Tôhoku Math. Journ.
Vol. 19, No. 4, 1967

CERTAIN HYPERSURFACES OF AN ODD DIMENSIONAL SPHERE

MASAFUMI OKUMURA

(Received March 4, 1967)

Introduction. Let us consider a hypersurface M^{n-1} of a Riemannian manifold $\tilde{M}^n$. For a linear transformation F of the tangent bundle $T(\tilde{M}^n)$ of $\tilde{M}^n$ we always define a linear transformation f of the tangent bundle $T(M^{n-1})$ of M^{n-1} in such a way that f is the correspondence $X \in T(M^{n-1})$ to the tangential components of FX to M^{n-1}. The study of such a transformation f is rather fundamental in studying the theory of hypersurfaces of a Riemannian manifold whose tangent bundle admits a remarkable transformation F. In fact, in the previous paper [3] the author has obtained the following result.

Let the linear transformation F be a natural almost complex structure of an even-dimensional Euclidean space and f the induced linear transformation of the tangent bundle of a hypersurface in the above sense. If the linear transformation f is commutative with the linear transformation h which is defined by the second fundamental tensor of the hypersurface, the hypersurface must be one of the followings:

1) $(2n-1)$-dimensional sphere S^{2n-1},
2) $(2n-1)$-dimensional Euclidean space E^{2n-1},
3) Product manifold of an odd dimensional sphere S^p with a Euclidean space E^{2n-p-1}.

In this paper, we assume that $\tilde{M}$ be an odd dimensional sphere S^{2n+1} and F be a linear transformation determined by the natural contact structure on an odd dimensional sphere S^{2n+1}. Then we can define the transformation f of the tangent bundle of a hypersurface in S^{2n+1} and discuss the problem that for what hypersurface the commutativity of f and h to be satisfied. To study this problem, in §1, we give some properties of the contact structure of S^{2n+1} and in §2, some preliminaries of the theory of hypersurface. In §3, we study a hypersurface of an odd dimensional sphere and prepare some identities for later use. In the last §4, we determine the hypersurface with constant mean curvature for which the commutativity $h \cdot f = f \cdot h$ to be valid.

1. Contact Riemannian structure on an odd dimensional sphere. A

$(2n+1)$-dimensional differentiable manifold M is said to have a contact structure and to be a contact manifold if there exists a 1-form η on M such that

$$\eta\wedge(d\eta)^{n}\neq 0 \tag{1.1}$$

everywhere on M, where $d\eta$ is the exterior derivative of η and the symbol $\wedge$ means the exterior multiplication. η is said to be a contact form of M.

Since (1.1) means that the two-form $d\eta$ is of rank $2n$ everywhere on M we can find a vector field E on M uniquely by

$$\eta(E)=1, \qquad d\eta(E, X)=0, \tag{1.2}$$

for an arbitrary $X\in T(M)$.

Let S^{2n+1} be an odd dimensional sphere which is represented by the equation

$$\sum_{A=1}^{2n+2}(x^{A})^{2}=1, \tag{1.3}$$

in a $(2n+2)$-dimensional Euclidean space E^{2n+2} with rectangular coordinates $x^{A}(A=1,2, \cdots, 2n+2)$. We put

$$\eta=\frac{1}{2}\sum_{\alpha=1}^{n+1}(x^{n+1+\alpha}dx^{\alpha}-x^{\alpha}dx^{n+1+\alpha}), \tag{1.4}$$

then the 1-form η defines a contact form of S^{2n+1} and so we find a vector field E on S^{2n+1} satisfying (1.2).

The Riemannian metric $\tilde{g}$ on S^{2n+1} is naturally induced from the Euclidean space E^{2n+2} in such a way that

$$\tilde{g}_{ji}=\tilde{g}\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right)=\delta_{ji}+\frac{x^{j}x^{i}}{(x^{2n+2})^{2}}, \qquad \tilde{g}^{ji}=\delta^{ji}-x^{j}x^{i}. \tag{1.5}$$

With respect to thus defined Riemannian metric, the Riemannian curvature tensor R of S^{2n+1} satisfies

$$R(X, Y)Z=\tilde{g}(Z, Y)X-\tilde{g}(Z, X)Y. \tag{1.6}$$

Using (1.5), we define a transformation F: $T(S^{2n+1})\rightarrow T(S^{2n+1})$ by

$$2\tilde{g}(FX, Y)=d\eta(X, Y), \tag{1.7}$$

for $X, Y \in T(S^{2n+1})$. Then the set $(F, E, \eta, \tilde{g})$ satisfies[1)]

$$\tilde{g}(E, X)=\eta(X), \tag{1.8}$$

$$\tilde{g}(FX, FY)=\tilde{g}(X, Y)-\eta(X)\eta(Y), \tag{1.9}$$

and consequently

$$\eta(FX)=0, \tag{1.10}$$

$$F^2X=-X+\eta(X)E. \tag{1.11}$$

In general, the set (F, E, η, g) which satisfies (1.1), (1.2), (1.7), (1.8) and (1.9) is called a contact Riemannian (or metric) structure. It is known[2)] that if the contact Riemannian structure on S^{2n+1} is defined by (1.4), (1.5) and (1.7), it satisfies further that

$$\frac{1}{2}(\tilde{\nabla}_Z d\eta)(X, Y)=\eta(X)\tilde{g}(Y, Z)-\eta(Y)\tilde{g}(X, Z), \tag{1.12}$$

$$\tilde{\nabla}_X E=FX, \tag{1.13}$$

for any vector fields X, Y, Z on M, where $\tilde{\nabla}_Z$ denotes the covariant derivative with respect to the Riemannian metric $\tilde{g}$.

2. Hypersurfaces of a Riemannian manifold. Let M^m be an m-dimensional orientable differentiable manifold and ϕ be a regular, differentiable mapping M^m into $\tilde{M}$, whose dimension is $m+1$.

By a regular mapping ϕ we mean a differentiable mapping, such that the $m\times(m+1)$ matrix of functions representing the first partial derivatives in any parametric representation of ϕ has rank m at every point of M^m.

The Riemannian metric $\tilde{g}$ of $\tilde{M}$ naturally induces a metric g on M^m by the mapping ϕ in such a way that $g(X, Y)=\tilde{g}(d\phi(X), d\phi(Y))$, where we denote $d\phi$ the differential map of ϕ, and X, Y tangent vectors to M^m. Since the image under $d\phi$ of the tangent vectors at each point $p\in M^m$ forms a hyperplane in the tangent space of $\tilde{M}$ at $\phi(p)$, one can choose either of the two opposite unit vectors orthogonal to the tangent space image. Since M^m is orientable, if we assume that $\tilde{M}$ is also orientable, we can choose the unit normal vector N to the hypersurface in such a way that, if $(B_1, \cdots, B_m)$ is a positively oriented frame of tangent vectors at p, then the frame $(N, d\phi(B_1),$

1) S. Sasaki and Y. Hatakeyama [4].
2) S. Sasaki and Y. Hatakeyama [4].

$\cdots, d\phi(B_m))$ at $\phi(p)$ is positively oriented.

The second fundamental form of the hypersurface M^m is defined as the components along N of the covariant derivative of $d\phi(X)$, that is,

$$\tilde{\nabla}_{d\phi(X)}d\phi(Y) = {}'\nabla_X Y + H(X, Y)N, \tag{2.1}$$

where we denote by $\tilde{\nabla}$ and ${}'\nabla_X Y$ the covariant derivative with respect to the Riemannian metric $\tilde{g}$ and the tangential component of $\tilde{\nabla}_{d\phi(X)}d\phi(Y)$ respectively.

It is easily verified that ${}'\nabla_X Y$ is identical with the covariant differentiation of Y with respect to the induced Riemannian metric g. Thus we write (2.1) as

$$\nabla_X Y = \tilde{\nabla}_X Y - H(X, Y)N, \tag{2.2}$$

where we identify, for each $p \in M^m$, the tangent space $T_p(M^m)$ with $d\phi(T_p(M^m)) \subset T_{\phi(p)}(\tilde{M})$ by means of the immersion ϕ.

Since N is the unit normal vector field to M^m we have

$$\tilde{g}(N, N) = 1, \tag{2.3}$$

and consequently

$$\tilde{g}(N, \tilde{\nabla}_X N) = 0. \tag{2.4}$$

This means that the covariant derivative $\tilde{\nabla}_X N$ of the unit normal N in the direction of the tangent vector field X is tangent to the hypersurface.

On the other hand, differentiating both members of the equation $\tilde{g}(N, Y) = 0$ covariantly in the direction of a tangent vector field X, we have

$$\tilde{g}(\tilde{\nabla}_X N, Y) + \tilde{g}(N, \tilde{\nabla}_X Y) = 0.$$

Substituting (2.2) into the second term of the above equation, we get easily

$$\tilde{g}(\tilde{\nabla}_X N, Y) = -H(X, Y). \tag{2.5}$$

This equation is called the equation of Weingarten.

Let $\{x^i\}, i = 1, 2, \cdots, m$ be local coordinates in an open neighborhood U of $p \in M^m$. The set of the vector fields $(B_1, \cdots, B_m)$, where $B_i = \dfrac{\partial}{\partial x^i}$ $(i = 1, 2, \cdots, m)$ is called the natural frame of M^m. We choose positively orientable frame

$(N, B_1, \cdots, B_m)$ at each point of $\phi(p) \in \tilde{M}$. Then $\tilde{\nabla}_X N$ can be expressed as a linear combination of B_i $(i=1, \cdots, m)$ and we put

$$\tilde{\nabla}_X N = -\sum_{i=1}^{m} h^i B_i$$

or

$$\tilde{\nabla}_{B_j} N = -\sum_{i=1}^{m} h_j{}^i B_i \,. \tag{2.6}$$

Consequently we have from (2.5),

$$H(B_i, B_h) = \sum_{h=1}^{m} h_j{}^h \tilde{g}(B_i, B_h) = \sum_{h=1}^{m} h_j{}^h g(B_i, B_h) = h_j{}^h g_{ih} = h_{ji}, \tag{2.7}$$

where and throughout the paper we put $g(B_i, B_j) = g_{ij}$ and use Einstein's summation convention for brevity. In the following discussions we identify $H(B_j, B_i)$ *and* $h_j{}^i$ by means of g_{ji} or of g^{ji} which is inverse matrix of g_{ji} and so we denote $H(B_j, B_i) = h_{ji}$.

Since the Riemannian connections $\tilde{\nabla}$ and ∇ are both torsionless, we easily see that $H(X, Y) = H(Y, X)$ and consequently $h_{ji} = h_{ij}$.

The mean curvature μ of M^m in $\tilde{M}$ is defined by

$$\mu = \frac{1}{m} g^{ji} h_{ji} = \frac{1}{m} h_i{}^i \,. \tag{2.8}$$

It is a scalar on M^m independent of the choice of the frame.

When at each point of the hypersurface M^m there exists a differentiable function α such that $H(X, Y) = \alpha g(X, Y)$ for any $X, Y \in T(M^m)$, or equivalently

$$h_{ji} = \alpha g_{ji}, \tag{2.9}$$

we call the hypersurface a totally umbilical hypersurface. Moreover, when the proportional function α vanishes identically we call the hypersurface a totally geodesic hypersurface.

LEMMA 2.1. *A necessary and sufficient condition for a hypersurface to be umbilical is that the following equation is satisfied*:

(2.10) $$h_{ji}h^{ji}=\frac{1}{m}(h_r{}^r)^2,$$

where $h^{ji}=g^{jr}g^{is}h_{rs}$ *and* $h_r{}^r=g^{ir}h_{ir}$.

PROOF. This follows from the identity

$$(h_{ji}-\frac{1}{m}h_r{}^r g_{ji})(h^{ji}-\frac{1}{m}h_r{}^r g^{ji})=h_{ji}h^{ji}-\frac{1}{m}(h_r{}^r)^2,$$

and the positive definiteness of the Riemannian metric g.

The relation of the curvature tensors M^m and $\widetilde{M}$ is given by the following Gauss equation

(2.11) $$R_{kjih}=\widetilde{g}(R(d\phi(B_k),\ d\phi(B_j))d\phi(B_i),\ d\phi(B_h))+h_{ji}h_{kh}-h_{ki}h_{jh},$$

where R_{kjih} denotes $g(R(B_k, B_j)B_i, B_h)$ for brevity.

We also give the equation of Codazzi.

(2.12) $$\nabla_j h_{ih}-\nabla_i h_{jh}=g(R(d\phi(B_j),\ d\phi(B_i))d\phi(B_h),\ N)$$

where

$$\nabla_j h_{ih}=\frac{\partial h_{ih}}{\partial x^j}-\left\{ {r \atop ji} \right\}h_{rh}-\left\{ {r \atop jh} \right\}h_{ir}.$$

If the Riemannian manifold $\widetilde{M}$ is a space of constant curvature, that is, if we have

(2.13) $\widetilde{R}(\widetilde{W},\widetilde{X})\widetilde{Y}=k\{\widetilde{g}(\widetilde{X},\widetilde{Y})\widetilde{W}-\widetilde{g}(\widetilde{Y},\widetilde{W})\widetilde{Z}\}$ for any $\widetilde{X},\widetilde{Y}$ *and* $\widetilde{W}\in T(\widetilde{M})$,

then, from (2.11), the curvature tensor R_{kjih} of M^m takes the form

(2.14) $$R_{kjih}=k(g_{ji}g_{kh}-g_{ki}g_{jh})+h_{ji}h_{kh}-h_{ki}h_{jh}.$$

In the same way, for a space of constant curvature $\widetilde{M}$, we have

(2.15) $$\nabla_j h_{ih}-\nabla_i h_{jh}=0,$$

because of (2.12).

3. **Hypersurfaces in an odd dimensional sphere**. We consider a hyper-

surface M^{2n} in an odd dimensional sphere S^{2n+1}. In this paper we regard S^{2n+1} as a contact manifold with contact Riemannian structure (F, E, η, g) defined by (1.4), (1.5) and (1.7).

The transform FX of a tangent vector field $X \in T(M^{2n}) \subset T(S^{2n+1})$ can be expressed as a sum of its tangential part $(FX)^T$ to M^{2n} and its normal part, that is,

$$FX=(FX)^T+\psi(X)N.$$

The correspondence $X \in T(M^{2n})$ to $(FX)^T$ and that $X \in T(M^{2n})$ to $\psi(X)$ define respectively a linear transformation $f: T(M^{2n}) \to T(M^{2n})$ and 1-form ψ on M^{2n}. So the above equation can be rewritten as

$$FX=fX+\psi(X)N, \tag{3.1}$$

from which we get

$$\psi(X)=\tilde{g}(FX, N)=-\tilde{g}(X, FN). \tag{3.2}$$

By means of definitions of F and f we have immediately that

$$g(fX, Y)=-g(X, fY). \tag{3.3}$$

Let $\{x^i\}$ be a local coordinates in a neighbourhood U of $p \in M^{2n}$. We choose a frame $(N, B_1, \cdots, B_{2n})$, $B_i=\dfrac{\partial}{\partial x^i}(i=1, \cdots, 2n)$ in $T(S^{2n+1})$, then from (3.1) we have

$$FB_i=\sum_{k=1}^{2n} f_i^h B_h+f_i N, \tag{3.4}$$

where f_i^h is the components of the matrix which defines the linear transformation f and f_i is that of the 1-form ψ.

On the other hand the transform FN of the unit normal N by F is perpendicular to N and consequently tangent to the hypersurface M^{2n}. Hence it follows that

$$FN=\sum_{h=1}^{2n} k^h B_h. \tag{3.5}$$

Substituting (3.5) into (3.2), we find

(3. 6) $$f_i=\psi(B_i)=-\sum_{h=1}^{2n} k^h\tilde{g}(B_i, B_h)=-k_i.$$

Thus, in what follows we identify f_i and $-k^i$ by means of the Riemannian metric g and denote k^h by $-f^h$.

The vector field E being tangent to S^{2n+1}, it is represented as

(3. 7) $$E=\sum_{h=1}^{2n} p^h B_h+qN,$$

from which we have immediately

(3. 8) $$\eta(B_i)=\tilde{g}(B_i, E)=p^h g_{ih},$$

and consequently we denote $\eta(B_i)$ by p_i.

We also get easily that

(3. 9) $$q=\tilde{g}(E, N)=\eta(N).$$

Transforming again the both members of (3.4) by F and making use of (1.11), (3.4), (3.5) and (3.6), we find

$$-B_i+p_iE=f_i^hf_h^jB_j+f_i^hf_hN-f_if^jB_j\,.$$

Substituting (3.7) into the above equation, we get

$$-B_i+p_ip^hB_h+qp_iN=f_i^hf_h^jB_j+f_i^hf_hN-f_if^jB_j\,,$$

from which

(3.10) $$f_i^hf_h^j=-\delta_i^j+p_ip^j+f_if^j,$$

(3.11) $$f_i^hg_h=qp_i.$$

Transforming again the both members of (3.5) by F and making use of (1.11), (3.4) and (3.6), we get

$$-N+\eta(N)E=-f^if_i^hB_h-f^if_iN.$$

Substituting (3.7) and (3.9) into the above equation, we have

$$-N+qp^hB_h+q^2N=-f^if_i{}^hB_h-f^if_iN,$$

from which

$$f^if_i{}^h=-qp^h, \tag{3.12}$$

$$f^if_i=1-q^2. \tag{3.13}$$

Since $g(FN, FN)=g(-f^iB_i, -f^jB_j)=g_{ji}f^jf^i$, the last equation shows us that the square of the length of FN is always less than 1.

Since the second condition of (1.2) is equivalent to $\tilde{g}(FE, X)=0$ for any $X\in T(M^{2n+1})$, it follows that $FE=0$ and consequently we have

$$FE=p^iFB_i+qFN=0,$$

because of (3.7). Substituting (3.4) and (3.5) into the above equation, we obtain

$$p^if_i{}^hB_h-qf^hB_h+p^if_iN=0,$$

from which

$$p^if_i{}^h=qf^h, \tag{3.14}$$

$$p^if_i=0. \tag{3.15}$$

From (3.7) the second condition of (1.2) is now rewritten as

$$\eta(E)=\tilde{g}(E, E)=\tilde{g}(p^iB_i+qN, p^jB_j+qN)=p^ip^jg_{ji}+q^2=1.$$

From this fact we have

$$g_{ji}p^jp^i=1-q^2. \tag{3.16}$$

Differentiating (1.7) covariantly in the direction of Z and taking account of (1.7) and (1.12), we find

$$\eta(\tilde{X})\tilde{g}(\tilde{Z}, \tilde{Y})-\eta(\tilde{Y})\tilde{g}(\tilde{Z}, \tilde{X})+\frac{1}{2}\,d\eta(\tilde{\nabla}_{\tilde{Z}}\tilde{X}, \tilde{Y})=\tilde{g}(\tilde{\nabla}_{\tilde{Z}}(F\tilde{X}), \tilde{Y}). \tag{3.17}$$

Substituting B_h, B_i *and* B_j for $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ respectively and regarding the fact that

$$d\eta(X, Y)=2\tilde{g}(FX, Y)=2g(fX, Y) \quad \text{for } X, Y\in T(M^{2n})$$

we obtain

$$2\{\eta(B_h)g(B_i, B_j)-\eta(B_i)g(B_h, B_j)\}+d\eta(\nabla_{B_j}B_h, B_i)+H(B_h, B_j)d\eta(N, B_i)$$
$$=2\widetilde{g}(\widetilde{\nabla}_{B_j}(f_h{}^r B_r), B_i)+2f_h\widetilde{g}(\widetilde{\nabla}_{B_j}N, B_i)$$

from which

$$\nabla_j f_{ih}=p_i g_{jh}-p_h g_{ji}+f_i h_{jh}-f_h h_{ji}, \tag{3.18}$$

where we put

$$\nabla_j f_{ih}=g_{hr}\nabla_j f_i{}^r=g_{hr}(\partial_j f_i{}^r+f_i{}^s\left\{{r \atop js}\right\}-f_s{}^r\left\{{s \atop ji}\right\}).$$

Substituting N, B_i and B_j for $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ respectively in (3.17), we get

$$\eta(N)\widetilde{g}(B_j, B_i)+d\eta(\widetilde{\nabla}_{B_j}N, B_i)=\widetilde{g}(\widetilde{\nabla}_{B_j}(FN), B_i).$$

This implies, together with (3.5), (3.6) and (3.9), that

$$\nabla_j f_i=-qg_{ji}+f_j{}^s h_s{}^r g_{ri}, \tag{3.19}$$

where

$$\nabla_j f_i=g_{ih}\nabla_j f^h=g_{ih}(\partial_j f^h+\left\{{h \atop jr}\right\}f^r).$$

Differentiating (3.8) covariantly in the direction of B_j, we obtain

$$\nabla_{B_j}p_i=g(\nabla_{B_j}E, B_i)+g(E, \nabla_{B_j}B_i).$$

Substituting (1.13) into the last equation, we get

$$\nabla_j p_i=(\partial_j p^h+p^r\left\{{h \atop rj}\right\})g_{hi}=f_{ji}+qh_{ji}. \tag{3.20}$$

In exactly the same way, we have

$$\nabla_j q=\nabla_{B_j}q=f_j-p^i h_{ji}, \tag{3.21}$$

because of (3.9).

4. Determination of the hypersurfaces with $\mathbf{f}\cdot\mathbf{h}=\mathbf{h}\cdot\mathbf{f}$. Let M^{2n} be a hypersurface of an odd dimensional sphere S^{2n+1} of radius 1. In this paragraph we always assume that the vector field E over S^{2n+1} is not tangent to the hypersurface at almost every point on S^{2n+1}.

LEMMA 4.1. *If the mapping f and h are commutative, the following identities are valid.*

$$H_{ji}f^j p^i = 0, \tag{4. 1}$$

$$H_{ji}f^j f^i = H_{ji}p^j p^i. \tag{4. 2}$$

PROOF. We operate $f{\cdot}h$ to $p^h B_h$, then $f{\cdot}h = h{\cdot}f$ and (3.14) show us that

$$f{\cdot}h(p^h B_h) = h{\cdot}f(p^h B_h) = h(p^h f_h{}^i B_i) = q f^i h(B_i), \tag{4. 3}$$

and consequently it follows that

$$g(f{\cdot}h(p^h B_h), p^j B_j) = q f^i p^j g(h(B_i), B_j) = q H_{ji} f^i p^j. \tag{4. 4}$$

On the other hand, from (3.3), we have

$$\begin{aligned} g(f{\cdot}h(p^h B_h), p^j B_j) &= -g(h(p^h B_h), f(p^j B_j)) \\ &= -H(p^h B_h, p^j f_j{}^i B_i) = -q H_{ih} p^h f^i. \end{aligned} \tag{4. 5}$$

Comparing (4.4) (4.5), we get (4.1). This proves the first assertion of the lemma.

Next we consider the inner product of $f{\cdot}h(p^h B_h)$ *and* $p^j(B_j)$. Then we get

$$g(f{\cdot}h(p^h B_h), f^j B_j) = g(q f^i h(B_i), f^j B_j) = q H_{ji} p^i f^j, \tag{4. 6}$$

because of (4.3).

On the other hand, from (3.3) and (3.12), it follows that

$$\begin{aligned} g(f{\cdot}h(p^h B_h), f^j B_j) &= -\, g(h(p^h B_h), f^j f(B_j)) \\ &= -\, g(h(p^h B_h), f^j f_j{}^i B_i) = q H_{ih} p^i p^h. \end{aligned}$$

This shows, together with (4.6), that (4.2) is valid. This completes the proof of the lemma.

A vector field X is called an infinitesimal conformal transformation if it satisfies for any Y and $Z \in T(M^{2n})$

$$(L(X)g)(Y, Z) = 2\rho g(X, Y), \tag{4. 7}$$

where ρ is a certain scalar function on M^{2n} and $L(X)$ is an operator of the Lie derivative.

LEMMA 4.2. *Assume that the mapping f and h are commutative. Then the vector field FN is an infinitesimal conformal transformation.*

PROOF. By definition of the Lie derivative the left hand member of (4.7) takes the form

$$(L(X)g)(Y, Z) = \nabla_X(g(Y, Z)) - g([X, Y], Z) + g(Y, [Z, X]) \\ = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),$$

from which

$$(4.8) \qquad (L(FN)g)(B_i, B_j) = g(\nabla_{B_i}(FN), B_j) + g(\nabla_{B_j}(FN), B_i) \\ = - g(\nabla_{B_i}(f^h B_h), B_j) - g(\nabla_{B_j}(f^h B_h), B_i) \\ = - \nabla_j f_i - \nabla_i f_j.$$

Substituting (3.19) into the last terms of the above equation, we obtain

$$(L(FN)g)(B_i, B_j) = h_j{}^r f_{ir} + h_i{}^r f_{jr} + 2q g_{j_i} = 2q g_{ji},$$

because of the assumption $h \cdot f = f \cdot h$. This shows that the vector field FN is an infinitesimal conformal transformation. This completes the proof of the lemma.

Owing to Lemma 4.2, FN is an infinitesimal conformal transformation and consequently we have[3)]

$$(4.9) \qquad \nabla_j \nabla_i f_h + R_{kjih} f^k = - g_{ih} \nabla_j q - g_{jh} \nabla_i q + g_{ji} \nabla_h q,$$

because of (4.8).

On the other hand covariant differentiation of (3.19) gives us that

$$(4.10) \qquad \nabla_j \nabla_i f_h = - \nabla_j h_i{}^r f_{kr} - p_h h_{ji} + p_r h_i{}^r g_{jh} \\ - f_h h_i{}^r h_{jr} + f_r h_i{}^r h_{jh} - g_{ih} \nabla_j q,$$

3) For example K. Yano [6].

because of (3.18). Since S^{2n+1} is a space of constant curvature $k=1$, we have

$$R_{kjih} = g_{ji}g_{kh} - g_{ki}g_{jh} + h_{ji}h_{kh} - h_{ki}h_{jh},$$

by virtue of (2.14), from which

$$R_{kjih}f^k = g_{ji}f_k - g_{jh}f_i + h_{ji}h_{kh}f^k - h_{jh}h_{ki}f^k. \tag{4.11}$$

Substituing (4.10), (4.11) into (4.9) and making use of (3.21), we get

$$-\nabla_j h_i{}^r f_{hr} - p_h h_{ji} - f_h h_i{}^r h_{jr} + h_{ji}h_{kh}f^k = -g_{ji}p^r h_{kr}.$$

Suppose now that M^{2n} has the constant mean curvature μ. Then the above equation gives us that

$$-p_h h_i{}^i - f_h h_{ji}h^{ji} + h_i{}^i h_{jh}f^j = -2np^i h_{ih}, \tag{4.12}$$

because of (2.15). The equations (3.13), (3.16), (4.12) and Lemma 4.1 imply that

$$(1-q^2)h_i{}^i = 2nh_{ih}p^i p^h, \tag{4.13}$$

and

$$(1-q^2)h_{ji}h^{ji} = h_i{}^i h_{jh}f^j f^h. \tag{4.14}$$

Again using Lemma 4.1 and eliminating $h_{jh}f^j f^h$, we get

$$(1-q^2)\left\{h_{ji}h^{ji} - \frac{1}{2n}(h_i{}^i)^2\right\} = 0. \tag{4.15}$$

Let M_0 be a set of non-umbilical points of M^{2n}. Then M_0 is necessarily an open set of M^{2n}. Furthermore by means of Lemma 2.1 and (4.15) it follows that $q^2=1$ in M_0, consequently we get $p_i p^i=0$ because of (3.16). The Riemannian metric g being positive definite, this shows that $p_i=0$ at each point of M_0. From this fact, in M_0, we have

$$\nabla_j p_i = f_{ji} + qh_{ji} = 0,$$

by virtue of (3.2). However since f_{ji} is skew-symmetric and h_{ji} symmetric with respect to their indices, the last equation means that $f_{ji}=0$, $h_{ji}=0$ in M_0. Thus we have

$$\mu = 0$$

at each point of M_0. The mean curvature μ being constant over M^{2n}, it follows that $h_{ji}=0$ in the complement of M_0 in M^{2n}. Thus we can deduce that if there exists a non-umbilical point in M^{2n}, the hypersurface is totally geodesic. From these consideration we can prove the following

THEOREM 4.3. *Let M^{2n} be a hypersurface with constant mean curvature of an odd dimensional sphere S^{2n+1} and E be the unit vector field on S^{2n+1} which is determined naturally from the contact form of S^{2n+1}. If E is not tangent to M^{2n} at almost everywhere and if the mapping f and h are commutative, the hypersurface is umbilical. Furthermore if M^{2n} is complete, it is a $2n$-dimensional sphere.*

PROOF. The proof of the first statement of the theorem has already given in the above discussions and so we give the proof of the latter statement of the theorem.

Since M^{2n} is totally umbilical, from (2.14), it follows that

$$R_{kjih}=(1+k^2)(g_{ji}g_{kh}-g_{ki}g_{jh}),\ k=\frac{1}{2n}h_r{}^r. \tag{4.16}$$

This shows that the hypersurface M^{2n} is a space of positive constant curvature. M^{2n} being connected, complete Riemannian manifold, we have from the Myers' theorem[4], M^{2n} is compact. Moreover, M^{2n} being orientable and even-dimensional, it is simply connected.[5] Thus we have the latter assertion of the theorem.[6] This completes the proof.

REMARK. Recently Y. Watanabe [7] have proved that in a normal contact Riemannian manifold, a complete, totally umbilical hypersurface with constant mean curvature μ is isometric with a sphere of radius $1/\sqrt{1+\mu^2}$. Using this result, we can also prove the latter assertion of the theorem.

BIBLIOGRAPHY

[1] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Interscience Tracts, John Wiley and Sons, New York, 1963.

[2] S. MYERS, Riemannian manifolds with positive mean curvature, Duke Math. Journ, 8 (1941) 401-404.

[3] M. OKUMURA, Certain almost contact hypersurfaces in Euclidean spaces, Kōdai Math. Sem. Rep., 16(1964), 44-54.

4) S. Myers [2].
5) J. L. Synge [5].
6) For example S. Kobayashi and K. Nomizu [1], p. 294.

[4] S. Sasaki AND Y. Hatakeyama, On differentiable manifolds with contact metric structure, Journ. Math. Soc. of Japan, 14(1962), 249-271.

[5] J. L. SYNGE, On the connectivity of spaces of positive curvature, Quart. Journ. Math. Oxford, 7(1936), 316-320.
[6] K. YANO, The theory of Lie derivatives and its applications, North Holland Publ. Co., Amsterdam, 1957.
[7] Y. WATANABE, Totally umbilical surfaces in a normal contact Riemannian manifolds, Kōdai Math. Sem. Rep. 19(1967), 474-487.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
TOKYO, JAPAN