axioms

Article

Certain Curvature Conditions on Kenmotsu Manifolds and $\star$-η-Ricci Solitons

Halil İbrahim Yoldaş [1], Abdul Haseeb [2,*] and Fatemah Mofarreh [3]

1 Department of Management Information Systems, Adana Alparslan Türkeş Science and Technology University, 01250 Adana, Turkey
2 Department of Mathematics, College of Science, Jazan University, P.O. Box 2097, Jazan 45142, Saudi Arabia
3 Mathematical Science Department, Faculty of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 13415, Riyadh 11546, Saudi Arabia
* Correspondence: malikhaseeb80@gmail.com or haseeb@jazanu.edu.sa

Abstract: The present paper deals with the investigations of a Kenmotsu manifold satisfying certain curvature conditions endowed with $\star$-η-Ricci solitons. First we find some necessary conditions for such a manifold to be φ-Einstein. Then, we study the notion of $\star$-η-Ricci soliton on this manifold and prove some significant results related to this notion. Finally, we construct a nontrivial example of three-dimensional Kenmotsu manifolds to verify some of our results.

Keywords: $\star$-Ricci tensor; $\star$-η-Ricci soliton; Kenmotsu manifolds; projective curvature tensor; concircular curvature tensor; $\mathcal{M}$-projective curvature tensor; weakly φ-Einstein manifold

MSC: 53C21; 53C25; 53D25

check for updates

Citation: Yoldaş, H.İ.; Haseeb, A.; Mofarreh, F. Certain Curvature Conditions on Kenmotsu Manifolds and $\star$-η-Ricci Solitons. *Axioms* **2023**, *12*, 140. https://doi.org/10.3390/axioms12020140

Academic Editor: Mica Stankovic

Received: 17 December 2022
Revised: 24 January 2023
Accepted: 26 January 2023
Published: 30 January 2023

1. Introduction

The study of manifolds is of high interest to geometers and physicists due to its wide applications in geometry, physics, and relativity. By studying the manifolds, the geometers have benefited from two fundamental tools—the Riemannian curvature tensor and the Ricci tensor–in understanding the differential geometric properties of the manifolds. Over the years, several new concepts have been introduced to the literature with the help of these tools in order to describe complex structures. One of these concepts is the $\star$-Ricci tensor $S^\star$, which was first introduced by Tachibana on almost Hermitian manifolds [1]. After Tachibana's work, Hamada [2] gave the definition of this concept for a contact metric manifold E as follows,

$$S^\star(X_1, X_2) = \frac{1}{2}(trace\{\varphi \circ R(X_1, \varphi X_2)\}),$$

for any vector fields $X_1, X_2 \in \mathfrak{X}(E)$. Here, R is the Riemannian curvature tensor, $S^\star$ is the $\star$-Ricci tensor of type $(0,2)$, φ is a tensor field of type $(1,1)$ and $\mathfrak{X}(E)$ denotes the set of all smooth vector fields of E. Hamada also took into account the concept of $\star$-Einstein manifold and gave a classification of $\star$-Einstein hypersurfaces. The $\star$-Einstein manifold is a Riemannian manifold whose $\star$-Ricci tensor is a constant multiple of its metric tensor g, that is, $S^\star = \lambda g$, where λ is a constant.

The $\star$-Ricci tensor, which has been very popular recently, carries important curvature properties, and these properties give helpful information regarding the geometry and structure of the manifold. Therefore, it has been the subject of interest of many mathematicians, and many studies have been done on this subject. For detailed information about this subject we recommend, in particular, references [1–6].

On the other hand, Hamilton [7] defined the notion of Ricci soliton as a natural generalization of Einstein manifolds in 1988. After Hamilton's definition, several classes of

Ricci solitons have been introduced in the literature. One important class is the notion of $\star$-Ricci soliton that was introduced by Kaimakamis et al. in 2014. They studied this notion in the context of real hypersurfaces of a complex space form [8]. A Riemannian metric g on a smooth manifold E is called $\star$-Ricci soliton, if there exists a smooth vector field J satisfying [8]

$$(\pounds_J g)(X_1, X_2) + 2S^\star(X_1, X_2) + 2\rho g(X_1, X_2) = 0, \qquad \rho \in \mathbb{R},$$

for any vector fields X_1, X_2 on E. Here, $\pounds_J$ indicates the Lie-derivative operator along the vector field J. If $\pounds_{\mathcal{J}} g = 0$, then the $\star$-Ricci soliton becomes a $\star$-Einstein manifold. Such a soliton is called steady, shrinking, or expanding according as $\rho = 0$, $\rho < 0$ or $\rho > 0$, respectively.

As a generalization of $\star$-Ricci soliton, Dey and Roy introduced the notion of $\star$-η-Ricci soliton as follows [9]:

$$(\pounds_J g)(X_1, X_2) + 2S^\star(X_1, X_2) + 2\rho g(X_1, X_2) + 2\sigma\eta(X_1)\eta(X_2) = 0, \quad \rho, \sigma \in \mathbb{R}, \tag{1}$$

for any vector fields X_1, X_2 on E. We denote the $\star$-η-Ricci soliton by (g, J, ρ, σ). If $\sigma = 0$ in (1), then such a soliton reduces to a $\star$-Ricci soliton (g, J, ρ). Recently, many geometers have made notable contributions to the Ricci, η-Ricci, $\star$-Ricci, $\star$-η-Ricci and $\star$-η-Ricci Yamabe solitons in the literature. Some of them are [9–24] and references therein.

The concept of soliton has been a current and popular topic for the last 20 years. In particular, this concept has become a more popular field of study for mathematicians after Perelman actively used Ricci solitons in his work to solve the Poincare conjecturé in 2002. The Ricci soliton and its generalizations have extensive applications, not only in mathematical physics but also in quantum cosmology, quantum gravity, and black holes as well. The Ricci soliton can be considered as a kinematic solution in fluid space-time, the profile of which develops a characterization of spaces of constant curvature along with the locally symmetric spaces. It also expresses geometrical and physical applications with relativistic viscous fluid space-time, admitting heat flux and stress, dark fluid, dust fluid, and radiation era in general relativistic space-time. A two-dimensional Ricci soliton can be used to discuss the behavior of mass under Ricci flow. The Ricci soliton is an important tool, as it can help in understanding the concepts of energy or entropy in general relativity. This property is the same as that of the heat equation due to which an isolated system loses the heat for thermal equilibrium. We, as a scholar in mathematics, study this subject theoretically, but this subject has many applications in physics. Therefore, it will be a field of study for physicists working on this topic.

During the last few years, one of the most active fields of study in differential geometry is the theory of contact manifolds. Contact metric manifolds are special Riemannian manifolds that have almost contact metric structure. Such manifolds have many applications in theoretical physics. They have several subclasses with various names and structures. One of these striking subclasses is the Kenmotsu manifold [25]. These manifolds were defined by K. Kenmotsu. He showed that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = ce^t$. Here, $c(\neq 0)$ is a constant. In recent years, Kenmotsu manifolds have been extensively studied by many geometers, such as ([6,13,15,23]).

In the present paper, we consider $\star$-Ricci solitons in the context of Kenmotsu manifolds satisfying certain curvature conditions. The present paper consists of five sections. Section 1 is the introductory section. In Section 2, some useful definitions and basic concepts of the contact metric manifolds are given. In Section 3, we study certain flatness and φ-semisymmetric conditions in Kenmotsu manifolds. In Section 4, we study $\star$-η-Ricci solitons in Kenmotsu manifolds and obtain some significant results. Finally, an example of three-dimensional Kenmotsu manifolds has been constructed to verify some of our results.

2. Preliminaries

In this section, we give the preliminary concepts and definitions that are required for the study of Kenmotsu manifolds. Moreover, this section helps the readers for a better understanding of the subsequent sections in the paper.

A $(2n+1)$-dimensional almost contact metric manifold E^{2n+1} is a differentiable manifold that have an almost contact metric structure $(\varphi, \zeta, \eta, g)$ satisfying [26,27]

$$\eta(\zeta) = 1, \quad \varphi^2 X_1 + X_1 = \eta(X_1)\zeta, \quad \eta(X_1) = g(X_1, \zeta) \tag{2}$$

and

$$g(\varphi X_1, \varphi X_2) + \eta(X_1)\eta(X_2) = g(X_1, X_2) \tag{3}$$

for any vector fields $X_1, X_2 \in \mathfrak{X}(E^{2n+1})$. Here, φ is a tensor field of type $(1,1)$ on E^{2n+1}, ζ is a vector field, η is a 1-form associated with the unit vector field ζ, such that $\eta(\zeta) = g(\zeta, \zeta) = 1$ and g is the Riemannian metric tensor.

One can easily see that the following are deducible from the equalities (2) and (3):

$$\eta \circ \varphi = 0, \quad \varphi\zeta = 0. \tag{4}$$

An almost contact metric manifold $(E^{2n+1}, \varphi, \zeta, \eta, g)$ is called a Kenmotsu manifold if [25]

$$(\nabla_{X_1}\varphi)X_2 = g(\varphi X_1, X_2)\zeta - \eta(X_2)\varphi X_1, \tag{5}$$

where ∇ appears for the Levi–Civta connection.

An almost contact metric manifold is a Kenmotsu manifold, if and only if

$$\nabla_{X_1}\zeta = X_1 - \eta(X_1)\zeta \tag{6}$$

for any vector field X_1 on E^{2n+1}.

For a $(2n+1)$-dimensional Kenmotsu manifold E^{2n+1}, we have

$$(\nabla_{X_1}\eta)X_2 = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{7}$$
$$R(\zeta, X_1)X_2 = \eta(X_2)X_1 - g(X_1, X_2)\zeta, \tag{8}$$
$$S(X_1, \zeta) = -2n\eta(X_1), \tag{9}$$
$$S(\zeta, \zeta) = -2n, \tag{10}$$
$$Q\zeta = -2n\zeta, \tag{11}$$
$$S(\varphi X_1, \varphi X_2) = S(X_1, X_2) + 2n\eta(X_1)\eta(X_2), \tag{12}$$
$$(\nabla_\zeta Q)X_1 = -2QX_1 - 4nX_1, \tag{13}$$
$$(\pounds_\zeta g)(X_1, X_2) = 2(g(X_1, X_2) - \eta(X_1)\eta(X_2)), \tag{14}$$

where Q is the Ricci operator related to the Ricci tensor S of type $(0,2)$ by $S(X_1, X_2) = g(QX_1, X_2)$.

A Kenmotsu manifold $(E^{2n+1}, \varphi, \zeta, \eta, g)$ is called a η-Einstein manifold if its Ricci tensor $S(\neq 0)$ satisfies [25]

$$S(X_1, X_2) = \lambda g(X_1, X_2) + \mu\eta(X_1)\eta(X_2),$$

where λ and μ are smooth functions on E^{2n+1}. If $\mu = 0$, then E^{2n+1} reduces to an Einstein manifold.

A Kenmotsu manifold E^{2n+1} is called weakly φ-Einstein if

$$S^\varphi(X_1, X_2) = \beta g^\varphi(X_1, X_2) \tag{15}$$

for some smooth function β. Here, g^{φ} is defined by $g^{\varphi}(X_1, X_2) = g(\varphi X_1, \varphi X_2)$ and S^{φ} (called the φ-Ricci tensor of E^{2n+1}) is the symmetric part of $S^{\star}$, such that

$$S^{\varphi}(X_1, X_2) = \frac{1}{2}\Big(S^{\star}(X_1, X_2) + S^{\star}(X_2, X_1)\Big), \tag{16}$$

where $S^{\star}$ is the $(0,2)$ type $\star$-Ricci tensor of E^{2n+1}. If the function β in (15) is a real number, then E^{2n+1} is called φ-Einstein manifold [28].

Now we recall some special curvature tensors, viz., the projective (or the Weyl projective), the concircular and the $\mathcal{M}$-projective curvature tensors that have many physical applications in geometry, physics and theory of relativity (see [29–31]).

The projective curvature tensor P, the concircular curvature tensor $\mathcal{Z}$ and the $\mathcal{M}$-projective curvature tensor $\mathcal{M}$ of a $(2n+1)$-dimensional Kenmotsu manifold E^{2n+1} are respectively defined by

$$P(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{2n}\Big\{S(X_2, X_3)X_1 - S(X_1, X_3)X_2\Big\}, \tag{17}$$

$$\mathcal{Z}(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{r}{2n(2n+1)}\Big\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big\}, \tag{18}$$

and

$$\begin{aligned}\mathcal{M}(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{4n}\Big\{&S(X_2, X_3)X_1 - S(X_1, X_3)X_2 \\ &+g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2\Big\}\end{aligned} \tag{19}$$

for any vector fields X_1, X_2, X_3 on E^{2n+1}. Here, r is the scalar curvature of E^{2n+1}. If $\mathcal{M} = 0$, $\mathcal{Z} = 0$ and $P = 0$, then the manifold E^{2n+1} is called an $\mathcal{M}$-projectively flat, a concircularly flat, and a projectively flat, respectively. Moreover, (17) implies that the manifold E^{2n+1} is projectively flat if and only if it is of constant curvature.

A Kenmotsu manifold $(E^{2n+1}, g), n > 1$, is said to be
(i) φ-projectively semisymmetric if $P(X_1, X_2) \cdot \varphi = 0$,
(ii) φ-$\mathcal{M}$-projectively semisymmetric if $\mathcal{M}(X_1, X_2) \cdot \varphi = 0$,
for all $X_1, X_2 \in \mathfrak{X}(E^{2n+1})$.

3. Kenmotsu Manifolds Satisfying Certain Flatness and φ-Semisymmetric Conditions

In this section, first we study the projectively flat, the concircularly flat and the $\mathcal{M}$-projectively flat Kenmotsu manifolds and prove that the $\star$-Ricci tensor of these flat Kenmotsu manifolds is symmetric and these flat manifolds are φ-Einstein, whereas the $\star$-Ricci tensor of the concircularly flat Kenmotsu manifold is symmetric and the manifold is weakly φ-Einstein. Moreover, we study φ-projectively semisymmetric and φ-$\mathcal{M}$-projectively semisymmetric Kenmotsu manifolds and prove that the $\star$-Ricci tensor of these semisymmetric Kenmotsu manifolds is symmetric and these semisymmetric manifolds are φ-Einstein.

First we prove the following result.

Theorem 1. *Let E^{2n+1} be a $(2n+1)$-dimensional projectively flat Kenmotsu manifold, then $S^{\star}$ of E^{2n+1} is symmetric and the manifold is φ-Einstein.*

Proof. We consider that the manifold E^{2n+1} is projectively flat, then the Equation (17) turns to

$$R(X_1, X_2)X_3 = \frac{1}{2n}\Big\{S(X_2, X_3)X_1 - S(X_1, X_3)X_2\Big\} \tag{20}$$

for any vector fields X_1, X_2, X_3 on E^{2n+1}. By replacing $X_3 = \varphi X_3$ in (20), we have

$$R(X_1, X_2)\varphi X_3 = \frac{1}{2n}\Big\{S(X_2, \varphi X_3)X_1 - S(X_1, \varphi X_3)X_2\Big\},$$

which by taking the inner product with φX_4 provides

$$g(R(X_1, X_2)\varphi X_3, \varphi X_4) = \frac{1}{2n}\Big\{S(X_2, \varphi X_3)g(X_1, \varphi X_4) - S(X_1, \varphi X_3)g(X_2, \varphi X_4)\Big\} \tag{21}$$

for any vector field X_4 on E^{2n+1}.

Let $\{\zeta = U_1, U_2, \ldots, U_{2n+1}\}$ be an orthonormal basis (called φ-basis) of the tangent space T_pE^{2n+1}, for all $p \in E^{2n+1}$. Putting $X_1 = X_4 = U_k$ in (21) and summing over k $(k = 1, 2, \ldots, 2n+1)$, we can easily compute

$$S^\star(X_2, X_3) = \frac{1}{2n}\Big\{S(X_2, X_3) + 2n\eta(X_2)\eta(X_3)\Big\}. \tag{22}$$

Replacing X_1 by ζ in (17) and then using (8) and (9) we arrive at

$$\eta(X_3)X_2 - g(X_2, X_3)\zeta = \frac{1}{2n}\Big\{S(X_2, X_3)\zeta + 2n\eta(X_3)X_2\Big\}. \tag{23}$$

Now, taking the inner product of (23) with ζ and using (2) one immediately has

$$S(X_2, X_3) = -2ng(X_3, X_3). \tag{24}$$

It follows from (22) and (24) that

$$S^\star(X_2, X_3) = -g(X_2, X_3) + \eta(X_2)\eta(X_3). \tag{25}$$

Interchanging the roles of X_2 and X_3 in (25) provides

$$S^\star(X_3, X_2) = -g(X_3, X_2) + \eta(X_3)\eta(X_2). \tag{26}$$

Subtracting (26) from (25) yields $S^\star(X_2, X_3) = S^\star(X_3, X_2)$. This means that $S^\star$ of E^{2n+1} is a symmetric tensor. Therefore, we obtain

$$S^\varphi(X_2, X_3) = -g(\varphi X_2, \varphi X_3),$$

where we have used the equations (16), (25) and (26). Thus, E^{2n+1} is a φ-Einstein manifold, which completes the proof. □

The next theorem gives a necessary condition for a Kenmotsu manifold to be weakly φ-Einstein.

Theorem 2. *If E^{2n+1} is a $(2n+1)$-dimensional concircularly flat Kenmotsu manifold, then $S^\star$ of E^{2n+1} is symmetric and the manifold is weakly φ-Einstein.*

Proof. Let E^{2n+1} be a concircularly flat Kenmotsu manifold. Then, the Equation (18) transforms to

$$R(X_1, X_2)X_3 = \frac{r}{2n(2n+1)}\Big\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big\} \tag{27}$$

for any vector fields X_1, X_2, X_3 on E^{2n+1}.

Taking the inner product on both sides of (27) with X_4 we have

$$g(R(X_1, X_2)X_3, X_4) = \frac{r}{2n(2n+1)}\Big\{g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\Big\} \tag{28}$$

for any vector fields X_4 on E^{2n+1}.

Now, by replacing X_3 by φX_3 and X_4 by φX_4 in (28) we have

$$g(R(X_1,X_2)\varphi X_3,\varphi X_4) = \frac{r}{2n(2n+1)}\Big\{g(X_2,\varphi X_3)g(X_1,\varphi X_4) - g(X_1,\varphi X_3)g(X_2,\varphi X_4)\Big\}. \tag{29}$$

Keeping in mind the φ-basis and putting $X_1 = X_4 = U_k$ in (29) and then summing over k ($k = 1, 2, \ldots, 2n+1$), we can easily compute

$$S^\star(X_2,X_3) = \frac{r}{2n(2n+1)} g(\varphi X_2,\varphi X_3), \tag{30}$$

from which it can be seen that $S^\star$ of E^{2n+1} is symmetric.

Now, from (15), (16) and (30) we find

$$S^\varphi(X_2,X_3) = \beta g^\varphi(X_2,X_3),$$

where $\beta = \frac{r}{2n(2n+1)}$. Therefore, E^{2n+1} is weakly φ-Einstein. □

As a direct consequence of the Theorem 2, we have the following.

Corollary 1. *If E^{2n+1} is a $(2n+1)$-dimensional concircularly flat Kenmotsu manifold whose scalar curvature is constant, then E^{2n+1} is φ-Einstein.*

Theorem 3. *If E^{2n+1} is a $(2n+1)$-dimensional $\mathcal{M}$-projectively flat Kenmotsu manifold, then $S^\star$ of E^{2n+1} is symmetric and the manifold is φ-Einstein.*

Proof. Let E^{2n+1} be an $\mathcal{M}$-projectively flat Kenmotsu manifold. Then, the Equation (19) takes the form

$$R(X_1,X_2)X_3 = \frac{1}{4n}\Big\{S(X_2,X_3)X_1 - S(X_1,X_3)X_2 + g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2\Big\} \tag{31}$$

for any vector fields X_1, X_2, X_3 on E^{2n+1}.

Taking the inner product of (31) with φX_4 we lead to

$$\begin{aligned} g(R(X_1,X_2)X_3,\varphi X_4) &= \frac{1}{4n}\Big\{S(X_2,X_3)g(X_1,\varphi X_4) - S(X_1,X_3)g(X_2,\varphi X_4) \\ &+ g(X_2,X_3)S(X_1,\varphi X_4) - g(X_1,X_3)S(X_2,\varphi X_4)\Big\}. \end{aligned} \tag{32}$$

Moreover, by replacing X_3 by φX_3 in (32), we get

$$\begin{aligned} g(R(X_1,X_2)\varphi X_3,\varphi X_4) &= \frac{1}{4n}\Big\{S(X_2,\varphi X_3)g(X_1,\varphi X_4) - S(X_1,\varphi X_3)g(X_2,\varphi X_4) \\ &+g(X_2,\varphi X_3)S(X_1,\varphi X_4) - g(X_1,\varphi X_3)S(X_2,\varphi X_4)\Big\}. \end{aligned} \tag{33}$$

Now, considering the φ-basis and putting $X_1 = X_4 = U_k$ in (33), then taking summation over k ($k = 1, 2, \ldots, 2n+1$), we easily obtain

$$S^\star(X_2,X_3) = \frac{1}{4n}\Big\{2S(X_2,X_3) + 4n\eta(X_2)\eta(X_3)\Big\}. \tag{34}$$

Taking ζ in place of X_1 in (31) and utilizing (8), (9), (11) we find

$$\begin{aligned} \eta(X_3)X_2 - g(X_2,X_3)\zeta &= \frac{1}{2n}\Big\{S(X_2,X_3)\zeta + 2n\eta(X_3)X_2 - 2ng(X_2,X_3)\zeta \\ &-\eta(X_3)QX_2\Big\}, \end{aligned}$$

which by taking the inner product with ζ and then using (2) and (9) leads to

$$S(X_2, X_3) = -2ng(X_2, X_3). \tag{35}$$

By means of (35), the Equation (34) turns to

$$S^\star(X_2, X_3) = -g(X_2, X_3) + \eta(X_2)\eta(X_3), \tag{36}$$

from which it can be seen that $S^\star$ of E^{2n+1} is symmetric. By considering (36) in (16), it follows that

$$S^\varphi(X_2, X_3) = -g(\varphi X_2, \varphi X_3).$$

Thus we get the desired result. □

Now we prove the following result.

Theorem 4. *If E^{2n+1} is a $(2n+1)$-dimensional φ-projectively semisymmetric Kenmotsu manifold, then $S^\star$ of E^{2n+1} is symmetric and the manifold is φ-Einstein.*

Proof. Let E^{2n+1} be a φ-projectively semisymmetric Kenmotsu manifold, i.e., E^{2n+1} satisfies the condition $P.\varphi = 0$. This implies that

$$P(X_1, X_2)\varphi X_3 = \varphi P(X_1, X_2)X_3 \tag{37}$$

for any vector fields X_1, X_2, X_3 on E^{2n+1}.

In view of (17), (37) takes the form

$$\begin{aligned} R(X_1, X_2)\varphi X_3 - \frac{1}{2n}\Big\{S(X_2, \varphi X_3)X_1 - S(X_1, \varphi X_3)X_2\Big\} = \\ \varphi R(X_1, X_2)X_3 - \frac{1}{2n}\Big\{S(X_2, X_3)\varphi X_1 - S(X_1, X_3)\varphi X_2\Big\}, \end{aligned} \tag{38}$$

which by taking the inner product with φX_4 becomes

$$\begin{aligned} & g(R(X_1, X_2)\varphi X_3, \varphi X_4) - \tfrac{1}{2n}\Big\{S(X_2, \varphi X_3)g(X_1, \varphi X_4) - S(X_1, \varphi X_3)g(X_2, \varphi X_4)\Big\} \\ & = g(R(X_1, X_2)X_3, X_4) - \eta(R(X_1, X_2)X_3)\eta(X_4) - \tfrac{1}{2n}\Big\{S(X_2, X_3)g(\varphi X_1, \varphi X_4) \\ & \qquad -S(X_1, X_3)g(\varphi X_2, \varphi X_4)\Big\}. \end{aligned} \tag{39}$$

By using the φ-basis and plugging $X_1 = X_4 = U_k$ in (39), then taking summation over k $(k = 1, 2, \ldots, 2n+1)$, after a straightforward calculation we find

$$S^\star(X_2, X_3) = \frac{1}{n}S(X_2, X_3) + g(X_2, X_3) + \eta(X_2)\eta(X_3). \tag{40}$$

Now by putting $X_1 = \zeta$ in (38) and then using (4), (8) and (9), we obtain

$$S(X_2, X_3) = -2ng(X_2, X_3),$$

which together with (40) takes the form

$$S^\star(X_4, X_2) = -g(X_4, X_2) + \eta(X_4)\eta(X_2). \tag{41}$$

Consequently, we have

$$S^\varphi(X_4, X_2) = -g(\varphi X_4, \varphi X_2). \tag{42}$$

From the Equations (41) and (42), our result follows. □

Theorem 5. *If E^{2n+1} is a $(2n+1)$-dimensional φ-$\mathcal{M}$-projectively semisymmetric Kenmotsu manifold, then $S^\star$ of E^{2n+1} is symmetric and the manifold is φ-Einstein.*

Proof. Let E^{2n+1} be a φ-$\mathcal{M}$-projectively semisymmetric Kenmotsu manifold, i.e., E^{2n+1} satisfies the condition $\mathcal{M}.\varphi = 0$. The condition $\mathcal{M}.\varphi = 0$ implies that

$$\mathcal{M}(X_1, X_2)\varphi X_3 = \varphi\mathcal{M}(X_1, X_2)X_3 \tag{43}$$

for any vector fields X_1, X_2, X_3 on E^{2n+1}.

Keeping in mind (19), the Equation (43) takes the form

$$\begin{aligned} R(X_1, X_2)\varphi X_3 - \tfrac{1}{4n}\Big\{S(X_2, \varphi X_3)X_1 - S(X_1, \varphi X_3)X_2 + g(X_2, \varphi X_3)QX_1 \\ -g(X_1, \varphi X_3)QX_2\Big\} = \varphi R(X_1, X_2)X_3 - \tfrac{1}{4n}\Big\{S(X_2, X_3)\varphi X_1 - S(X_1, X_3)\varphi X_2 \\ +g(X_2, X_3)\varphi QX_1 - g(X_1, X_3)\varphi QX_2\Big\}. \end{aligned} \tag{44}$$

By putting $X_1 = \zeta$ in (44) and recalling (4), (8), (9), (11), we have

$$\begin{aligned} -g(X_2, \varphi X_3)\zeta - \frac{1}{4n}\Big\{S(X_2, \varphi X_3)\zeta - 2ng(X_2, \varphi X_3)\zeta\Big\} \\ = \eta(X_3)\varphi X_2 - \frac{1}{4n}\Big\{2n\eta(X_3)\varphi X_2 - \eta(X_3)\varphi QX_2\Big\}, \end{aligned}$$

which by taking the inner product with ζ provides

$$S(X_2, \varphi X_3) = -2ng(X_2, \varphi X_3). \tag{45}$$

Replacing X_3 by φX_3 in (45) and by virtue of (2), (9) we deduce

$$S(X_2, X_3) = -2ng(X_2, X_3), \tag{46}$$

which yields

$$QX_2 = -2nX_2. \tag{47}$$

Now taking the inner product of (44) with φX_4 we have

$$\begin{aligned} g(R(X_1, X_2)\varphi X_3, \varphi X_4) - \tfrac{1}{4n}\Big\{S(X_2, \varphi X_3)g(X_1, \varphi X_4) - S(X_1, \varphi X_3)g(X_2, \varphi X_4) \\ +g(X_2, \varphi X_3)g(QX_1, \varphi X_4) - g(X_1, \varphi X_3)g(QX_2, \varphi X_4)\Big\} = g(R(X_1, X_2)X_3, X_4) \\ -\eta(R(X_1, X_2)X_3)\eta(X_4) - \tfrac{1}{4n}\Big\{S(X_2, X_3)g(\varphi X_1, \varphi X_4) - S(X_1, X_3)g(\varphi X_2, \varphi X_4) \\ +g(X_2, X_3)g(\varphi QX_1, \varphi X_4) - g(X_1, X_3)g(\varphi QX_2, \varphi X_4)\Big\} \end{aligned} \tag{48}$$

for any vector field X_4 on E^{2n+1}.

By substituting (47) in (48) we have

$$\begin{aligned} g(R(X_1,X_2)\varphi X_3,\varphi X_4) - \tfrac{1}{4n}\Big\{ S(X_2,\varphi X_3)g(X_1,\varphi X_4) - S(X_1,\varphi X_3)g(X_2,\varphi X_4) \\ -2ng(X_2,\varphi X_3)g(X_1,\varphi X_4) + 2ng(X_1,\varphi X_3)g(X_2,\varphi X_4)\Big\} = g(R(X_1,X_2)X_3,X_4) \\ -\eta(R(X_1,X_2)X_3)\eta(X_4) - \tfrac{1}{4n}\Big\{ S(X_2,X_3)g(\varphi X_1,\varphi X_4) - S(X_1,X_3)g(\varphi X_2,\varphi X_4) \\ -2ng(X_2,X_3)g(\varphi X_1,\varphi X_4) + 2ng(X_1,X_3)g(\varphi X_2,\varphi X_4)\Big\}. \end{aligned} \tag{49}$$

Considering the φ-basis and putting $X_1 = X_4 = U_k$ in (49) and then taking summation over k $(k = 1,2,\ldots,2n+1)$, after straightforward computation we obtain

$$\begin{aligned} S^\star(X_2,X_3) - \tfrac{1}{4n}\Big\{ S(\varphi X_3,\varphi X_2) - 2ng(\varphi X_3,\varphi X_2)\Big\} = S(X_2,X_3) \\ -\eta(X_2)\eta(X_3) + g(X_2,X_3) - \tfrac{1}{4n}\Big\{ 2nS(X_2,X_3) - S(\varphi X_2,\varphi X_3) \\ -4n^2 g(X_2,X_3) + 2ng(\varphi X_2,\varphi X_3)\Big\}, \end{aligned}$$

from which, in view of (12) and (46), we easily obtain

$$S^\star(X_2,X_3) = -g(X_2,X_3) + \eta(X_2)\eta(X_3),$$

which by using in (16) gives

$$S^\varphi(X_2,X_3) = -g(\varphi X_2,\varphi X_3).$$

Our claim follows from the last two equations. □

4. $\star$-η-Ricci Solitons on Kenmotsu Manifolds

There are many classes of manifolds studied in differential geometry. One of them includes Einstein manifolds. The manifolds where the Ricci tensor is proportional to the metric tensor are called Einstein manifolds. Because the Ricci tensor is a part of Einstein's famous field equations, thus the manifolds endowed with the Ricci tensor are closely related to Einstein's field equations. That's why these manifolds are very important in both Riemannian geometry and the general theory of relativity. It is emphasized that the results of the present paper are important because they are reduced to the Einstein manifold.

In this section, we consider Kenmotsu manifolds endowed with $\star$-η-Ricci solitons, and we obtain some significant results concerning such manifolds. To prove our next theorems, we use the following lemma.

Lemma 1 ([6])**.** *In a $(2n+1)$-dimensional Kenmotsu manifold, the $\star$-Ricci tensor $S^\star$ is given by*

$$S^\star(X_1,X_2) = S(X_1,X_2) + (2n-1)g(X_1,X_2) + \eta(X_1)\eta(X_2) \tag{50}$$

for any vector fields X_1, X_2 on E^{2n+1}.

Now, first we prove the following theorem:

Theorem 6. *Let E^{2n+1} be a $(2n+1)$-dimensional Kenmotsu manifold endowed with $\star$-η-Ricci soliton (g,ζ,ρ,σ). Then the manifold is Einstein as well as φ-Einstein.*

Proof. Considering (g,ζ,ρ,σ) as a $\star$-η-Ricci soliton on E^{2n+1}, then in view of (1), we have

$$(\pounds_\zeta g)(X_1,X_2) + 2S^\star(X_1,X_2) + 2\rho g(X_1,X_2) + 2\sigma\eta(X_1)\eta(X_2) = 0 \tag{51}$$

for any vector fields X_1, X_2 on E^{2n+1}.

By using (6), (14) and (50), the Equation (51) turns to

$$S(X_1, X_2) = -(\rho + 2n)g(X_1, X_2) - \sigma\eta(X_1)\eta(X_2). \tag{52}$$

Taking $X_1 = X_2 = \zeta$ in (52) and making use of (10), we get

$$\rho + \sigma = 0. \tag{53}$$

By taking the covariant derivative of (52) with respect to X_4 and keeping in mind (6), we are led to

$$(\nabla_{X_4} S)(X_1, X_2) = -\sigma(g(X_1, X_4)\eta(X_2) + g(X_2, X_4)\eta(X_1) - 2\eta(X_1)\eta(X_2)\eta(X_4)), \tag{54}$$

where

$$(\nabla_{X_4} S)(X_1, X_2) = \nabla_{X_4} S(X_1, X_2) - S(\nabla_{X_4} X_1, X_2) - S(X_1, \nabla_{X_4} X_2).$$

With the help of (6), (7) and (9) it can be easily seen that

$$(\nabla_{X_4} S)(X_1, \zeta) = -S(X_1, X_4) - 2ng(X_1, X_4). \tag{55}$$

By the cyclic rearrangement of X_1, X_2 and X_4 in (54), we have

$$(\nabla_{X_1} S)(X_2, X_4) = -\sigma(g(X_2, X_1)\eta(X_4) + g(X_4, X_1)\eta(X_2) - 2\eta(X_1)\eta(X_2)\eta(X_4)), \tag{56}$$

and

$$(\nabla_{X_2} S)(X_4, X_1) = -\sigma(g(X_1, X_2)\eta(X_4) + g(X_4, X_2)\eta(X_1) - 2\eta(X_1)\eta(X_2)\eta(X_4)). \tag{57}$$

By adding the Equations (54)–(57), we have

$$\begin{aligned} &(\nabla_{X_4} S)(X_1, X_2) + (\nabla_{X_1} S)(X_2, X_4) + (\nabla_{X_2} S)(X_4, X_1) = \\ &-2\sigma(g(X_1, X_4)\eta(X_2) + g(X_2, X_4)\eta(X_1) + g(X_1, X_2)\eta(X_4) - 3\eta(X_1)\eta(X_2)\eta(X_4)). \end{aligned} \tag{58}$$

Now taking $X_4 = \zeta$ in (58) and using (13), we arrive at

$$4ng(X_1, X_2) + 2S(X_1, X_2) = \sigma(g(X_1, X_2) - \eta(X_1)\eta(X_2)). \tag{59}$$

Keeping in mind the φ-basis and putting $X_1 = X_2 = U_k$ in (59) and then summing over k $(k = 1, 2, \ldots, 2n + 1)$, we obtain

$$4n(2n + 1) + 2r = 2n\sigma. \tag{60}$$

Setting $X_2 = \zeta$ in (54) and using (55), we obtain

$$2ng(X_1, X_4) + S(X_1, X_4) = \sigma(g(X_1, X_4) - \eta(X_1)\eta(X_4)), \tag{61}$$

from which we obtain

$$2n(2n + 1) + r = 2n\sigma. \tag{62}$$

By combining (60) and (62), it follows that $2n\sigma = 0$, and hence $\sigma = 0$. By using this fact together with (53), the Equation (52) leads to

$$S(X_1, X_2) = -2ng(X_1, X_2), \tag{63}$$

which informs us that E^{2n+1} is Einstein.

Furthermore, substituting (63) in (50) we are led to

$$S^{\star}(X_1, X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2).$$

Consequently, we have

$$S^{\varphi}(X_1, X_2) = -g(\varphi X_1, \varphi X_2).$$

Hence, the proof is completed. □

Theorem 7. *Let E^{2n+1} be a $(2n+1)$-dimensional Kenmotsu manifold endowed with $\star$-η-Ricci soliton (g, J, ρ, σ) such that J is pointwise collinear with ζ. Then E is weakly φ-Einstein.*

Proof. Let J be pointwise collinear with ζ, that is, $J = b\zeta$ for some function b. Then, one can calculate

$$(\pounds_J g)(X_1, X_2) = X_1(b)\eta(X_2) + X_2(b)\eta(X_1) + 2b(g(X_1, X_2) - \eta(X_1)\eta(X_2)) \tag{64}$$

for any vector fields X_1, X_2 on E^{2n+1}.

As (g, J, ρ, σ) is $\star$-η-Ricci soliton on E^{2n+1}, then from (1) and (64) we have

$$\begin{aligned} X_1(b)\eta(X_2) + X_2(b)\eta(X_1) + 2b(g(X_1, X_2) - \eta(X_1)\eta(X_2)) + 2S^{\star}(X_1, X_2) \\ +2\rho g(X_1, X_2) + 2\sigma\eta(X_1)\eta(X_2) = 0. \end{aligned} \tag{65}$$

Putting $X_1 = X_2 = \zeta$ in (65) and then utilizing (10) and (50) we find

$$\zeta(b) = -(\rho + \sigma). \tag{66}$$

Again putting $X_2 = \zeta$ in (65) and then from (9), (50) and (66), we obtain

$$X_1(b) = -(\rho + \sigma)\eta(X_1), \tag{67}$$

which together with (65) leads to

$$S^{\star}(X_1, X_2) = -(\rho + b)g(X_1, X_2) + (\rho + b)\eta(X_1)\eta(X_2). \tag{68}$$

Thus, from (16) and (68) we have

$$S^{\varphi}(X_1, X_2) = (\rho + b)g^{\varphi}(X_1, X_2),$$

which completes the proof. □

Now, we present an example of three-dimensional Kenmotsu manifold to verify some of our results.

Example 1 ([32])**.** *We consider a three-dimensional Riemannian manifold $E^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the usual coordinates in $\mathbb{R}^3$. Let U_1, U_2 and U_3 be the linearly independent vector fields on E^3 given by*

$$U_1 = z\frac{\partial}{\partial x}, \quad U_2 = z\frac{\partial}{\partial y}, \quad U_3 = -z\frac{\partial}{\partial z} = \zeta.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(U_i, U_i) &= 1, \quad for \quad i = 1, 2, 3, \\ g(U_i, U_j) &= 0, \quad for \quad i \neq j, \end{aligned}$$

and given by

$$g = \frac{1}{z^2}\Big\{dx \otimes dx + dy \otimes dy + dz \otimes dz\Big\}.$$

Now let the 1-form η and the $(1,1)$-tensor field φ be defined by

$$\eta(X_1) = g(X_1, U_3) \quad , \varphi(U_1) = -U_2, \quad \varphi(U_2) = U_1, \quad \varphi(U_3) = 0$$

for all X_1 on E^3.

The linearity property of φ and g yields

$$\eta(U_3) = 1, \;\; \varphi^2 X_1 = -X_1 + \eta(X_1)\zeta, \;\; g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2),$$

for all X_1, X_2 on E^3. Therefore, $(E^3, \varphi, \zeta, \eta, g)$ is an almost contact metric manifold of dimension 3 for $U_3 = \zeta$.

By direct calculations, we obtain

$$[U_1, U_2] = 0, \;\; [U_1, U_3] = U_1, \;\; [U_2, U_3] = U_2.$$

By the use of Koszul's formula for the Riemannian metric g, we obtain

$$\begin{gathered}\nabla_{U_1} U_3 = U_1, \quad \nabla_{U_2} U_3 = U_2, \quad \nabla_{U_1} U_1 = \nabla_{U_2} U_2 = -U_3, \\ \nabla_{U_3} U_3 = \nabla_{U_1} U_2 = \nabla_{U_2} U_1 = \nabla_{U_3} U_1 = \nabla_{U_3} U_2 = 0.\end{gathered}$$

Therefore, by using the above values, it can be easily verified that E^3 is a 3-dimensional Kenmotsu manifold.

The following components of R can be easily obtain

$$\begin{array}{lll} R(U_1, U_2)U_3 = 0, & R(U_1, U_3)U_2 = 0, & R(U_2, U_3)U_1 = 0, \\ R(U_1, U_2)U_2 = -U_1, & R(U_1, U_2)U_1 = U_2, & R(U_1, U_3)U_3 = -U_1, \\ R(U_1, U_3)U_1 = U_3, & R(U_2, U_3)U_3 = -U_2, & R(U_3, U_2)U_2 = -U_3, \end{array}$$

which gives

$$S(U_1, U_1) = S(U_2, U_2) = S(U_3, U_3) = -2, \;\; S(U_i, U_j) = 0, \; i \neq j.$$

With the help of (50), we find

$$S^\star(U_1, U_1) = S^\star(U_2, U_2) = -1, \;\; S^\star(U_3, U_3) = 0, \;\; S^\star(U_i, U_j) = 0, \; i \neq j \tag{69}$$

for all $i, j = 1, 2, 3$.

Now taking $J = \zeta$ in (1), we have

$$(\pounds_\zeta g)(X_1, X_2) + 2S^\star(X_1, X_2) + 2\rho g(X_1, X_2) + 2\sigma\eta(X_1)\eta(X_2) = 0 \tag{70}$$

for any vector fields X_1, X_2 on E^3. Taking $X_1 = X_2 = U_i$ in (70) and summing over $i(1 \leq i \leq 3)$ and then using (69) yields

$$\sigma = 0 \quad and \quad \rho = 0.$$

Then, the data (g, ζ, ρ, σ) satisfying the Equation (1) is a $\star$-η-Ricci soliton with $\sigma = 0$ and $\rho = 0$ on E^3. This result verifies Theorem 6. Moreover, we remark that this example supports Theorems 1–3.

Author Contributions: Conceptualization, H.İ.Y., A.H. and F.M.; methodology, H.İ.Y., A.H. and F.M.; investigation, H.İ.Y., A.H. and F.M.; writing—original draft preparation, H.İ.Y., A.H. and F.M.; writing—review and editing, H.İ.Y., A.H. and F.M. All authors have read and agreed to the published version of the manuscript.

Funding: Fatemah Mofarreh expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the editor and anonymous referees for the constructive comments given to improve the quality of the paper. The third author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Tachibana, S. On almost-analytic vectors in almost Kahlerian manifolds. *Tohoku Math. J.* **1959**, *11*, 247–265. [CrossRef]
2. Hamada, T. Real hypersurfaces of complex space forms in terms of Ricci $\star$-tensor. *Tokyo J. Math.* **2002**, *25*, 473–483. [CrossRef]
3. Ivey, T.; Ryan, P.J. The $\star$-Ricci tensor for hypersurfaces in CP_n and CH_n. *Tokyo J. Math.* **2011**, *34*, 445–471.
4. Amruthalakshmi, M.R.; Prakasha, D.G.; Turki, N.B.; Unal, I. $*$-Ricci tensor on α-cosymplectic manifolds. *Adv. Math. Phys.* **2022**, *2022*, 7939654. [CrossRef]
5. Unal, I.; Sari, R.; Prakasha, D.G. $\star$-Ricci tensor on normal metric contact pair manifolds. *Balkan J. Geom. Appl.* **2022**, *27*, 178–189.
6. Venkatesha, V.; Naik, D.M.; Kumara, H.A. $\star$-Ricci solitons and gradient almost $\star$-Ricci solitons on Kenmotsu manifolds. *Math. Slovaca* **2019**, *69*, 1447–1458. [CrossRef]
7. Hamilton, R.S. The Ricci Flow on Surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986). *Contemp. Math.* **1988**, *71*, 237–262.
8. Kaimakamis, G.; Panagiotidou, K. $\star$-Ricci solitons of real hypersurface in non-flat complex space forms. *J. Geom. Phy.* **2014**, *76*, 408–413. [CrossRef]
9. Dey, S.; Roy, S. $\star$-η-Ricci soliton within the framework of Sasakian manifold. *J. Dyn. Syst. Geom. Theor.* **2020**, *18*, 163–181. [CrossRef]
10. Chen, X. Real hypersurfaces with $\star$-Ricci solitons of non-flat complex space forms. *Tokyo J. Math.* **2018**, *41*, 433–451. [CrossRef]
11. Dai, X.; Zhao, Y.; De, U.C. $\star$-Ricci soliton on $(\kappa,\mu)'$-almost Kenmotsu manifold. *Open Math.* **2019**, *17*, 874–882. [CrossRef]
12. Dey, S.; Sarkar, S.; Bhattacharyya, A. $\star$-η-Ricci soliton and contact geometry. *Ric. Mat.* **2021**, 1–19. [CrossRef]
13. Ghosh, A. Ricci soliton and Ricci almost soliton within the framework of Kenmotsu manifold. *Carpathian Math. Publ.* **2019**, *11*, 59–69. [CrossRef]
14. Ghosh, A.; Patra, D.S. $\star$-Ricci soliton within the framework of Sasakian and (κ,μ)-contact manifold. *Int. J. Geom. Methods Modern Phys.* **2018**, *15*, 1850120. [CrossRef]
15. Haseeb, A.; Bilal, M.; Chaubey, S.K.; Khan, M.N.I. Geometry of indefinite Kenmotsu manifolds as $*\eta$-Ricci-Yamabe solitons. *Axioms* **2022**, *11*, 461. [CrossRef]
16. Haseeb, A.; Chaubey, S.K. Lorentzian para-Sasakian manifolds and $*$-Ricci solitons. *Krag. J. Math.* **2024**, *48*, 167–179.
17. Haseeb, A.; De, U.C. η-Ricci solitons in ϵ-Kenmotsu manifolds. *J. Geom.* **2019**, *110*, 34. [CrossRef]
18. Haseeb, A.; Harish, H.; Prakasha, D.G. 3-dimensional trans-Sasakian manifolds admitting $*$-Ricci solitons. *Jordan J. Math. Stat.* **2022**, *15*, 105–121.
19. Haseeb, A.; Prasad, R.; Mofarreh, F. Sasakian manifolds admitting $*$-η-Ricci-Yamabe solitons. *Adv. Math. Phys.* **2022**, *2022*, 5718736. [CrossRef]
20. Li, Y.; Haseeb, A.; Ali, M. LP-Kenmotsu manifolds admitting η-Ricci solitons and spacetime. *J. Math.* **2022**, *2022*, 6605127. [CrossRef]
21. Majhi, P.; De, U.C.; Suh, Y.J. $\star$-Ricci solitons on Sasakian 3-manifolds. *Publ. Math. Debrecen* **2018**, *93*, 241–252. [CrossRef]
22. Patra, D.S.; Ali, A.; Mofarreh, F. Geometry of almost contact metrics as almost $\star$-Ricci solitons. *arXiv* **2021**, arXiv:2101.01459v1.
23. Yoldaş, H.İ. On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 2150189. [CrossRef]
24. Yoldaş, H.İ. Remarks on some soliton types with certain vector fields. *Fundam. Contemp. Math. Sci.* **2022**, *3*, 146–159. [CrossRef]
25. Kenmotsu, K. A class of almost contact Riemannian manifolds. *Tohoku Math. J.* **1972**, *24*, 93–103. [CrossRef]
26. Blair, D.E. *Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany, 1976.
27. Yano, K.; Kon, M. *Structures on Manifolds, Series in Mathematics*; World Scientific Publishing: Singapore; Springer: Berlin/Heidelberg, Germany, 1984.
28. Cho, J.T.; Inoguchi, J.I. On φ-Einstein contact Riemannian manifolds. *Mediterr. J. Math.* **2011**, *7*, 143–167. [CrossRef]
29. Mishra, R.S. *Structures on a Differentiable Manifold and Their Applications*; Chandrama Prakashan: Allahabad, India, 1984.
30. Pokhariyal, G.P.; Mishra, R.S. Curvature tensors and their relativistic significance. *Yokohama Math. J.* **1971**, *19*, 97–103.

31. Yano, K. Concircular geometry I. concircular transformations. *Proc. Imp. Acad.* **1940**, *16*, 195–200. [CrossRef]
32. Hui, S.K.; Yadav, S.K.; Patra, A. Almost conformal Ricci solitons on f-Kenmotsu manifolds. *Khayyam J. Math.* **2019**, *5*, 89–104.