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Certain new dynamic nonlinear inequalities in two independent variables and applications

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Abstract

Several inequalities were proved in 2018 by Boudeliou, in 2015 by Abdeldain and El-Deeb and in 1998 by Pachpatte. It is our aim in this paper to generalize these inequalities to time scales. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Furthermore, we study the boundedness of some problems by applying our results.

Keywords: Gronwall-type inequality; Boundedness; Time scales

1 Introduction

In 2018, Boudeliou [9] discussed the following inequalities.

Theorem 1.1 *Suppose $a \in C(\hat{\Omega}, \mathbb{R}_+)$ is nondecreasing with respect to $(\breve{x}, \breve{y}) \in \hat{\Omega} = I_1 \times I_2$, let $\hat{\alpha}(\breve{x}) \in C^1(I_1, I_2)$ and $\hat{\beta}(\breve{y}) \in C^1(I_2, I_2)$ be nondecreasing functions with $\hat{\alpha}(\breve{x}) \le \breve{x}$ on I_1, $\hat{\beta}(\breve{y}) \le \breve{y}$, and g, u, p, $f \in C(\hat{\Omega}, \mathbb{R}_+)$. Furthermore, suppose $\bar{\psi}$, $\bar{\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\bar{\psi}, \bar{\varphi}\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \bar{\psi}(u) = +\infty$. If $u(\breve{x}, \breve{y})$ satisfies*

$$\bar{\psi}\big(u(\breve{x}, \breve{y})\big) \le a(\breve{x}, \breve{y}) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} \big[f(\breve{s}, \breve{t})\bar{\varphi}\big(u(\breve{s}, \breve{t})\big) + p(\breve{s}, \breve{t})\big]\, d\breve{t}\, d\breve{s}$$
$$+ \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \bar{\varphi}\big(u(\breve{\tau}, \breve{t})\big)\, d\breve{\tau} \right) d\breve{t}\, d\breve{s},$$

for $(\breve{x}, \breve{y}) \in \hat{\Omega}$, then

$$u(\breve{x}, \breve{y}) \le \bar{\psi}^{-1} \left\{ \breve{G}^{-1} \left[\breve{G}\big(q(\breve{x}, \breve{y})\big) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \left(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t})\, d\breve{\tau} \right) d\breve{t}\, d\breve{s} \right] \right\},$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where

$$q(\breve{x}, \breve{y}) = a(\breve{x}, \breve{y}) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} p(\breve{s}, \breve{t})\, d\breve{t}\, d\breve{s},$$

$$\breve{G}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})}, \quad r \geq r_0 > 0, \qquad \breve{G}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})} = +\infty,$$

and $(\breve{x}_1, \breve{y}_1) \in \hat{\Omega}$ *is chosen so that*

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s},\breve{t}) \left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\, d\breve{\tau}\right) d\breve{t}\, d\breve{s}\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Theorem 1.2 *Assume that* $g, a, f, u, \hat{\beta}, \hat{\alpha}, , \bar{\psi}$ *and* $\bar{\varphi}$ *be as in Theorem 1.1. If* $u(\breve{x},\breve{y})$ *satisfies*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \leq a(\breve{x},\breve{y}) + \left(\int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\, d\breve{t}\, d\breve{s}\right)^2 + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\, d\breve{\tau}\right) d\breve{t}\, d\breve{s},$$

for $(\breve{x},\breve{y}) \in \hat{\Omega}$, *then*

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s},\breve{t})\, d\breve{t}\, d\breve{s}\right)^2\right]\right\},$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\, d\breve{\tau}\right) d\breve{t}\, d\breve{s},$$

$$\breve{H}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})}, \quad r \geq r_0 > 0, \qquad \breve{H}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})} = +\infty,$$

and $(\breve{x}_1, \breve{y}_1) \in \hat{\Omega}$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s},\breve{t})\, d\breve{t}\, d\breve{s}\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

In 1988, Hilger [33] presented time scale theory to unify continuous and discrete analysis. For some Gronwall–Bellman-type integral, dynamic inequalities and other type inequalities on time scales, see Refs. [1–8, 13, 14, 16–32, 34–41]. For more details on time scales calculus see [15].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We suppose throughout the article that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined for any $t \in \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined for any $t \in \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In the previous two definitions, we set $\inf \emptyset = \sup \mathbb{T}$ (i.e., if t is the maximum of $\mathbb{T}$, then $\sigma(t) = t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if t is the minimum of $\mathbb{T}$, then $\rho(t) = t$), where $\emptyset$ is the empty set.

A point $t \in \mathbb{T}$ with $\inf \mathbb{T} < t < \sup \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$. Points that are simultaneously right-dense and left-dense are called dense points. Points that are simultaneously right-scattered and left-scattered are called isolated points.

We define the forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ for any $t \in \mathbb{T}$ by $\mu(t) := \sigma(t) - t$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$, $\forall t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$. In a similar manner, the function $f^\rho : \mathbb{T} \to \mathbb{R}$ is defined by $f^\rho(t) = f(\rho(t))$, $\forall t \in \mathbb{T}$, that is, $f^\rho = f \circ \rho$.

We introduce the set $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

The interval $[a,b]$ in $\mathbb{T}$ is defined by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-closed interval are defined similarly.

Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$. Then we say that $f^\Delta(t) \in \mathbb{R}$ is the delta derivative of f at t if for any $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in U$, we have

$$\left| \left[f\big(\sigma(t)\big) - f(s)\right] - f^\Delta(t)\left[\sigma(t) - s\right] \right| \leq \varepsilon \left|\sigma(t) - s\right|.$$

Furthermore, f is said to be delta differentiable on $\mathbb{T}^\kappa$ if it is delta differentiable at each $t \in \mathbb{T}^\kappa$.

If $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable functions at $t \in \mathbb{T}^\kappa$, then

1. $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$;
2. $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$;
3. $(\frac{f}{g})^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}$ provided $g(t)g(\sigma(t)) \neq 0$.

A function $g : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if g is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. In this case, the definite delta integral of f is defined by

$$\int_a^b f(\eta)\Delta\eta = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

If $g \in C_{\mathrm{rd}}(\mathbb{T})$ and $t, t_0 \in \mathbb{T}$, then the definite integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, and $G^\Delta(t) = g(t)$ holds.

Assume that $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and f, g be continuous functions on $[a,b]_{\mathbb{T}}$. Then

1. $\int_a^b [f(t) + g(t)]\Delta\eta = \int_a^b f(\eta)\Delta\eta + \int_a^b g(\eta)\Delta\eta$;
2. $\int_a^b \alpha f(\eta)\Delta\eta = \alpha \int_a^b f(\eta)\Delta\eta$;
3. $\int_a^b f(\eta)\Delta\eta = \int_a^c f(\eta)\Delta\eta + \int_c^b f(\eta)\Delta\eta$;
4. $\int_a^b f(\eta)\Delta\eta = -\int_b^a f(\eta)\Delta\eta$;
5. $\int_a^a f(\eta)\Delta\eta = 0$;
6. if $f(t) \geq g(t)$ on $[a,b]_{\mathbb{T}}$, then $\int_a^b f(\eta)\Delta\eta \geq \int_a^b g(\eta)\Delta\eta$.

We will need the following important relations between calculus on time scales $\mathbb{T}$ and either continuous calculus on $\mathbb{R}$ or discrete calculus on $\mathbb{Z}$. Note that:

1. If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \qquad \mu(t) = 0, \qquad f^{\Delta}(t) = f'(t), \qquad \int_a^b f(\eta)\Delta\eta = \int_a^b f(t)\,dt.$$

2. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \qquad \mu(t) = 1, \qquad f^{\Delta}(t) = f(t+1) - f(t), \qquad \int_a^b f(\eta)\Delta\eta = \sum_{t=a}^{b-1} f(t).$$

In the following, we present the basic theorems that will be needed in the proofs of our main results.

Theorem 1.3 *If $\hat{f}$ is $\hat{\Delta}$-integrable on $[a,b]$, then so is $|\hat{f}|$, and*

$$\left|\int_a^b \hat{f}(\check{t})\hat{\Delta}\check{t}\right| \leq \int_a^b \left|\hat{f}(\check{t})\right|\hat{\Delta}\check{t}.$$

Theorem 1.4 (Chain rule on time scales [15]) *Assume $\hat{g}: \mathbb{R} \to \mathbb{R}$ is continuous, $\hat{g}: \check{\mathbb{T}} \to \mathbb{R}$ is $\hat{\Delta}$-differentiable on $\mathbb{T}^{\kappa}$, and $\hat{f}: \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists $c \in [\check{t}, \sigma(\check{t})]$ with*

$$(\hat{f} \circ \hat{g})^{\hat{\Delta}}(\check{t}) = \hat{f}'\big(\hat{g}(c)\big)\hat{g}^{\hat{\Delta}}(\check{t}). \tag{1.1}$$

Theorem 1.5 (Chain rule on time scales [15]) *Let $\hat{f}: \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $\hat{g}: \check{\mathbb{T}} \to \mathbb{R}$ is $\hat{\Delta}$-differentiable. Then $f \circ \hat{g}: \check{\mathbb{T}} \to \mathbb{R}$ is $\hat{\Delta}$-differentiable and the formula*

$$(\hat{f} \circ \hat{g})^{\hat{\Delta}}(\check{t}) = \left\{\int_0^1 \big[\hat{f}'\big(h\hat{g}^{\sigma}(\check{t}) + (1-h)\hat{g}(\check{t})\big)\big]\,dh\right\}\hat{g}^{\hat{\hat{\Delta}}}(\check{t}) \tag{1.2}$$

holds.

This paper gives us the time scale versions of the results provided in [9]. These inequalities, proved here, extend some known results in the literature, and they are also unify the continuous and the discrete case.

2 Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$, $\check{\mathbb{T}}_1$, $\check{\mathbb{T}}_2$ are two time scales and we put $\Omega = \check{\mathbb{T}}_1 \times \check{\mathbb{T}}_2 = \{(\check{t}, \check{s}) : \check{t} \in \check{\mathbb{T}}_1, \check{s} \in \check{\mathbb{T}}_2\}$ which is a complete metric space with the metric $\breve{\rho}$ defined by

$$\breve{\rho}\big((\check{t}, \check{s}), (\acute{t}, \acute{s})\big) = \sqrt{(\check{t} - \acute{t})^2 + (\check{s} - \acute{s})^2}, \quad \forall (\check{t}, \check{s}), (\acute{t}, \acute{s}) \in \check{\mathbb{T}}_1 \times \check{\mathbb{T}}_2.$$

$C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ denotes the set of all right-dense continuous functions from Ω into $\mathbb{R}_+$ and $C^1_{\mathrm{rd}}(\check{\mathbb{T}}_i, \check{\mathbb{T}}_i)$ denotes the set of all right-dense continuously delta-differentiable functions from $\check{\mathbb{T}}_i$ into $\check{\mathbb{T}}_i$, $i = 1, 2$. The two-variables time scales calculus and multiple integration on time scales were introduced in [10, 11] (see also [12]).

Theorem 2.1 *Suppose that $a \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ is nondecreasing with respect to $(\breve{x}, \breve{y}) \in \Omega$, and g, u, p, $f \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$. Furthermore, suppose that $\bar{\psi}$, $\bar{\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\bar{\psi}, \bar{\varphi}\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \bar{\psi}(u) = +\infty$. If $u(\breve{x}, \breve{y})$ satisfies*

$$\bar{\psi}\bigl(u(\breve{x},\breve{y})\bigr) \le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bigl[f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr) + p(\breve{s},\breve{t})\bigr]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\biggl(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},$$

for $(\breve{x}, \breve{y}) \in \Omega$, then

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\biggl\{\breve{G}^{-1}\biggl[\breve{G}\bigl(q(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\biggl(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\biggr]\biggr\} \tag{2.1}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where

$$q(\breve{x},\breve{y}) = a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} p(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{2.2}$$

$$\breve{G}(r) = \int_{r_0}^{r} \frac{\hat{\Delta}\breve{s}}{\bar{\varphi}\circ\bar{\psi}^{-1}(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{G}(+\infty) = \int_{r_0}^{+\infty} \frac{\hat{\Delta}\breve{s}}{\bar{\varphi}\circ\bar{\psi}^{-1}(\breve{s})} = +\infty, \tag{2.3}$$

and $(\breve{x}_1, \breve{y}_1) \in \Omega$ is chosen so that

$$\biggl(\breve{G}\bigl(q(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\biggl(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\biggr) \in \operatorname{Dom}\bigl(G^{-1}\bigr).$$

Proof Assume that $a(\breve{x}, \breve{y}) > 0$. Since $q \ge 0$ and it is nondecreasing, fixing an arbitrary point $(\breve{\xi}, \breve{\zeta}) \in \Omega$ and defining $z(\breve{x}, \breve{y})$ by

$$z(\breve{x},\breve{y}) = q(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\biggl(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},$$

which is a positive and nondecreasing function for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{x} \le \breve{\zeta} \le \breve{y}_1$, we have $z(0, \breve{y}) = z(\breve{x}, 0) = q(\breve{\xi}, \breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{y})\bigr). \tag{2.4}$$

Differentiating $z(\breve{x}, \breve{y})$, with respect to $\breve{x}$ and using (2.4), we get

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} f(\breve{x},\breve{t})\biggl[\bar{\varphi}\bigl(u(\breve{x},\breve{t})\bigr) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr]\hat{\Delta}\breve{t} \\ &\le \int_0^{\breve{y}} f(\breve{x},\breve{t})\biggl[\bar{\varphi}\circ\bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr]\hat{\Delta}\breve{t}, \end{aligned}$$

since $\bar{\varphi} \circ \bar{\psi}^{-1}$ is nondecreasing with respect to $(\breve{x},\breve{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &\le \int_0^{\breve{y}} f(\breve{x},\breve{t})\left[\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr) + \bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr) \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t} \\ &\le \bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{y})\bigr) \int_0^{\breve{y}} f(\breve{x},\breve{t})\left[1 + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}, \end{aligned} \tag{2.5}$$

and from (2.5) we get

$$\frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{\bar{\varphi} \circ \bar{\psi}^{-1}(z(\breve{x},\breve{y}))} \le \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(1 + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}. \tag{2.6}$$

From (2.6) we get

$$\breve{G}\bigl(z(\breve{x},\breve{y})\bigr) \le \breve{G}\bigl(q(\breve{\xi},\breve{\zeta})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\breve{x},\breve{y}) \le \breve{G}^{-1}\left[\breve{G}\bigl(q(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]. \tag{2.7}$$

So from (2.7) and (2.4) we get the desired inequality in (2.1). For $a(\breve{x},\breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x},\breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.2 *If we take* $\breve{\mathbb{T}} = \mathbb{R}$ *in Theorem* 2.1, *then the inequality*

$$\begin{aligned} \bar{\psi}\bigl(u(\breve{x},\breve{y})\bigr) &\le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bigl[f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr) + p(\breve{s},\breve{t})\bigr]\,d\breve{t}\,d\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\,d\breve{\tau}\right)d\breve{t}\,d\breve{s}, \end{aligned}$$

for $(\breve{x},\breve{y}) \in \Omega$, *implies*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\bigl(q(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\,d\breve{\tau}\right)d\breve{t}\,d\breve{s}\right]\right\}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$\begin{aligned} &q(\breve{x},\breve{y}) = a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} p(\breve{s},\breve{t})\,d\breve{t}\,d\breve{s}, \\ &\breve{G}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{G}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})} = +\infty, \end{aligned} \tag{2.8}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{G}\bigl(q(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\,d\breve{\tau}\right)d\breve{t}\,d\breve{s}\right) \in \operatorname{Dom}\bigl(G^{-1}\bigr).$$

Corollary 2.3 *The discrete form can be obtained by letting* $\breve{\mathbb{T}} = \mathbb{Z}$ *in Theorem* 2.1:

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \leq a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big] + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right)$$

for $(\breve{x},\breve{y}) \in \Omega$, *which implies*

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(1 + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right)\right]\right\}$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, *where*

$$q(\breve{x},\breve{y}) = a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} p(\breve{s},\breve{t}),$$

$$\breve{G}(r) = \sum_{\breve{s}=r_0}^{r-1} \frac{1}{\bar{\varphi}\circ\bar{\psi}^{-1}(\breve{s})}, \quad r \geq r_0 > 0, \qquad \breve{G}(+\infty) = \sum_{\breve{s}=r_0}^{+\infty} \frac{1}{\bar{\varphi}\circ\bar{\psi}^{-1}(\breve{s})} = +\infty, \tag{2.9}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(1 + \sum_{\breve{\tau}_0}^{\breve{s}} g(\breve{\tau},\breve{t})\right)\right) \in \mathrm{Dom}\big(G^{-1}\big).$$

Theorem 2.4 *Assume that* $h, b \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$. *Let* $g, f, p, a, u, \bar{\psi}$ *and* $\bar{\varphi}$ *be as in Theorem* 2.1, *if* $u(\breve{x},\breve{y})$ *satisfies*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \leq a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}}\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} + \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}$$

for $(\breve{x},\breve{y}) \in \Omega$, *then*

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + A(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]\right\} \tag{2.10}$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, *where* q, $\breve{G}$ *are defined by* (2.2) *and* (2.3), *respectively, and*

$$\breve{A}(\breve{x},\breve{y}) = \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t}) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \tag{2.11}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right) \in \mathrm{Dom}\big(\breve{G}^{-1}\big).$$

Proof Assume that $a(\breve{x},\breve{y}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define positive and non-decreasing function $z(\breve{x},\breve{y})$ by

$$\begin{aligned} z(\breve{x},\breve{y}) &= q(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\biggl[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\biggr]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \end{aligned}$$

for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{\zeta} \le y_1$, then $z(0,\breve{y}) = z(\breve{x},0) = q(\breve{\xi},\breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big);$$

then we have

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\big(u(\breve{x},\breve{t})\big)\hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} b(\breve{x},\breve{t})\biggl(h(\breve{x},\breve{t})\bar{\varphi}\big(u(\breve{x},\breve{t})\big) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t} \\ &\le \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big)\hat{\Delta}\breve{t} + \int_0^{\breve{y}} b(\breve{x},\breve{t}) \\ &\quad \times \biggl(h(\breve{x},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t} \\ &\le \bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big)\biggl[\int_0^{\breve{y}} f(\breve{x},\breve{t})\hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} b(\breve{x},\breve{t})\biggl(h(\breve{x},\breve{t}) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\biggr)\biggr]\hat{\Delta}\breve{t}, \end{aligned}$$

then

$$\frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{\bar{\varphi}\circ\bar{\psi}^{-1}(z(\breve{x},\breve{y}))} \le \biggl[\int_0^{\breve{y}} f(\breve{x},\breve{t})\hat{\Delta}\breve{t} + \int_0^{\breve{y}} b(\breve{x},\breve{t})\biggl(h(\breve{x},\breve{t}) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\biggr)\biggr]\hat{\Delta}\breve{t}. \tag{2.12}$$

Integrating (2.12) and using (2.3) and (2.11), we get

$$\breve{G}\big(z(\breve{x},\breve{y})\big) \le \breve{G}\big(q(\breve{\xi},\breve{\zeta})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\breve{x},\breve{y}) \le \breve{G}^{-1}\biggl[\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\biggr]. \tag{2.13}$$

From (2.13) and $u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}(z(\breve{x},\breve{y}))$, we get the required inequality in (2.10). For $a(\breve{x},\breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x},\breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.5 *If we take $\breve{\mathbb{T}} = \mathbb{R}$ in Theorem 2.4, then the inequality*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\,d\breve{t}\,d\breve{s}$$
$$+ \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\,d\breve{\tau}\right]d\breve{t}\,d\breve{s},$$

for $(\breve{x},\breve{y}) \in \Omega$, implies

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + A(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\,d\breve{t}\,d\breve{s}\right]\right\}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where $\breve{G}$ is defined by (2.8) and

$$\breve{A}(\breve{x},\breve{y}) = \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t}) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\,d\breve{\tau}\right]d\breve{t}\,d\breve{s}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ is chosen so that

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\,d\breve{t}\,d\breve{s}\right) \in \operatorname{Dom}\big(\breve{G}^{-1}\big).$$

Corollary 2.6 *The discrete form can be obtained by letting $\breve{\mathbb{T}} = \mathbb{Z}$ in Theorem 2.4:*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} \big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]$$
$$+ \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right],$$

for $(\breve{x},\breve{y}) \in \Omega$, implies

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{G^{-1}\left[G\big(q(\breve{x},\breve{y})\big) + A(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right]\right\}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where $\breve{G}$ is defined by (2.9) and

$$\breve{A}(\breve{x},\breve{y}) = \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t}) + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right]$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ is chosen so that

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right) \in \operatorname{Dom}\big(\breve{G}^{-1}\big).$$

Theorem 2.7 *Assume that g, a, u, f, p, $\bar{\psi}$ and $\bar{\varphi}$ are as in Theorem 2.1. If $u(\breve{x},\breve{y})$ satisfies*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\bigg(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},$$

for $(\breve{x},\breve{y}) \in \Omega$, then

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\bigg\{\breve{G}^{-1}\bigg(\breve{F}^{-1}\bigg[\breve{F}\big(q_1(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bigg(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\bigg]\bigg)\bigg\}, \tag{2.14}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where $\breve{G}$ is defined in (2.3) and

$$q_1(\breve{x},\breve{y}) = \breve{G}\big(a(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} p(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{2.15}$$

$$\breve{F}(r) = \int_{r_0}^{r} \frac{\hat{\Delta}\breve{s}}{((\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1})(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{F}(+\infty) = \int_{r_0}^{+\infty} \frac{\hat{\Delta}\breve{s}}{(\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1}(\breve{s})} = +\infty, \tag{2.16}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ is chosen so that

$$\bigg(\breve{F}\big(q_1(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bigg(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\bigg) \in \operatorname{Dom}\big(\breve{F}^{-1}\big).$$

Proof Suppose that $a(\breve{\xi},\breve{\zeta}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define a positive and nondecreasing function $z(\breve{x},\breve{y})$ by

$$z(\breve{x},\breve{y}) = a(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\bigg(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},$$

for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{\zeta} \le \breve{y}_1$, then $z(0,\breve{y}) = z(\breve{x},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big),$$

then we have

$$z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) = \int_0^{\breve{y}} \bar{\varphi}\big(u(\breve{x},\breve{t})\big)\big[f(\breve{x},\breve{t})\bar{\varphi}\big(u(\breve{x},\breve{t})\big) + p(\breve{x},\breve{t})\big]\hat{\Delta}\breve{t} + \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\big(u(\breve{x},\breve{t})\big)\bigg(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}$$

$$\begin{aligned}
&\leq \int_0^{\breve{y}} \bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr)\bigl[f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr) + p(\breve{x},\breve{t})\bigr]\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr)\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t} \\
&\leq \bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{y})\bigr) \int_0^{\breve{y}} \bigl[f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr) + p(\breve{x},\breve{t})\bigr]\hat{\Delta}\breve{t} \\
&\quad + \bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{y})\bigr) \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t},
\end{aligned}$$

or

$$\begin{aligned}
\frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{\bar{\varphi} \circ \bar{\psi}^{-1}(z(\breve{x},\breve{y}))} &\leq \int_0^{\breve{y}} \bigl[f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr) + p(\breve{x},\breve{t})\bigr]\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}.
\end{aligned} \tag{2.17}$$

Integrating (2.17) and using (2.3), we get

$$\begin{aligned}
\breve{G}\bigl(z(\breve{x},\breve{y})\bigr) &\leq \breve{G}\bigl(a(\breve{\xi},\breve{\zeta})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bigl[f(\breve{s},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{s},\breve{t})\bigr) + p(\breve{s},\breve{t})\bigr]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.
\end{aligned}$$

$(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily, then from (2.15) we have

$$\begin{aligned}
\breve{G}\bigl(z(\breve{x},\breve{y})\bigr) &\leq q_1(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{s},\breve{t})\bigr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.
\end{aligned}$$

Since $q_1(\breve{x},\breve{y}) > 0$ is a nondecreasing function, fixing an arbitrary point $(\breve{\xi},\breve{\zeta}) \in \Omega$ and defining $\nu(\breve{x},\breve{y}) > 0$ to be a nondecreasing function by

$$\begin{aligned}
\nu(\breve{x},\breve{y}) &= q_1(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{s},\breve{t})\bigr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},
\end{aligned}$$

for $0 \leq \breve{x} \leq \breve{\xi} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{\zeta} \leq y_1$, we have $\nu(0,\breve{y}) = \nu(\breve{x},0) = q_1(\breve{\xi},\breve{\zeta})$ and

$$z(\breve{x},\breve{y}) \leq \breve{G}^{-1}\bigl(\nu(\breve{x},\breve{y})\bigr); \tag{2.18}$$

then we have

$$\begin{aligned}
\nu^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{t})\bigr)\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\bigl(z(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(G^{-1}\big(v(\breve{x},\breve{t})\big)\big)\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(G^{-1}\big(v(\breve{\tau},\breve{t})\big)\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t} \\
&\leq \big(\bar{\varphi}\circ\bar{\psi}^{-1}\big)\circ\breve{G}^{-1}\big(v(\breve{x},\breve{y})\big)\left[\int_0^{\breve{y}} f(\breve{x},\breve{t})\hat{\Delta}\breve{t} + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\right],
\end{aligned}$$

or

$$\frac{v^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{(\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1}(v(\breve{x},\breve{y}))} \leq \left[\int_0^{\breve{y}} f(\breve{x},\breve{t})\hat{\Delta}\breve{t} + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\right]. \tag{2.19}$$

Integrating (2.19) and using (2.16), we get

$$\breve{F}\big(v(\breve{x},\breve{y})\big) \leq \breve{F}\big(q_1(\breve{\xi},\breve{\zeta})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left[1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.$$

Since we can choose $(\breve{\xi},\breve{\zeta}) \in \Omega$ arbitrarily, we have

$$v(\breve{x},\breve{y}) \leq \breve{F}^{-1}\left[\breve{F}\big(q_1(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left[1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]. \tag{2.20}$$

From (2.20), (2.18) and $u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}(z(\breve{x},\breve{y}))$ we get the desired inequality in (2.14). For $a(\breve{x},\breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x},\breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.8 *If we take $\breve{\mathbb{T}} = \mathbb{R}$ in Theorem 2.7, then the inequality*

$$\begin{aligned}
\bar{\psi}\big(u(\breve{x},\breve{y})\big) &\leq a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\,d\breve{t}\,d\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)d\breve{t}\,d\breve{s},
\end{aligned}$$

for $(\breve{x},\breve{y}) \in \Omega$, implies

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left(\breve{F}^{-1}\left[\breve{F}\big(q_2(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\,d\breve{\tau}\right)d\breve{t}\,d\breve{s}\right]\right)\right\},$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, where $\breve{G}$ is as defined in (2.8) and

$$\begin{aligned}
q_2(\breve{x},\breve{y}) &= \breve{G}\big(a(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} p(\breve{s},\breve{t})\,d\breve{t}\,d\breve{s}, \\
\breve{F}(r) &= \int_{r_0}^{r} \frac{d\breve{s}}{((\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1})(\breve{s})}, \quad r \geq r_0 > 0, \\
\breve{F}(+\infty) &= \int_{r_0}^{+\infty} \frac{d\breve{s}}{(\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1}(\breve{s})} = +\infty,
\end{aligned}$$

and $(\breve{x}_1,\breve{y}_1)\in\Omega$ *is chosen so that*

$$\left(\breve{F}\big(q_2(\breve{x},\breve{y})\big)+\int_0^{\breve{x}}\int_0^{\breve{y}}f(\breve{s},\breve{t})\left(1+\int_0^{\breve{s}}g(\breve{\tau},\breve{t})\,d\breve{\tau}\right)d\breve{t}\,d\breve{s}\right)\in\operatorname{Dom}\big(\breve{F}^{-1}\big).$$

Corollary 2.9 *The discrete form of Theorem 2.7 can be obtained by letting* $\breve{\mathbb{T}}=\mathbb{Z}$:

$$\begin{aligned}\bar{\psi}\big(u(\breve{x},\breve{y})\big)\le{}& a(\breve{x},\breve{y})+\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)+p(\breve{s},\breve{t})\big]\\&+\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\sum_{\breve{\tau}=0}^{\breve{s}-1}g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right),\end{aligned}$$

for $(\breve{x},\breve{y})\in\Omega$, *implies*

$$u(\breve{x},\breve{y})\le\bar{\psi}^{-1}\left\{\bar{G}^{-1}\left(\bar{F}^{-1}\left[\bar{F}\big(\bar{q}_2(\breve{x},\breve{y})\big)+\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}f(\breve{s},\breve{t})\left(1+\sum_{\breve{\tau}=0}^{\breve{s}-1}g(\breve{\tau},\breve{t})\right)\right]\right)\right\},$$

for $0\le\breve{x}\le\breve{x}_1$, $0\le\breve{y}\le\breve{y}_1$, *where* $\breve{G}$ *is as defined in* (2.9) *and*

$$\begin{aligned}&\bar{q}_2(\breve{x},\breve{y})=\breve{G}\big(a(\breve{x},\breve{y})\big)+\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}p(\breve{s},\breve{t}),\\&\bar{F}(r)=\sum_{\breve{s}=r_0}^{r-1}\frac{1}{((\bar{\varphi}\circ\bar{\psi}^{-1})\circ\bar{G}^{-1})(\breve{s})},\qquad r\ge r_0>0,\\&\bar{F}(+\infty)=\sum_{\breve{s}=r_0}^{+\infty}\frac{1}{(\bar{\varphi}\circ\bar{\psi}^{-1})\circ\bar{G}^{-1}(\breve{s})}=+\infty,\end{aligned}$$

and $(\breve{x}_1,\breve{y}_1)\in\Omega$ *is chosen so that*

$$\left(\bar{F}\big(\bar{q}_2(\breve{x},\breve{y})\big)+\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}f(\breve{s},\breve{t})\left(1+\sum_{\breve{\tau}=0}^{\breve{s}-1}g(\breve{\tau},\breve{t})\right)\right)\in\operatorname{Dom}\big(\bar{F}^{-1}\big).$$

Theorem 2.10 *Assume that* $g,a,f,u,,\bar{\psi}$ *and* $\bar{\varphi}$ *be as in Theorem* 2.1. *If* $u(\breve{x},\breve{y})$ *satisfies*

$$\begin{aligned}\bar{\psi}\big(u(\breve{x},\breve{y})\big)\le{}& a(\breve{x},\breve{y})+\left(\int_0^{\breve{x}}\int_0^{\breve{y}}f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right)^2\\&+\int_0^{\breve{x}}\int_0^{\breve{y}}f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\int_0^{\breve{s}}g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned}$$

for $(\breve{x},\breve{y})\in\Omega$, *then*

$$u(\breve{x},\breve{y})\le\bar{\psi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big)+\breve{B}(\breve{x},\breve{y})+\left(\int_0^{\breve{x}}\int_0^{\breve{y}}f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right)^2\right]\right\},\tag{2.21}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s}, \tag{2.22}$$

$$\begin{aligned} \breve{H}(r) &= \int_{r_0}^{r} \frac{\hat{\Delta}\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})}, \quad r \ge r_0 > 0, \\ \breve{H}(+\infty) &= \int_{r_0}^{+\infty} \frac{\hat{\Delta}\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})} = +\infty, \end{aligned} \tag{2.23}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right)^2 \right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Proof Assume that $a(\breve{x},\breve{y}) > 0$. Taking $(\breve{\xi},\breve{\zeta}) \in \Omega$ as a fixed arbitrary point, we define $z(\breve{x},\breve{y}) > 0$ to be a nondecreasing function by

$$\begin{aligned} z(\breve{x},\breve{y}) = a(\breve{\xi},\breve{\zeta}) &+ \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right)^2 \\ &+ \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s}, \end{aligned} \tag{2.24}$$

for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{\zeta} \le \breve{y}_1$, hence $z(0,\breve{y}) = z(\breve{x},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big).$$

From (2.24), and applying the chain rule on time scales, Theorem 1.4, we get

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= 2\left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi}\big(u(\breve{x},\breve{t})\big) \hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi}\big(u(\breve{x},\breve{t})\big) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \\ &\le 2\left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{s},\breve{t})\big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) \hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \\ &\le 2\big(\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big)\big)^2 \left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \hat{\Delta}\breve{t} \\ &\quad + \big(\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big)\big)^2 \int_0^{\breve{y}} f(\breve{x},\breve{t}) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t}, \end{aligned}$$

thus, we have

$$\begin{aligned} \frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{(\bar{\varphi} \circ \bar{\psi}^{-1}(z(\breve{x},\breve{y})))^2} &\le 2\left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \end{aligned}$$

$$= \left[\left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \right)^2 \right]^{\hat{\Delta}_{\breve{x}}} + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t}. \tag{2.25}$$

Integrating (2.25) and using (2.23), we get

$$\breve{H}\big(z(\breve{x},\breve{y})\big) \le \breve{H}\big(a(\breve{\xi},\breve{\zeta})\big) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \right)^2 + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\breve{x},\breve{y}) \le \breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \right)^2 \right]. \tag{2.26}$$

From (2.26) and $u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}(z(\breve{x},\breve{y}))$, we get the desired inequality (2.21). For $a(\breve{x},\breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x},\breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.11 *If we take* $\breve{\mathbb{T}} = \mathbb{R}$ *in Theorem* 2.10, *then the inequality*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\, d\breve{t}\, d\breve{s} \right)^2 + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\, d\breve{\tau} \right) d\breve{t}\, d\breve{s},$$

for $(\breve{x},\breve{y}) \in \Omega$, *implies*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{ \breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t})\, d\breve{t}\, d\breve{s} \right)^2 \right] \right\},$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\, d\breve{\tau} \right) d\breve{t}\, d\breve{s},$$

$$\breve{H}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{H}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})} = +\infty,$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t})\, d\breve{t}\, d\breve{s} \right)^2 \right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Corollary 2.12 *The discrete form can be obtained by letting* $\breve{\mathbb{T}} = \mathbb{Z}$ *in Theorem* 2.10:

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \leq a(\breve{x},\breve{y}) + \left(\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\right)^2 + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right),$$

for $(\breve{x},\breve{y}) \in \Omega$, *implies*

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right)^2\right]\right\},$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}} f(\breve{s},\breve{t})\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right),$$

$$\breve{H}(r) = \sum_{\breve{s}=r_0}^{r-1}\frac{1}{(\bar{\varphi}\circ\bar{\psi}^{-1})^2(\breve{s})}, \qquad r \geq r_0 > 0, \qquad \breve{H}(+\infty) = \sum_{\breve{s}=r_0}^{+\infty}\frac{1}{(\bar{\varphi}\circ\bar{\psi}^{-1})^2(\breve{s})} = +\infty,$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

3 Applications

The present section illustrates how Theorems 2.7 and 2.1 can be used to study the boundedness of the solutions of some initial boundary value problem for partial dynamic equations in two independent variables.

Let us consider the problem

$$u^{\hat{\Delta}\breve{x}\hat{\Delta}\breve{y}}(\breve{x},\breve{y}) = \breve{F}\left(\breve{x},\breve{y},u(\breve{x},\breve{y}),\int_0^{\breve{x}}\breve{k}\big(\breve{s},\breve{y},u(s,\breve{y})\big)\hat{\Delta}\breve{s}\right), \tag{3.1}$$

$$u(\breve{x},0) = a_1(\breve{x}), \qquad u(0,\breve{y}) = a_2(\breve{y}), \qquad a_1(0) = a_2(0) = 0, \tag{3.2}$$

for any $(\breve{x},\breve{y}) \in \Omega$, where $\breve{k} \in C_{\mathrm{rd}}(\Omega \times \mathbb{R},\mathbb{R})$, $\breve{F} \in C_{\mathrm{rd}}(\Omega \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, $a_1 \in C_{\mathrm{rd}}(\breve{\mathbb{T}}_1,\mathbb{R})$ and $a_2 \in C_{\mathrm{rd}}(\breve{\mathbb{T}}_2,\mathbb{R})$.

Theorem 3.1 *Suppose that the functions* $\breve{k}, \breve{F}, a_2, a_1$ *in* (3.1) *and* (3.2) *satisfy the conditions*

$$\big|\breve{F}(\breve{x},\breve{y},u(\breve{x},\breve{y},v)\big| \leq \bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big)\big[f(\breve{x},\breve{y})\bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big) + p(\breve{x},\breve{y})\big] + f(\breve{x},\breve{y})\bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big)v, \tag{3.3}$$

$$\big|\breve{k}\big(\breve{x},\breve{y},u(\breve{x},\breve{y})\big)\big| \leq g(\breve{x},\breve{y})\bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big), \tag{3.4}$$

$$\big|a_1(\breve{x}) + a_2(\breve{y})\big| \leq a(\breve{x},\breve{y}), \tag{3.5}$$

where the functions p, g, a, f, and $\bar{\varphi}$ *are defined as in Theorem 2.7 with* $a(\breve{x},\breve{y}) > 0$, *for all* $(\breve{x},\breve{y}) \in \Omega$, *then*

$$\left|u(\breve{x},\breve{y})\right| \le \breve{G}^{-1}\left(\breve{F}^{-1}\left[\breve{F}\left(q_2(\breve{x},\breve{y})\right) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left[1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]\right), \tag{3.6}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where F,* q_2 *and G are defined as in Theorem 2.7.*

Proof If the problem (3.1) and (3.2) has a solution $u(\breve{x},\breve{y})$, it can be written as

$$u(\breve{x},\breve{y}) = a_1(\breve{x}) + a_2(\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \breve{F}\left(\breve{s},\breve{t},u(\breve{s},\breve{t}),\int_0^{\breve{s}}\breve{k}\left(\breve{\tau},\breve{t},u(\breve{\tau},t)\right)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{3.7}$$

for any $(\breve{x},\breve{y}) \in \Omega$. Using the conditions (3.3), (3.4) and (3.5) in (3.7), we get

$$\begin{aligned}\left|u(\breve{x},\breve{y})\right| \le{}& a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\left(\left|u(\breve{s},\breve{t})\right|\right)\left[f(\breve{s},\breve{t})\bar{\varphi}\left(\left|u(\breve{s},\breve{t})\right|\right) + p(\breve{s},\breve{t})\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(s,t)\bar{\varphi}\left(\left|u(s,t)\right|\right)\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\left(\left|u(\breve{\tau},\breve{t})\right|\right)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned} \tag{3.8}$$

for any $(\breve{x},\breve{y}) \in \Omega$. Now, an application of Theorem 2.7 to (3.8) yields the required inequality in (3.6) where $\bar{\psi}(u) = u$. □

Let us consider the initial boundary value problem of the form

$$\left(z^q\right)^{\hat{\Delta}\breve{x}\hat{\Delta}\breve{y}}(\breve{x},\breve{y}) = \breve{A}\left(\breve{x},\breve{y},z(\breve{x},\breve{y}),\int_0^{\breve{x}} h\left(\breve{s},\breve{y},z(\breve{s},\breve{y})\right)\hat{\Delta}\breve{s}\right) \tag{3.9}$$

$$z(\breve{x},0) = a_1(\breve{x}), \qquad z(0,\breve{y}) = a_2(\breve{y}), \qquad a_1(0) = a_2(0) = 0, \tag{3.10}$$

for any $(\breve{x},\breve{y}) \in \Omega$.

Theorem 3.2 *Assume that the functions h, A,* a_2, a_1 *in (3.9) and (3.10) satisfy the conditions*

$$\left|A(\breve{x},\breve{y},z(\breve{x},\breve{y},v)\right| \le f(\breve{x},\breve{y})\left|z^r(\breve{x},\breve{y})\right| + f(\breve{x},\breve{y})v, \tag{3.11}$$

$$\left|h\left(\breve{x},\breve{y},z(\breve{x},\breve{y})\right)\right| \le g(\breve{x},\breve{y})\left|z^r(\breve{x},\breve{y})\right|, \tag{3.12}$$

$$\left|a_1(\breve{x}) + a_2(\breve{y})\right| \le a(\breve{x},\breve{y}), \tag{3.13}$$

where $r \ge q > 0$, *then*

$$\left|z(\breve{x},\breve{y})\right| \le \left[\left(a(\breve{x},\breve{y})\right)^{\frac{q-r}{q}} + \frac{q-r}{q}\int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]^{\frac{1}{q-r}}, \tag{3.14}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$.

Proof If the problem (3.9) and (3.10), has a solution $z(\breve{x},\breve{y})$ it can be written as

$$z^q(\breve{x},\breve{y}) = a_1(x) + a_2(y) + \int_0^{\breve{x}}\int_0^{\breve{y}} \breve{F}\left(\breve{s},\breve{s}\breve{t},u(\breve{s},\breve{t}),\int_0^{\breve{s}}\breve{k}\left(\breve{\tau},\breve{t},u(\breve{\tau},\breve{t})\right)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{3.15}$$

for any $(\breve{x},\breve{y}) \in \Omega$. Using the conditions (3.11), (3.12) and (3.13) in (3.15), we get

$$\begin{aligned} \left|z^q(\breve{x},\breve{y})\right| \leq a(\breve{x},\breve{y}) &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left|z^r(s,t)\right|\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\left|z^r(\breve{\tau},\breve{t})\right|\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \end{aligned} \tag{3.16}$$

from (3.16), we get

$$\begin{aligned} \left|z^q(\breve{x},\breve{y})\right| \leq a(\breve{x},\breve{y}) &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left|z^r(\breve{t},\breve{t})\right|\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\left|z^r(\breve{\tau},\breve{t})\right|\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \end{aligned} \tag{3.17}$$

for any $(\breve{x},\breve{y}) \in \Omega$. A suitable application of Theorem 2.1 to (3.17) with $\bar{\psi}(u) = u^q$, $\bar{\varphi}(u) = u^r$ and $p(\breve{x},\breve{y}) = 0$ gives the required inequality in (3.14). □

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