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Certain graphs arising from Hadamard matrices

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We establish several infinite classes of regular graphs with the property that any two distinct vertices have a fixed number of other vertices joined to both of them. The graphs are found by constructing their incidence matrices, which correspond to certain Hadamard matrices.

1. Preliminaries

We assume such standard ideas as (v,k,λ) - *configuration* and Hadamard *matrix* (see, for example, [2]). By a (v,k,λ)-graph G we mean a regular graph with v points, of valency k, such that for any pair of points there are exactly λ points joined to both by arcs of G. In other words, if B_i is the set of all points joined to point i, then

$$B_i \text{ has } k \text{ elements for every } i \text{ ;}$$
$$B_i \cap B_j \text{ has } \lambda \text{ elements whenever } i \neq j \text{ .}$$

It is clear that $B_1, \ldots, B_v$ are the blocks of a (v,k,λ)-configuration whose varieties are the points of G; this configuration has the properties

$$i \notin B_i \textit{ for any } i \text{ ;}$$
$$i \in B_j \Leftrightarrow j \in B_i \text{ .}$$

On the other hand, if we have such a configuration, we can form a graph by "join points i and j when $i \in B_j$". Recasting this in terms of the incidence matrix of the configuration, we have the following

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characterization:

(1) *There is a* (v,k,λ)*-graph if and only if there is a* (v,k,λ)*-configuration whose incidence matrix is symmetric and has diagonal* $(0,0,\ldots,0)$.

Consequently the necessary conditions for the existence of a (v,k,λ)-configuration [*2*,p. 107]:

(2) $\lambda(v-1) = k(k-1)$;

(3) (i) *if* v *is even then* $(k-\lambda)$ *is a perfect square,*

(ii) *if* v *is odd then*

$$z^2 = (k-\lambda)x^2 + (-1)^{(v-1)/2}\lambda y^2$$

has a non-trivial integer solution;

are also necessary for a (v,k,λ)-graph.

Ahrens and Szekeres [*1*] have proposed the question: for what triples (v,k,λ) is there a (v,k,λ)-graph? We shall establish the existence of certain infinite classes of graphs.

I would like to thank Mr Ahrens and Professor Szekeres for their helpful correspondence, and for sending me a prepublication copy of [*1*].

2. Correspondence between graphs and Hadamard matrices

It is known [*2*, p. 206] that if there is a $(4u^2,2u^2-u,u^2-u)$-configuration with incidence matrix A , then there is an Hadamard matrix H of order $4u^2$,

$$H = J - 2A ,$$

where J is the matrix with every element $+1$. On the other hand, suppose H is an Hadamard matrix of order $4u^2$ with exactly $2u^2 - u$ elements -1 in each row. Then $H\,J = J\,H^T = 2u\,J$; so if $A = \frac{1}{2}(J - H)$,

$$A\,A^T = \tfrac{1}{4}(4u^2\,J - H\,J - J\,H^T + 4u^2\,I)$$
$$= u^2I + (u^2-u)\,J .$$

A is a $(0,1)$-matrix, so it is the incidence matrix of a $(4u^2,2u^2-u,u^2-u)$-configuration. The 1's in A are in the same positions as the -1's in H. Therefore, if H is symmetric and has +1 in every diagonal position, A will be the incidence matrix corresponding in (1) to a $(4u^2, 2u^2-u, u^2-u)$-graph. We will refer to a graph with these parameters as being "of type A_u"; we have proven

THEOREM 1. *There is a graph of type* A_u *if and only if there is a symmetric* Hadamard *matrix of order* $4u^2$, *with diagonal elements all* +1 *and with exactly* $2u^2 + u$ *elements* +1 *in every row.*

We will call a $(4u^2, 2u^2+u, u^2+u)$-graph one "of type B_u". Analogously to the above, we can prove

THEOREM 2. *There is a graph of type* B_u *if and only if there is a symmetric* Hadamard *matrix of order* $4u^2$, *with diagonal elements all* +1 *and with exactly* $2u^2 - u$ *elements* +1 *in every row.*

Another well-known correspondence between configurations and Hadamard matrices is as follows: suppose A is the incidence matrix of a $(4n-1, 2n, n)$-configuration. Construct a matrix B by replacing every 0 of A by +1 and every 1 of A by -1. Then put

$$H = \left[\begin{array}{c|c} 1 & 1 \ldots 1 \\ \hline 1 & \\ \vdots & B \\ 1 & \end{array}\right].$$

H is Hadamard of order $4n$. Conversely, if H is a Hadamard matrix whose first row and column consist of entries +1, the process can be reversed to produce the incidence matrix of a $(4n-1, 2n, n)$-configuration. If, further, H has diagonal entries all +1 and is symmetric, then by (1) the configuration corresponds to a $(4n-1, 2n, n)$-graph. We say a graph is "of type C_u" if it is a $(4u^2-1, 2u^2, u^2)$-graph.

THEOREM 3. *There is a graph of type* C_u *if and only if there is a symmetric* Hadamard *matrix of order* $4u^2$ *with diagonal entries all* +1.

This follows from the preceding remarks.

THEOREM 4. *If there is a graph of type* A_u *or a graph of type* B_u *then there is a graph of type* C_u.

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Proof. Suppose there is a graph of type A_u. Then by Theorem 1 there is a symmetric Hadamard matrix A with all diagonal entries $+1$ and with exactly $2u^2 + u$ entries $+1$ in every row. Negate all rows of A with first element -1, and negate the corresponding columns. The resulting matrix C is Hadamard, is symmetric, has every diagonal entry $+1$ and has every entry in the first row and column $+1$. The existence of C together with Theorem 3 imply the existence of a graph of type C_u. If we assume the existence of a B_u, a similar proof applies.

3. Kronecker products

Given two matrices A and B, where A is $p \times q$, the Kronecker product of A and B is

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & \vdots & & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{bmatrix}.$$

It is well-known that if A and B are Hadamard then so is $A \otimes B$.

THEOREM 5. *If there are graphs of types* A_u *and* B_v*, or there are graphs of types* B_u *and* A_v*, then there is a graph of type* B_{2uv}.

Proof. Suppose A and B are symmetric Hadamard matrices corresponding to graphs of types A_u and B_v respectively. Then $A \otimes B$ is a symmetric Hadamard matrix of order $16u^2v^2$;

A and B have all diagonal entries $+1$, so $A \otimes B$ has the same;
A has exactly $2u^2 + u$ elements $+1$ and $2u^2 - u$ elements -1 per row;
B has $2v^2 - v$ positive and $2v^2 + v$ negative, so the number of entries $+1$ in any row of $A \otimes B$ is

$$\begin{aligned}(2u^2+u)(2v^2-v) + (2u^2-u)(2v^2+v) &= 8u^2v^2 - 2uv \\ &= 2(2uv)^2 - (2uv).\end{aligned}$$

Therefore (using Theorem 2) $A \otimes B$ corresponds to a graph of type B_{2uv}.

If we assume that there are graphs of types B_u and A_v, the proof is

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similar.

In the same way we can prove

THEOREM 6. *If there are graphs of types* A_u *and* A_v *, or there are graphs of types* B_u *and* B_v *, then there is a graph of type* A_{2uv} .

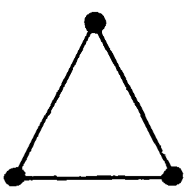
If A and B are symmetric Hadamard matrices of order $4u^2$ and $4v^2$ with every diagonal entry +1 , then $A \otimes B$ will be a matrix of order $16\ u^2\ v^2$ with the same properties. Therefore

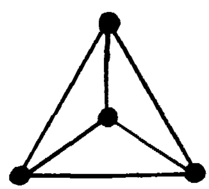
THEOREM 7. *If there are graphs of types* C_u *and* C_v *, then there is a graph of type* C_{2uv} .

4. Some particular graphs.

In order to use Theorems 5, 6 and 7, we need to show that at least some (v,k,λ)-graphs of the various types exist. In fact there are graphs of types A_1 , B_1 , C_1 , A_3 , B_3 , C_3 , C_5 and C_7 .

For any n it is clear that the complete graph of order n is an $(n,n-1,n-2)$-graph. For example, the (3,2,1) and (4,3,2) graphs exist:





These are graphs of types C_1 and B_1 respectively; their incidence matrices are

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and the corresponding Hadamard matrices are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

If n is even, then we can form an $(n,1,0)$ graph by joining points 1 to 2, 3 to 4, and so on: the pairs joined by the arcs are the pairs $\{2i-1,2i\}$. The $(4,1,0)$ and $(6,1,0)$ graphs are

The first of these is of type A_1; its incidence matrix and Hadamard matrix are respectively

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

In [3] there is exhibited the incidence matrix of a $(36,15,6)$-configuration. The matrix is symmetric and has zero diagonal, so it corresponds to a $(36,15,6)$-graph; therefore there is a graph of class A_3. In this particular matrix, if we carry out the operations

(i) for every element in columns 10-27, replace every 0 by a 1 and every 1 by a 0,

(ii) for every element in rows 10-27 of the resulting matrix, replace every 0 by a 1 and every 1 by a 0,

then the resulting matrix corresponds to a $(36,21,12)$-graph. So there is a graph of type B_3. By Theorem 4 there is a graph of type C_3.

R.W. Ahrens (personal communication) has shown that if there is a balanced incomplete block design of parameters

$$(4u^2-1, 2u^2-u, 2u+1, u, 1)$$

then there is a graph of type C_u: the vertices of the graph correspond to

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the blocks of the design, and two vertices are joined by an arc if and only if the corresponding blocks have non-empty intersection. Block designs of the required parameters are given in Appendix I of [2] for $u = 2, 3, 4, 5$ and 7, so there are graphs of types C_2, C_3, C_4, C_5 and C_7.

Combining these results and using Theorems 5, 6 and 7, we have

THEOREM 8. *There are graphs of types* A_u *and* B_u *for any* u *of the form* $2^a 3^b$, *where* a *and* b *are non-negative integers such that* $a \geqq b-1$. *There are graphs of types* C_u *for any* u *of the form* $2^c 3^d 5^e 7^f$, *where* c, d, e, *and* f *are non-negative integers such that* $c \geqq d+e+f-1$.

References

[1] R.W. Ahrens and G. Szekeres, "On a combinatorial generalization of twenty-seven lines associated with a cubic surface", *J. Austral. Math. Soc.* (to appear).

[2] Marshall Hall, Jr, *Combinatorial theory* (Blaisdell, Waltham, Massachusetts, 1967).

[3] Jennifer Wallis, "Two new block designs", *J. Combinatorial Theory* (to appear).

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