

## CERTAIN CURVATURE CONDITIONS ON AN LP-SASAKIAN MANIFOLD WITH A COEFFICIENT $\alpha$

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ABSTRACT. The object of the present paper is to study certain curvature restriction on an LP-Sasakian manifold with a coefficient  $\alpha$ . Among others it is shown that if an LP-Sasakian manifold with a coefficient  $\alpha$  is a manifold of constant curvature, then the manifold is the product manifold. Also it is proved that a 3-dimensional Ricci semisymmetric LP-Sasakian manifold with a constant coefficient  $\alpha$  is a spaceform.

### 1. Introduction

In 1989, Matsumoto [6] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [7] introduced the same notion independently and they obtained several results in this manifold. In a recent paper, De, Shaikh, and Sengupta [3] introduced the notion of LP-Sasakian manifolds with a coefficient  $\alpha$  which generalizes the notion of LP-Sasakian manifolds. Recently, T. Ikawa and his coauthors [4], [5] studied Sasakian manifolds with Lorentzian metric and obtained several results in this manifold. The object of the present paper is to study certain curvature restriction on an LP-Sasakian manifold with a coefficient  $\alpha$ . After preliminaries, in Section 3 it is shown that if an LP-Sasakian manifold  $M^n$  with a coefficient  $\alpha$  is of constant curvature, then the vector field  $\xi$  is a concircular vector field and as an important consequence of this theorem we prove that such a manifold is the product manifold. In the last section we study a 3-dimensional LP-Sasakian manifold with a constant coefficient  $\alpha$ .

### 2. Preliminaries

Let  $M^n$  be an  $n$ -dimensional differentiable manifold endowed with a  $(1, 1)$  tensor field  $\phi$ , a contravariant vector field  $\xi$ , a covariant vector field  $\eta$  and a Lorentzian metric  $g$  of type  $(0, 2)$  such that for each point  $p \in M$ , the tensor  $g_p : T_p M \times T_p M \rightarrow R$  is a non-degenerate inner product of signature

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$(-, +, +, \dots, +)$ , where  $T_pM$  denotes the tangent vector space of  $M$  at  $p$  and  $R$  is the real number space, which satisfies

$$(2.1) \quad \eta(\xi) = -1, \quad \phi^2 X = X + \eta(X)\xi,$$

$$(2.2) \quad g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vectors fields  $X$  and  $Y$ . Then such a structure  $(\phi, \xi, \eta, g)$  is termed as Lorentzian almost paracontact structure and the manifold  $M^n$  with the structure  $(\phi, \xi, \eta, g)$  is called Lorentzian almost paracontact manifold [6]. In a Lorentzian almost paracontact manifold  $M^n$ , the following relations hold good [6]:

$$(2.3) \quad \phi \xi = 0, \quad \eta(\phi X) = 0,$$

$$(2.4) \quad \Omega(X, Y) = \Omega(Y, X), \quad \text{where } \Omega = g(X, \phi Y).$$

In the Lorentzian almost paracontact manifold  $M^n$ , if the relations

$$(2.5) \quad (\nabla_Z \Omega)(X, Y) = \alpha[\{g(X, Z) + \eta(X)\eta(Z)\}\eta(Y) + \{g(Y, Z) + \eta(Y)\eta(Z)\}\eta(X)], \quad (\alpha \neq 0)$$

$$(2.6) \quad \Omega(X, Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y),$$

hold where  $\nabla$  denotes the operator of covariant differentiation with respect to the Lorentzian metric  $g$ , then  $M^n$  is called an LP-Sasakian manifold with a coefficient  $\alpha$  [3]. An LP-Sasakian manifold with a coefficient 1 is an LP-Sasakian manifold [6]. If a vector field  $V$  satisfies the equation of the following form:

$$\nabla_X V = \beta X + T(X)V,$$

where  $\beta$  is a non-zero scalar function and  $T$  is a non-zero 1-form, then  $V$  is called a torse-forming vector field [9]. In a Lorentzian manifold  $M^n$ , if we assume that  $\xi$  is a unit torse-forming vector field, then we have the equation:

$$(2.7) \quad (\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)],$$

where  $\alpha$  is a non-zero scalar function. Hence the manifold admitting a unit torse-forming vector field satisfying (2.7) is an LP-Sasakian manifold with a coefficient  $\alpha$ . Especially, if  $\eta$  satisfies

$$(2.8) \quad (\nabla_X \eta)(Y) = \epsilon[g(X, Y) + \eta(X)\eta(Y)], \quad \epsilon^2 = 1,$$

then  $M^n$  is called an LSP-Sasakian manifold [6]. In particular, if  $\alpha$  satisfies (2.7) and the equation of the form:

$$(2.9) \quad \alpha(X) = p\eta(X), \quad \alpha(X) = \nabla_X \alpha,$$

where  $p$  is a scalar function. Then  $\xi$  is called a concircular vector field. A Riemannian manifold satisfying the condition  $\nabla S = 0$ , where  $S$  denotes the Ricci tensor is called Ricci-symmetric. A Riemannian manifold satisfying the condition  $R(X, Y).S = 0$  is called Ricci-symmetric [8] where  $R(X, Y)$  denotes

the derivation of the tensor algebra at each point of the tangent space. Let us consider an LP-Sasakian manifold  $M^n (\phi, \xi, \eta, g)$  with a coefficient  $\alpha$ . Then we have the following relations [3]:

$$(2.10) \quad \eta(R(X, Y)Z) = -\alpha(X)\Omega(Y, Z) + \alpha(Y)\Omega(X, Z) + \alpha^2\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\},$$

$$(2.11) \quad S(X, \xi) = -\psi\alpha(X) + (n - 1)\alpha^2\eta(X) + \alpha(\phi X),$$

where  $R, S$  denote respectively the curvature tensor and the Ricci tensor of the manifold and  $\psi = \text{Trace}(\phi)$ . We state the following results which will be needed in latter sections.

**Lemma 2.1** ([3]). *In an LP-Sasakian manifold  $M^n$  with a non-constant coefficient  $\alpha$ , one of the following cases occurs:*

- (i)  $\psi^2 = (n - 1)^2$ .
- (ii)  $\alpha(Y) = -p\eta(Y)$ , where  $p = \alpha(\xi)$ .

**Lemma 2.2** ([3]). *In a Lorentzian almost paracontact manifold  $M^n$  with structure  $(\phi, \xi, \eta, g)$  satisfying  $\Omega(X, Y) = \frac{1}{\alpha}(\nabla_X\eta)(Y)$ , where  $\alpha$  is a non-zero scalar function, the vector field  $\xi$  is torse-forming if and only if the relation  $\psi^2 = (n - 1)^2$  holds good.*

### 3. LP-Sasakian manifolds with a coefficient $\alpha$ which is of constant curvature

We consider an LP-Sasakian manifold which is of constant curvature. Then we have

$$(3.1) \quad R(X, Y, Z, W) = \frac{r}{n(n - 1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

From (3.1) we have

$$(3.2) \quad S(Y, Z) = \frac{r}{n}g(Y, Z)$$

which implies that the manifold is Einstein and hence the scalar curvature  $r$  of the manifold is given by [3]

$$(3.3) \quad r = n\{p\psi + (n - 1)\alpha^2\}.$$

Putting  $Z = \xi$  in (3.2) we have by virtue of (2.11)

$$(3.4) \quad \alpha(\phi Y) = \psi\alpha(Y) + \left\{ \frac{r}{n} - (n - 1)\alpha^2 \right\} \eta(Y).$$

Again from (3.1) we have by virtue of (2.2)

$$(3.5) \quad \sum_{i=1}^n \epsilon_i R(e_i, Z, \phi Y, \phi e_i) = \frac{r}{n(n - 1)}[\psi\Omega(Y, Z) - g(Y, Z) - \eta(Y)\eta(Z)],$$

where  $\{e_i\}$  is an orthonormal basis of the tangent space at any point of the manifold and  $\epsilon_i = g(e_i, e_i)$ . Now in an LP-Sasakian manifold with a coefficient  $\alpha$  we have the following relation [3]:

$$(3.6) \quad \begin{aligned} S(Y, Z) - \sum_{i=1}^n \epsilon_i R(e_i, Z, \phi Y, \phi e_i) \\ = \{\psi\alpha(Z) - \alpha(\phi Z) - (2n-3)\alpha^2\eta(Z)\}\eta(Y) \\ - (n-2)\alpha^2 g(Y, Z) + (p + \psi\alpha^2)\Omega(Y, Z). \end{aligned}$$

Using (3.2), (3.3), (3.4) and (3.5) in (3.6) we obtain

$$(3.7) \quad \begin{aligned} \left\{2(n-1)\alpha^2 + \frac{np\psi}{n-1}\right\} g(Y, Z) - \left\{2\psi\alpha^2 + \left(1 + \frac{\psi^2}{n-1}\right)p\right\} \Omega(Y, Z) \\ + \left\{2(n-1)\alpha^2 + \frac{np\psi}{n-1}\right\} \eta(Y)\eta(Z) = 0. \end{aligned}$$

We consider the case when  $\alpha$  is not constant. In this case, taking a frame field and contracting over  $Y$  and  $Z$  we obtain from (3.7) that

$$[(n-1)^2 - \psi^2] \left\{2\alpha^2 + \frac{p\psi}{n-1}\right\} = 0.$$

From this equation we find either

$$(3.8) \quad \psi^2 = (n-1)^2,$$

or

$$(3.9) \quad p\psi = -2(n-1)\alpha^2.$$

If (3.9) holds, then from (3.3) we obtain

$$r = -n(n-1)\alpha^2,$$

from which we find that  $\alpha$  is constant, since  $r$  is constant, which contradicts our assumption that  $\alpha$  is non constant.

On the other hand, from (3.8) by virtue of Lemma 2.2 we conclude that  $\xi$  is torse-forming. We have that

$$(\nabla_X \eta)(Y) = \beta\{g(X, Y) + \eta(X)\eta(Y)\}.$$

Then from (2.6) we get

$$\begin{aligned} \Omega(X, Y) &= \frac{\beta}{\alpha}\{g(X, Y) + \eta(X)\eta(Y)\} \\ &= g\left(\frac{\beta}{\alpha}(X + \eta(X)\xi), Y\right) \end{aligned}$$

$$\text{and } \Omega(X, Y) = g(\phi X, Y).$$

Now from (3.4) and using  $\phi(X) = X + \eta(X)\xi$  we obtain

$$\alpha(Y + \eta(Y)\xi) = \psi\alpha(Y) + \left\{\frac{n(p\psi + (n-1)\alpha^2)}{n} - (n-1)\alpha^2\right\} \eta(Y)$$

$$\begin{aligned} \text{or, } \alpha(Y) + p\eta(Y) &= \psi\alpha(Y) + p\psi\eta(Y) \\ \text{or, } \alpha(Y) - \psi\alpha(Y) &= p\psi\eta(Y) - p\eta(Y) \\ \text{or, } (1 - \psi)\alpha(Y) &= p(-1 + \psi)\eta(Y) \\ \text{or, } \alpha(Y) &= p\left(\frac{-1 + \psi}{1 - \psi}\right)\eta(Y) = -p\eta(Y). \end{aligned}$$

In a similar way using  $\phi(X) = -X + \eta(X)\xi$  in (3.4) we obtain  $\alpha(X) = -p\eta(Y)$ . Since  $g$  is non-singular, we have

$$\phi(X) = \frac{\beta}{\alpha}(X + \eta(X)\xi)$$

and

$$\phi^2(X) = \left(\frac{\beta}{\alpha}\right)^2 (X + \eta(X)\xi).$$

It follows from (2.1) that  $(\frac{\beta}{\alpha})^2 = 1$  and hence,  $\alpha = \pm\beta$ . Thus we have

$$(3.10) \quad \phi(X) = \pm(X + \eta(X)\xi).$$

Thus in both the cases we obtain

$$\alpha(Y) = -p\eta(Y).$$

Hence we can state the following:

**Theorem 3.1.** *If an LP-Sasakian manifold  $M^n$  with a coefficient  $\alpha$  is a manifold of constant curvature, then the vector field  $\xi$  is a concircular vector field.*

Again since  $\xi$  is a concircular vector field, we have

$$(3.11) \quad \nabla_X \xi = \alpha[X + \eta(X)\xi],$$

where  $\alpha(Y) = p\eta(Y)$ , where  $p$  is a scalar function.

Let  $\xi^\perp$  denote the  $(n - 1)$ -dimensional distribution in an LP-Sasakian manifold with coefficient  $\alpha$  orthogonal to  $\xi$ . If  $X$  and  $Y$  belong to  $\xi^\perp$ , where  $Y \neq \lambda X$ , then

$$(3.12) \quad g(X, \xi) = 0$$

and

$$(3.13) \quad g(Y, \xi) = 0.$$

Since  $(\nabla_X g)(Y, \xi) = 0$ , it follows from (3.11) and (3.13) that

$$g(\nabla_X Y, \xi) = g(\nabla_X \xi, Y) = \alpha g(X, Y).$$

Similarly, we get

$$g(\nabla_Y X, \xi) = g(\nabla_Y \xi, X) = \alpha g(X, Y).$$

Hence

$$(3.14) \quad g(\nabla_X Y, \xi) = g(\nabla_Y X, \xi).$$

Now  $[X, Y] = \nabla_X Y - \nabla_Y X$ . Therefore

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi) = 0 \text{ by (3.14).}$$

Hence  $[X, Y]$  is orthogonal to  $\xi$ , i.e.,  $[X, Y]$  belong to  $\xi^\perp$ . Thus the distribution  $\xi^\perp$  is involutive [2]. Hence from Frobenius' theorem [2] it follows that  $\xi^\perp$  is integrable. This implies that if an LP-Sasakian manifold with a coefficient  $\alpha$  is a manifold of constant curvature, then it is a product manifold. We can therefore state the following theorem.

**Theorem 3.2.** *If an LP-Sasakian manifold with a coefficient  $\alpha$  is a manifold of constant curvature, then the manifold is the product manifold.*

#### 4. 3-dimensional LP-Sasakian manifold with a constant coefficient $\alpha$

Let us consider a 3-dimensional LP-Sasakian manifold with a constant coefficient  $\alpha$ . In a 3-dimensional Riemannian manifold we have

$$(4.1) \quad \begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\ &\quad - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where  $Q$  is the Ricci operator, i.e.,  $g(QX, Y) = S(X, Y)$  and  $r$  is the scalar curvature of the manifold.

Since  $\alpha$  is constant and dimension of the manifold is 3, equations (2.10) and (2.11) reduce to

$$(4.2) \quad \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

$$(4.3) \quad S(X, \xi) = 2\alpha^2\eta(X).$$

From (4.2) we get

$$(4.4) \quad R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y].$$

Putting  $Z = \xi$  in (4.1) and using (4.4) we have

$$(4.5) \quad \eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} - \alpha^2\right)[\eta(Y)X - \eta(X)Y].$$

Putting  $Y = \xi$  in (4.5) and using (2.1) and (4.3), we get

$$(4.6) \quad QX = \frac{1}{2}\{(r - 2\alpha^2)X + (r - 6\alpha^2)\eta(X)\xi\}$$

i.e.,

$$S(X, Y) = \frac{1}{2}\{(r - 2\alpha^2)g(X, Y) + (r - 6\alpha^2)\eta(X)\eta(Y)\}.$$

An LP-Sasakian manifold is said to be a space form if the manifold is a space of constant curvature. We assume that  $\psi = \text{trace of } \phi \neq 0$ , i.e.,  $\xi$  is not harmonic [1].

Using (4.6) in (4.1), we get

$$(4.7) \quad R(X, Y)Z = \frac{r - 4\alpha^2}{2}[g(Y, Z)X - g(X, Z)Y] + \frac{r - 6\alpha^2}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

Hence we can state the following:

**Theorem 4.1.** *A 3-dimensional LP-Sasakian manifold with a constant coefficient  $\alpha$  is a space form if and only if the scalar curvature  $r = 6\alpha^2$ .*

Next we consider a 3-dimensional LP-Sasakian manifold with constant coefficient  $\alpha$  which satisfies the condition

$$(4.8) \quad R(X, Y).S = 0.$$

From (4.8) we have

$$(4.9) \quad S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.$$

Again from (4.2) we get

$$(4.10) \quad R(X, \xi)Z = \alpha^2[\eta(Z)X - g(X, Z)\xi].$$

Putting  $Y = \xi$  in (4.9) and using (4.10) we get

$$(4.11) \quad \eta(U)S(X, V) - g(X, U)S(\xi, V) + \eta(V)S(U, X) - g(X, V)S(U, \xi) = 0.$$

Since  $\alpha^2 \neq 0$  using (4.3) in (4.11) we have

$$(4.12) \quad \eta(U)S(X, V) - 2\alpha^2g(X, U)\eta(V) + \eta(V)S(U, X) - 2\alpha^2g(X, V)\eta(V) = 0.$$

Taking a frame field and contracting over  $X$  and  $U$  from (4.12) we obtain

$$(4.13) \quad S(\xi, V) - 8\alpha^2\eta(V) + r\eta(V) = 0.$$

Using (4.3) in (4.13) we obtain

$$(r - 6\alpha^2)\eta(V) = 0.$$

This gives  $r = 6\alpha^2$  (since  $\eta(V) \neq 0$ ), which implies by Theorem 4.1 that the manifold is a space form.

Hence we can state the following:

**Theorem 4.2.** *A 3-dimensional Ricci semi-symmetric LP-Sasakian manifold with a constant coefficient  $\alpha$  is a space form.*

Since  $\nabla S = 0$  implies  $R(X, Y).S = 0$ , we get the following:

**Corollary.** *A 3-dimensional Ricci symmetric LP-Sasakian manifold with a constant coefficient  $\alpha$  is a space form.*

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