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Certain integral transforms concerning the product of family of polynomials and generalized incomplete functions

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Abstract. In this article, we have derived some integral transforms of the polynomial weighted incomplete H-functions and incomplete $\overline{H}$-functions. The obtained image formulas are of general nature and may, as special cases, give rise to integral transforms involved with the H-functions and $\overline{H}$-functions.

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1. Introduction and Definitions

Special functions are significant in the analysis of differential equation solutions and are correlated with a broad variety of issues in several fields of mathematical physics, such as acoustics, radio physics, hydrodynamics, and atomic and nuclear physics [2, 5, 6, 7, 23]. Particularly, H-function and its applications in different sub-fields of related mathematical research have been defined as significant, see [3, 8, 10, 12, 13]. Incomplete special functions have recently

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been applicable to a broad spectrum of problems relating to reaction, combustion, reaction-diffusion, electronics and communication, fractional differential and integral equations, other fields of theoretical mechanics, statistical probability theory, etc. Consequently, several research papers [4, 9, 15, 16, 17, 19, 24, 25, 26] on incomplete special functions have recently been published by various researchers along with related other higher transcendental special functions.

On the other hand, integral transformations are extensively used and therefore a huge amount of work has already been accomplished on the concept and its implementations, one can refer [1, 14, 18, 20, 27, 28]. The computing of image formulas under integral transformations of the special functions of one or maybe more variables is significant from the perspective of the effectiveness regarding these outcomes in the solution for differential and integral equations. Inspired by such directions of use, several researchers have produced a relatively large number of image formulas for fundamental transformations concerning a range of special functions. In this article, we take into account and derive the various integral transforms involved with the product of a general family of polynomials and incomplete H-function or incomplete $\overline{H}$-functions.

The Gamma functions, H-functions and $\overline{H}$-functions of incomplete type, that are to be used in the sequel, are described below:

The familiar lower and upper gamma functions of incomplete type $\gamma(s,x)$ and $\Gamma(s,x)$, respectively, are characterized as:

$$\gamma(s,u) := \int_0^u v^{s-1}\, e^{-v}\, dv, \qquad \big(\Re(s) > 0;\ u \geqq 0\big), \tag{1.1}$$

and

$$\Gamma(s,u) := \int_u^\infty v^{s-1}\, e^{-v}\, dv, \qquad \big(u \geqq 0;\ \Re(s) > 0 \quad \text{if} \quad u = 0\big). \tag{1.2}$$

These functions fulfill the corresponding relation:

$$\gamma(s,u) + \Gamma(s,u) = \Gamma(s),\ (\Re(s) > 0). \tag{1.3}$$

By the use of above defined incomplete gamma functions, Srivastava et al. [22] characterized the incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$. The incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ are widely used in science and engineering problems (see [11, 12, 10]).

$$\begin{aligned} {}_p\gamma_q \left[\begin{matrix} (a_1,x), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} \ z \right] &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \sum_{l=0}^\infty \frac{\gamma(a_1+l,x) \prod_{j=2}^p \Gamma(a_j+l)}{\prod_{j=1}^q \Gamma(b_j+l)} \frac{z^l}{l!} \\ &= \frac{1}{2\pi i} \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \int_{\mathcal{L}} \frac{\gamma(a_1+s,x) \prod_{j=2}^p \Gamma(a_j+s)}{\prod_{j=1}^q \Gamma(b_j+s)} \Gamma(-s)(-z)^s ds,\ (|\arg(-z)| < \pi), \end{aligned} \tag{1.4}$$

and

$${}_p\Gamma_q \left[\begin{matrix} (a_1,x), a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} \ z \right] = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \sum_{l=0}^\infty \frac{\Gamma(a_1+l,x) \prod_{j=2}^p \Gamma(a_j+l)}{\prod_{j=1}^q \Gamma(b_j+l)} \frac{z^l}{l!}$$

$$= \frac{1}{2\pi i}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\int_{\mathcal{L}}\frac{\Gamma(a_1+s,x)\prod_{j=2}^{p}\Gamma(a_j+s)}{\prod_{j=1}^{q}\Gamma(b_j+s)}\Gamma(-s)(-z)^s ds,\ (|\arg(-z)|<\pi), \tag{1.5}$$

with the existence and convergence conditions setout in [22].

Inspired by the applications of ${}_p\gamma_q$ and ${}_p\Gamma_q$ functions and their representation as Mellin-Barnes contour integrals, Srivastava et al. [24] presented and researched the incomplete H-functions as follows:

$$\gamma_{p,\,q}^{m,\,n}(z) = \gamma_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} g(s,x)\, z^{-s}\, ds, \tag{1.6}$$

and

$$\Gamma_{p,\,q}^{m,\,n}(z) = \Gamma_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} G(s,x)\, z^{-s}\, ds, \tag{1.7}$$

where

$$g(s,x) = \frac{\gamma(1-a_1-A_1 s,x)\ \prod_{j=1}^{m}\Gamma(b_j+B_j s)\ \prod_{j=2}^{n}\Gamma(1-a_j-A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j+A_j s)}, \tag{1.8}$$

and

$$G(s,x) = \frac{\Gamma(1-a_1-A_1 s,x)\ \prod_{j=1}^{m}\Gamma(b_j+B_j s)\ \prod_{j=2}^{n}\Gamma(1-a_j-A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j+A_j s)}. \tag{1.9}$$

These incomplete H-functions fulfill the following relation (known as decomposition formula):

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z). \tag{1.10}$$

The incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ characterized in (1.6) and (1.7) exist for $x \geq 0$, under the set of conditions given by Srivastava et al. [24], with

$$\Omega > 0,\ |\arg(z)| < \frac{1}{2}\Omega\pi,\ \text{and}\ \Delta > 0,$$

where

$$\Omega = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j,\ \text{ and }\ \Delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j.$$

Following Srivastava et al. [24], incomplete $\overline{H}$-functions have been implemented and described as:

$$\overline{\gamma}_{p,q}^{m,n}(z) = \overline{\gamma}_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{c} (a_1, A_1; \zeta_1; x),\, (a_j, A_j; \zeta_j)_{2,n},\, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m},\, (b_j, B_j; \eta_j)_{m+1,q} \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \overline{g}(s,x)\, z^{-s}\, ds, \tag{1.11}$$

and

$$\overline{\Gamma}_{p,q}^{m,n}(z) = \overline{\Gamma}_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{c} (a_1, A_1; \zeta_1; x),\, (a_j, A_j; \zeta_j)_{2,n},\, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m},\, (b_j, B_j; \eta_j)_{m+1,q} \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \overline{G}(s,x)\, z^{-s}\, ds, \tag{1.12}$$

where

$$\overline{g}(s,x) = \frac{[\gamma(1-a_1-A_1 s, x)]^{\zeta_1}\prod_{j=1}^{m}\Gamma(b_j+B_j s)\prod_{j=2}^{n}[\Gamma(1-a_j-A_j s)]^{\zeta_j}}{\prod_{j=m+1}^{q}[\Gamma(1-b_j-B_j s)]^{\eta_j}\prod_{j=n+1}^{p}\Gamma(a_j+A_j s)}, \tag{1.13}$$

and

$$\overline{G}(s,x) = \frac{[\Gamma(1-a_1-A_1 s, x)]^{\zeta_1}\prod_{j=1}^{m}\Gamma(b_j+B_j s)\prod_{j=2}^{n}[\Gamma(1-a_j-A_j s)]^{\zeta_j}}{\prod_{j=m+1}^{q}[\Gamma(1-b_j-B_j s)]^{\eta_j}\prod_{j=n+1}^{p}\Gamma(a_j+A_j s)}. \tag{1.14}$$

The general class of polynomials $S_v^u(z)$ proposed by Srivastava et al. [29] and lay it out as

$$S_v^u(z) = \sum_{r=0}^{[v/u]} \frac{(-v)_{ur}}{r!} A_{v,r}\, z^r, \qquad (v = 0, 1, 2, \cdots), \tag{1.15}$$

where, u is positive integer and $A_{v,r} \in \mathbb{R}$ (or $\mathbb{C}$) are arbitrary positive constants. The notations $(-v)_u$ and "[,]", respectively, represent Pochhammer symbol and greatest integer function. Srivastava's polynomials provide as their special cases a number of established polynomials on customizing the coefficient $A_{v,r}$ appropriately.

2. Integral Transforms

In this segment, some integral transformations of that same class of polynomials weighted incomplete H and $\overline{H}$-functions are obtained.

2.1. Laplace Transform. The traditional Laplace transformation of the $f(z)$ function is known as:

$$F(\omega) = L\{f(z); \omega\} = \int_0^\infty e^{-\omega z} f(z) dz, \;\; (\Re(\omega) > 0), \tag{2.1}$$

if the above integral exists.

Theorem 2.1. *If* $\Omega > 0$, $\mu > 0$, $|arg(z)| < \frac{1}{2}\Omega\pi$, $\Delta > 0$, $\Re\left(\lambda + \mu \min_{1\leq j\leq m}\left(\frac{\Re(b_j)}{B_j}\right)\right) > 0$, $\Re(\omega) > 0$ *and* $\vartheta > 0$. *Thereupon the laplace transform formula of incomplete H-function* $\Gamma^{m,n}_{p,q}$ *holds for* $x \geq 0$:

$$L\left\{z^{\lambda-1}\, S^u_v[c_1 z^{\vartheta}]\, \Gamma^{m,\,n}_{p,\,q}\left[c_2 z^{\mu}\,\middle|\, \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix}\right]; \omega\right\}$$

$$= \omega^{-\lambda} \sum_{r=0}^{[v/u]} \frac{(-v)_{ur}\, A_{v,r}}{r!}\, (c_1 \omega^{-\vartheta})^r$$

$$\times\, \Gamma^{m,\,n+1}_{p+1,\,q}\left[c_2 \omega^{-\mu}\,\middle|\, \begin{matrix} (1-\lambda-\vartheta r, \mu), (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix}\right], \tag{2.2}$$

if every member in (2.2) *exist.*

Proof. To proof the result (2.2), use the definition of Laplace transform defined in (2.1), we obtain

$$L\left\{z^{\lambda-1}\, S^u_v[c_1 z^{\vartheta}]\, \Gamma^{m,\,n}_{p,\,q}\left[c_2 z^{\mu}\,\middle|\, \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix}\right]; \omega\right\}$$

$$= \int_0^\infty e^{-\omega z} z^{\lambda-1} \sum_{r=0}^{[v/u]} (-v)_{ur}\, A_{v,r}\, \frac{(c_1 z^{\vartheta})^r}{r!}\, \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x)(c_2 z^{\mu})^{-s} ds\, dz$$

(change the order of integration)

$$= \sum_{r=0}^{[v/u]} (-v)_{ur}\, A_{v,r}\, \frac{c_1^r}{r!}\, \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x) c_2^{-s} \left\{\int_0^\infty e^{-\omega z}\, z^{\lambda+\vartheta r-\mu s-1} dz\right\} ds$$

$$= \sum_{r=0}^{[v/u]} (-v)_{ur} A_{v,r}\, \frac{c_1^r}{r!}\, \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x) c_2^{-s} \frac{\Gamma(\vartheta r + \lambda - \mu s)}{\omega^{\lambda+\vartheta r-\mu s}} ds.$$

Finally, with the help of (1.7) and (1.9), the right hand side of result (2.2) can be obtained easily.

Theorem 2.2. *If* $\overline{\Omega} > 0$, $\mu > 0$, $|arg(z)| < \frac{1}{2}\overline{\Omega}\pi$, $\vartheta > 0$, $\Re\left(\lambda + \mu \min_{1\leq j\leq m}\left(\frac{\Re(b_j)}{B_j}\right)\right) > 0$, $\Re(\omega) > 0$. *Thereupon the laplace transform formula holds for incomplete* $\overline{H}$*-function* $\overline{\Gamma}^{m,n}_{p,q}$:

$$L\left\{z^{\lambda-1}\, S^u_v[c_1 z^{\vartheta}]\, \overline{\Gamma}^{m,\,n}_{p,\,q}\left[c_2 z^{\mu}\,\middle|\, \begin{matrix} (a_1, A_1; \zeta_1; x), (a_j, A_j; \zeta_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \eta_j)_{m+1,q} \end{matrix}\right]; \omega\right\}$$

$$= \omega^{-\lambda} \sum_{r=0}^{[v/u]} \frac{(-v)_{ur}\, A_{v,r}}{r!} (c_1 \omega^{-\vartheta})^r$$

$$\times\, \overline{\Gamma}^{m,\,n+1}_{p+1,\,q}\left[c_2 \omega^{-\mu}\,\middle|\, \begin{matrix} (1-\lambda-\vartheta r, \mu; 1), (a_1, A_1; \zeta_1; x), (a_j, A_j; \zeta_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \eta_j)_{m+1,q} \end{matrix}\right], \tag{2.3}$$

if every member in (2.3) exist.

Proof. To proof the result (2.3), use the definition of Laplace transform defined in (2.1), we obtain

$$L\left\{z^{\lambda-1}\, S_v^u[c_1 z^{\vartheta}]\, \overline{\Gamma}_{p,\,q}^{m,\,n}\left[c_2 z^{\mu}\,\middle|\,\begin{matrix}(a_1, A_1;\zeta_1;x),(a_j,A_j;\zeta_j)_{2,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j;\eta_j)_{m+1,q}\end{matrix}\right];\omega\right\}$$

$$= \int_0^{\infty} e^{-\omega z} z^{\lambda-1}\sum_{r=0}^{[v/u]}(-v)_{ur}\, A_{v,r}\,\frac{(c_1 z^{\vartheta})^r}{r!}\,\frac{1}{2\pi i}\int_{\mathcal{L}}\overline{G}(s,x)(c_2 z^{\mu})^{-s} ds\, dz$$

(change the order of the integration)

$$= \sum_{r=0}^{[v/u]}(-v)_{ur}\, A_{v,r}\,\frac{c_1^r}{r!}\,\frac{1}{2\pi i}\int_{\mathcal{L}}\overline{G}(s,x)c_2^{-s}\left\{\int_0^{\infty} e^{-\omega z}\, z^{\lambda+\vartheta r-\mu s-1}dz\right\}\, ds$$

$$= \sum_{r=0}^{[v/u]}(-v)_{ur} A_{v,r}\,\frac{c_1^r}{r!}\,\frac{1}{2\pi i}\,\int_{\mathcal{L}}\overline{G}(s,x)c_2^{-s}\frac{\Gamma(\vartheta r+\lambda-\mu s)}{\omega^{\lambda+\vartheta r-\mu s}}ds.$$

Finally, with the help of (1.12) and (1.14), we arrive at the right hand side of result (2.3).

2.2. Hankel Transform. The Hankel Transform of order $\nu \in \mathbb{C}$ is defined for a suitably constrained function $f(z)$:

$$H_{\nu}\left\{f(z);\omega\right\} = \int_0^{\infty} z\, J_{\nu}(\omega z)\, f(z)\, dz,\ \ (\Re(\omega) > 0)\,, \tag{2.4}$$

if that the improper integral exist and $J_{\nu}(\omega z)$ is the Bessel function of order ν.

Theorem 2.3. *If $\Omega > 0$, $\mu > 0$, $|arg(z)| < \frac{1}{2}\Omega\pi$, $\Delta > 0$, $\Re(\omega) > 0$, $-1 < \Re(\lambda+\nu) + \mu \min\limits_{1\le j\le m}\left(\frac{\Re(b_j)}{B_j}\right) < \Re(\lambda+\nu) + \mu \min\limits_{1\le j\le n}\left(\frac{\Re(1-a_j)}{A_j}\right)$, $c_1 > 0$, $c_2 > 0$ and $\vartheta > 0$. Thereupon the Hankel transform formula holds for $x \ge 0$:*

$$H_{\nu}\left\{z^{\lambda-2}\, S_v^u[c_1 z^{\vartheta}]\Gamma_{p,\,q}^{m,\,n}\left[c_2 z^{\mu}\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\(b_j,B_j)_{1,q}\end{matrix}\right];\omega\right\}$$

$$= \frac{2^{\lambda-1}}{\omega^{\lambda}}\sum_{r=0}^{[v/u]}\frac{(-v)_{ur}\, A_{v,r}}{r!}\left(c_1\left(\frac{2}{\omega}\right)^{\vartheta}\right)^r \times$$

$$\Gamma_{p+2,\,q}^{m,\,n+1}\left[c_2\left(\frac{2}{\omega}\right)^{\mu}\,\middle|\,\begin{matrix}\left(1-\frac{\lambda+\vartheta r+\nu}{2},\frac{\mu}{2}\right),(a_1,A_1,x),(a_j,A_j)_{2,p},\left(1-\frac{\lambda+\vartheta r-\nu}{2},\frac{\mu}{2}\right)\\(b_j,B_j)_{1,q}\end{matrix}\right], \tag{2.5}$$

if every member in (2.5) exist.

Proof. We start with the left handed part of (2.5) and using the definition (1.7), we obtain

$$\text{L.H.S} = \int_0^{\infty} z^{\lambda-1} J_{\nu}(\omega z)\sum_{r=0}^{[v/u]}(-v)_{ur}\, A_{v,r}\,\frac{(c_1 z^{\vartheta})^r}{r!}\,\frac{1}{2\pi i}\int_{\mathcal{L}} G(s,x)c_2^{-s}z^{-\mu s}dsdz$$

(change the order of the integration)

$$= \sum_{r=0}^{[v/u]} (-v)_{ur}\ A_{v,r}\ \frac{(c_1)^r}{r!} \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x) c_2^{-s} \int_0^\infty z^{\lambda+\vartheta r-\mu s-1} J_\nu(\omega z) dz ds,$$

applying the following formula [5, Vol. II, p. 49, Eq. 7.3.3(19)]

$$\int_0^\infty z^{\lambda-1} J_\nu(\omega z) dz = \frac{2^{\lambda-1}}{z^\lambda} \frac{\Gamma\left(\frac{\lambda+\nu}{2}\right)}{\Gamma\left(1+\frac{\nu-\lambda}{2}\right)}, \quad (\omega>0;\ -\Re(\nu)<\Re(\lambda)<\frac{3}{2}),$$

we get,

$$= \sum_{r=0}^{[v/u]} (-v)_{ur}\ A_{v,r}\ \frac{(c_1)^r}{r!} \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x) c_2^{-s} \frac{2^{\lambda+\vartheta r-\mu s-1}}{\omega^{\lambda+\vartheta r-\mu s}} \frac{\Gamma(\frac{\lambda+\vartheta r-\mu s+\nu}{2})}{\Gamma(1+\frac{\nu-\lambda-\vartheta r+\mu s}{2})} ds,$$

finally, as a consequence of (1.7), we obtain the required result.

The concrete evidence below theorem would also be parallel to Theorem 2.3, so it is given here without proof.

Theorem 2.4. *If* $\overline{\Omega}>0$, $\mu>0$, $|arg(z)|<\frac{1}{2}\overline{\Omega}\pi$, $\Re(\omega)>0$, $-1< \Re(\lambda+\nu)+\mu \min\limits_{1\le j\le m}\left(\frac{\Re(b_j)}{B_j}\right) < \Re(\lambda+\nu)+\mu \min\limits_{1\le j\le n}\left(\frac{\Re(1-a_j)}{A_j}\right)$, $c_1>0$, $c_2>0$ *and* $\vartheta>0$. *Thereupon the Hankel transform formula holds for incomplete* $\overline{H}$*-function* $\overline{\Gamma}_{p,q}^{m,n}$*:*

$$H_\nu\left\{z^{\lambda-2}\, S_v^u[c_1 z^\vartheta]\overline{\Gamma}_{p,\,q}^{m,\,n}\left[c_2 z^\mu \,\middle|\, \begin{matrix}(a_1,A_1;\zeta_1;x),(a_j,A_j;\zeta_j)_{2,n},(a_j,A_j)_{n+1,p}\\ (b_j,B_j)_{1,m},(b_j,B_j;\eta_j)_{m+1,q}\end{matrix}\right];\omega\right\}$$
$$= \frac{2^{\lambda-1}}{\omega^\lambda}\sum_{r=0}^{[v/u]}\frac{(-v)_{ur}\ A_{v,r}}{r!}\left(c_1\left(\frac{2}{\omega}\right)^\vartheta\right)^r \times$$
$$\overline{\Gamma}_{p+2,\,q}^{m,\,n+1}\left[c_2\left(\frac{2}{\omega}\right)^\mu \,\middle|\, \begin{matrix}\left(1-\frac{\lambda+\vartheta r+\nu}{2},\frac{\mu}{2}\right),(a_1,A_1;\zeta_1;x),(a_j,A_j;\zeta_j)_{1,n},(a_j,A_j)_{n+1,p},\left(1-\frac{\lambda+\vartheta r-\nu}{2},\frac{\mu}{2}\right)\\ (b_j,B_j)_{1,m},(b_j,B_j;\eta_j)_{m+1,q}\end{matrix}\right], \tag{2.6}$$

if every member in (2.6) *exist.*

2.3. Euler's Beta Transform. The integral transform of Euler's Beta type for a given function $f(z)$ is characterized as:

$$\mathcal{B}\{f(z);\zeta,\eta\} = \int_0^1 z^{\zeta-1}\,(1-z)^{\eta-1}\,f(z)\,dz \tag{2.7}$$

with $\Re(\zeta)>0$ and $\Re(\eta)>0$.

Theorem 2.5. *Let $\Delta > 0$, $\Omega > 0$, $\sigma > 0$, $\mu > 0$, $|arg\ z| > \Omega\ \frac{\pi}{2}$, $\Re(\zeta) + \sigma \min\limits_{1\le j\le m} \Re\left(\frac{b_j}{B_j}\right) > 0$, $\Re(\eta) > 0$, $c_1 > 0$, $c_2 > 0$, and $x \ge 0$. Then the transform of the incomplete H-function of Euler's Beta type is given as:*

$$\mathcal{B}\left\{ S_v^u[c_1 z^\sigma]\ \Gamma_{p,\,q}^{m,\,n}\left[c_2 z^\mu \left| \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array} \right. \right]; \zeta, \eta \right\}$$
$$= \Gamma(\eta) \sum_{r=0}^{[v/u]} \frac{(-v)_{ur}\, A_{v,r}}{r!} (c_1)^r \times$$
$$\Gamma_{p+1,\,q+1}^{m,\,n+1}\left[c_2 \left| \begin{array}{c} (a_1, A_1, x), (1-\zeta-\sigma r, \mu), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q}, (1-\zeta-\eta-\sigma r, \mu) \end{array} \right. \right], \tag{2.8}$$

if every member in (2.8) exist.

Proof. To prove the result, we begin with the left-hand side of the assertion (2.8); in part of (2.8), use the definitions (2.7) and (1.7), we get

$$\mathcal{B}\left\{ S_v^u[c_1 z^\sigma]\ \Gamma_{p,\,q}^{m,\,n}\left[c_2 z^\mu \left| \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array} \right. \right]; \zeta, \eta \right\}$$
$$= \int_0^1 z^{\zeta-1}(1-z)^{\eta-1} \sum_{r=0}^{[v/u]} (-v)_{ur} A_{v,r} \frac{(c_1 z^\sigma)^r}{r!} \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x)\ (c_2 z^\mu)^{-s}\, ds\, dz$$

change the order of the integration

$$= \sum_{r=0}^{[v/u]} (-v)_{ur} A_{v,r} \frac{(c_1)^r}{r!} \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x) c_2^{-s} \int_0^1 z^{\zeta+\sigma r-\mu s-1}(1-z)^{\eta-1} dz\, ds$$

apply Beta formula (2.7), we obtain

$$= \sum_{r=0}^{[v/u]} (-v)_{ur} A_{v,r} \frac{(c_1)^r}{r!} \frac{1}{2\pi i} \int_{\mathcal{L}} G(s,x) c_2^{-s} \frac{\Gamma(\zeta+\sigma r-\mu\, s)\Gamma(\eta)}{\Gamma(\zeta+\eta+\sigma r-\mu\, s)}\, ds$$

finally, as a consequence of (1.7), we get the desired result.

Theorem 2.6. *Let $\overline{\Omega} > 0$, $\sigma > 0$, $\mu > 0$, $|arg\ z| > \overline{\Omega}\ \frac{\pi}{2}$, $\Re(\zeta) + \sigma \min\limits_{1\le j\le m} \Re\left(\frac{b_j}{B_j}\right) > 0$, $\Re(\eta) > 0$, $c_1 > 0$ and $c_2 > 0$. Thereupon the transform formula holds:*

$$\mathcal{B}\left\{ S_v^u[c_1 z^\sigma]\ \overline{\Gamma}_{p,\,q}^{m,\,n}\left[c_2 z^\mu \left| \begin{array}{c} (a_1, A_1; \zeta_1; x), (a_j, A_j; \zeta_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \eta_j)_{m+1,q} \end{array} \right. \right]; \zeta, \eta \right\}$$

$$= \Gamma(\eta) \sum_{r=0}^{[v/u]} \frac{(-v)_{ur}\, A_{v,r}}{r!} (c_1)^r \times$$

$$\overline{\Gamma}^{m,\, n+1}_{p+1,\, q+1} \left[c_2 \left| \begin{array}{c} (a_1, A_1; \zeta_1; x), (1-\zeta-\sigma r, \mu), (a_j, A_j; \zeta_j)_{1,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \eta_j)_{m+1,q}, (1-\zeta-\eta-\sigma r, \mu) \end{array} \right. \right], \tag{2.9}$$

if every member in (2.9) *exist.*

2.4. Srivastava-Whittaker Transform. For the Whittaker function $W_{k,\mu}(z)$ of the second kind it is known that

$$W_{0,\mu}(z) = \left(\frac{z}{\pi}\right)^{\frac{1}{2}} K_\mu\left(\frac{z}{2}\right), \text{ and } W_{\mu+\frac{1}{2},\pm\mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}}, \tag{2.10}$$

where $K_\nu(z)$ denotes the modified Bessel function (or the Macdonald function) of order ν.

In fact, we also have

$$K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}. \tag{2.11}$$

Such reduction formulas as those depicted by (2.10) and (2.11) have led to several generalizations of the classical laplace transform. We recall here the following unification and further generalization of all these generalized laplace transforms with their kernels involving the Whittaker function $W_{k,\mu}(z)$ or the modified Bessel function $K_\nu(z)$, which was given by Srivastava [21]:

$$\mathcal{S}^{(\rho,\sigma)}_{\lambda,k,\mu}\{F(t); \mathfrak{s}\} = \int_0^\infty (\mathfrak{s}t)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\lambda \mathfrak{s} t}\, W_{k,\mu}(\rho \mathfrak{s} t) f(t) dt, \quad (\Re(\mathfrak{s}) > 0), \tag{2.12}$$

if that the improper integral in (2.12) exists.

Theorem 2.7. *Let* $\Omega > 0$, $\nu > 0$, $\delta > 0$, $|arg\ z| < \Omega\frac{\pi}{2}$, $\Delta > 0$, $\Re((\rho+\lambda)\mathfrak{s}) > 0$, *and* $\Re(\sigma) + \nu \min\limits_{1\le j\le m}\left(\frac{\Re(b_j)}{B_j}\right) > |\Re(\mu)| - 1$. *Thereupon the Srivastava-Whittaker transform holds true for* $x \geq 0$*:*

$$\mathcal{S}^{(\rho,\sigma)}_{\lambda,k,\mu}\left\{ S^u_v[c_1 t^\delta] \Gamma^{m,\, n}_{p,\, q}\left[c_2 t^\nu \left| \begin{array}{c} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array} \right. \right]; \mathfrak{s} \right\}$$

$$= \frac{\rho^{\mu+1/2}}{\mathfrak{s}^{1/2}} \left(\frac{2}{\lambda+\rho}\right)^{1+\sigma+\mu} \sum_{r=0}^{[v/u]} \frac{(-v)_{ur}\, A_{v,r}}{r!} \left(c_1 \left(\frac{2}{(\lambda+\rho)\mathfrak{s}}\right)^\delta \right)^r \times$$

$$\sum_{l=0}^\infty \frac{\Gamma(\mu-k+l+1/2)}{\Gamma(\mu-k+1/2)} \left(\frac{\lambda-\rho}{\lambda+\rho}\right)^l \times$$

$$\Gamma^{m,\, n+2}_{p+2,\, q+1}\left[c_2 \left(\frac{2}{(\rho+\lambda)\mathfrak{s}}\right)^\nu \left| \begin{array}{c} (\mu-\sigma-\delta r, \nu), (-\mu-\sigma-\delta r-l, \nu), (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q}, (k-\sigma-\delta r-l-1/2, \nu) \end{array} \right. \right], \tag{2.13}$$

if every member in (2.13) *exist.*

Proof. Just as in our demonstration of the other theorems in this section, we begin with the left-hand side of the assertion (2.13), make use of the Mellin-barnes type contour integral in (1.7) which defines the incomplete H-function $\Gamma_{p,q}^{m,n}(z)$, change the order of the integration and then apply the following known formula (see, for example, [6, Vol.**II**, p. 337, Entry 6.9 (8)]):

$$\begin{aligned}
&\int_0^\infty t^{\rho-1}\, e^{-\zeta t} W_{k,\mu}(\eta\, t)\, dt \\
&= \frac{\eta^{\mu+\frac{1}{2}}\Gamma(\rho+\mu+\frac{1}{2})\Gamma(\rho-\mu+\frac{1}{2})}{(\zeta+\frac{1}{2}\eta)^{\rho+\mu+\frac{1}{2}}\Gamma(\rho-k+1)}\, {}_2F_1\left[\begin{matrix}\rho+\mu+\frac{1}{2}, & \mu-k+\frac{1}{2}; \\ & \rho-k+1;\end{matrix}\ \frac{2\zeta-\eta}{2\zeta+\eta}\right] \\
&= \frac{\eta^{\mu+\frac{1}{2}}\Gamma(\rho-\mu+\frac{1}{2})}{(\zeta+\frac{1}{2}\eta)^{\rho+\mu+\frac{1}{2}}\Gamma(\mu-k+1/2)} \sum_{l=0}^{\infty} \frac{\Gamma(\rho+\mu+l+\frac{1}{2})\Gamma(\mu-k+l+\frac{1}{2})}{l!\,\Gamma(\rho-k+l+1)} \left(\frac{2\zeta-\eta}{2\zeta+\eta}\right)^l, \\
&\qquad \left(\Re(\zeta+\eta/2)>0;\ \ \Re(\rho)>|\Re(\mu)|-\frac{1}{2}\right).
\end{aligned} \tag{2.14}$$

We are thus led easily to the right-hand side of the assertion (2.13) of Theorem 2.7.

Theorem 2.8. *Let* $\overline{\Omega}>0$, $\nu>0$, $\delta>0$, $|arg\ z|<\overline{\Omega}\frac{\pi}{2}$, $\Re((\rho+\lambda)\mathfrak{s})>0$, *and* $\Re(\sigma)+\nu\min\limits_{1\le j\le m}\left(\frac{\Re(b_j)}{B_j}\right)>|\Re(\mu)|-1$. *Thereupon the Srivastava-Whittaker transform holds true for* $x\geq 0$*:*

$$\begin{aligned}
&\mathcal{S}_{\lambda,k,\mu}^{(\rho,\sigma)}\left\{S_v^u[c_1t^\delta]\overline{\Gamma}_{p,q}^{m,n}\left[c_2t^\nu\ \middle|\ \begin{matrix}(a_1,A_1;\zeta_1;x),(a_j,A_j;\zeta_j)_{2,n},(a_j,A_j)_{n+1,p}\\ (b_j,B_j)_{1,m},(b_j,B_j;\eta_j)_{m+1,q}\end{matrix}\right];\mathfrak{s}\right\} \\
&= \frac{\rho^{\mu+1/2}}{\mathfrak{s}^{1/2}}\left(\frac{2}{\lambda+\rho}\right)^{1+\sigma+\mu}\sum_{r=0}^{[v/u]}\frac{(-v)_{ur}\,A_{v,r}}{r!}\left(c_1\left(\frac{2}{(\lambda+\rho)\mathfrak{s}}\right)^\delta\right)^r \times \\
&\quad \sum_{l=0}^{\infty}\frac{\Gamma(\mu-k+l+1/2)}{\Gamma(\mu-k+1/2)}\left(\frac{\lambda-\rho}{\lambda+\rho}\right)^l \times \\
&\quad \overline{\Gamma}_{p+2,q+1}^{m,n+2}\left[c_2\left(\frac{2}{(\rho+\lambda)\mathfrak{s}}\right)^\nu\ \middle|\ \begin{matrix}(\mu-\sigma-\delta r,\nu),(-\mu-\sigma-\delta r-l,\nu),(a_1,A_1;\zeta_1;x),(a_j,A_j;\zeta_j)_{2,n},\\ (b_j,B_j)_{1,m},(b_j,B_j;\eta_j)_{m+1,q},\end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix}(a_j,A_j)_{n+1,p}\\ (k-\sigma-\delta r-l-1/2,\nu)\end{matrix}\right],
\end{aligned} \tag{2.15}$$

if every member in (2.15) *exist.*

Remark 2.1. *If we set* $\zeta_j\,(j=1,\cdots,n)=1$ *and* $\eta_j\,(j=m+1,\cdots,q)=1$ *in Theorem 2.2, Theorem 2.4, Theorem 2.6 and Theorem 2.8, then we get Theorem 2.1, Theorem 2.3, Theorem 2.5 and Theorem 2.7, respectively.*

3. Special Cases

By specialization the parameters in incomplete H-functions, incomplete $\overline{H}$-functions and general polynomial class, we may obtain the established findings that are accessible in the literature. For example, if we take

(i) $S_v^u(z) = \text{constant} = 1$, then Theorem 2.1, Theorem 2.3, Theorem 2.5 and Theorem 2.7 depleted to known outcomes of Srivastava et al. [24],

(ii) $S_v^u(z) = \text{constant} = 1$, $\zeta_j\,(j = 1, \cdots, n) = 1$ and $\eta_j\,(j = m+1, \cdots, q) = 1$, then Theorem 2.2, Theorem 2.4, Theorem 2.6 and Theorem 2.8 depleted to known outcomes of Srivastava et al. [24],

(iii) $x = 0$ and $S_v^u(z) = \text{constant} = 1$, then Theorem 2.1, Theorem 2.3, Theorem 2.5 and Theorem 2.7 depleted to known outcomes of integral transforms involving H-function,

(iv) $x = 0$, $S_v^u(z) = \text{constant} = 1$, $\zeta_j\,(j = 1, \cdots, n) = 1$ and $\eta_j\,(j = m+1, \cdots, q) = 1$, then Theorem 2.2, Theorem 2.4, Theorem 2.6 and Theorem 2.8 depleted to known outcomes of integral transforms involving H-function,

(v) $x = 0$, $S_v^u(z) = \text{constant} = 1$, $\zeta_j\,(j = 1, \cdots, n) = 1$, $\eta_j\,(j = m+1, \cdots, q) = 1$ and $A_j = B_j = 1$ then Theorem 2.2, Theorem 2.4, Theorem 2.6 and Theorem 2.8 depleted to known outcomes of integral transforms involving Meijer's G-function.

Here, we have derived certain image formulas under certain integral transforms of the polynomials weighted incomplete H-functions and incomplete $\overline{H}$-functions. It is to note that the incomplete $\overline{H}$-function generalizes incomplete H-function, incomplete Meijer G-function, incomplete Wright function, incomplete hypergeometric functions and many other classical special functions. In addition, the polynomials family produces a number of known polynomials as their particular cases on a properly specialized bound sequence. As a consequence, by assigning correct specific values to arbitrary sequences and constraints, our key findings can be used to obtain a range of image formulas containing polynomials and a variety of specific functions.

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