

CERTAIN CURVATURE CONDITIONS ON KENMOTSU MANIFOLDS ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

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ABSTRACT. We study certain curvature properties of Kenmotsu manifolds with respect to the quarter-symmetric metric connection. First we consider Ricci semisymmetric Kenmotsu manifolds with respect to a quarter-symmetric metric connection. Next, we study ξ -conformally flat and ξ -concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Moreover, we study Kenmotsu manifolds satisfying the condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$, where \tilde{Z} and \tilde{S} are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Then, we prove the non-existence of ξ -projectively flat and pseudo-Ricci symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection for illustration.

1. Introduction

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said to be a quarter symmetric connection [7] if the torsion tensor T of the connection $\tilde{\nabla}$

$$(1.1) \quad T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

satisfies

$$(1.2) \quad T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a $(1, 1)$ tensor field. A linear connection $\tilde{\nabla}$ is called a metric connection of M if

$$(1.3) \quad (\tilde{\nabla}_X g)(Y, Z) = 0,$$

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where $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M . A linear connection $\bar{\nabla}$ satisfying (1.2) and (1.3) is called a quarter-symmetric metric connection [7]. If we change ϕX by X , then the connection is called a semi-symmetric metric connection [30]. Thus the notion of quarter-symmetric connection generalizes the notion of the semi-symmetric connection. Semi-symmetric metric connections have been studied by several authors such as Özgür and Sular [18, 19], Ozen et al [20, 21], Prvanović [23], Smaranda and Andonie [26], Singh and Pandey [27] and many others.

A transformation in an n -dimensional Riemannian manifold M , which transforms every geodesic circle of M into a geodesic circle of M , is said to be a concircular transformation [31, 14]. A concircular transformation is always a conformal transformation [14]. Here, we mean a geodesic circle by a curve in M whose first curvature is constant and second curvature is identically zero. Thus, the geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism [2]. An important invariant of a concircular transformation is the concircular curvature tensor Z , defined by [31]

$$(1.4) \quad Z(X, Y)W = R(X, Y)W - \frac{r}{n(n-1)}[g(Y, W)X - g(X, W)Y],$$

for all $X, Y, W \in \chi(M)$, where R is the Riemannian curvature tensor and r is the scalar curvature with respect to the Levi-Civita connection. The importance of concircular transformation and concircular curvature tensor is very well known in the differential geometry of certain F -structure such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structure etc., [3, 34, 33].

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The Weyl conformal curvature tensor is defined by [33]

$$(1.5) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\},$$

where S is the Ricci tensor of type $(0, 2)$ and Q is the Ricci operator defined by

$$S(X, Y) = g(QX, Y).$$

Let M be an n -dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [28]

$$(1.6) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y],$$

for all $X, Y, Z \in \chi(M)$. In fact, M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

A non-flat n -dimensional Riemannian manifold (M, g) , $n > 3$ is called pseudo Riccissymmetric [5] with respect to the quarter-symmetric metric connection if there exists a non-zero 1-form α on M such that

$$(1.7) \quad (\tilde{\nabla}_X \tilde{S})(Y, U) = 2\alpha(X)\tilde{S}(Y, U) + \alpha(Y)\tilde{S}(X, U) + \alpha(U)\tilde{S}(Y, X),$$

where $X, Y, U \in \chi(M)$.

Quarter-symmetric metric connection in a Riemannian manifold have been studied by several authors such as Mandal and De [15], Rastogi [24, 25], Yano and Imai [32], Mukhopadhyay, Roy and Barua [16], Han et al [8, 9], Biswas and De [4] and many others. Recently, Sular, Özgür and De [29] studied quarter-symmetric metric connection in a Kenmotsu manifold.

Motivated by the above studies in the present paper, we study quarter-symmetric metric connection in a Kenmotsu manifold. The paper is organized as follows: In section 2, we give a brief account of Kenmotsu manifolds. In section 3, we give the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. Next, in section 4 we consider Ricci semisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection is an Einstein manifold with respect to the Levi-Civita connection. Section 5 is devoted to study ξ -conformally flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that a ξ -conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an η -Einstein manifold with respect to the Levi-Civita connection. Section 6 deals with ξ -concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that if a Kenmotsu manifold is ξ -concircularly flat then the scalar curvature $r = -n(n-1)$. Next, we study Kenmotsu manifolds satisfying the condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$, where \tilde{Z} and \tilde{S} are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection and prove that in this case the manifold is η -Einstein with respect to the Levi-Civita connection. In section 8, we consider ξ -projectively flat and pseudo Riccissymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. We obtain the non-existence of these type manifolds. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection to verify some theorems.

2. Kenmotsu Manifolds

Let M be an $n=(2m+1)$ -dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) , where ϕ is a (1,1) tensor field, ξ is the associated vector field, η is a 1-form and g is the Riemannian metric satisfying [1]

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.3) \quad g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X),$$

for any vector fields X, Y on M . If an almost contact metric manifold satisfies $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, where ∇ denotes the Levi-Civita connection of g , then M is called a Kenmotsu manifold [12]. From the above equations it follows

$$(2.4) \quad \nabla_X \xi = X - \eta(X)\xi,$$

$$(2.5) \quad (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y).$$

In addition to the above results in a Kenmotsu manifold the following conditions hold [12, 29, 11]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

$$S(X, \xi) = -(n-1)\eta(X), \quad Q\xi = -(n-1)\xi, \quad R(\xi, X)\xi = X - \eta(X)\xi,$$

$$(2.6) \quad S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y),$$

$$(2.7) \quad S(\phi X, Y) = -S(X, \phi Y),$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator. A Kenmotsu manifold is normal, that is, the Nijenhuis tensor of ϕ equals $-2d\eta \otimes \xi$ but not Sasakian. Moreover, Kenmotsu manifold is not compact since from the equation (2.4) we have $\operatorname{div} \xi = n-1$. In [12], Kenmotsu showed (1) that locally a Kenmotsu manifold is a warped product $I \times_f N$, where I is an interval, N is a Kähler manifold and f is a warping function defined by $f(t) = se^t$, s is a nonzero constant; (2) that a Kenmotsu manifold of constant ϕ -sectional curvature is a space of constant curvature -1 , hence it is hyperbolic space. Let M be a Kenmotsu manifold. M is said to be an η -Einstein manifold if there exists the real valued functions a and b such that $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$. For $b = 0$, the manifold M is an Einstein manifold. Now we state the following:

LEMMA 2.1. [12] *Let M be an η -Einstein Kenmotsu manifold of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$. Then $a + b = -(n-1)$. If $b = \text{constant}$ (or, $a = \text{constant}$), then M is an Einstein one.*

Kenmotsu manifolds have been studied by several authors such as Pitis [22], De and Pathak [6], Jun, De and Pathak [11], Özgür and De [17], Kirichenko [13], Hong et al [10] and many others.

3. Curvature Tensor

The quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on a Kenmotsu manifold are related by [29]

$$(3.1) \quad \tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y,$$

for all vector fields X, Y on M . Let \tilde{R} and R be the Riemannian curvature tensor with respect to the quarter-symmetric metric connection and Levi-Civita connection respectively of a Kenmotsu manifold. Then the relation between \tilde{R} and R is

given by [29]

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + \eta(X)g(\phi Y, Z)\xi - \eta(Y)g(\phi X, Z)\xi \\ &\quad - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \end{aligned}$$

Also in a Kenmotsu manifold with respect to the quarter-symmetric metric connection the following relations hold [29]

$$(3.2) \quad \tilde{R}(X, Y)\xi = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X,$$

$$(3.3) \quad \tilde{R}(X, \xi)Y = g(X, Y)\xi - \eta(Y)X - g(\phi X, Y)\xi + \eta(Y)\phi X,$$

$$(3.4) \quad \tilde{R}(\xi, X)\xi = X - \eta(X)\xi - \phi X,$$

$$(3.5) \quad \tilde{S}(X, Y) = S(X, Y) + g(\phi X, Y),$$

$$(3.6) \quad \tilde{S}(X, \xi) = S(X, \xi) = -(n - 1)\eta(X),$$

$$(3.7) \quad \tilde{r} = r,$$

where \tilde{S} and \tilde{r} are the Ricci tensor and the scalar curvature respectively with respect to the quarter-symmetric metric connection. Moreover, it is noted that the Ricci tensor \tilde{S} with respect to the quarter-symmetric metric connection is not symmetric. Using expressions (3.2) and (3.3), the following are easily obtained from (1.4)

$$(3.8) \quad \tilde{Z}(\xi, Y)U = \left[1 + \frac{\tilde{r}}{n(n - 1)}\right][\eta(U)Y - g(Y, U)\xi] + g(\phi Y, U)\xi - \eta(U)\phi Y$$

$$(3.9) \quad \tilde{Z}(\xi, Y)\xi = \left[1 + \frac{\tilde{r}}{n(n - 1)}\right][Y - \eta(Y)\xi] - \phi Y.$$

4. Ricci Semisymmetric Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

In this section we consider Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$.

DEFINITION 4.1. A manifold M is said to be Ricci semisymmetric manifold if it satisfies $R(X, Y) \cdot S = 0$, where R is (1, 3)-type curvature tensor and S is the (0, 2)-type Ricci tensor.

Let us consider an n -dimensional Kenmotsu manifold which is Ricci semisymmetric with respect to the quarter-symmetric metric connection, that is,

$$\tilde{R}(X, Y) \cdot \tilde{S} = 0,$$

where $\tilde{R}(X, Y)$ denotes the derivation of the tensor algebra at each point of the manifold. This implies

$$(4.1) \quad \tilde{S}(\tilde{R}(X, Y)U, V) + \tilde{S}(U, \tilde{R}(X, Y)V) = 0.$$

Substituting $X = \xi$ in (4.1) and using (3.5) we get

$$S(\tilde{R}(\xi, Y)U, V) + S(U, \tilde{R}(\xi, Y)V) - g(\tilde{R}(\xi, Y)U, \phi V) + g(\phi U, \tilde{R}(\xi, Y)V) = 0.$$

Putting $U = \xi$ in the above equation and in view of (3.3) and (3.4) yields

$$(4.2) \quad S(Y, V) - S(\phi Y, V) + (n - 2)g(\phi V, Y) + ng(Y, V) - \eta(Y)\eta(V) = 0.$$

Interchanging Y and V in the above equation we have

$$(4.3) \quad S(V, Y) - S(\phi V, Y) + (n-2)g(\phi Y, V) + ng(V, Y) - \eta(V)\eta(Y) = 0.$$

Adding (4.2) and (4.3) and using the facts (2.3) and (2.7) we obtain

$$(4.4) \quad S(Y, V) = -ng(Y, V) + \eta(Y)\eta(V).$$

Hence by Lemma 2.1, (4.4) implies that the manifold is an η -Einstein manifold. Now, in this position we can state the following:

THEOREM 4.1. *Let M be a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold is an η -Einstein manifold with respect to the Levi-Civita connection.*

Now, using Lemma 2.1 and Theorem 4.1 we can state the following:

THEOREM 4.2. *Let M be a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

Since Ricci symmetric manifold ($\tilde{\nabla}\tilde{S} = 0$) with respect to the quarter-symmetric metric connection implies $\tilde{R} \cdot \tilde{S} = 0$, therefore we obtain the following:

COROLLARY 4.1. *If a Kenmotsu manifold is Ricci symmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

5. ξ -conformally Flat Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

ξ -conformally flat K -contact manifolds have been studied by Zhen et al [35]. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p , the conformal curvature tensor C is a map

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \rightarrow \phi(T_p(M^n)) \oplus \{\xi_p\}.$$

DEFINITION 5.1. [35] An almost contact metric manifold M^n is called ξ -conform-ally flat if the projection of the image of C onto $\{\xi_p\}$ is zero, that is, $C(X, Y)\xi = 0$, where C is the conformal curvature tensor defined in (1.5).

This section deals with ξ -conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$, that is, $\tilde{C}(X, Y)\xi = 0$. From this and (1.5) we have

$$(5.1) \quad \tilde{R}(X, Y)\xi - \frac{1}{n-2}[\tilde{S}(Y, \xi)X - \tilde{S}(X, \xi)Y + \eta(Y)\tilde{Q}X - \eta(X)\tilde{Q}Y] \\ + \frac{\tilde{r}}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y] = 0.$$

With the help of (3.2), (3.5) and (3.6) we have from (5.1)

$$\eta(Y)QX - \eta(X)QY = \left[1 + \frac{\tilde{r}}{n-1}\right][\eta(Y)X - \eta(X)Y]$$

$$+ (n - 3)[\eta(Y)\phi X - \eta(X)\phi Y].$$

Substituting $Y = \xi$ in the above equation and using (3.7) we have

$$(5.2) \quad QX = \left(1 + \frac{r}{n-1}\right)X - \left(n + \frac{r}{n-1}\right)\eta(Y)\xi + (n-3)\phi X.$$

Taking the inner product of (5.2) with Y , we have

$$(5.3) \quad S(X, Y) = \left(1 + \frac{r}{n-1}\right)g(X, Y) - \left(n + \frac{r}{n-1}\right)\eta(X)\eta(Y) + (n-3)g(\phi X, Y),$$

where $S(X, Y) = g(QX, Y)$. Interchanging X and Y in the above equation we obtain

$$(5.4) \quad S(Y, X) = \left(1 + \frac{r}{n-1}\right)g(Y, X) - \left(n + \frac{r}{n-1}\right)\eta(Y)\eta(X) + (n-3)g(\phi Y, X).$$

Adding (5.3) and (5.4) and using fact (2.3) yields

$$(5.5) \quad S(X, Y) = \left(1 + \frac{r}{n-1}\right)g(X, Y) - \left(n + \frac{r}{n-1}\right)\eta(X)\eta(Y).$$

Therefore by Lemma 2.1, (5.5) shows that the manifold under consideration is an η -Einstein manifold. This leads to the following:

THEOREM 5.1. *A ξ -conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an η -Einstein manifold.*

Since the conformally flatness implies ξ -conformally flat, therefore from the above theorem we state the following:

COROLLARY 5.1. *A conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an η -Einstein manifold.*

6. ξ -concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogously to the definition of ξ -conformally flat almost contact metric manifold, we define ξ -concircularly flat Kenmotsu manifolds.

DEFINITION 6.1. A Kenmotsu manifold M is said to be ξ -concircularly flat with respect to the quarter-symmetric metric connection if it satisfies

$$(6.1) \quad \tilde{Z}(X, Y)\xi = 0,$$

for any vector fields $X, Y \in \chi(M)$ and \tilde{Z} is the concircular curvature tensor defined by (1.4) with respect to the quarter-symmetric metric connection.

In this section we study ξ -concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then from (1.4) and (6.1), we have

$$(6.2) \quad \tilde{R}(X, Y)\xi - \frac{\tilde{r}}{n(n-1)}[\eta(Y)X - \eta(X)Y] = 0.$$

Using (3.2) and (3.7) we obtain from (6.2)

$$(6.3) \quad \left[1 + \frac{r}{n(n-1)}\right][\eta(X)Y - \eta(Y)X] - \eta(X)\phi Y + \eta(Y)\phi X = 0.$$

Taking the inner product of (6.3) with U we obtain

$$(6.4) \quad \left[1 + \frac{r}{n(n-1)}\right] [\eta(X)g(Y, U) - \eta(Y)g(X, U)] \\ - \eta(X)g(\phi Y, U) + \eta(Y)g(\phi X, U) = 0.$$

Now putting $Y = U = e_i$ in (6.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i = 1, 2, \dots, n$ we get $\left[1 + \frac{r}{n(n-1)}\right](n-1)\eta(X) = 0$. Hence for $n \geq 3$, the scalar curvature $r = -n(n-1)$. Therefore we conclude the following:

THEOREM 6.1. *If an n -dimensional ($n \geq 3$) Kenmotsu manifold is ξ -concurrently flat with respect to the quarter-symmetric metric connection, then the scalar curvature $r = -n(n-1)$.*

7. Kenmotsu Manifolds Satisfying the Condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$

This section is devoted to study Kenmotsu manifold satisfying the condition

$$(7.1) \quad \tilde{Z}(\xi, Y) \cdot \tilde{S} = 0,$$

where \tilde{Z} and \tilde{S} are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Equation (7.1) implies

$$(7.2) \quad \tilde{S}(\tilde{Z}(\xi, Y)U, V) + \tilde{S}(U, \tilde{Z}(\xi, Y)V) = 0.$$

In view of (3.5) and (7.2) we have

$$(7.3) \quad S(\tilde{Z}(\xi, Y)U, V) - g(\tilde{Z}(\xi, Y)U, \phi V) \\ + S(U, \tilde{Z}(\xi, Y)V) + g(\phi U, \tilde{Z}(\xi, Y)V) = 0.$$

Putting $U = \xi$ in (7.3), we get

$$(7.4) \quad S(\tilde{Z}(\xi, Y)\xi, V) - g(\tilde{Z}(\xi, Y)\xi, \phi V) + S(\xi, \tilde{Z}(\xi, Y)V) = 0.$$

With the help of (3.7), (3.8) and (3.9) we have from (7.4)

$$(7.5) \quad \left[1 + \frac{r}{n(n-1)}\right] [S(Y, V) + (n-1)\eta(Y)\eta(V)] \\ - S(\phi Y, V) - \left\{ \left[1 + \frac{r}{n(n-1)}\right] g(Y, \phi V) - g(\phi Y, \phi V) \right\} \\ - (n-1) \left\{ \left[1 + \frac{r}{n(n-1)}\right] [\eta(V)\eta(Y) - g(Y, V)] + g(\phi Y, V) \right\} = 0.$$

Interchanging Y and V in (7.5) yields

$$(7.6) \quad \left[1 + \frac{r}{n(n-1)}\right] [S(V, Y) + (n-1)\eta(V)\eta(Y)] \\ - S(\phi V, Y) - \left\{ \left[1 + \frac{r}{n(n-1)}\right] g(V, \phi Y) - g(\phi V, \phi Y) \right\} \\ - (n-1) \left\{ \left[1 + \frac{r}{n(n-1)}\right] [\eta(Y)\eta(V) - g(V, Y)] + g(\phi V, Y) \right\} = 0.$$

Subtracting (7.5) from (7.6), we get

$$(7.7) \quad S(\phi Y, V) - S(\phi V, Y) - \left[1 + \frac{r}{n(n-1)}\right][g(V, \phi Y) - g(Y, \phi V)] - (n-1)[g(\phi V, Y) - g(\phi Y, V)] = 0.$$

In view of (2.7), (2.3) and (7.7) we have

$$(7.8) \quad S(\phi Y, V) = \left[2 - n + \frac{r}{n(n-1)}\right]g(\phi Y, V).$$

Substituting $V = \phi V$ in (7.8) and using (2.6) and (2.2) we obtain

$$(7.9) \quad S(Y, V) = \left[2 - n + \frac{r}{n(n-1)}\right]g(Y, V) - \left[1 + \frac{r}{n(n-1)}\right]\eta(Y)\eta(V).$$

Hence by Lemma 2.1, (7.9) shows the manifold under consideration is an η -Einstein manifold.

From the above discussions we have the following:

THEOREM 7.1. *If a Kenmotsu manifold satisfies the condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$ with respect to the quarter-symmetric metric connection, then the manifold is an η -Einstein manifold with respect to the Levi-Civita connection.*

8. Non-existence of Certain Kinds of Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

THEOREM 8.1. *There is no ξ -projectively flat Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$.*

PROOF. Let us suppose that there exists a ξ -projectively flat Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then we have $\tilde{P}(X, Y)\xi = 0$, for any vector fields X and Y on M . From this and (1.6) we have

$$(8.1) \quad \tilde{R}(X, Y)\xi - \frac{1}{n-1}[\tilde{S}(Y, \xi)X - \tilde{S}(X, \xi)Y] = 0.$$

With the help of (3.2) and (3.6) we obtain from (8.1)

$$(8.2) \quad \eta(Y)\phi X - \eta(X)\phi Y = 0.$$

Taking $Y = \xi$ in (8.2) and using the facts $\phi\xi = 0$ and $\eta(\xi) = 1$, we get $\phi X = 0$, which is a contradiction. Therefore the statement of this theorem follows. \square

THEOREM 8.2. *There is no pseudo Riccissymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$.*

PROOF. Suppose that there exists a pseudo Riccissymmetric Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then from (1.7) we have

$$(8.3) \quad (\tilde{\nabla}_X \tilde{S})(Y, U) = 2\alpha(X)\tilde{S}(Y, U) + \alpha(Y)\tilde{S}(X, U) + \alpha(U)\tilde{S}(Y, X).$$

Taking $U = \xi$ in (8.3) and using (3.6) we obtain

$$(8.4) \quad (\tilde{\nabla}_X \tilde{S})(Y, \xi) = -2(n-1)\alpha(X)\eta(Y) - (n-1)\alpha(Y)\eta(X) + \alpha(\xi)\tilde{S}(Y, X).$$

On the other hand, by the covariant differentiation of the Ricci tensor \tilde{S} with respect to the quarter-symmetric metric connection $\tilde{\nabla}$, we have

$$(8.5) \quad (\tilde{\nabla}_X \tilde{S})(Y, U) = \tilde{\nabla}_X \tilde{S}(Y, U) - \tilde{S}(\tilde{\nabla}_X Y, U) - \tilde{S}(Y, \tilde{\nabla}_X U).$$

So putting $U = \xi$ in (8.5) and using (3.5), (3.1) and (2.4) we get

$$(8.6) \quad (\tilde{\nabla}_X \tilde{S})(Y, \xi) = -(n-1)g(X, Y) - S(X, Y) - g(X, \phi Y).$$

Then comparing the right hand sides of equations (8.4) and (8.6), we obtain

$$\begin{aligned} -2(n-1)\alpha(X)\eta(Y) - (n-1)\alpha(Y)\eta(X) + \alpha(\xi)\tilde{S}(Y, X) \\ = -(n-1)g(X, Y) - S(X, Y) - g(X, \phi Y). \end{aligned}$$

Substituting X and Y with ξ in the above equation we find (since $n > 3$)

$$(8.7) \quad \alpha(\xi) = 0.$$

Now we show that $\alpha = 0$ holds for any vector field on M . Taking $Y = \xi$ in (8.4) and using (8.7) we have $(\tilde{\nabla}_X \tilde{S})(\xi, \xi) = -2(n-1)\alpha(X)$. By the use of (8.6) we find $\alpha(X) = 0$ for every vector field X on M , which implies that $\alpha = 0$ on M . This contradicts to the definition of pseudo Riccisymmetry. \square

9. Example of a 5-dimensional Kenmotsu Manifold Admitting a Quarter-symmetric Metric Connection

We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in \mathbb{R}^5 . We choose the vector fields

$$e_1 = e^{-v} \frac{\partial}{\partial x}, \quad e_2 = e^{-v} \frac{\partial}{\partial y}, \quad e_3 = e^{-v} \frac{\partial}{\partial z}, \quad e_4 = e^{-v} \frac{\partial}{\partial u}, \quad e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M .

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_i, e_j) &= 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5 \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_5)$, for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ -tensor field defined by

$$\phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = -e_1, \quad \phi e_4 = -e_2, \quad \phi e_5 = 0.$$

Using the linearity of ϕ and g , we have

$$\eta(e_5) = 1, \quad \phi^2(Z) = -Z + \eta(Z)e_5, \quad g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Thus, for $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form η is closed. We have

$$\Omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, \phi \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right) = -e^{2v}.$$

Hence, we obtain $\Omega = -e^{2v} dx \wedge dz$. Thus, $d\Omega = -2e^{2v} dv \wedge dx \wedge dz = 2\eta \wedge \Omega$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, [e_1, e_5] = e_1, \\ [e_4, e_5] = e_4, [e_2, e_4] = [e_3, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_5] = e_3.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_5, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= 0, & \nabla_{e_1} e_4 &= 0, & \nabla_{e_1} e_5 &= e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= -e_5, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= 0, & \nabla_{e_2} e_5 &= e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= -e_5, & \nabla_{e_3} e_4 &= 0, & \nabla_{e_3} e_5 &= e_3, \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= -e_5, & \nabla_{e_4} e_5 &= e_4, \\ \nabla_{e_5} e_1 &= 0, & \nabla_{e_5} e_2 &= 0, & \nabla_{e_5} e_3 &= 0, & \nabla_{e_5} e_4 &= 0, & \nabla_{e_5} e_5 &= 0. \end{aligned}$$

Using the above relations in (3.1), we obtain

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= -e_5, & \tilde{\nabla}_{e_1} e_2 &= 0, & \tilde{\nabla}_{e_1} e_3 &= 0, & \tilde{\nabla}_{e_1} e_4 &= 0, & \tilde{\nabla}_{e_1} e_5 &= e_1, \\ \tilde{\nabla}_{e_2} e_1 &= 0, & \tilde{\nabla}_{e_2} e_2 &= -e_5, & \tilde{\nabla}_{e_2} e_3 &= 0, & \tilde{\nabla}_{e_2} e_4 &= 0, & \tilde{\nabla}_{e_2} e_5 &= e_2, \\ \tilde{\nabla}_{e_3} e_1 &= 0, & \tilde{\nabla}_{e_3} e_2 &= 0, & \tilde{\nabla}_{e_3} e_3 &= -e_5, & \tilde{\nabla}_{e_3} e_4 &= 0, & \tilde{\nabla}_{e_3} e_5 &= e_3, \\ \tilde{\nabla}_{e_4} e_1 &= 0, & \tilde{\nabla}_{e_4} e_2 &= 0, & \tilde{\nabla}_{e_4} e_3 &= 0, & \tilde{\nabla}_{e_4} e_4 &= -e_5, & \tilde{\nabla}_{e_4} e_5 &= e_4, \\ \tilde{\nabla}_{e_5} e_1 &= -e_3, & \tilde{\nabla}_{e_5} e_2 &= -e_4, & \tilde{\nabla}_{e_5} e_3 &= e_1, & \tilde{\nabla}_{e_5} e_4 &= e_2, & \tilde{\nabla}_{e_5} e_5 &= 0. \end{aligned}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$\begin{aligned} R(e_1, e_2)e_2 &= R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -e_1, \\ R(e_1, e_2)e_1 &= e_2, R(e_1, e_3)e_1 = R(e_5, e_3)e_5 = R(e_2, e_3)e_2 = e_3, \\ R(e_2, e_3)e_3 &= R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -e_2, R(e_3, e_4)e_4 = -e_3, \\ R(e_2, e_5)e_2 &= R(e_1, e_5)e_1 = R(e_4, e_5)e_4 = R(e_3, e_5)e_3 = e_5, \\ R(e_1, e_4)e_1 &= R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = R(e_5, e_4)e_5 = e_4, \\ \tilde{R}(e_1, e_2)e_2 &= \tilde{R}(e_1, e_3)e_3 = \tilde{R}(e_1, e_4)e_4 = -e_1, \\ \tilde{R}(e_1, e_2)e_1 &= e_2, \tilde{R}(e_1, e_3)e_1 = \tilde{R}(e_2, e_3)e_2 = e_3, \\ \tilde{R}(e_2, e_3)e_3 &= \tilde{R}(e_2, e_4)e_4 = -e_2, \tilde{R}(e_2, e_5)e_5 = e_4 - e_2, \\ \tilde{R}(e_3, e_4)e_4 &= -e_3, \tilde{R}(e_2, e_5)e_2 = \tilde{R}(e_1, e_5)e_1 = \tilde{R}(e_4, e_5)e_4 = e_5, \\ \tilde{R}(e_3, e_5)e_3 &= e_5, \tilde{R}(e_1, e_4)e_1 = \tilde{R}(e_2, e_4)e_2 = \tilde{R}(e_3, e_4)e_3 = e_4, \\ \tilde{R}(e_1, e_5)e_5 &= e_3 - e_1, \tilde{R}(e_3, e_5)e_5 = -e_1 - e_3, \tilde{R}(e_4, e_5)e_5 = -e_2 - e_4. \end{aligned}$$

With the help of the above results we get the Ricci tensors as follows:

$$(9.1) \quad S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4$$

$$(9.2) \quad \tilde{S}(e_1, e_1) = \tilde{S}(e_2, e_2) = \tilde{S}(e_3, e_3) = \tilde{S}(e_4, e_4) = \tilde{S}(e_5, e_5) = -4.$$

Therefore $r = \sum_{i=1}^5 S(e_i, e_i) = -20$ and $\tilde{r} = \sum_{i=1}^5 \tilde{S}(e_i, e_i) = -20$.

From (9.2) it can be easily verified that the manifold is Ricci semisymmetric with respect to the quarter-symmetric metric connection. Also from (9.1) it follows that the manifold is Einstein with respect to the Levi-Civita connection. Therefore Theorem 4.2 is verified.

Also $\tilde{r} = r = -20$. Again from the expressions of the curvature tensor we can easily verify that the manifold is ξ -concurvally flat with respect to the quarter-symmetric metric connection. Hence Theorem 6.1 is verified.

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