis. Mat. Natur., in print.

[3] J. Deprez, R. Deszcz and L. Verstraelen, *Pseudosymmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kählerian manifolds*, Ann. Fac. Sci. Toulouse 9 (1988), 183–192.

[4] —,—,—, *Examples of pseudosymmetric conformally flat warped products*, Chinese J. Math. 17 (1989), 51–65.

[5] R. Deszcz, *Notes on totally umbilical submanifolds*, in: Geometry and Topology of Submanifolds, Proc. Luminy, May 1987, World Sci., Singapore 1989, 89–97.

[6] —, *On Ricci-pseudosymmetric warped products*, Demonstratio Math. 22 (1989), 1053–1065.

[7] —, *Examples of four dimensional Riemannian manifolds satisfying some pseudosymmetry curvature condition*, in: Differential Geometry and its Applications, II, Proc. Avignon, May/June 1988, World Sci., Singapore 1990, 134–143.

[8] —, *On conformally flat Riemannian manifolds satisfying certain curvature conditions*, Tensor (N.S.) 49 (1990), 134–145.

[9] —, *On four-dimensional Riemannian warped product manifolds satisfying certain pseudosymmetry curvature conditions*, Colloq. Math. 62 (1991), 103–120.

[10] —, *On pseudosymmetric warped product manifolds*, J. Geom., to appear.

[11] —, *On pseudosymmetric totally umbilical submanifolds of Riemannian manifolds admitting some types of generalized curvature tensors*, Zeszyty Nauk. Politech. Śląsk., in print.

[12] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 19 (1987), 271–282.

[13] R. Deszcz and W. Grycak, *On manifolds satisfying some curvature conditions*, Colloq. Math. 57 (1989), 89–92.
[14] —,—, *On certain curvature conditions on Riemannian manifolds*, ibid. 58 (1990), 259–268.
[15] R. Deszcz and M. Hotloś, *On geodesic mappings in pseudosymmetric manifolds*, Bull. Inst. Math. Acad. Sinica 16 (1988), 251–262.
[16] —,—, *Notes on pseudosymmetric manifolds admitting special geodesic mappings*, Soochow J. Math. 15 (1989), 19–27.
[17] —,—, *Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor*, Prace Nauk. Politech. Szczec. 11 (1988), 23–34.
[18] —,—, *On conformally related four-dimensional pseudosymmetric metrics*, Rend. Sem. Fac. Univ. Cagliari 59 (1989), 165–175.
[19] —,—, *On conformally related pseudosymmetric metrics*, ibid., to appear.
[20] R. Deszcz, L. Verstraelen and L. Vrancken, *On the symmetry of warped product spacetimes*, Gen. Rel. Gravit. 23 (1991), 671–681.
[21] K. Nomizu, *On the decomposition of generalized curvature tensor fields*, in: Differential Geometry in honor of K. Yano, Kinokuniya, Tokyo 1972, 335–345.
[22] —, *What is affine differential geometry*?, in: Proc. Differential Geom., Münster 1982, 42–43.
[23] —, *Introduction to Affine Differential Geometry*, Part I, Lecture Notes, MPI preprint 88-37.
[24] K. Nomizu and Ü. Pinkall, *On the geometry of affine immersions*, Math. Z. 195 (1987), 165–178.
[25] Z. Olszak, *Bochner flat Kählerian manifolds with certain condition on the Ricci tensor*, Simon Stevin 63 (1989), 295–303.
[26] B. Opozda, *New affine curvature tensor and its properties*, lecture given during the meeting "Current Topics in Affine Differential Geometry", Leuven 1989.
[27] B. Opozda and L. Verstraelen, *On a new curvature tensor in affine differential geometry*, in: Geometry and Topology of Submanifolds, II, Avignon, May/June 1988, World Sci., Singapore 1990, 271–293.
[28] U. Simon, *Hypersurfaces in equiaffine differential geometry*, Geom. Dedicata 17 (1984), 157–168.
[29] —, *The fundamental theorem in affine hypersurface theory*, ibid. 26 (1988), 125–137.
[30] P. Verheyen, *Hyperoppervlakken in een affiene ruimte*, Ph.D. thesis, Katholieke Universiteit Leuven, 1983.
[31] P. Verheyen and L. Verstraelen, *Locally symmetric affine hypersurfaces*, Proc. Amer. Math. Soc. 93 (1985), 101–105.

DEPARTMENT OF MATHEMATICS
AGRICULTURAL UNIVERSITY OF WROCŁAW
NORWIDA 25
50-375 WROCŁAW, POLAND

Reçu par la Rédaction le 13.6.1990