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Certain inequalities via generalized proportional Hadamard fractional integral operators

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Abstract

In the article, we introduce the generalized proportional Hadamard fractional integrals and establish several inequalities for convex functions in the framework of the defined class of fractional integrals. The given results are generalizations of some known results.

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1 Introduction

Fractional calculus has gained more attention due to its applications in distinct fields. In the development of fractional calculus, researchers focus on developing several fractional integral operators and their applications in diverse fields. The idea of fractional conformable derivative operators was introduced by Khalil et al. [23] with a deficiency that the new derivative operator does not tend to the original function when the order $\rho \to 0$. In [1], Abdeljawad studied studied various properties of the fractional conformable derivative operators and raised the problem of how to use conformable derivative operators to generate more general nonlocal fractional derivative operators; the method was demonstrated in [22]. Later on in [12], Anderson and Unless improved the idea of the fractional conformable derivative by introducing the idea of local proportional derivatives. In [2, 3, 13, 14, 26], some researchers defined new continuous and discrete fractional derivative operators by using the exponential and Mittag-Leffler functions in the kernels. Generalizations of such type promote future research to establish new ideas to unify the fractional derivative and integral operators and obtain fractional integral inequalities via such generalized fractional derivative and integral operators. Integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics.

Certain weighted Grüss-type inequalities and new inequalities involving Riemann–Liouville fractional integrals are found in [17, 19]. In [29], the authors define several inequalities for the extended gamma and confluent hypergeometric k-functions. In [30], the authors established certain Gronwall inequalities for Riemann–Liouville and Hadamard k-fractional derivatives with applications. Inequalities involving the generalized (k,ρ)-

fractional integral operators can be found in [35]. The generalized fractional integral and its applications and Grüss-type inequalities via generalized fractional integrals can be found in [40, 42]. In [39], Sarikaya and Budak have studied the (k,s)-Riemann–Liouville fractional integral and applications. In [41], Set et al. established generalized Hermite–Hadamard-type inequalities via fractional integral operators. In [9], Agarwal et al. introduced Hermite–Hadamard-type inequalities by employing the k-fractional integrals operators. In [18], Dahmani introduced certain classes of fractional integral inequalities by utilizing a family of n positive functions.

In [10], the authors established fractional integral inequalities for a class of family of n ($n \in \mathbb{N}$) positive continuous and decreasing functions on $[a,b]$ by employing the (k,s)-fractional integral operators. Recently, the fractional conformable integrals have attracted the attention of many researchers. In particular, many remarkable inequalities, properties, and applications for the fractional conformable integrals can be found in the literature [4–8, 16, 20, 24, 25, 27, 28, 31, 32, 34, 36, 37, 43].

2 Preliminaries

In [21], Jarad et al. introduced the left and right generalized proportional integral operators which are respectively defined by

$$\left({}_a\Im^{\beta,\mu}f\right)(x)=\frac{1}{\mu^{\beta}\Gamma(\beta)}\int_a^x \exp\left[\frac{\mu-1}{\mu}(x-t)\right](x-t)^{\beta-1}f(t)\,dt,\quad a<x \tag{2.1}$$

and

$$\left(\Im_b^{\beta,\mu}f\right)(x)=\frac{1}{\mu^{\beta}\Gamma(\beta)}\int_x^b \exp\left[\frac{\mu-1}{\mu}(t-x)\right](t-x)^{\beta-1}f(t)\,dt,\quad x<b, \tag{2.2}$$

where the proportionality index $\mu \in (0,1]$, $\beta \in \mathbb{C}$, $\Re(\beta)>0$, and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ is the gamma function [44–46].

Remark 2.1 If we consider $\mu=1$ in (2.1) and (2.2), then we get the left and right Riemann–Liouville integrals, which are respectively defined as

$$\left({}_a\Im^{\beta}f\right)(x)=\frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}f(t)\,dt,\quad a<x, \tag{2.3}$$

and

$$\left(\Im_b^{\beta}f\right)(x)=\frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}f(t)\,dt,\quad x<b, \tag{2.4}$$

where $\beta \in \mathbb{C}$ and $\Re(\beta)>0$.

Recently Alzabut et al. and Rahman et al. [11, 34] studied generalized proportional derivative and integral operators and established certain Gronwall and Minkowski inequalities involving the operators mentioned above.

Next, motivated by the above, we define the following generalized version of Hadamard fractional integrals. Rahman et al. [33] presented certain new classes of integral inequalities for a class of n ($n \in \mathbb{N}$) positive continuous and decreasing functions on $[a,b]$ by employing generalized proportional fractional integrals.

Definition 2.1 The left-sided generalized proportional Hadamard fractional integral of order $\beta > 0$ and proportionality index $\mu \in (0,1]$ is defined by

$$\left({}_a\mathcal{H}^{\beta,\mu} f\right)(x) = \frac{1}{\mu^\beta \Gamma(\beta)} \int_a^x \exp\left[\frac{\mu-1}{\mu}(\ln x - \ln t)\right](\ln x - \ln t)^{\beta-1}\frac{f(t)}{t}\,dt, \quad a < x. \tag{2.5}$$

Definition 2.2 The right-sided generalized proportional Hadamard fractional integral of order $\beta > 0$ and proportionality index $\mu \in (0,1]$ is defined by

$$\left(\mathcal{H}_b^{\beta,\mu} f\right)(x) = \frac{1}{\mu^\beta \Gamma(\beta)} \int_x^b \exp\left[\frac{\mu-1}{\mu}(\ln t - \ln x)\right](\ln t - \ln x)^{\beta-1}\frac{f(t)}{t}\,dt, \quad x < b. \tag{2.6}$$

Definition 2.3 The one-sided generalized proportional Hadamard fractional integral of order $\beta > 0$ and proportionality index $\mu \in (0,1]$ is defined by

$$\left(\mathcal{H}_{1,x}^{\beta,\mu} f\right)(x) = \frac{1}{\mu^\beta \Gamma(\beta)} \int_1^x \exp\left[\frac{\mu-1}{\mu}(\ln x - \ln t)\right](\ln x - \ln t)^{\beta-1}\frac{f(t)}{t}\,dt, \quad t > 1. \tag{2.7}$$

Remark 2.2 If we consider $\mu = 1$, then (2.5)–(2.7) will lead to the following well-known Hadamard fractional integrals:

$$\left({}_a\mathcal{H}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_a^x (\ln x - \ln t)^{\beta-1}\frac{f(t)}{t}\,dt, \quad a < x, \tag{2.8}$$

$$\left(\mathcal{H}_b^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_x^b (\ln t - \ln x)^{\beta-1}\frac{f(t)}{t}\,dt, \quad x < b, \tag{2.9}$$

and

$$\left(\mathcal{H}_{1,x}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_1^x (\ln x - \ln t)^{\beta-1}\frac{f(t)}{t}\,dt, \quad t > 1. \tag{2.10}$$

One can easily prove the following results:

Lemma 2.1

$$\begin{aligned}&\left(\mathcal{H}_{1,x}^{\beta,\mu} \exp\left[\frac{\mu-1}{\mu}(\ln x)\right](\ln x)^{\lambda-1}\right)(x)\\ &\quad= \frac{\Gamma(\lambda)}{\mu^\beta \Gamma(\beta+\lambda)} \exp\left[\frac{\mu-1}{\mu}(\ln x)\right](\ln x)^{\beta+\lambda-1}\end{aligned} \tag{2.11}$$

and the semigroup property holds:

$$\left(\mathcal{H}_{1,x}^{\beta,\mu}\right)\left(\mathcal{H}_{1,x}^{\lambda,\mu}\right) f(x) = \left(\mathcal{H}_{1,x}^{\beta+\lambda,\mu}\right) f(x). \tag{2.12}$$

Remark 2.3 If $\mu = 1$, then (2.11) will reduce to the result of [38] as defined by

$$\left(\mathcal{H}_{1,x}^{\beta}(\ln x)^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\beta+\lambda)}(\ln x)^{\beta+\lambda-1}. \tag{2.13}$$

3 Main results

In this section, we employ the generalized proportional Hadamard fractional integral operator to establish generalizations of some classical inequalities.

Theorem 3.1 *Let f and h be two positive continuous functions on the interval $[1,\infty)$ and $f \le h$ on $[1,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[1,\infty)$, then for a convex function Φ with $\Phi(0) = 0$, the generalized proportional Hadamard fractional integral operator given by (2.7) satisfies the inequality*

$$\frac{\mathcal{H}_{1,x}^{\beta,\mu}[f(x)]}{\mathcal{H}_{1,x}^{\beta,\mu}[h(x)]} \ge \frac{\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(f(x))]}{\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(h(x))]}, \tag{3.1}$$

where $\mu \in (0,1]$, $\beta \in \mathbb{C}$, and $\Re(\beta) > 0$.

Proof Since Φ is convex with $\Phi(0) = 0$, the function $\frac{\Phi(f(x))}{x}$ is increasing. As f is increasing, so is the function $\frac{\Phi(f(x))}{f(x)}$. Obviously, the function $\frac{f(x)}{h(x)}$ is decreasing. Thus for all $t, \vartheta \in [1,\infty)$, we have

$$\left(\frac{\Phi(f(t))}{f(t)} - \frac{\Phi(f(\vartheta))}{f(\vartheta)}\right)\left(\frac{f(\vartheta)}{h(\vartheta)} - \frac{f(t)}{h(t)}\right) \ge 0. \tag{3.2}$$

It follows that

$$\frac{\Phi(f(t))}{f(t)}\frac{f(\vartheta)}{h(\vartheta)} + \frac{\Phi(f(\vartheta))}{f(\vartheta)}\frac{f(t)}{h(t)} - \frac{\Phi(f(\vartheta))}{f(\vartheta)}\frac{f(\vartheta)}{h(\vartheta)} - \frac{\Phi(f(t))}{f(t)}\frac{f(t)}{h(t)} \ge 0. \tag{3.3}$$

Multiplying (3.3) by $h(t)h(\vartheta)$, we have

$$\frac{\Phi(f(t))}{f(t)}f(\vartheta)h(t) + \frac{\Phi(f(\vartheta))}{f(\vartheta)}f(t)h(\vartheta) - \frac{\Phi(f(\vartheta))}{f(\vartheta)}f(\vartheta)h(t) - \frac{\Phi(f(t))}{f(t)}f(t)h(\vartheta) \ge 0. \tag{3.4}$$

Multiplying (3.4) by $\frac{1}{\mu^\beta \Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}$, which is positive because $t \in (1,x)$, $x > 1$ and integrating the resulting identity from 1 to x, we have

$$\begin{aligned}
&\frac{1}{\mu^\beta \Gamma(\beta)}\int_1^x \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t}\frac{\Phi(f(t))}{f(t)}f(\vartheta)h(t)\,dt \\
&\quad + \frac{1}{\mu^\beta \Gamma(\beta)}\int_1^x \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t}\frac{\Phi(f(\vartheta))}{f(\vartheta)}f(t)h(\vartheta)\,dt \\
&\quad - \frac{1}{\mu^\beta \Gamma(\beta)}\int_1^x \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t}\frac{\Phi(f(\vartheta))}{f(\vartheta)}f(\vartheta)h(t)\,dt \\
&\quad - \frac{1}{\mu^\beta \Gamma(\beta)}\int_1^x \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t}\frac{\Phi(f(t))}{f(t)}f(t)h(\vartheta)\,dt \ge 0.
\end{aligned} \tag{3.5}$$

This follows that

$$\begin{aligned}
&f(\vartheta)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)\right) + \left(\frac{\Phi(f(\vartheta))}{f(\vartheta)}h(\vartheta)\right)\mathcal{H}_{1,x}^{\beta,\mu}(f(x)) \\
&\quad - \left(\frac{\Phi(f(\vartheta))}{f(\vartheta)}f(\vartheta)\right)\mathcal{H}_{1,x}^{\beta,\mu}(h(x)) - h(\vartheta)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}f(x)\right) \ge 0.
\end{aligned} \tag{3.6}$$

Again, multiplying both sides of (3.6) by $\frac{1}{\mu^{\beta}\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln\vartheta)](\ln x-\ln\vartheta)^{\beta-1}}{\vartheta}$, which is positive because $\vartheta\in(1,x)$, $x>1$ and integrating the resulting identity from 1 to x, we get

$$\begin{aligned}&\mathcal{H}_{1,x}^{\beta,\mu}\big(f(x)\big)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)\right)+\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)\right)\mathcal{H}_{1,x}^{\beta,\mu}\big(f(x)\big)\\&\quad\ge\mathcal{H}_{1,x}^{\beta,\mu}\big(h(x)\big)\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(f(x)\big)\big)+\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(f(x)\big)\big)\mathcal{H}_{1,x}^{\beta,\mu}\big(h(x)\big).\end{aligned}\tag{3.7}$$

It follows that

$$\frac{\mathcal{H}_{1,x}^{\beta,\mu}(f(x))}{\mathcal{H}_{1,x}^{\beta,\mu}(h(x))}\ge\frac{\mathcal{H}_{1,x}^{\beta,\mu}(\Phi(f(x)))}{\mathcal{H}_{1,x}^{\beta,\mu}(\frac{\Phi(f(x))}{f(x)}h(x))}.\tag{3.8}$$

Now, since $f\le h$ on $[1,\infty)$ and $\frac{\Phi(x)}{x}$ is an increasing function, for $t,\vartheta\in[1,x)$, we have

$$\frac{\Phi(f(t))}{f(t)}\le\frac{\Phi(h(t))}{h(t)}.\tag{3.9}$$

Multiplying both sides of (3.9) by $\frac{1}{\mu^{\beta}\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}h(t)$, which is positive because $t\in(1,x)$, $x>1$ and integrating the resulting identity from 1 to x, we get

$$\begin{aligned}&\frac{1}{\mu^{\beta}\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(f(t))}{f(t)}h(t)\,dt\\&\quad\le\frac{1}{\mu^{\beta}\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(h(t))}{h(t)}h(t)\,dt,\end{aligned}\tag{3.10}$$

which, in view of (2.7), can be written as

$$\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)\right)\le\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(h(x)\big)\big).\tag{3.11}$$

Hence from (3.8) and (3.11), we get (3.1). □

Theorem 3.2 *Let f and h be two positive continuous functions on the interval $[1,\infty)$ and $f\le h$ on $[1,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[1,\infty)$, then for a convex function Φ with $\Phi(0)=0$, the generalized proportional Hadamard fractional integral operator given by (2.7) satisfies the inequality*

$$\frac{\mathcal{H}_{1,x}^{\beta,\mu}[f(x)]\mathcal{H}_{1,x}^{\rho,\mu}[\Phi(h(x))]+\mathcal{H}_{1,x}^{\rho,\mu}[f(x)]\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(h(x))]}{\mathcal{H}_{1,x}^{\beta,\mu}[h(x)]\mathcal{H}_{1,x}^{\rho,\mu}[\Phi(f(x))]+\mathcal{H}_{1,x}^{\rho,\mu}[h(x)]\mathcal{H}_{1,x}^{\rho,\mu}[\Phi(f(x))]}\ge1,\tag{3.12}$$

where $\mu\in(0,1]$, $\beta,\rho\in\mathbb{C}$, $\Re(\beta)>0$, and $\Re(\rho)>0$.

Proof Since Φ is convex with $\Phi(0)=0$, the function $\frac{\Phi(f(x))}{x}$ is increasing. As f is increasing, so is the function $\frac{\Phi(f(x))}{f(x)}$. Obviously, the function $\frac{f(x)}{h(x)}$ is decreasing for all $t,\vartheta\in[1,x)$. Multiplying (3.6) by $\frac{1}{\mu^{\rho}\Gamma(\rho)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln\vartheta)](\ln x-\ln\vartheta)^{\rho-1}}{\vartheta}$, which is positive because $\vartheta\in(1,x)$,

$\vartheta > 1$ and integrating the resulting identity from 1 to x, we get

$$\begin{aligned} &\mathcal{H}_{1,x}^{\rho,\mu}\bigl(f(x)\bigr)\mathcal{H}_{1,x}^{\beta,\mu}\biggl(\frac{\Phi(f(x))}{f(x)}h(x)\biggr) + \mathcal{H}_{1,x}^{\rho,\mu}\biggl(\frac{\Phi(f(x))}{f(x)}h(x)\biggr)\mathcal{H}_{1,x}^{\beta,\mu}\bigl(f(x)\bigr) \\ &\quad \geq \mathcal{H}_{1,x}^{\beta,\mu}\bigl(h(x)\bigr)\mathcal{H}_{1,x}^{\rho,\mu}\biggl(\frac{\Phi(f(x)}{f(x)}f(x)\biggr) + \mathcal{H}_{1,x}^{\beta,\mu}\biggl(\frac{\Phi(f(x)}{f(x)}f(x)\biggr)\mathcal{H}_{1,x}^{\rho,\mu}\bigl(h(x)\bigr). \end{aligned} \tag{3.13}$$

Now, since $f \leq h$ on $[1,\infty)$ and $\frac{\Phi(x)}{x}$ is an increasing function, for $t, \vartheta \in [1,x)$, we have

$$\frac{\Phi(f(t))}{f(t)} \leq \frac{\Phi(h(t))}{h(t)}. \tag{3.14}$$

Multiplying both sides of (3.14) by $\frac{1}{\mu^\beta \Gamma(\beta)} \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t} h(t)$, which is positive because $t \in (1,x)$, $x > 1$ and integrating the resulting identity with respect to t from 1 to x, we get

$$\begin{aligned} &\frac{1}{\mu^\beta \Gamma(\beta)} \int_1^x \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t} \frac{\Phi(f(t))}{f(t)} h(t)\,dt \\ &\quad \leq \frac{1}{\mu^\beta \Gamma(\beta)} \int_1^x \frac{\exp[\frac{\mu-1}{\mu}(\ln x - \ln t)](\ln x - \ln t)^{\beta-1}}{t} \frac{\Phi(h(t))}{h(t)} h(t)\,dt, \end{aligned} \tag{3.15}$$

which, in view of (2.7), can be written as

$$\mathcal{H}_{1,x}^{\beta,\mu}\biggl(\frac{\Phi(f(x))}{f(x)}h(x)\biggr) \leq \mathcal{H}_{1,x}^{\beta,\mu}\bigl(\Phi\bigl(h(x)\bigr)\bigr). \tag{3.16}$$

Hence, from (3.11), (3.13), and (3.16), we get the desired result. □

Remark 3.1 If we consider $\beta = \rho$, then Theorem 3.2 will lead to Theorem 3.1.

Theorem 3.3 *Let f, h, and g be positive continuous functions on the interval $[1,\infty)$ and $f \leq h$ on $[1,\infty)$. If $\frac{f}{h}$ is decreasing and f and g are increasing on $[1,\infty)$, then for a convex function Φ with $\Phi(0) = 0$, the generalized proportional Hadamard fractional integral operator given by (2.7) satisfies the inequality*

$$\frac{\mathcal{H}_{1,x}^{\beta,\mu}[f(x)]}{\mathcal{H}_{1,x}^{\beta,\mu}[h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(f(x))g(x)]}{\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(h(x))g(x)]}, \tag{3.17}$$

where $\mu \in (0,1]$, $\beta \in \mathbb{C}$ and $\Re(\beta) > 0$.

Proof Since $f \leq h$ on $[1,\infty)$ and $\frac{\Phi(x)}{x}$ is increasing, for $t, \vartheta \in [1,x)$, we have

$$\frac{\Phi(f(t))}{f(t)} \leq \frac{\Phi(h(t))}{h(t)}. \tag{3.18}$$

Multiplying both sides of (3.18) by $\frac{1}{\mu^\beta\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}h(t)g(t)$, which is positive because $t\in(1,x)$, $x>1$ and integrating the resulting identity from 1 to x, we get

$$\begin{aligned}&\frac{1}{\mu^\beta\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(f(t))}{f(t)}h(t)g(t)\,dt\\&\quad\le\frac{1}{\mu^\beta\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(h(t))}{h(t)}h(t)g(t)\,dt,\end{aligned}\tag{3.19}$$

which, in view of (2.7), can be written as

$$\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)\le\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(h(x)\big)g(x)\big).\tag{3.20}$$

Also, since the function Φ is convex and such that $\Phi(0)=0$, $\frac{\Phi(t)}{t}$ is increasing. Since f is increasing, so is $\frac{\Phi(f(t))}{f(t)}$. Clearly, the function $\frac{f(t)}{h(t)}$ is decreasing for all $t,\vartheta\in[1,x)$, $x>1$. Thus,

$$\left(\frac{\Phi(f(t))}{f(t)}g(t)-\frac{\Phi(f(\vartheta))}{f(\vartheta)}g(\vartheta)\right)\big(f(\vartheta)h(t)-f(t)h(\vartheta)\big)\ge0.\tag{3.21}$$

It follows that

$$\begin{aligned}&\frac{\Phi(f(t))g(t)}{f(t)}f(\vartheta)h(t)+\frac{\Phi(f(\vartheta))g(\vartheta)}{f(\vartheta)}f(t)h(\vartheta)\\&\quad-\frac{\Phi(f(\vartheta))g(\vartheta)}{f(\vartheta)}f(\vartheta)h(t)-\frac{\Phi(f(t))g(t)}{f(t)}f(t)h(\vartheta)\ge0.\end{aligned}\tag{3.22}$$

Multiplying (3.22) by $\frac{1}{\mu^\beta\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}$, which is positive because $t\in(1,x)$, $x>1$ and integrating the resulting identity from 1 to x, we have

$$\begin{aligned}&\frac{1}{\mu^\beta\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(f(t))}{f(t)}f(\vartheta)h(t)g(t)\,dt\\&\quad+\frac{1}{\mu^\beta\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(f(\vartheta))}{f(\vartheta)}f(t)h(\vartheta)g(\vartheta)\,dt\\&\quad-\frac{1}{\mu^\beta\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(f(\vartheta))}{f(\vartheta)}f(\vartheta)g(\vartheta)h(t)\,dt\\&\quad-\frac{1}{\mu^\beta\Gamma(\beta)}\int_1^x\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}\frac{\Phi(f(t))}{f(t)}f(t)g(t)h(\vartheta)\,dt\\&\ge0.\end{aligned}\tag{3.23}$$

This follows that

$$\begin{aligned}&f(\vartheta)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)+\left(\frac{\Phi(f(\vartheta))}{f(\vartheta)}h(\vartheta)g(\vartheta)\right)\mathcal{H}_{1,x}^{\beta,\mu}\big(f(x)\big)\\&\quad-\left(\frac{\Phi(f(\vartheta))}{f(\vartheta)}f(\vartheta)g(\vartheta)\right)\mathcal{H}_{1,x}^{\beta,\mu}\big(h(x)\big)-h(\vartheta)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}f(x)g(x)\right)\ge0.\end{aligned}\tag{3.24}$$

Again, multiplying both sides of (3.24) by $\frac{1}{\mu^\beta\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln\vartheta)](\ln x-\ln\vartheta)^{\beta-1}}{\vartheta}$, which is positive because $\vartheta\in(1,x)$, $\vartheta>1$ and integrating the resulting identity from 1 to x, we get

$$\begin{aligned}&\mathcal{H}_{1,x}^{\beta,\mu}\big(f(x)\big)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)+\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)\mathcal{H}_{1,x}^{\beta,\mu}\big(f(x)\big)\\&\quad\geq\mathcal{H}_{1,x}^{\beta,\mu}\big(h(x)\big)\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(f(x)\big)g(x)\big)+\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(f(x)\big)g(x)\big)\mathcal{H}_{1,x}^{\beta,\mu}\big(h(x)\big).\end{aligned}\tag{3.25}$$

It follows that

$$\frac{\mathcal{H}_{1,x}^{\beta,\mu}(f(x))}{\mathcal{H}_{1,x}^{\beta,\mu}(h(x))}\geq\frac{\mathcal{H}_{1,x}^{\beta,\mu}(\Phi(f(x))g(x))}{\mathcal{H}_{1,x}^{\beta,\mu}(\frac{\Phi(f(x))}{f(x)}h(x)g(x))}.\tag{3.26}$$

Hence, from (3.20) and (3.26), we obtain the required result. □

Theorem 3.4 *Let f, h, and g be positive continuous functions on the interval $[1,\infty)$ and $f\leq h$ on $[1,\infty)$. If $\frac{f}{h}$ is decreasing and f and g are increasing on $[1,\infty)$, then for a convex function Φ with $\Phi(0)=0$, the generalized proportional Hadamard fractional integral operator given by (2.7) satisfies the inequality*

$$\frac{\mathcal{H}_{1,x}^{\beta,\mu}[f(x)]\mathcal{H}_{1,x}^{\rho,\mu}[\Phi(f(x))g(x)]+\mathcal{H}_{1,x}^{\rho,\mu}[f(x)]\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(f(x))g(x)]}{\mathcal{H}_{1,x}^{\beta,\mu}[h(x)]\mathcal{H}_{1,x}^{\rho,\mu}[\Phi(f(x))g(x)]+\mathcal{H}_{1,x}^{\rho,\mu}[h(x)]\mathcal{H}_{1,x}^{\beta,\mu}[\Phi(f(x))g(x)]}\geq1,\tag{3.27}$$

where $\mu\in(0,1]$, $\beta,\rho\in\mathbb{C}$, $\Re(\beta)>0$, and $\Re(\rho)>0$.

Proof Multiplying both sides of (3.27) by $\frac{1}{\mu^\beta\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln\vartheta)](\ln x-\ln\vartheta)^{\rho-1}}{\vartheta}$, which is positive because $\vartheta\in(1,x)$, $\vartheta>1$ and integrating the resulting identity from 1 to x, we get

$$\begin{aligned}&\mathcal{H}_{1,x}^{\rho,\mu}\big(f(x)\big)\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)+\mathcal{H}_{1,x}^{\rho,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)\mathcal{H}_{1,x}^{\beta,\mu}\big(f(x)\big)\\&\quad\geq\mathcal{H}_{1,x}^{\beta,\mu}\big(h(x)\big)\mathcal{H}_{1,x}^{\rho,\mu}\big(\Phi\big(f(x)\big)g(x)\big)+\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(f(x)\big)g(x)\big)\mathcal{H}_{1,x}^{\rho,\mu}\big(h(x)\big).\end{aligned}\tag{3.28}$$

Since $f\leq h$ on $[1,\infty)$ and $\frac{\Phi(x)}{x}$ is increasing, for $t,\vartheta\in[1,x)$, we have

$$\frac{\Phi(f(t))}{f(t)}\leq\frac{\Phi(h(t))}{h(t)}.\tag{3.29}$$

Multiplying both sides of (3.29) by $\frac{1}{\mu^\beta\Gamma(\beta)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\beta-1}}{t}h(t)g(t)$, $t\in(1,x)$, $x>1$ and integrating the resulting identity from 1 to x, we get

$$\mathcal{H}_{1,x}^{\beta,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)\leq\mathcal{H}_{1,x}^{\beta,\mu}\big(\Phi\big(h(x)\big)g(x)\big).\tag{3.30}$$

Similarly, multiplying both sides of (3.29) by $\frac{1}{\mu^\rho\Gamma(\rho)}\frac{\exp[\frac{\mu-1}{\mu}(\ln x-\ln t)](\ln x-\ln t)^{\rho-1}}{t}h(t)g(t)$, $t\in(1,x)$, $x>1$ and integrating the resulting identity from 1 to x, we get

$$\mathcal{H}_{1,x}^{\rho,\mu}\left(\frac{\Phi(f(x))}{f(x)}h(x)g(x)\right)\leq\mathcal{H}_{1,x}^{\rho,\mu}\big(\Phi\big(h(x)\big)g(x)\big).\tag{3.31}$$

Hence, from (3.28), (3.30), and (3.31), we obtain the required inequality (3.31). □

Remark 3.2 If we consider $\beta = \rho$, then Theorem 3.4 will lead to Theorem 3.3.

4 Concluding remarks

In this paper, first we defined nonlocal generalized proportional Hadamard fractional integral operators and then we established certain inequalities by employing the generalized proportional Hadamard fractional integral operator. The inequalities obtained in this present paper will lead to the classical inequalities which are established earlier by Chinchane and Pachpatte [15]. The results established in this paper give some contribution in the field of fractional calculus and Hadamard fractional integral inequalities. One can establish various integral inequalities by employing the newly defined Hadamard fractional integral operators.

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