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# CERTAIN RANKIN-SELBERG INTEGRALS FOR UNITARY GROUPS

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**Abstract.** We consider the real rank one unitary group $G$ and its subgroup $H$ obtained as the stabilizer of an anisotropic vector in the skew-hermitian space defining $G$. We compute the inner-product of an Eisenstein series on $H$ and a non-holomorphic cuspidal Hecke eigenform on $G$ restricted to $H$ to obtain an integral representation of the standard $L$-function of the eigenform. We also discuss some consequences of the integral representation.

**1. Introduction.** The Poincaré dual forms of special cycles on a Shimura variety yield an interesting class of non-holomorphic automorphic forms of many variables, and had been investigated by several people in different ways ([4], [5], [17], [18], [11]). In order to deepen our understanding of the arithmetic nature of such forms, the study of the associated $L$-series is indispensable. However, for application to arithmetic, many of the existing works on $L$-functions seem to lack the local theory for the ramified factors and the gamma factors; one may need a heavy and sophisticated apparatus of the representation theory to handle them thoroughly. The aim of this paper is to deduce basic properties of the $L$-functions for a narrow but important class of automorphic forms on a unitary group by taking advantage from the special feature of our targeting automorphic forms.

As a generalization of the work of Andrianov on the $L$-functions of Siegel modular forms of genus two, Sugano studied the Dirichlet series and the Rankin-Selberg integrals associated with holomorphic cusp forms on the type IV tube domain in connection with the standard $L$-functions of orthogonal groups ([14]). In this paper, we carry out a unitary analogue of the study. Let $R$ be a non-degenerate skew-hermitian form on a vector space $V$ of finite dimension $m$ over an imaginary quadratic field $E(\subset \boldsymbol{C})$ and $\tilde{R} = R \oplus \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$ its extension by a hyperbolic plane with a Witt basis $\{e, e'\}$. If we assume that $\sqrt{-1}R$ is positive definite, then the unitary group $G = \mathrm{U}(\tilde{R})$, regarded as a $\boldsymbol{Q}$-algebraic group, is of $\boldsymbol{R}$-rank one and the symmetric space $\mathfrak{D}$ associated with the real points of $G$ is realized as a complex hyperball in $\boldsymbol{C}^{m+1}$. Let $\mathcal{O}$ be the maximal order of $E$ and fix a maximal $\mathcal{O}$-integral lattice $\mathcal{M}$ in $(R, V)$. Then, $K_{\mathrm{f}}$, the stabilizer of the extended $\mathcal{O}$-lattice $\tilde{\mathcal{M}} = \mathcal{M} \oplus \langle e, e' \rangle_{\mathcal{O}}$, yields a maximal compact subgroup of $G_{\mathrm{f}}$, the group of finite adeles of $G$. Let $Y$ be a reduced vector for $(R, \mathcal{M})$ (see 3.4), and $\tilde{Y} = (Y; 0, 0)$ its image in the space of $\tilde{R}$. Since $G_0^Y \times \mathrm{GL}_1$ is regarded as a Levi subgroup of the parabolic subgroup $P^{\tilde{Y}}$ of $G^{\tilde{Y}}$ stabilizing the isotropic line $Ee$, a Hecke eigenfunction $f$ on the finite space $G_{0,\boldsymbol{Q}}^Y \backslash G_{0,\boldsymbol{A}}^Y / G_{0,\infty}^Y (G_{0,\mathrm{f}}^Y \cap K_{\mathrm{f}})$ yields an Eisenstein series $E(f; s; g)$ on $G_{\boldsymbol{A}}^{\tilde{Y}}$. Let $F$ be a $K_{\mathrm{f}}$-invariant Hecke eigen cusp form on $G_{\boldsymbol{Q}} \backslash G_{\boldsymbol{A}}$. Then we consider the inner

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product $Z^{F}_{f,Y}(s)$ of $F$ restricted to $G^{\tilde{Y}}_{\boldsymbol{Q}}\backslash G^{\tilde{Y}}_{\boldsymbol{A}}$ and the Eisenstein series $E(f;s)$. We investigate the integral $Z^{F}_{f,Y}(s)$ for two types of non-holomorphic cusp forms $F$; one is the wave cusp forms corresponding to Laplace eigenfunctions on the symmetric space $\mathfrak{D}$, and the other the cohomological cusp forms corresponding to harmonic differential forms of type (1, 1) on $\mathfrak{D}$. We calculate the integral $Z^{F}_{f,Y}(s)$ and obtain an identity which equates $Z^{F}_{f,Y}(s)$ with a ratio of standard $L$-functions of $f$ and $F$ up to a certain proportionality constant $c_{f,Y}(F)$ called the Whittaker coefficient (Theorem 58 and 61). We should mention that the same integral is studied by Gelbart and Piatetski-Shapiro ([1]) for generic cusp forms on the quasi-split unitary group of degree 3.

For the proof, we closely follow the method of [14] and [15] to calculate the non-archimedean zeta-integrals, and use the explicit formula of Whittaker functions to calculate the archimedean zeta-integrals. For the latter, we examine the differential equations satisfied by Whittaker functions which have already been discussed by Taniguchi [16] for the discrete series Whittaker functions. We prove a multiplicity one theorem of Whittaker functions (Proposition 51), which enables us to define the Whittaker coefficients $c_{f,Y}(F)$ for a cusp form $F$. As an application of the main identity, we show the functional equation of the standard $L$-function $L(s, F)$ attached to $F$ with a non-zero Whittaker coefficient, and also have a criterion for the right-most possible pole of $L(s, F)$ to occur actually (Theorem 59 and 62).

We are going to use the results obtained in this paper to study a fine structure of the Hecke module generated by the Poincaré dual forms of special divisors on a unitary Shimura variety with full level.

NOTATIONS. The number 0 is included in the set of natural numbers $\boldsymbol{N}$: $\boldsymbol{N} = \{0, 1, 2, \dots, \}$. We use the usual notations $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ to denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively.

The ring of finite adeles of $\boldsymbol{Q}$ is denoted by $\boldsymbol{A}_{\mathrm{f}}$; the adele ring $\boldsymbol{A}$ of $\boldsymbol{Q}$ is then the direct product of $\boldsymbol{A}_{\mathrm{f}}$ and $\boldsymbol{R}$, i.e., $\boldsymbol{A} = \boldsymbol{R} \times \boldsymbol{A}_{\mathrm{f}}$. For an idele $a \in \boldsymbol{A}^{\times}$, $|a|_{\boldsymbol{A}}$ denotes its idele norm. For an algebraic group $H$ defined over a field $k$ and a $k$-algebra $A$, the group of $A$-valued points of $H$ is denoted by $H_A$.

For $r$ matrices $A_1, \dots, A_r$ with coefficients in a commutative ring, $\mathrm{diag}(A_1, \dots, A_r)$ denotes the block-diagonal matrix $A_1 \oplus \cdots \oplus A_r$. For $m \in \boldsymbol{N}$ and a commutative ring $A$ with the identity 1, we denote by $1_m = \mathrm{diag}(1, \dots, 1)$ the unit matrix of size $m$. We denote by $A^m$ the set of column vectors with entries in $A$ of size $m$, and by $0_m$ the zero vector in $A^m$.

For $n, m \in \boldsymbol{N}$, we denote by $\mathrm{U}(n, m)$ the real Lie group $\{g \in \mathrm{GL}_{n+m}(\boldsymbol{C}) \mid {}^t\bar{g}\,\mathrm{diag}(1_n, -1_m)g = \mathrm{diag}(1_n, -1_m)\}$. In particular, $\mathrm{U}(n, 0)$, the compact unitary group of matrix size $n$, is denoted by $\mathrm{U}(n)$.

For a condition P, we use the 'Kronecker symbol' $\delta(\mathrm{P})$ in an extended sense that $\delta(\mathrm{P}) \in \{0, 1\}$ equals 1 if and only if the condition P is true.

**2. Preliminaries.** In this section, $k$ denotes the rational number field $\boldsymbol{Q}$ or one of its localizations $\boldsymbol{Q}_p$ at prime numbers $p$; $F/k$ denotes a quadratic field extension of $\boldsymbol{Q}$ if $k = \boldsymbol{Q}$, and a quadratic algebra over $\boldsymbol{Q}_p$ if $k = \boldsymbol{Q}_p$ with a prime $p$. We denote by $a \mapsto \bar{a}$ the unique non-trivial $k$-automorphism of $F$. Set $\mathrm{N}(a) = a\bar{a}$ and $\mathrm{tr}(a) = a + \bar{a}$ for $a \in F$. Let $\mathcal{O}_F$ and $\mathcal{O}_k$ be the maximal orders of $F$ and $k$, respectively.

2.0.1. A *skew-hermitian space over* $F$ is a pair $(R, V)$ of a free $F$-module $V$ of finite rank and a bi $k$-linear form $R : V \times V \to F$ such that $R(\lambda v, \mu w) = \lambda\bar{\mu}R(v, w)$ for all $\lambda, \mu \in F$ and all $v, w \in V$, $R(v, w) = -\overline{R(w, v)}$ for all $v, w \in V$; we always assume $R$ is non-degenerate, i.e., $R(V, v) \neq \{0\}$ if $v \neq 0$. The unitary group of $(R, V)$ is defined to be a $k$-algebraic group $\mathrm{U}(R)$ whose set of $k$-points is given by

$$\mathrm{U}(R)_k = \{g \in \mathrm{GL}_F(V) \mid R(gv, gw) = R(v, w) \ \text{ for all } v, w \in V\}\,.$$

If $k = \boldsymbol{Q}$ and $(R, V)$ is a skew-hermitian space over $F$, then the natural extension $R_p : V_p \times V_p \to F_p$ yields a skew-hermitian space $(R_p, V_p)$ over $F_p$ for each prime $p$. Here $F_p = F \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$, $V_p = V \otimes_F F_p$ for a prime $p$.

Given an $\mathcal{O}_F$-lattice $\mathcal{L}$ in $V$, we say $\mathcal{L}$ is an *$\mathcal{O}_F$-integral lattice in* $(R, V)$ if $R(\mathcal{L}, \mathcal{L}) \subset \mathcal{O}_F$ and $R[\mathcal{L}] \subset \{a - \bar{a} \mid a \in \mathcal{O}_F\}$. An $\mathcal{O}_F$-integral lattice $\mathcal{M}$ in $(R, V)$ is said to be *maximal* if there exists no $\mathcal{O}_F$-integral lattice in $(R, V)$ which contains $\mathcal{M}$ properly.

An $\mathcal{O}_F$-lattice $\mathcal{L}$ in a skew-hermitian space $(R, V)$ over a quadratic extension $F$ of $\boldsymbol{Q}$ is maximal $\mathcal{O}_F$-integral if and only if $\mathcal{L}_p$ is maximal $\mathcal{O}_{F_p}$-integral in $(R_p, V_p)$ for all prime numbers $p$. Here $\mathcal{L}_p = \mathcal{L} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$ for a prime $p$.

Given an $\mathcal{O}_F$-lattice $\mathcal{L}$ and a vector $\xi \in \mathcal{L}$, we say $\xi$ is *$\mathcal{O}_F$-primitive* in $\mathcal{L}$ if $\xi \in \mathcal{L} - \mathfrak{m}\mathcal{L}$ for any maximal ideal $\mathfrak{m}$ of $\mathcal{O}_F$. The set of $\mathcal{O}_F$-primitive vectors in $\mathcal{L}$ is denoted by $\mathcal{L}_{\mathrm{prim}}$.

Given an $\mathcal{O}_F$-lattice $\mathcal{L}$ in $V$, we define the $\mathcal{O}_F$-ideal $\mathfrak{d}_R(\mathcal{L})$ following way. When $F$ is a quadratic $\boldsymbol{Q}_p$-algebra, $\mathfrak{d}_R(\mathcal{L})$ is defined to be $\det(R(v_i, v_j))\mathcal{O}_F$ with $\{v_i\}$ an $\mathcal{O}_F$-basis of $\mathcal{L}$; the $\mathcal{O}_F$-ideal is independent of the choice of $\{v_i\}$. When $F$ is a quadratic extension of $\boldsymbol{Q}$, $\mathfrak{d}_R(\mathcal{M})$ is defined to be the $\mathcal{O}_F$-ideal such that $\mathfrak{d}_R(\mathcal{M})\mathcal{O}_{F_p} = \mathfrak{d}_{R_p}(\mathcal{M}_p)$ for all prime numbers $p$.

LEMMA 1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be $\mathcal{O}_F$-lattices in $V$ such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Then there exists an $\mathcal{O}_F$-ideal $I$ such that $\mathfrak{d}_R(\mathcal{L}_1) = \mathrm{N}(I)\mathfrak{d}_R(\mathcal{L}_2)$. Here $\mathrm{N}(I)$ denotes the norm of $I$, i.e., $\mathrm{N}(I) = \sharp(\mathcal{O}_F/I)$.*

PROOF. It suffices to show the claim when $F$ is a quadratic $\boldsymbol{Q}_p$-algebra with a prime $p$. By the elementary divisor theory, there exists an $\mathcal{O}_F$-basis $\{e_j\}$ of $\mathcal{L}_2$ and integers $\lambda_j \in \mathcal{O}_F$ such that $\{\lambda_j e_j\}$ is an $\mathcal{O}_F$-basis of $\mathcal{L}_1$. Set $a = \prod_i \lambda_i$. Then the relation $\mathfrak{d}_R(\mathcal{L}_1) = \mathrm{N}(a)\mathfrak{d}_R(\mathcal{L}_2)$ follows from the obvious equation $\det(R(\lambda_i e_i, \lambda_j e_j)) = \mathrm{N}(\prod_i \lambda_i)\det(R(e_i, e_j))$. □

The dual of an $\mathcal{O}_F$-lattice $\mathcal{L}$ is denoted by $\mathcal{L}^*$, i.e.,

$$\mathcal{L}^* = \{v \in V \mid R(v, \mathcal{L}) \subset \mathcal{O}_F\}\,.$$

LEMMA 2. *Let $\mathcal{L}$ be an $\mathcal{O}_F$-integral lattice in $(R, V)$. Then $\mathcal{L} \subset \mathcal{L}^*$ and* $\mathrm{N}(\mathfrak{d}_R(\mathcal{L})) = \sharp(\mathcal{L}^*/\mathcal{L})$.

PROOF. The inclusion $\mathcal{L} \subset \mathcal{L}^*$ results from the assumption that $\mathcal{L}$ is $\mathcal{O}$-integral. To prove the second assertion, it suffices to show the claim when $F$ is a quadratic $\boldsymbol{Q}_p$-algebra with a prime $p$. Let $\{e_j\}$ be an $\mathcal{O}_F$-basis of $\mathcal{L}$ and set $S = (R(e_i, e_j))$. Then by the elementary divisor theory, there exist unimodular matrices $A, B \in \mathrm{GL}_n(\mathcal{O}_F)$ such that $ASB$ is a diagonal matrix : $ASB = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. The basis $\{e_j\}$ affords the identifications $\mathcal{L} \cong \mathcal{O}_F^n$ and $\mathcal{L}^* \cong S^{-1}\mathcal{O}_F^n$, which induce the first map in the sequence of $\mathcal{O}_F$-isomorphisms:

$$\mathcal{L}^*/\mathcal{L} \cong S^{-1}\mathcal{O}_F^n/\mathcal{O}_F^n \cong \mathcal{O}_F^n/S\mathcal{O}_F^n \cong \prod_{j=1}^{n} \mathcal{O}_F/\lambda_j\mathcal{O}_F\,.$$

This gives us $\sharp(\mathcal{L}^*/\mathcal{L}) = \prod_{j=1} \mathrm{N}(\lambda_j\mathcal{O}_F) = \mathrm{N}(\det(S)\mathcal{O}_F) = \mathrm{N}(\mathfrak{d}_R(\mathcal{L}))$. □

For matrices $X, Y, Z$ with coefficients in $F$, we denote by $X(Y, Z)$ (resp. $X[Y]$) the matrix ${}^t\bar{Z}XY$ (resp. ${}^t\bar{Y}XY$) whenever the product is defined.

A matrix $S \in \mathrm{GL}_n(F)$ is called a skew-hermitian matrix if ${}^t\bar{S} = -S$. We always use the same notation $S$ to denote the function $(X, X') \mapsto S(X, X')$ on $F^n \times F^n$.

2.0.2. For a skew-hermitian matrix $R \in \mathrm{GL}_m(F)$ of size $m \geqslant 1$, set $\tilde{R} = \begin{bmatrix} & & -1 \\ & R & \\ 1 & & \end{bmatrix}$. Put $V = F^m$ and $\tilde{V} = \begin{bmatrix} F \\ V \\ F \end{bmatrix}$. Then we have skew-hermitian spaces $(R, V)$ and $(\tilde{R}, \tilde{V})$ over $F$. Let $G$ and $G_0$ denote the unitary groups $\mathrm{U}(\tilde{R})$ and $\mathrm{U}(R)$, respectively.

2.0.3. Consider the $k$-subgroups $M$ and $N$ of $G$ such that

$$M_A = \{\mathsf{m}(t; g_0) := \mathrm{diag}(t, g_0, \bar{t}^{-1}) \mid t \in (F \otimes_k A)^\times,\, g_0 \in G_{0,A}\},$$

$$N_A = \left\{ \mathsf{n}(X; \zeta) := \begin{bmatrix} 1 & -{}^t\bar{X}R & \zeta - 2^{-1}R[X] \\ 0 & 1_m & X \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, X \in V \otimes_k A,\, \zeta \in A \right\}$$

for an $k$-algebra $A$. Then $P = MN$ is a parabolic $k$-subgroup of $G$ and $M$ (resp. $N$) is a Levi subgroup (resp. the unipotent radical) of $P$.

2.0.4. For a non-isotropic vector $Y \in V$, set $\tilde{Y} = \begin{bmatrix} 0 \\ Y \\ 0 \end{bmatrix} \in \tilde{V}$ and $\Delta = R[Y]$. The form $\tilde{R}$ induces a non-degenerate skew-hermitian form $\tilde{R}|\tilde{Y}^\perp$ on the orthogonal complement $\tilde{Y}^\perp$ of $\tilde{Y}$ in $\tilde{V}$, whose unitary group $\mathrm{U}(\tilde{R}|\tilde{Y}^\perp)$ is identified with $G^{\tilde{Y}}$, the stabilizer of $\tilde{Y}$ in $G$.

2.0.5. The intersection $P^{\tilde{Y}} = P \cap G^{\tilde{Y}}$ is a parabolic $k$-subgroup of $G^{\tilde{Y}}$ with the unipotent radical $N^{\tilde{Y}} = N \cap G^{\tilde{Y}}$ and $M^{\tilde{Y}} = M \cap G^{\tilde{Y}}$ is a Levi part of $P^{\tilde{Y}}$. We also note that

$$M_A^{\tilde{Y}} = \{\mathsf{m}(t; g_0) \mid t \in (F \otimes_k A)^\times,\, g_0 \in G_{0,A}^Y\}, \quad N_A^{\tilde{Y}} = \{\mathsf{n}(X; \zeta) \mid X \in Y_A^\perp,\, \zeta \in A\}$$

for $A$ as above. Here $G_0^Y$ is the stabilizer of $Y$ in $G_0$ and $Y^\perp$ is the orthogonal complement of $Y$ in $V$. We usually regard $G_0$ as a closed $k$-subgroup of $G$ by the inclusion $g_0 \mapsto \mathsf{m}(1; g_0)$.

**3. Local fine structure of Hermitian lattices and reduced vectors.** All materials in this section are adapted from the similar results for orthogonal group obtained by Sugano [14], [15].

In this section, we fix a prime $p$ and denote by $E_p = \boldsymbol{Q}_p(\sqrt{D})$ a quadratic field extension of $\boldsymbol{Q}_p$ with discriminant $D$. Set $\tau(a) = \sqrt{D}^{-1}(a - \bar{a})$ for $a \in E_p$. Let $\mathcal{O}_p$ be the maximal order of $E_p$, $\pi$ a prime element of $\mathcal{O}_p$, $e$ the ramification index of $E_p/\boldsymbol{Q}_p$ and $q$ the order of the residue field $\mathcal{O}_p/\pi\mathcal{O}_p$.

3.1. Classification of skew-hermitian spaces.

LEMMA 3. *$\tau(\mathcal{O}_p) = \boldsymbol{Z}_p$ and $\tau(\pi^{-1}\mathcal{O}_p) = p^{-1}\boldsymbol{Z}_p$.*

PROOF. There exists $\theta \in \mathcal{O}_p$ such that $\tau(\theta) = 1$ and $\mathcal{O}_p = \boldsymbol{Z}_p + \boldsymbol{Z}_p\theta$; from this fact the relation $\tau(\mathcal{O}_p) = \boldsymbol{Z}_p$ is obvious. When $e = 1$, we obtain $\tau(\pi^{-1}\mathcal{O}_p) = p^{-1}\boldsymbol{Z}_p$ from $\tau(\mathcal{O}_p) = \boldsymbol{Z}_p$ taking $\pi = p$. Suppose $e = 2$. Then, to prove $\tau(\pi^{-1}\mathcal{O}_p) = p^{-1}\boldsymbol{Z}_p$, it suffices to show $\tau(\pi\mathcal{O}_p) = \boldsymbol{Z}_p$. We may take $\pi = \sqrt{D}/2 - 1$ if $p = 2$, $D/4 \equiv -1 \pmod 4$, and may take $\pi = \sqrt{D}/2$ otherwise. Then $\tau(\pi) = 1$. Since $\tau(\mathcal{O}_p) = \boldsymbol{Z}_p$, the set $\tau(\pi\mathcal{O}_p)$ is an ideal of $\boldsymbol{Z}_p$. Therefore, $\tau(\pi\mathcal{O}_p) = \boldsymbol{Z}_p$. □

We record two fundamental lemmas on the classification of maximal integral lattices in a skew-hermitian space over $E_p$.

LEMMA 4. *Let $(R_0, V_0)$ be an anisotropic skew-hermitian space of dimension $n_0$. Then $n_0 \in \{0, 1, 2\}$. For an $l \in \boldsymbol{Z}$, the set $\mathcal{M}_0(l) = \{z \in V_0 \mid R_0[z]/\sqrt{D} \in p^l\boldsymbol{Z}_p\}$ is an $\mathcal{O}_p$-lattice in $V_0$. The $\mathcal{O}_p$-lattice $\mathcal{M}_0 = \mathcal{M}_0(0)$ yields the unique maximal $\mathcal{O}_p$-integral lattice in $(R_0, V_0)$.*

In the remaining part of this subsection, we denote by $(R, V)$ a skew-hermitian space over $E_p$ and by $\mathcal{M}$ a maximal $\mathcal{O}_p$-integral lattice in $(R, V)$. The Witt index of $(R, V)$ is denoted by $\nu(R)$; the dimension of a maximal anisotropic subspace of $V$ is denoted by $n_0(R)$.

LEMMA 5. *Let $(R, V)$ and $\mathcal{M}$ be as above and set $\nu = \nu(R)$ and $n_0 = n_0(R)$. Then there exists a system of isotropic vectors $\{e_j, e'_j\}_{1 \leqslant j \leqslant \nu}$ in $\mathcal{M}$ such that $R(e_j, e'_i) = \delta_{ij}$ which satisfies the condition: $V_0 = \{v \in V \mid R(v, e_j) = R(v, e'_j) = 0$ for all $j\}$ is a maximal anisotropic subspace, $\mathcal{M}_0 = V_0 \cap \mathcal{M}$ is the maximal $\mathcal{O}_p$-integral lattice in $(R \mid V_0, V_0)$ and*

$$\mathcal{M} = \bigoplus_{j=1}^{\nu} \langle e_j, e'_j \rangle_{\mathcal{O}_p} \oplus \mathcal{M}_0 . \tag{3.1}$$

*Moreover, when an isotropic vector $e \in \mathcal{M}_{\mathrm{prim}}$ is given, we can choose the decomposition* (3.1) *so that $e_1 = e$.*

PROOF. cf. [7, Lemma 3.2 (p. 37)]. □

The decomposition (3.1) is called a Witt decomposition of $\mathcal{M}$. If the form is isotropic, a special form of Witt decompositions is available. Indeed,

LEMMA 6. *Let $Y \in \mathcal{M}^*_{\mathrm{prim}}$. If $\nu(R) \geqslant 1$, then there exists a Witt decomposition* (3.1) *of $\mathcal{M}$ such that $R(e_1, Y) = 1$, $R(e_j, Y) = R(e'_j, Y) = 0$ $(2 \leqslant j \leqslant \nu(R))$.*

PROOF. Take a Witt decomposition $\mathcal{M}=\bigoplus_{j=1}^{\nu}\langle v_j, v_j'\rangle_{\mathcal{O}_p}\oplus\mathcal{M}_1$ and choose an $\mathcal{O}_p$-basis $\{f_k\}$ of the $\mathcal{O}_p$-lattice $\mathcal{M}_1$. Set $\tilde{f_k}=f_k-a_kv_1-v_1'$ with $a_k\in\mathcal{O}_p$ such that $R[f_k]/\sqrt{D}=-\tau(a_k)$. Then $\{v_j, v_j', \tilde{f_k}\}$ yields an $\mathcal{O}_p$-basis of $\mathcal{M}$ consisting of isotropic vectors. Since $Y$ is $\mathcal{O}_p$-primitive in $\mathcal{M}^*$, the $\mathcal{O}_p$-ideal $R(Y,\mathcal{M})=\langle R(Y,v_j), R(Y,v_j'), R(Y,\tilde{f_k})\mid j,k\rangle_{\mathcal{O}_p}$ coincides with $\mathcal{O}_p$. From this, we conclude the existence of an isotropic vector $\tilde{e}_1\in\mathcal{M}$ such that $R(Y,\tilde{e}_1)=1$. Since $Y\in\mathcal{M}^*$, it is forced that $\tilde{e}_1\in\mathcal{M}_{\rm prim}$; hence we can take a Witt decomposition $\mathcal{M}=\sum_{j=1}^{\nu}\langle\tilde{e}_j,\tilde{e}_j'\rangle_{\mathcal{O}_p}\oplus\mathcal{M}_0$ extending $\tilde{e}_1$. For $2\leqslant j\leqslant\nu$, set $\alpha_j=R(Y,\tilde{e}_j)$, $\beta_j=R(Y,\tilde{e}_j')$ and consider the vectors $e_j=\tilde{e}_j-\bar{\alpha}_j\tilde{e}_1, e_j'=\tilde{e}_j-\bar{\beta}_j\tilde{e}_1$ $(2\leqslant j\leqslant\nu)$, $e_1=\tilde{e}_1, e_1'=\tilde{e}_1'+\sum_{i=2}^{\nu}(\alpha_i\tilde{e}_i'+\beta_i\tilde{e}_i-\alpha_i\bar{\beta}_i\tilde{e}_1)$. Then $e_j, e_j'$ $(1\leqslant j\leqslant\nu)$ are isotropic vectors in $\mathcal{M}$ which yields a desired Witt decomposition. □

We recall here the basic notations and facts on $\mathcal{O}_p$-lattices. For $\mathcal{M}$ as above, we set

$$\mathcal{M}'=\{X\in\mathcal{M}^*\mid \sqrt{D}^{-1}R[X]\in\tau(\pi^{-1}\mathcal{O}_p)\}\,.$$

LEMMA 7. *The set $\mathcal{M}'$ is an $\mathcal{O}_p$-lattice in $V$. We have the inclusions of $\mathcal{O}_p$-lattices*:

$$\mathcal{M}\subset\mathcal{M}'\subset\mathcal{M}^*\,,\qquad \mathcal{M}\subset(\mathcal{M}')^*\subset\mathcal{M}^*\,,$$
$$\pi\mathcal{M}'\subset\mathcal{M}\,,\qquad \pi\mathcal{M}^*\subset(\mathcal{M}')^*\,.$$

PROOF. By Lemma 4 and the Witt decomposition (3.1), $\mathcal{M}'=\bigoplus_{j=1}^{\nu}\langle e_j, e_j'\rangle_{\mathcal{O}_p}\oplus\mathcal{M}_0(-1)$ is an $\mathcal{O}_p$-lattice. We prove $\pi\mathcal{M}'\subset\mathcal{M}$ first. Let $X\in\mathcal{M}'$. Then $\pi X\in\mathcal{M}^*$ on one hand. On the other hand, by Lemma 3, we have the relation $R[\pi X]/\sqrt{D}={\rm N}(\pi)R[X]/\sqrt{D}\in{\rm N}(\pi)p^{-1}\mathbf{Z}_p$, which yields $R[\pi X]/\sqrt{D}\in\mathbf{Z}_p$. Since $\mathcal{M}$ is a maximal $\mathcal{O}_p$-integral lattice in $(R,V)$, we obtain $\pi X\in\mathcal{M}$. This shows $\pi\mathcal{M}'\subset\mathcal{M}$. The remaining inclusions are obvious or are deduced easily from the proved ones by taking duals. □

Let $\partial_R(\mathcal{M})$ be the dimension of the $\mathcal{O}_p/\pi\mathcal{O}_p$-vector space $\mathcal{M}'/\mathcal{M}$. It is easy to see that $\partial_R(\mathcal{M})=\partial_{R\mid V_0}(\mathcal{M}_0)$ for the decomposition (3.1).

LEMMA 8. *Let $(R_0,V_0)$ be an anisotropic skew-hermitian space of dimension $n_0$ and $\mathcal{M}_0$ the maximal $\mathcal{O}_p$-integral lattice in $(R_0,V_0)$.*

- *Assume $n_0=1$. Then there exists an $\mathcal{O}_p$-basis of $\mathcal{M}_0$ such that $R_0$ is given by the matrix $S_0=a\sqrt{D}$ with some $a\in\mathbf{Z}_p\cap(\mathcal{O}_p^\times\cup\pi\mathcal{O}_p^\times)$. We have*

$$\partial_{a\sqrt{D}}(\mathcal{O}_p)=\begin{cases}0 & (e=1)\,,\\ 1 & (e=2\ \ or\ \ a\in p\mathbf{Z}_p^\times)\,.\end{cases}$$

- *Assume $n_0=2$. Then there exists an $\mathcal{O}_p$-basis of $\mathcal{M}_0$ with respect to which $R_0$ is given by the matrix $S_0=s\sqrt{D}\left[\begin{smallmatrix}1 & b\\ \bar{b} & c\end{smallmatrix}\right]$ with some $(b,c,s)\in\sqrt{D}^{-1}\mathcal{O}_p\times\mathbf{Z}_p\times\mathbf{Z}_p^\times$ such that $b\bar{b}-c\in pD^{-1}\mathbf{Z}_p^\times$, $b\bar{b}-c\notin{\rm N}(E_p^\times)$. We have*

$$\partial_{s\sqrt{D}\left[\begin{smallmatrix}1 & b\\ \bar{b} & c\end{smallmatrix}\right]}(\mathcal{O}_p^2)=\begin{cases}1 & (e=1)\,,\\ 2 & (e=2)\,.\end{cases}$$

PROOF. cf. [13], [12]. We follow the formulation in [8]. □

3.2. Maximal lattices. Let $(S, E_p^m)$ be a skew-hermitian space; by the standard basis of $E_p^m$, $S$ is identified with the representing matrix. From the relation $S = -{}^t\bar{S}$, we obtain $\det(S) = (-1)^m\overline{\det(S)}$, which implies $\det(S) \in \boldsymbol{Q}_p$ if $m$ is even and $\det(S)/\sqrt{D} \in \boldsymbol{Q}_p$ if $m$ is odd. Note $\mathfrak{d}_S(\mathcal{O}_p^m) = \det(S)\mathcal{O}_p$. Here is a criterion for the $\mathcal{O}_p$-lattice $\mathcal{O}_p^m$ to be maximal $\mathcal{O}_p$-integral in $(S, E_p^m)$.

PROPOSITION 9. *Suppose $\mathcal{O}_p^m$ is $\mathcal{O}_p$-integral in $(S, E_p^m)$. Suppose the extension $E_p/\boldsymbol{Q}_p$ is tame, i.e.,* $\operatorname{ord}_p(D) \in \{0, 1\}$. *Then the $\mathcal{O}_p$-lattice $\mathcal{O}_p^m$ is maximal $\mathcal{O}_p$-integral in $(S, E_p^m)$ if and only if one of the following two conditions is satisfied.*

(1) *$m$ is even and* $\det(S) \in \boldsymbol{Z}_p^\times \cup (p\boldsymbol{Z}_p^\times - \mathrm{N}(E_p^\times))$.

(2) *$m$ is odd and* $\det(S)/\sqrt{D} \in \boldsymbol{Z}_p \cap (\mathcal{O}_p^\times \cup \pi\mathcal{O}_p^\times)$.

PROOF. First we prove the direct part. Assume $m$ is even and $\mathcal{O}_p^m$ is maximal $\mathcal{O}_p$-integral. Then, by Lemma 5, we take a Witt decomposition $\mathcal{O}_p^m = \bigoplus_{j=1}^{\nu}\langle e_j, e'_j\rangle_{\mathcal{O}_p} \oplus \mathcal{L}_0$. The rank $n_0$ of $\mathcal{L}_0$ equals 0 or 2. If $n_0 = 0$, then $\det(S) = 1 \in \boldsymbol{Z}_p^\times$. If $n_0 = 2$, then by Lemma 8, $S|\mathcal{L}_0$ is represented by a matrix of the form $S_0 = s\sqrt{D}\begin{bmatrix} 1 & b \\ \bar{b} & c \end{bmatrix}$ with $(b, c) \in \sqrt{D}^{-1}\mathcal{O}_p \times \boldsymbol{Z}_p$ such that $b\bar{b} - c \in pD^{-1}\boldsymbol{Z}_p^\times$, $s \in \boldsymbol{Z}_p^\times$, $b\bar{b} - c \notin \mathrm{N}(E_p^\times)$. We have $\det(S)^{-1}\det(S_0) \in \mathrm{N}(\mathcal{O}_p^\times)$ and $\det(S_0) = -s^2D(b\bar{b} - c) \in p\boldsymbol{Z}_p^\times - \mathrm{N}(E_p^\times)$. Hence $\det(S) \in p\boldsymbol{Z}_p^\times - \mathrm{N}(E_p^\times)$. The odd case is similar.

We prove the converse part. Let $\Lambda$ be the set of $\mathcal{O}_p$-integral lattices $\mathcal{L}$ in $(R, V)$ such that $\mathcal{O}_p^m \subset \mathcal{L}$. By assumption, $\mathcal{O}_p^m \in \Lambda$, and $\mathcal{L} \subset \mathcal{L}^* \subset (\mathcal{O}_p^m)^*$ for all $\mathcal{L} \in \Lambda$. Since $(\mathcal{O}_p^m)^*$ is Noetherian, $\Lambda$ has a maximal element $\mathcal{M}$, which is a maximal $\mathcal{O}_p$-integral lattice in $(S, E_p^m)$ containing $\mathcal{O}_p^m$. To complete the proof, it suffices to show $\mathcal{M} = \mathcal{O}_p^m$.

From $\mathcal{O}_p^m \subset \mathcal{M}$, noting $\mathcal{M}$ is $\mathcal{O}_p$-integral and by taking duals, we obtain

$$\mathcal{O}_p^m \subset \mathcal{M} \subset \mathcal{M}^* \subset (\mathcal{O}_p^m)^*. \tag{3.2}$$

Suppose $m$ is even. If $\det(S) \in \boldsymbol{Z}_p^\times$, then by Lemma 1, Lemma 2 and (3.2), the equality $\mathcal{M} = \mathcal{O}_p^m$ follows. Assume $\det(S) \in p\boldsymbol{Z}_p^\times - \mathrm{N}(E_p^\times)$; then $\mathrm{N}(\mathfrak{d}_S(\mathcal{O}_p^m)) = [(\mathcal{O}_p^m)^* : \mathcal{O}_p^m] = p^2$. By Lemma 1, Lemma 2 and (3.2), we have the two cases: $\mathrm{N}(\mathfrak{d}_S(\mathcal{M})) = 1$ or $p^2$. If the first case occurs, then $\mathcal{M}^* = \mathcal{M}$ by Lemma 2. Since $\mathcal{M}$ is a maximal $\mathcal{O}_p$-integral lattice with even rank, the equality $\mathcal{M}^* = \mathcal{M}$ is possible only when $n_0(S) = 0$ by Lemma 8 and Lemma 5. Hence $\det(S) \in \mathrm{N}(E_p^\times)$, contradictory to the assumption. Thus $\mathrm{N}(\mathfrak{d}_S(\mathcal{M})) = \mathrm{N}(\mathfrak{d}_S(\mathcal{O}_p^m)) = p^2$, or equivalently $[(\mathcal{O}_p^m)^* : \mathcal{O}_p^m] = [\mathcal{M}^* : \mathcal{M}] = p^2$, which, combined with (3.2), yields $\mathcal{M} = \mathcal{O}_p^m$.

Suppose $m$ is odd. If $\det(S)/\sqrt{D} \in \boldsymbol{Z}_p^\times$, then, by Lemma 2, the index $[(\mathcal{O}_p^m)^* : \mathcal{O}_p^m]$ equals $|D|_p^{-1}$, which is 1 or $p$ by the assumption $\operatorname{ord}_p(D) \in \{0, 1\}$. Since $[(\mathcal{O}_p^m)^* : \mathcal{M}^*]$ and $[\mathcal{M} : \mathcal{O}_p^m]$ divide $[(\mathcal{O}_p^m)^* : \mathcal{O}_p^m]$, we must have $[(\mathcal{O}_p^m)^* : \mathcal{M}^*] = 1$ or $[\mathcal{M} : \mathcal{O}_p^m] = 1$, which in turn give us the equality $\mathcal{M} = \mathcal{O}_p^m$. Assume $\det(S)/\sqrt{D} \in p\boldsymbol{Z}_p$, $e = 1$; then $\mathrm{N}(\det(S)/\sqrt{D}) = p^2$, which implies $[(\mathcal{O}_p^m)^* : \mathcal{O}_p^m] = p^2$. Combined with (3.2), this yields that the order of any subquotient of (3.2) is 1 or $p^2$. (Note the order of the $\mathcal{O}_p$-module

$\mathcal{O}_p/p\mathcal{O}_p$, which is simple since $e = 1$, is $p^2$.) If $\mathcal{M} \neq \mathcal{O}_p^m$, then $\mathcal{M} = \mathcal{M}^* = (\mathcal{O}_p^m)^*$ and a contradictory equality $\mathcal{M} = \mathcal{O}_p^m$ follows. Hence $\mathcal{M} = \mathcal{O}_p^m$. □

3.3. Witt towers of skew-hermitian spaces. Let $S_0$ be a matrix given in Lemma 8. For $\nu \in \boldsymbol{N}$, consider the matrix

$$(3.3) \qquad S_\nu = \begin{bmatrix} & & -J_\nu \\ & S_0 & \\ J_\nu & & \end{bmatrix}, \quad J_\nu = (\delta_{i,\nu-j+1})_{ij}$$

of size $m = 2\nu + n_0$; it defines a skew-hermitian form with the Witt index $\nu$ on the $m$-dimensional $E_p$-vector space $V_\nu = \begin{bmatrix} E_p^\nu \\ E_p^{n_0} \\ E_p^\nu \end{bmatrix}$. The standard $\mathcal{O}_p$-lattice $L_\nu = \begin{bmatrix} \mathcal{O}_p^\nu \\ \mathcal{O}_p^{n_0} \\ \mathcal{O}_p^\nu \end{bmatrix}$ affords a maximal $\mathcal{O}_p$-integral lattice in $(S_\nu, V_\nu)$.

We call the family $\{(S_\nu, V_\nu)\}_{\nu\in \boldsymbol{N}}$ the *Witt tower* associated with $S_0$.

3.4. Reduced vectors. Recall that a vector $Y \in V$ is said to be *reduced* for $(R, \mathcal{M})$ if $Y$ is $\mathcal{O}_p$-primitive in $\mathcal{M}^*$ and $Y^\perp \cap \mathcal{M}$ is a maximal $\mathcal{O}_p$-integral lattice in the skew-hermitian space $(R|Y^\perp, Y^\perp)$.

A skew-hermitian matrix $S \in \mathrm{GL}_n(E_p)$ is said to be $\mathcal{O}_p$-*integral* if $\mathcal{O}_p^n$ is an $\mathcal{O}_p$-integral lattice in $(S, E_p^n)$.

LEMMA 10. *Let $\{(S_\nu, V_\nu)\}_{\nu\in \boldsymbol{N}}$ be a Witt tower. Let $\nu \in \boldsymbol{N}$ and $Y$ a vector in $L_{\nu+1}^*$ of the form $Y = \begin{bmatrix} a \\ \mathsf{a} \\ 1 \end{bmatrix}$ $(a \in \mathcal{O}_p, \mathsf{a} \in L_\nu^*)$. Set $S_{\nu+1}^\sim = \begin{bmatrix} S_\nu & -S_\nu \mathsf{a} \\ -{}^t\bar{\mathsf{a}} S_\nu & \bar{a} - a \end{bmatrix}$. Then the following conditions on $Y$ are mutually equivalent.*

(1) *$Y$ is reduced for $(S_{\nu+1}, L_{\nu+1})$.*

(2) *The skew-hermitian matrix $S_{\nu+1}^\sim$ is $\mathcal{O}_p$-integral, and $S_{\nu+1}^\sim\left[\begin{bmatrix} 1 & x \\ 0 & \pi^{-1}\end{bmatrix}\right]$ is not $\mathcal{O}_p$-integral for all $x \in V_\nu$.*

(3) *The $\mathcal{O}_p$-lattice $L_{\nu+1}^\sim = \begin{bmatrix} L_\nu \\ \mathcal{O}_p \end{bmatrix}$ is a maximal $\mathcal{O}_p$-integral lattice in $(S_{\nu+1}^\sim, V_{\nu+1}^\sim)$ with $V_{\nu+1}^\sim = L_{\nu+1}^\sim \otimes E_p$.*

PROOF. cf. [15, Lemma 2.5 (p. 8)]. □

LEMMA 11. *Let $\{(S_\nu, V_\nu)\}_{\nu\in \boldsymbol{N}}$ be a Witt tower. Let $Y \in L_{\nu+1}^*$ be a reduced vector for $(S_{\nu+1}, L_{\nu+1})$ and set $n_0' = n_0(S_{\nu+1}|Y^\perp)$, $\partial' = \partial_{S_{\nu+1}|Y^\perp}(L_{\nu+1} \cap Y^\perp)$ and $d_Y = \mathrm{ord}_p(S_{\nu+1}[Y]/\sqrt{D})$. Then the possible values of $(n_0, \partial)$, $(n_0', \partial')$ and $(e, d_Y)$ are given in the Table* 1.

PROOF. By Lemma 6, we may assume $\nu = 0$ and $Y = \begin{bmatrix} a \\ \mathsf{a} \\ 1 \end{bmatrix}$ $(a \in \mathcal{O}_p, \mathsf{a} \in L_0^*)$ without loss of generality. By Lemma 10, in order for $Y$ to be reduced in $(S_1, L_1)$, it is necessary and sufficient for the $\mathcal{O}_p$-lattice $L_1^\sim$ to be maximal $\mathcal{O}_p$-integral in $(S_1^\sim, V_1^\sim)$. We examine the latter condition for each anisotropic form $S_0$ classified in Lemma 8.

For example, consider the case when $e = 2$, $L_0 = \mathcal{O}_p$ and $S_0 = s\sqrt{D}$ $(s \in \boldsymbol{Z}_p^\times)$. In this case $(n_0, \partial) = (1, 1)$ and $L_0^* = \sqrt{D}^{-1}\mathcal{O}_p$. By a direct computation, $\det(S_1^\sim) = sD(S_1[Y]/\sqrt{D})$. Since the size of $S_1^\sim$ is 2, by Lemma 9, $L_1^\sim$ is maximal $\mathcal{O}_p$-integral in

TABLE 1.

| $(n_0, \partial)$ | $(n'_0, \partial')$ | $(e, d_Y)$ | $\beta_Y$ | $\rho_Y$ |
|---|---|---|---|---|
| $(0,0)$ | $(1,0)$ | $(1,0)$ | $-1$ | $0$ |
| $(0,0)$ | $(1,1)$ | $(1,1), (2,0)$ | $0$ | $0$ |
| $(1,0)$ | $(0,0)$ | $(1,0)$ | $q^{1/2}$ | $0$ |
| $(1,0)$ | $(2,1)$ | $(1,1)$ | $0$ | $0$ |
| $(1,1)$ | $(0,0)$ | $(1,-1), (2, -\mathrm{ord}_p(D))$ | $q^{e/2} - q$ | $q^{1-e/2}$ |
| $(1,1)$ | $(2,1)$ | $(1,0)$ | $-q$ | $0$ |
| $(1,1)$ | $(2,2)$ | $(2, 1-\mathrm{ord}_p(D))$ | $0$ | $0$ |
| $(2,1)$ | $(1,0)$ | $(1,-1)$ | $q^{3/2} - q$ | $q^{1/2}$ |
| $(2,1)$ | $(1,1)$ | $(1,0)$ | $q^{3/2}$ | $0$ |
| $(2,2)$ | $(1,1)$ | $(2,-1)$ | $0$ | $q$ |

$(S_1^{\sim}, V_1^{\sim})$ if and only if $\det(S_1^{\sim}) \in \boldsymbol{Z}_p^{\times}$ in which case $n'_0 = \partial' = 0$, $d_Y = -\mathrm{ord}_p(D)$, or $\det(S_1^{\sim}) \in p\boldsymbol{Z}_p^{\times} - \mathrm{N}(E_p^{\times})$ in which case $n'_0 = \partial' = 2$, $d_Y = 1 - \mathrm{ord}_p(D)$. This affords the 5-th line and the 7-th line of the Table 1 when $e = 2$. The remaining parts of the Table 1 are proved similarly. □

3.5. Iwasawa decomposition of fundamental double cosets. Fix a Witt tower $\{(S_\nu, V_\nu)\}_{\nu \in \boldsymbol{N}}$ and set $G_\nu = \mathrm{U}(S_\nu)$, $K_\nu = G_\nu \cap \mathrm{GL}_{n_0+2\nu}(\mathcal{O}_p)$.

LEMMA 12. *Let $\nu \in \boldsymbol{N}$. The set $\tilde{c}_\nu^{(r)} = \{g \in G_\nu \mid \mathrm{rank}_{\mathcal{O}_p/\pi\mathcal{O}_p}(\pi g \pmod{\pi\mathcal{O}_p}) = r\}$ is non-empty if and only if $0 \leqslant r \leqslant \nu$, in which case $\tilde{c}_\nu^{(r)} = K_\nu c_\nu^{(r)} K_\nu$ with $c_\nu^{(r)} =$* $\mathrm{diag}(\pi 1_r, 1_{n_0+2\nu-2r}, \bar{\pi}^{-1} 1_r)$.

PROOF. This follows from the elementary divisor theory. □

For $0 \leqslant r \leqslant \nu$, let $R_\nu^{(r)}$ be a complete set of representatives for $K_\nu / K_\nu \cap c_\nu^{(r)} K_\nu c_\nu^{(r)-1}$, i.e., $\tilde{c}_\nu^{(r)} = \bigcup_{u \in R_\nu^{(r)}} u c_\nu^{(r)} K_\nu$.

For each $\nu \in \boldsymbol{N}$, set

$$\mathcal{U}_\nu = \{X \in \pi^{-1} L_\nu / L_\nu \mid \sqrt{D}^{-1} S_\nu[X] \in \tau(\pi^{-1}\mathcal{O}_p)\},$$
$$L'_\nu = \{X \in L_\nu^* \mid \sqrt{D}^{-1} S_\nu[X] \in \tau(\pi^{-1}\mathcal{O}_p)\}.$$

Moreover, we need the notation:

$$\mathsf{m}_\nu(t; g_0) := \mathrm{diag}(t, g_0, \bar{t}^{-1}), \quad (t \in E_p^{\times}, g_0 \in G_\nu),$$
$$\mathsf{n}_\nu(X; \zeta) := \begin{bmatrix} 1 & -{}^t\bar{X} S_\nu & \zeta - 2^{-1} S_\nu[X] \\ 0 & 1_{n_0+2\nu} & X \\ 0 & 0 & 1 \end{bmatrix}, \quad (X \in V_\nu, \zeta \in \boldsymbol{Q}_p).$$

The following lemma, which describes explicit Iwasawa decompositions of the double $K_{\nu+1}$ cosets $\tilde{c}_{\nu+1}^{(r)}$, plays a fundamental role in the paragraph 4.1.1 and Subsection 6.2.

LEMMA 13. *Let $\nu \in \boldsymbol{N}$. The double coset $\tilde{c}_{\nu+1}^{(r)}$ is a disjoint union of the following left $K_{\nu+1}$-cosets*:

• $\mathsf{m}_\nu(\pi; uc_\nu^{(r-1)})\mathsf{n}_\nu(X_1;\zeta_1)K_{\nu+1}$ *with* $u \in R_\nu^{(r-1)}$, $(X_1,\zeta_1) \in \mathsf{X}_{\nu,1}^{(r)}$, *where* $\mathsf{X}_{\nu,1}^{(r)}$ *is the set of pairs* $\left(\begin{bmatrix} x \\ X' \\ 0 \end{bmatrix}, \zeta_1\right)$ *satisfying*

$$x \in (\pi^{-2}\mathcal{O}_p/\mathcal{O}_p)^{r-1}\,, \quad X' \in \pi^{-1}L_{\nu-r+1}/L_{\nu-r+1}\,,$$
$$\zeta_1 \in (\boldsymbol{Q}_p \cap (\pi^{-2}\mathcal{O}_p + 2^{-1}S_{\nu-r+1}[X']))/\boldsymbol{Z}_p\,.$$

• $\mathsf{m}_\nu(1; uc_\nu^{(r-2)})\mathsf{n}_\nu(X_2;\zeta_2)K_{\nu+1}$ *with* $u \in R_\nu^{(r-2)}$, $(X_2,\zeta_2) \in \mathsf{X}_{\nu,2}^{(r)}$, *where* $\mathsf{X}_{\nu,2}^{(r)}$ *is the set of pairs* $\left(\begin{bmatrix} x \\ X' \\ 0 \end{bmatrix}, \zeta_2\right)$ *satisfying*

$$x \in (\pi^{-1}\mathcal{O}_p/\mathcal{O}_p)^{r-2}\,, \quad X' \in \mathcal{U}_{\nu-r+2} - L'_{\nu-r+2}/L_{\nu-r+2}\,,$$
$$\zeta_2 \in (\boldsymbol{Q}_p \cap (\pi^{-1}\mathcal{O}_p + 2^{-1}S_{\nu-r+2}[X']))/\boldsymbol{Z}_p\,.$$

• $\mathsf{m}_\nu(1; uc_\nu^{(r-1)})\mathsf{n}_\nu(X_3;\zeta_3)K_{\nu+1}$ *with* $u \in R_\nu^{(r-1)}$, $(X_3,\zeta_3) \in \mathsf{X}_{\nu,3}^{(r)}$, *where* $\mathsf{X}_{\nu,3}^{(r)}$ *is the set of pairs* $\left(\begin{bmatrix} x \\ X' \\ 0 \end{bmatrix}, \zeta_3\right)$ *satisfying*

$$x \in (\pi^{-1}\mathcal{O}_p/\mathcal{O}_p)^{r-1}\,, \quad X' \in (L'_{\nu-r+1} - L_{\nu-r+1})/L_{\nu-r+1}\,,$$
$$\zeta_3 \in (\boldsymbol{Q}_p \cap (\pi^{-1}\mathcal{O}_p + 2^{-1}S_{\nu-r+1}[X']))/\boldsymbol{Z}_p\,.$$

• $\mathsf{m}_\nu(1; uc_\nu^{(r-1)})\mathsf{n}_\nu(X_4;\zeta_4)K_{\nu+1}$ *with* $u \in R_\nu^{(r-1)}$, $(X_4,\zeta_4) \in \mathsf{X}_{\nu,4}^{(r)}$, *where* $\mathsf{X}_{\nu,4}^{(r)}$ *is the set of pairs* $\left(\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \zeta_4\right)$ *satisfying*

$$x \in (\pi^{-1}\mathcal{O}_p/\mathcal{O}_p)^{r-1}\,,$$
$$\zeta_4 \in (\boldsymbol{Q}_p \cap (\pi^{-1}\mathcal{O}_p - \mathcal{O}_p))/\boldsymbol{Z}_p\,.$$

• $\mathsf{m}_\nu(1; uc_\nu^{(r)})\mathsf{n}_\nu(X_5;0)K_{\nu+1}$ *with* $u \in R_\nu^{(r)}$, $X_5 \in \mathsf{X}_{\nu,5}^{(r)}$, *where* $\mathsf{X}_{\nu,5}^{(r)}$ *is the set of all vectors of the form* $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$ $(x \in (\pi^{-1}\mathcal{O}_p/\mathcal{O}_p)^r)$.

• $\mathsf{m}_\nu(\pi^{-1}; uc_\nu^{(r-1)})K_{\nu+1}$ *with* $u \in R_\nu^{(r-1)}$.

PROOF. cf. [14, Lemma 2 (p. 342)]. □

3.6. Cardinalities of some basic sets. Fix a Witt tower $\{(S_\nu, V_\nu)\}_{\nu\in \boldsymbol{N}}$ and set $n_0 = n_0(S_0)$, $\partial = \partial_{S_0}(L_0)$.

First we show an auxiliary lemma.

LEMMA 14. *Assume $E_p/\boldsymbol{Q}_p$ is unramified. For $u \in \mathcal{O}_p^\times$ and $a \in \boldsymbol{Z}_p$*,

$$\sharp\{\xi \in \mathcal{O}_p/\pi\mathcal{O}_p \mid \tau(u\xi) \equiv a \pmod{p\boldsymbol{Z}_p}\} = p\,.$$

PROOF. We may assume $u = 1$. There exists $\theta \in \mathcal{O}_p$ such that $\tau(\theta) = 1$ and $\mathcal{O}_p = \boldsymbol{Z}_p \oplus \theta\boldsymbol{Z}_p$. Let $\xi \in \mathcal{O}_p$. If we write $\xi = x + \theta y$ with $x, y \in \boldsymbol{Z}_p$, then $\tau(\xi) = y$. Hence $\{\xi \in \mathcal{O}_p \mid \tau(u\xi) \equiv a \pmod{p\boldsymbol{Z}_p}\} = \boldsymbol{Z}_p \oplus \theta(a + p\boldsymbol{Z}_p)$. Since $e = 1$, we may assume

$\pi = p$. Therefore,

$$\{\xi \in \mathcal{O}_p/\pi\mathcal{O}_p \mid \tau(u\xi) \equiv a \pmod{p\boldsymbol{Z}_p}\}$$
$$= \{\boldsymbol{Z}_p \oplus \theta(a + p\boldsymbol{Z}_p)\}/\{p\boldsymbol{Z}_p \oplus \theta p\boldsymbol{Z}_p\} \cong \boldsymbol{Z}_p/p\boldsymbol{Z}_p\,.$$

This proves the assertion. □

PROPOSITION 15. *Let $\nu, r \in \boldsymbol{N}$ and $0 \leqslant r \leqslant \nu$. We have*

$$\sharp\mathcal{U}_\nu = q^{\nu+n_0-1+e/2}(q^\nu - 1) + q^{\nu+\partial}\,,$$

*and*

$$\sharp\mathsf{X}^{(r)}_{\nu,1} = q^{2\nu+n_0+1}\,, \quad \sharp\mathsf{X}^{(r)}_{\nu,2} = q^{r-2+1-e/2}(\sharp\mathcal{U}_{\nu-r+2} - q^\partial)\,,$$
$$\sharp\mathsf{X}^{(r)}_{\nu,3} = q^{r-e/2}(q^\partial - 1)\,, \quad \sharp\mathsf{X}^{(r)}_{\nu,4} = q^{r-1}(q^{1-e/2} - 1)\,, \quad \sharp\mathsf{X}^{(r)}_{\nu,5} = q^r\,.$$

PROOF. For a vector $X = \begin{bmatrix} x \\ z \\ y \end{bmatrix} \in \pi^{-1}L_\nu$ with $x, y \in E_p^\nu$, $z \in V_0$, the condition $X \pmod{L_\nu} \in \mathcal{U}_\nu$ is equivalent to

$$\sqrt{D}^{-1} S_0[\pi z] + \tau({}^t\overline{(\pi y)} J_\nu(\pi x)) \in \mathrm{N}(\pi) p^{-1} \boldsymbol{Z}_p\,. \tag{3.4}$$

Let $(\xi, \eta, \zeta)$ be the reduction of $(J_\nu \pi x, \pi y, \pi z) \in \mathcal{O}_p^{2\nu+n_0}$ modulo $\pi\mathcal{O}_p$.

Assume $e = 1$ and $\pi = p$. The condition (3.4) is written as a congruence equation:

$$\sqrt{D}^{-1} S_0[\zeta] + \tau({}^t\bar{\eta}\xi) \equiv 0 \pmod{\pi\mathcal{O}_p}\,. \tag{3.5}$$

If $\eta = (\eta_j) \neq 0$, then $\eta_j \neq 0$ for some $j$. Suppose $\eta_1 \neq 0$. Then for given $\zeta \in (\mathcal{O}_p/\pi\mathcal{O}_p)^{n_0}$ and for $\xi_j \in \mathcal{O}_p/\pi\mathcal{O}_p$ $(2 \leqslant j \leqslant \nu)$, the condition (3.5) is regarded as a condition on $\xi_1$. From Lemma 14, the number of $\xi_1$ satisfying (3.5) is exactly $p$. Hence the number of the solutions $(\xi, \eta, \zeta)$ of (3.5) such that $\eta \neq 0$ is $p \cdot q^{\nu-1} \cdot (q^\nu - 1) \cdot q^{n_0} = q^{n_0+\nu-1/2}(q^\nu - 1)$. If $\eta = 0$, then the condition (3.5) is equivalent to $S_0[\zeta]/\sqrt{D} \in p\boldsymbol{Z}_p$. In terms of $z$, this means $S_0[z]/\sqrt{D} \in p^{-1}\boldsymbol{Z}_p = \tau(\pi^{-1}\mathcal{O}_p)$, or equivalently $z \in L_0'$. Thus the number of the solutions $(\xi, \eta, \zeta)$ of (3.5) such that $\eta = 0$ is $q^\nu \cdot q^\partial = q^{\nu+\partial}$. Summing up, we obtain $\sharp\mathcal{U}_\nu = q^{\nu+n_0-1/2}(q^\nu - 1) + q^{\nu+\partial}$, which settles the case $e = 1$.

Assume $e = 2$. Then $\mathrm{N}(\pi) \in p\boldsymbol{Z}_p^\times$ and the condition (3.4) becomes $S_0[\zeta]/\sqrt{D} + \tau({}^t\bar{\eta}\xi) \in \boldsymbol{Z}_p$, which holds for arbitrary $(\xi, \eta, \zeta) \in (\mathcal{O}_p/\pi\mathcal{O}_p)^{2\nu+n_0}$. Hence $\sharp\mathcal{U}_\nu = q^{2\nu+n_0}$.

The formulas of $\sharp\mathsf{X}^{(r)}_{\nu,j}$ are obtained by a straightforward consideration by Lemma 13. □

LEMMA 16. *For $\nu, r \in \boldsymbol{N}$ such that $0 \leqslant r \leqslant \nu$, we have $\sharp R_\nu^{(r)} = \prod_{j=1}^r f_{\nu,j}$ with*

$$f_{\nu,j} = \frac{q^{j-1}(q^{\nu-j+1} - 1)(q^{\nu-j+n_0+1} + q^{\partial+1-e/2})}{q^j - 1}\,.$$

PROOF. From Lemma 13 and Proposition 15, we obtain a recurrence formula among the numbers $\sharp R_\nu^{(r)}$:

$$\sharp R_{\nu+1}^{(r)} = \{q^{2\nu+n_0+1} + q^{r-1}(q^{\partial+1-e/2} - 1)\}\sharp R_\nu^{(r-1)}$$
$$+ q^{r-2}(q^{\nu-r+2} - 1)(q^{\nu+1-r+(n_0+1)} + q^{\partial+1-e/2})\sharp R_\nu^{(r-2)} + q^r \sharp R_\nu^{(r)}\,.$$

By this, the formula is proved by induction on $\nu$. □

REMARK. It is observed that the formula in Lemma 16 is obtained from the orthogonal group counterpart [15, (7.11) p. 44] by substitutions $n_0 \mapsto n_0 + 1$, $\partial \mapsto \partial + 1 - e/2$.

LEMMA 17. *Let* $\nu \in \boldsymbol{N}$. *For* $\mathsf{a} \in L_\nu^*$, *the cardinality of the set*

$$\mathcal{F}_{\nu,\mathsf{a}} = \{X \in L_\nu^*/L_\nu \mid \sqrt{D}^{-1}\{S_\nu[\mathsf{a}] - S_\nu[X - \mathsf{a}]\} \in \tau(\mathcal{O}_p)\}$$

*is* $\sharp\mathcal{F}_{\nu,\mathsf{a}} = 1 + \rho_{\mathsf{a}}$ *with*

$$\rho_{\mathsf{a}} = q^{\partial - e/2}\delta \ \ (\mathsf{a} \notin L_\nu'^*)\,. \tag{3.6}$$

PROOF. First we prove

$$\mathcal{F}_{\nu,\mathsf{a}} = \{X \in L_\nu'/L_\nu \mid \sqrt{D}^{-1} S_\nu[X] \equiv \tau S_\nu(X, \mathsf{a}) \pmod{\boldsymbol{Z}_p}\}\,. \tag{3.7}$$

Since $\tau(\mathcal{O}_p) = \boldsymbol{Z}_p$, the condition $S_\nu[\mathsf{a}]/\sqrt{D} \equiv S_\nu[X - \mathsf{a}]/\sqrt{D} \pmod{\tau(\mathcal{O}_p)}$ is equivalent to $S_\nu[X]/\sqrt{D} \equiv \tau S_\nu(X, \mathsf{a}) \pmod{\boldsymbol{Z}_p}$. Hence to show (3.7), it suffices to have $X \in L_\nu'/L_\nu$ for $X \in \mathcal{F}_{\nu,\mathsf{a}}$. Let $X \in \mathcal{F}_{\nu,\mathsf{a}}$. Since $L_\nu$ is an $\mathcal{O}_p$-lattice there exists $l \in \boldsymbol{N}$ such that $p^l X \in L_\nu$; choose the smallest one among such $l$'s. Then $p^l S_\nu[X]/\sqrt{D} \in \boldsymbol{Z}_p$ since $p^l S_\nu[X]/\sqrt{D} \equiv \tau S_\nu(p^l X, \mathsf{a}) \pmod{\boldsymbol{Z}_p}$ and $S_\nu(p^l X, \mathsf{a}) \in \boldsymbol{Z}_p$. Suppose $l \geqslant 2$. Then $S_\nu[p^{l-1}X]/\sqrt{D} = p^l S_\nu[X]/\sqrt{D} \cdot p^{l-2} \in \boldsymbol{Z}_p$. By the maximality of $L_\nu$, we then obtain $p^{l-1}X \in L_\nu$, a contradiction to the minimality of $l$. Thus $l = 1$ and $pX \in L_\nu$. Hence $pS_\nu[X]/\sqrt{D} \equiv \tau S_\nu(pX, \mathsf{a}) \equiv 0 \pmod{\boldsymbol{Z}_p}$, which in turn yields $S_\nu[X]/\sqrt{D} \in p^{-1}\boldsymbol{Z}_p = \tau(\pi^{-1}\mathcal{O}_p)$, or equivalently $X \in L_\nu'$.

Assume $\mathsf{a} \in L_\nu'^*$. Then $\tau(S_\nu(X, \mathsf{a})) \in \boldsymbol{Z}_p$ for all $X \in L_\nu'$. Hence for $X \in L_\nu'$ the condition $X \in \mathcal{F}_{\nu,\mathsf{a}}$ is equivalent to $S_\nu[X]/\sqrt{D} \in \boldsymbol{Z}_p$, which implies $X \in L_\nu$ by the maximality of $L_\nu$. Thus $\mathcal{F}_{\nu,\mathsf{a}} = \{0\}$ and $\sharp\mathcal{F}_{\nu,\mathsf{a}} = 1$.

Assume $\mathsf{a} \notin L_\nu'^*$. In this case we can easily show that the map $X \mapsto (S_\nu[X]/\sqrt{D})^{-1}X$ is a bijection

$$\mathcal{F}_{\nu,\mathsf{a}} - \{0\} \stackrel{\cong}{\to} \{Z \in pL_\nu'/pL_\nu \mid \tau S_\nu(Z, \mathsf{a}) \equiv 1 \pmod{p}\}\,. \tag{3.8}$$

Since $\mathsf{a} \notin L_\nu'^*$, we have $Z_0' \in L_\nu'$ such that $\tau S_\nu(Z_0', \mathsf{a}) \notin \boldsymbol{Z}_p$ on one hand. On the other hand, the inclusion $pL_\nu' \subset L_\nu$ (cf. Lemma 7) and the assumption $\mathsf{a} \in L_\nu^*$ yield $\tau S_\nu(Z_0', \mathsf{a}) \in p^{-1}\boldsymbol{Z}_p^\times$. Hence $\tau S_\nu(Z_0', \mathsf{a}) = p^{-1}u$ for some $u \in \boldsymbol{Z}_p^\times$. The element $Z_0 = pu^{-1}Z_0'$ satisfies $Z_0 \in pL_\nu'$ and $\tau S_\nu(Z_0, \mathsf{a}) = 1$. The map $\tilde{Z} = Z - Z_0$ defines a bijection from the set on the right-hand side of (3.8) onto the set

$$\mathfrak{K} = \{\tilde{Z} \in pL_\nu'/pL_\nu \mid \tau S_\nu(\tilde{Z}, \mathsf{a}) \equiv 0 \pmod{p}\}\,.$$

Since the condition $\mathsf{a} \notin L_\nu'^*$ means the map $\zeta \mapsto \tau S_\nu(\zeta, \mathsf{a}) \bmod p$ is a non-zero linear form on the $\boldsymbol{Z}_p/p\boldsymbol{Z}_p$-vector space $pL_\nu'/pL_\nu \cong L_\nu'/L_\nu$, we get $\sharp\mathfrak{K} = p^{\dim(L_\nu'/L_\nu)-1} = q^\partial p^{-1}$. Thus we obtain $\sharp(\mathcal{F}_{\nu,\mathsf{a}} - \{0\}) = q^{\partial - e/2}$, and hence $\sharp\mathcal{F}_{\nu,\mathsf{a}} = 1 + q^{\partial - e/2}$. □

REMARK. If $Y \in L_{\nu+1}^*$ is a reduced vector for $(S_{\nu+1}, L_{\nu+1})$, the possible values of $\rho_Y$ are assembled in Table 1 (for notations see Lemma 11).

For a pair of natural numbers $n \geqslant n'$ and a vector $\mathsf{a} = \begin{bmatrix} a' \\ \mathsf{a}' \\ b' \end{bmatrix} \in V_n$ with $a', b' \in E_p^{n-n'}$, $\mathsf{a}' \in V_{n'}$, we set $\Pi_{n'}(\mathsf{a}) = \mathsf{a}'$.

LEMMA 18. *Let $\nu \in \boldsymbol{N}$. For a vector $Y \in L_{\nu+1}^*$ and an $n \in \boldsymbol{N}$ such that $n \leqslant \nu$, the cardinality of the set*

$$\mathcal{V}_{n,Y} = \{X \in L_n/\pi L_n \mid \sqrt{D}^{-1}\{S_{\nu+1}[Y] - S_n[X - \Pi_n(Y)]\} \in \tau(\pi\mathcal{O}_p)\} \tag{3.9}$$

*is given by*

$$\sharp\mathcal{V}_{n,Y} = \begin{cases} q^{n+n_0-1/2}(q^n - 1) + q^n \sharp\mathcal{V}_{0,Y} & (e = 1)\,, \\ q^{2n}\sharp\mathcal{V}_{0,Y} & (e = 2)\,. \end{cases} \tag{3.10}$$

PROOF. This can be proved by an argument similar to the proof of Proposition 15. □

LEMMA 19. *Let $\nu \in \boldsymbol{N}$. Assume $Y = \begin{bmatrix} a \\ \mathsf{a} \\ 1 \end{bmatrix} \in L_{\nu+1}^*$ is a reduced vector for $(S_{\nu+1}, L_{\nu+1})$. Set $n_0' = n_0(S_{\nu+1}|Y^\perp)$ and $\partial' = \partial_{S_{\nu+1}|Y^\perp}(Y^\perp \cap L_{\nu+1})$. Then $\sharp\mathcal{V}_{0,Y} = q^\partial + \beta_Y$ with*

$$\beta_Y = \frac{q^{n_0+1/2} - q^{(n_0+n_0')/2} + q^{\partial'+1+(n_0-n_0'-e)/2} - q^{\partial+(3-e)/2}}{q-1}\,.$$

*For every $n \in \boldsymbol{N}$ such that $0 \leqslant n \leqslant \nu$, we have*

$$\sharp\mathcal{V}_{n,Y} = \sharp\mathcal{U}_n + q^n\beta_Y\,.$$

PROOF. We follow the argument of [15, Lemma 2.11 (p. 10)] and use the notation in Lemma 10. Since $S_{\nu+1}^\sim\left[\left[\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right]\right] = S_\nu[\xi - \mathsf{a}] - S_{\nu+1}[Y]$,

$$\mathcal{V}_{\nu,Y} = \left\{\xi \in L_\nu/\pi L_\nu \mid \sqrt{D}^{-1} S_{\nu+1}^\sim\left[\left[\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right]\right] \in \tau(\pi\mathcal{O}_p)\right\}. \tag{3.11}$$

By Lemma 10, $L_{\nu+1}^\sim$ is maximal $\mathcal{O}_p$-integral for $S_{\nu+1}^\sim$. Hence we can find an anisotropic skew-hermitian matrix $S_0'$ of size $n_0'$ such that $S_{\nu+1}^\sim \cong \begin{bmatrix} & & -J_{\nu'} \\ & S_0' & \\ J_{\nu'} & & \end{bmatrix}$ and $L_{\nu+1}^\sim \cong \begin{bmatrix} \mathcal{O}_p^{\nu'} \\ \mathcal{O}_p^{n_0'} \\ \mathcal{O}_p^{\nu'} \end{bmatrix}$. By Proposition 15, noting $n_0' = \partial'$, we have

$$\begin{aligned} &\sharp\{z \in L_{\nu+1}^\sim/\pi L_{\nu+1}^\sim \mid \sqrt{D}^{-1} S_{\nu+1}^\sim[z] \in \tau(\pi\mathcal{O}_p)\} \\ &\qquad = \begin{cases} q^{\nu'+n_0'-1/2}(q^{\nu'} - 1) + q^{\nu'+\partial'} & (e = 1)\,, \\ q^{2\nu'+n_0'} & (e = 2)\,. \end{cases} \end{aligned} \tag{3.12}$$

On the other hand,

$$
\begin{aligned}
&\sharp\{z \in L^{\sim}_{\nu+1}/\pi L^{\sim}_{\nu+1} \mid \sqrt{D}^{-1} S^{\sim}_{\nu+1}[z] \in \tau(\pi\mathcal{O}_p)\} \\
&\quad = \sharp\big\{(\xi, x) \in L_\nu/\pi L_\nu \times \mathcal{O}_p/\pi\mathcal{O}_p \mid x \notin \pi\mathcal{O}_p, \sqrt{D}^{-1} S_\nu\big[\big[\begin{smallmatrix} x^{-1}\xi \\ 1 \end{smallmatrix}\big]\big] \in \tau(\pi\mathcal{O}_p)\big\} \\
&\qquad + \sharp\{\xi \in L_\nu/\pi L_\nu \mid \sqrt{D}^{-1} S_\nu[\xi] \in \tau(\pi\mathcal{O}_p)\} \qquad (3.13)\\
&\quad = (q-1)\sharp\big\{\xi \in L_\nu/\pi L_\nu \mid \sqrt{D}^{-1} S^{\sim}_{\nu+1}\big[\big[\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\big]\big] \in \tau(\pi\mathcal{O}_p)\big\} \\
&\qquad + \begin{cases} q^{\nu+n_0-1/2}(q^\nu - 1) + q^{\nu+\partial} & (e=1), \\ q^{2\nu'+\partial'} & (e=2). \end{cases}
\end{aligned}
$$

From (3.11), (3.12) and (3.13), we have the formula of $\sharp\mathcal{V}_{\nu,Y}$. By comparing this with (3.10), we obtain the formula of $\sharp\mathcal{V}_{0,Y}$. Then the formula of $\sharp\mathcal{V}_{n,Y}$ for $n \leqslant \nu$ follows from $\sharp\mathcal{V}_{0,Y}$ and Proposition 15. □

REMARK. We assemble the explicit values of $\beta_Y$ in Table 1. Note $\beta_Y = 0$ if $e = 2$.

3.7. Evaluations of some exponential sums. Let $\psi_p$ be an additive character of $\boldsymbol{Q}_p$ such that $\psi_p$ is trivial on $\boldsymbol{Z}_p$ and non-trivial on $p^{-1}\boldsymbol{Z}_p$. Fix a Witt tower $\{(S_\nu, V_\nu)\}_{\nu\in\boldsymbol{N}}$. For $X \in L_n^*$ with $n \in \boldsymbol{N}$, set

$$\theta'_n(X) = \sum_{Z \in L'_n/L_n} \psi_p(\tau S_n(X, Z)).$$

When $n \geqslant 1$, we also consider the sum

$$\theta_n(X) = \sum_{Z \in \mathcal{U}_n} \psi_p(\tau S_n(X, Z)), \quad X \in L_n^*.$$

For the orthogonal case, the evaluation of similar sums is stated in [15, p. 49] without proof.

LEMMA 20. *Let $n \in \boldsymbol{N}$.*
(1) $\theta'_n(X) = q^\partial \delta(X \in L'^*_n)$.
(2) *If $n \geqslant 1$, then*

$$\theta_n(X) = \delta(X \in \pi L_n^*)\sharp\mathcal{U}_n + \delta(X \notin \pi L_n^*)(-q^{n+n_0-1+e/2} + q^n \sharp\mathcal{V}_{0,X}).$$

PROOF. We give a proof for completeness.

(1) follows from the orthogonal relation of characters of the finite abelian group $L'_n/L_n$, whose order is $q^\partial$.

(2) If $X \in \pi L_n^*$, then $S_n(X, \mathcal{U}_n) \subset \mathcal{O}_p$; hence $\theta_n(X) = \sharp\mathcal{U}_n$. Assume $X \in L_n^* - \pi L_n^*$. If we write $X = \left[\begin{smallmatrix} x_1 \\ x_0 \\ x_2 \end{smallmatrix}\right]$, $(x_1, x_2 \in \mathcal{O}_p^n, x_0 \in L_0^*)$ and $Z = \left[\begin{smallmatrix} z_1 \\ z_0 \\ z_2 \end{smallmatrix}\right]$, $(z_1, z_2 \in (\pi^{-1}\mathcal{O}_p)^n, z_0 \in \pi^{-1}L_0)$, then the condition $Z \in \mathcal{U}_n$ is equivalent to $S_0[z_0]/\sqrt{D} + \tau({}^t\bar{z}_2 J_n z_1) \in \tau(\pi^{-1}\mathcal{O}_p) =$

$p^{-1}\mathbf{Z}_p$. Hence

$$\begin{aligned}
\theta_n(X) &= \sum_{\substack{z_1,z_2\in(\pi^{-1}\mathcal{O}_p/\mathcal{O}_p)^n \\ z_0\in\pi^{-1}L_0/L_0 \\ S_0[z_0]/\sqrt{D}+\tau({}^t\bar{z}_2J_nz_1)\in p^{-1}\mathbf{Z}_p}} \psi_p(\tau(-{}^t\bar{z}_1J_nx_2+{}^t\bar{z}_2J_nx_1+S_0(x_0,z_0))) \\
&= \sum_{\substack{z_1,z_2\in(\mathcal{O}_p/\pi\mathcal{O}_p)^n \\ z_0\in L_0/\pi L_0 \\ -S_0[z_0]/\sqrt{D}\equiv\tau({}^t\bar{z}_2z_1) \mod p^{-1}\mathrm{N}(\pi)}} \psi_p(\tau(\bar{\pi}^{-1}\{-{}^t\bar{z}_1J_nx_2+{}^t\bar{z}_2x_1+S_0(x_0,z_0)\})) \\
&= \sum_{z_0\in L_0/\pi L_0} \psi_p(\tau(\bar{\pi}^{-1}S_0(x_0,z_0)))g(-S_0[z_0]/\sqrt{D})
\end{aligned}$$

with

$$g(d) = \sum_{\substack{z_1,z_2\in(\mathcal{O}_p/\pi\mathcal{O}_p)^n \\ \tau({}^t\bar{z}_2z_1)\equiv d \mod p^{-1}\mathrm{N}(\pi)}} \psi_p(\tau(\bar{\pi}^{-1}\{-{}^t\bar{z}_1J_nx_2+{}^t\bar{z}_2x_1\}))\,.$$

First we assume $e = 1$ and take $\pi = p$. A straightforward calculation of the Fourier transform $\hat{g}(\varepsilon) = \sum_{d\in\mathbf{Z}_p/p\mathbf{Z}_p} g(d)\psi_p(d\varepsilon/p)$ of $g(d)$ yields its evaluation:

$$\hat{g}(\varepsilon) = \begin{cases} q^n\psi_p(-(p\varepsilon)^{-1}\tau({}^t\bar{x}_2J_nx_1))\,, & (\varepsilon\neq 0)\,, \\ q^{2n}\delta\left(\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]\in\pi\mathcal{O}_p^{2n}\right), & (\varepsilon = 0)\,. \end{cases}$$

By the Fourier inversion formula $g(d) = p^{-1}\sum_{\varepsilon\in\mathbf{Z}_p/p\mathbf{Z}_p}\hat{g}(\varepsilon)\psi_p(-d\varepsilon/p)$ we have

$$\begin{aligned}
\theta_n(X) = p^{-1}\Bigg\{ & q^{2n}\delta\left(\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]\in\pi\mathcal{O}_p^{2n}\right)\sum_{z_0\in L_0/\pi L_0}\psi_p(\tau(p^{-1}S_0(x_0,z_0))) \\
& + q^n\sum_{\varepsilon\in(\mathbf{Z}_p/p\mathbf{Z}_p)^\times}\sum_{z_0\in L_0/\pi L_0}\psi_p\Bigg(\frac{-\varepsilon^{-1}\sqrt{D}\tau({}^t\bar{x}_1J_nx_2)+\varepsilon S_0[z_0]}{p\sqrt{D}} \\
& \qquad + \frac{\tau(S_0(x_0,z_0))}{p}\Bigg)\Bigg\}
\end{aligned} \tag{3.14}$$

The first summation on the right-hand side of (3.14) gives us $\delta(x_0\in\pi L_0^*)q^{n_0}$ by the orthogonal relation of characters. Since $X\notin\pi L_n^*$ by assumption, we have $\delta(x_1,x_2\in\pi\mathcal{O}_p^n)\delta(x_0\in\pi L_0^*) = \delta(X\in\pi L_n^*) = 0$. Hence the first term on the right-hand side of (3.14) vanishes.

In the second term, since $\varepsilon S_0[z_0]+\sqrt{D}\tau S_0(x_0,z_0) = \varepsilon^{-1}S_0[\varepsilon z_0+x_0]-\varepsilon^{-1}S_0[x_0]$, we have

$$\theta_n(X) = q^{n-1/2} \sum_{\varepsilon\in(\boldsymbol{Z}_p/p\boldsymbol{Z}_p)^\times} \psi_p\Big(-\frac{\sqrt{D}\tau({}^t\bar{x}_1 J_n x_2) + S_0[x_0]}{p\varepsilon\sqrt{D}}\Big) \sum_{z_0\in L_0/\pi L_0} \psi_p\Big(\frac{S_0[\varepsilon z_0 + x_0]}{p\varepsilon\sqrt{D}}\Big)$$

$$= q^{n-1/2}\Big\{\sum_{\varepsilon\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p}\sum_{z_0\in L_0/\pi L_0} \psi_p\Big(\frac{\varepsilon(-S_n[X] + S_0[z_0+x_0])}{p\sqrt{D}}\Big) - q^{n_0}\Big\}$$

making the change of variables $\varepsilon z_0 = z_0'$, $\varepsilon^{-1} = \varepsilon'$ to prove the second equality. Since the orthogonal relation of characters, combined with the definition (3.9) of the set $\mathcal{V}_{0,X}$, yields

$$\sum_{z_0\in L_0/\pi L_0}\sum_{\varepsilon\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p} \psi_p\Big(\frac{\varepsilon(-S_n[X] + S_0[z_0+x_0])}{p\sqrt{D}}\Big) = p\sharp\mathcal{V}_{0,X}\,, \tag{3.15}$$

we have the desired formula. This settles the case $e = 1$. The other case $e = 2$ is similar. □

3.8. A double coset decomposition. Let $\{(S_\nu, V_\nu)\}_{\nu\in\boldsymbol{N}}$ be a Witt tower. Let $Y = \begin{bmatrix} a \\ \mathrm{a} \\ 1 \end{bmatrix} \in L_{\nu+1}^*$ be a reduced vector for $(S_{\nu+1}, L_{\nu+1})$ and $G_{\nu+1}^Y$ the stabilizer of $Y$ in $G_{\nu+1}$. Set $K_{\nu+1}^* = \{k \in K_{\nu+1} \,|\, kX - X \in L_{\nu+1}$ (for all $X \in L_{\nu+1}^*)\}$; $K_{\nu+1}^*$ is an open normal subgroup of $K_{\nu+1}$.

LEMMA 21. *We have*

$$G_{\nu+1} = G_{\nu+1}^Y K_{\nu+1} \cup \bigcup_{l\geqslant 1} G_{\nu+1}^Y M_l K_{\nu+1}^*\,,$$

*where* $M_l = \mathrm{diag}(\bar{\pi}^{-l}, 1_{2\nu+n_0}, \pi^l)$.

PROOF. Similar to [15, Lemma 7.2 (p. 45)], [7, Proposition 3.9 (p. 41)]. □

**4. Local $L$-factors.** In this section, we shall recall the definition of the local $L$-factor attached to a character of the local Hecke algebra ([8]).

4.1. The non-split case. In this subsection, we retain the notation introduced at the beginning of Section 3. Let $\{(S_\nu, V_\nu)\}_{\nu\in\boldsymbol{N}}$ be a Witt tower (see 3.3) and set $n_0 = n_0(S_0)$, $\partial = \partial_{S_0}(L_0)$. The unitary group $G_\nu := \mathrm{U}(S_\nu)$ has the torus $A_\nu$ formed by all the points of the form $a = \mathrm{diag}(a_1, \dots, a_\nu, 1_{n_0}, \bar{a}_\nu^{-1}, \dots, \bar{a}_1^{-1})$ ($a_i \in E_p^\times$), whose $\boldsymbol{Q}_p$-rational character group $X^*(A_\nu)$ is generated by $\alpha_j : a(\in A_\nu) \mapsto a_j\bar{a}_j$, $1 \leqslant j \leqslant \nu$. Set $m = 2\nu + n_0$. The subgroup $A_\nu^+ = A_\nu \cap \mathrm{GL}_m(\boldsymbol{Q}_p)$ is a maximal $\boldsymbol{Q}_p$-split torus of $G_\nu$, and the root system $\Sigma_\nu = \Sigma(G_\nu, A_\nu^+)$ is of type $\mathrm{BC}_\nu$ if $n_0 > 0$ and of type $\mathrm{C}_\nu$ if $n_0 = 0$. By restriction, $X^*(A_\nu) \hookrightarrow X^*(A_\nu^+)$ and the image of $\alpha_j$ can be written as $2\eta_j$ with a unique $\eta_j \in X^*(A_\nu^+)$. Let $N_\nu$ be the unipotent algebraic subgroup of $G_\nu$ such that the roots of $A_\nu^+$ in the Lie algebra of $N_\nu$ are $\eta_i - \eta_j$ ($1 \leqslant i < j \leqslant \nu$), $\eta_i + \eta_j$ ($1 \leqslant i \leqslant j \leqslant \nu$) and $\eta_j$ ($1 \leqslant j \leqslant \nu$).

Let $\{\alpha_j^\vee\}_{1\leqslant j\leqslant\nu}$ be the dual of $\{\alpha_j\}$. Then the Weyl group $W_\nu$ of $\Sigma_\nu$ acts naturally on the coordinate functions $X_j = q^{-\alpha_{\nu+1-j}^\vee}$ ($1 \leqslant j \leqslant \nu$) on the dual torus

$$\check{A}_\nu(\boldsymbol{C}) = X^*(A_\nu)_{\boldsymbol{C}}/2\pi i(\log q)^{-1}X^*(A_\nu) \cong (\boldsymbol{C}^\times)^\nu\,.$$

We have the Iwasawa decomposition $G_\nu = N_\nu A_\nu K_\nu$ and the Cartan decomposition $G_\nu = K_\nu A_\nu K_\nu$ with respect to the maximal compact subgroup $K_\nu := G_\nu \cap \mathrm{GL}_m(\mathcal{O}_p)$. For each $\mathbf{r} = (r_j)_{1\leqslant j\leqslant \nu} \in \boldsymbol{Z}^\nu$, set

$$\pi^{\mathbf{r}} := \mathrm{diag}(\pi^{r_1}, \dots, \pi^{r_\nu}, 1_{n_0}, \bar{\pi}^{-r_\nu}, \dots, \bar{\pi}^{-r_1}) \in A_\nu .$$

For a double $K_\nu$-coset $K_\nu g K_\nu$ in $G_\nu$, take a complete set of representatives $\{n_i \pi^{\mathbf{r}_i}\}_{i\in I}$ of $K_\nu g K_\nu / K_\nu$ in the set $N_\nu \pi^{\boldsymbol{Z}^\nu}$. Let $\mathcal{H}$ be the Hecke algebra of the pair $(G_\nu, K_\nu)$ with respect to the Haar measure of $G_\nu$ such that $\mathrm{vol}(K_\nu) = 1$. Then the main result of [12] tells that there exists the unique $\boldsymbol{C}$-algebra isomorphism $\Phi_\nu : \mathcal{H} \to \boldsymbol{C}[X_1^\pm, \dots, X_\nu^\pm]^{W_\nu}$ such that

$$\Phi_\nu(\phi_{K_\nu g K_\nu}; X) = \sum_{i\in I} \prod_{j=1}^{\nu} (q^{(1-n_0)/2-j} X_j)^{r_{\nu+1-j,i}} , \tag{4.1}$$

for all $K_\nu g K_\nu = \bigcup_{i\in I} n_i \pi^{\mathbf{r}_i} K_\nu$ with $\mathbf{r}_i = (r_{j,i})_{1\leqslant j\leqslant\nu}$, where $\phi_{K_\nu g K_\nu}$ denotes the characteristic function of $K_\nu g K_\nu$ (We follow the formulation of [14] and [3].). Let $\Lambda : \mathcal{H} \to \boldsymbol{C}$ be a $\boldsymbol{C}$-algebra homomorphism. The Satake parameter of $\Lambda$ is defined to be the unique element $\mathbf{s} \in \check{A}_\nu(\boldsymbol{C})/W_\nu$ such that $\Phi_\nu(\phi; \mathbf{s}) = \Lambda(\phi)$ for any $\phi \in \mathcal{H}$. Let $T$ be an indeterminate and consider the polynomial $P_\nu(T; X) = \prod_{j=1}^{\nu}(1 - X_j T)(1 - X_j^{-1} T)$ with coefficients in $\boldsymbol{C}[X_1^\pm, \dots, X_\nu^\pm]^{W_\nu}$. Then the local $L$-factor of $\Lambda$ is defined by

$$L(s, \Lambda) = P_\nu(q^{-s}; \mathbf{s})^{-1} A(s)$$

where $A(s)$ is given as follows ([8]).

- Suppose $e = 1$. Then

$$A(s) = \begin{cases} 1 & (n_0, \partial) = (0,0) , \\ (1-q^{-s})^{-1} & (n_0, \partial) = (1,0) , \\ (1-q^{-s})^{-1}(1+q^{-(s-1/2)}) & (n_0, \partial) = (1,1) , \\ (1-q^{-(s+1/2)})^{-1} & (n_0, \partial) = (2,1) . \end{cases}$$

- Suppose $e = 2$. Then

$$A(s) = \begin{cases} 1 & n_0 = 0 , \\ (1-q^{-s})^{-1} & n_0 = 1 , \\ (1-q^{-(s+1/2)})^{-1}(1+q^{-(s-1/2)}) & n_0 = 2 . \end{cases}$$

REMARK. When $G_\nu$ is unramified, the $L$-factor given above is the usual one corresponding to the $2m$-dimensional complex representation of the $L$-group ${}^L G_\nu$, which is a semi-direct product of $\mathrm{GL}_m(\boldsymbol{C})$ with the Weil group of $\boldsymbol{Q}_p$. When $G_\nu$ is not unramified, the modified factor $A(s)$ is introduced by Murase and Sugano ([8], cf. [9] for orthogonal case).

4.1.1. Recurrence relations of Hecke polynomials. The image of the double coset $\tilde{c}_n^{(r)}$ (see Lemma 12) by the Satake isomorphism $\Phi_n$ satisfies the following recurrence relation.

LEMMA 22. *For $n \geqslant 0, 0 \leqslant r \leqslant n$,*

$$\Phi_{n+1}(\tilde{c}_{n+1}^{(r)}) = q^{n+(n_0+1)/2}(X_{n+1} + X_{n+1}^{-1})\Phi_n(\tilde{c}_n^{(r-1)}) + C_n^{(r-2)}\Phi_n(\tilde{c}_n^{(r-2)}) + D^{(r-1)}\Phi_n(\tilde{c}_n^{(r-1)}) + q^r\Phi_n(\tilde{c}_n^{(r)}) .$$

*Here*

$$C_n^{(r)} = q^{r+1-e/2}(q^{n-r} - 1)(q^{n+n_0-r-1+e/2} + q^{\partial}) , \quad D^{(r)} = q^r(q^{\partial+1-e/2} - 1) . \tag{4.2}$$

PROOF. This follows from Lemma 13. □

We have an additive expression of the polynomial $P_n(T; X)$:

LEMMA 23. *For each $n \in \mathbf{N}$, there exists a family of complex numbers $\{a_{n,k}(r) \mid 0 \leqslant k \leqslant 2n, 0 \leqslant r \leqslant n\}$ such that*

$$P_n(T; X) = \sum_{k=0}^{2n}(-1)^k\left(\sum_{r=0}^{n} a_{n,k}(r)\Phi_n(\tilde{c}_n^{(r)})\right)T^k .$$

*Moreover $\{a_{n,k}(r)\}$ satisfies the following recurrence formulas.*

(1) (i) *For $n \geqslant 0, k \geqslant 1, r \geqslant 1$,*

$$a_{n+1,k}(r) = q^{-(n+(n_0+1)/2)}a_{n,k-1}(r - 1) .$$

(ii) *For $n \geqslant 0, k \geqslant 1$,*

$$a_{n+1,k}(0) = a_{n,k}(0) + a_{n,k-2}(0) - q^{-(n+(n_0+1)/2)}(a_{n,k-1}(1)C_n^{(0)} + a_{n,k-1}(0)D^{(0)}) .$$

(2) *For $n \geqslant 0, 0 \leqslant k \leqslant 2n + 2, 1 \leqslant r \leqslant n$,*

$$a_{n,k}(r) + a_{n,k-2}(r) = q^{-(n+(n_0+1)2)}(a_{n,k-1}(r + 1)C_n^{(r)} + a_{n,k-1}(r)D^{(r)} + a_{n,k-1}(r - 1)q^r) .$$

*Here we understand $a_{n,k'}(r') = 0$ unless $0 \leqslant k' \leqslant 2n$ or unless $0 \leqslant r' \leqslant n$.*

PROOF. cf. [14, Lemma 4 (p. 345)]. □

LEMMA 24. *Let $0 \leqslant k \leqslant 2n, 0 \leqslant r \leqslant n$. Then we have the following relations.*

$$a_{n,k}(r) = a_{n,2n-k}(r) , \tag{4.3}$$

$$a_{n,k}(r) = 0 , \quad \text{(for all } k \in [0, r - 1] \cup [2n - r + 1, 2n]) , \tag{4.4}$$

$$a_{n,2n}(0) = 1 , \tag{4.5}$$

$$a_{n,2n-1}(1) = q^{-(n-1+(n_0+1)/2)} , \tag{4.6}$$

$$a_{n,2n-1}(0) = -q^{-(n+(n_0-1)/2)}\frac{(q^n - 1)(q^{\partial+1-e/2} - 1)}{q - 1} ,$$

$$a_{n,2n-2}(1) = -q^{-(2n-2+n_0)}\frac{(q^{n-1} - 1)(q^{\partial+1-e/2} - 1)}{q - 1} . \tag{4.7}$$

PROOF. This results from Lemma 23. □

4.2. The split case. In this subsection, we set $E_p = \boldsymbol{Q}_p \oplus \boldsymbol{Q}_p$ and $\mathcal{O}_p = \boldsymbol{Z}_p \oplus \boldsymbol{Z}_p$. Let $(R, V)$ be a skew-hermitian space over $E_p$ and $\mathcal{M}$ a maximal $\mathcal{O}_p$-integral lattice in $(R, V)$. Set $m = \mathrm{rk}_{E_p}(V)$. Then by choosing an $\mathcal{O}_p$-basis of $\mathcal{M}$, we may assume $\mathcal{M} = \mathcal{O}_p^m = \boldsymbol{Z}_p^m \oplus \boldsymbol{Z}_p^m$, $V = E_p^m = \boldsymbol{Q}_p^m \oplus \boldsymbol{Q}_p^m$ and $R(\mathsf{v}, \mathsf{w}) = {}^t\bar{\mathsf{w}}(T, -{}^tT)\mathsf{v}$ for any $\mathsf{v}, \mathsf{w} \in V$ for a $T \in \mathrm{GL}_m(\boldsymbol{Q}_p)$. By the maximality of $\mathcal{M}$, the matrix $T$ has to belong to $\mathrm{GL}_m(\boldsymbol{Z}_p)$. Since $\mathrm{U}(R) = \{(g_1, g_2) \in \mathrm{GL}_m(\boldsymbol{Q}_p)^2 \,|\, {}^tg_2 T g_1 = T\}$, the first projection $\mathrm{GL}_m(\boldsymbol{Q}_p)^2 \to \mathrm{GL}_m(\boldsymbol{Q}_p)$ yields an isomorphism $\mathrm{U}(R) \cong \mathrm{GL}_m(\boldsymbol{Q}_p)$ which maps $\mathrm{U}(R) \cap \mathrm{GL}(\mathcal{M})$ onto $K_m := \mathrm{GL}_m(\boldsymbol{Z}_p)$. Let $A_m = \{\mathrm{diag}(a_1, \ldots, a_m) \,|\, a_i \in \boldsymbol{Q}_p^\times\}$, and $N_m$ the unipotent subgroup formed by all the upper triangular unipotent matrices in $\mathrm{GL}_m(\boldsymbol{Q}_p)$. We have the Iwasawa decomposition $\mathrm{GL}_m(\boldsymbol{Q}_p) = N_m A_m K_m$ and the Cartan decomposition $\mathrm{GL}_m(\boldsymbol{Q}_p) = K_m A_m K_m$. For $\mathbf{r} = (r_j)_{1 \leqslant j \leqslant m} \in \boldsymbol{Z}^m$, set $p^{\mathbf{r}} := \mathrm{diag}(p^{r_1}, \ldots, p^{r_m})$. For a double coset $K_m g K_m$ we fix a representative $\{n_i p^{\mathbf{r}_i}\}_{i \in I}$ of $K_m g K_m / K_m$ in the set $N_m p^{\mathbf{Z}^m}$. The symmetric group $S_m$ acts on the algebra $\boldsymbol{C}[X_1^{\pm}, \ldots, X_m^{\pm}]$ by the permutations of the indeterminates $X_j$. Let $\mathcal{H}$ be the Hecke algebra of the pair $(\mathrm{GL}_m(\boldsymbol{Q}_p), K_m)$ with respect to the Haar measure of $\mathrm{GL}_m(\boldsymbol{Q}_p)$ such that $\mathrm{vol}(K_m) = 1$. By [12], there exists the unique $\boldsymbol{C}$-algebra isomorphism $\Psi_m : \mathcal{H} \to \boldsymbol{C}[X_1^{\pm}, \ldots, X_m^{\pm}]^{S_m}$ such that

$$\Psi_m(\phi_{K_m g K_m}; X) = \sum_{i \in I} \prod_{j=1}^{m} (p^{(1+m)/2 - j} X_j)^{r_{m+1-j,i}} \tag{4.8}$$

for all $K_m g K_m = \bigcup_{i \in I} n_i p^{\mathbf{r}_i} K_m$ with $\mathbf{r}_i = (r_{j,i})_{1 \leqslant j \leqslant m}$. Let $\Lambda : \mathcal{H} \to \boldsymbol{C}$ be a $\boldsymbol{C}$-algebra homomorphism. The Satake parameter of $\Lambda$ is defined to be the unique element $\mathbf{s} \in (\boldsymbol{C}^\times)^m / S_m$ such that $\Psi_m(\phi; \mathbf{s}) = \Lambda(\phi)$ for any $\phi \in \mathcal{H}$. Let $T$ be an indeterminate and consider the polynomials $P_m^{(1)}(T; X) = \prod_{j=1}^m (1 - X_j T)$ and $P_m^{(2)}(T; X) = \prod_{j=1}^m (1 - X_j^{-1} T)$ with coefficients in $\boldsymbol{C}[X_1^{\pm}, \ldots, X_m^{\pm}]^{S_m}$. Then the $L$-factor of $\Lambda$ is defined by

$$L(s, \Lambda) := P_m^{(1)}(p^{-s}; \mathbf{s})^{-1} P_m^{(2)}(p^{-s}; \mathbf{s})^{-1}.$$

**5. Automorphic forms and Rankin-Selberg integrals.** For an algebraic $\boldsymbol{Q}$-group $H$ and a prime number $p$, we use a simpler notation $H_p$ for $H_{\boldsymbol{Q}_p}$. The group of real points $H_{\boldsymbol{R}}$ and the group of finite adele points $H_{\boldsymbol{A}_\mathrm{f}}$ are denoted by $H_\infty$ and $H_\mathrm{f}$, respectively. Then the adele group $H_{\boldsymbol{A}}$ is identified with the direct product of $H_\infty$ and $H_\mathrm{f}$, i.e., $H_{\boldsymbol{A}} \cong H_\infty \times H_\mathrm{f}$.

Let $E = \boldsymbol{Q}(\sqrt{D})$ ($\subset \boldsymbol{C}$) be an imaginary quadratic field with discriminant $D$ and $\mathcal{O}$ the integer ring of $E$. For $a \in E$, set $\tau(a) = \sqrt{D}^{-1}(a - \bar{a})$. Then $\tau(\mathcal{O}) = \boldsymbol{Z}$. Let $\mathrm{I}(E)$ (resp. $\mathrm{S}(E)$, $\mathrm{R}(E)$) be the set of primes which are inert (resp. split, ramify) for the extension $E/\boldsymbol{Q}$. Let $\omega$ be the quadratic character of $\boldsymbol{A}^\times/\boldsymbol{Q}^\times$ corresponding to the extension $E/\boldsymbol{Q}$. We set $E_\infty = E \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ and $E_{\boldsymbol{A}} = E \otimes_{\boldsymbol{Q}} \boldsymbol{A}$. Note that $E_\infty \cong \boldsymbol{C}$.

We use the notations introduced in Section 2 with $F = E$ and $k = \boldsymbol{Q}$.

5.1. Let $(R, V)$ and $(\tilde{R}, \tilde{V})$ be as in 2.0.2 and consider their unitary groups $G_0 = \mathrm{U}(R)$, $G = \mathrm{U}(\tilde{R})$. We fix a non-isotropic vector $Y \in V$ and consider the stabilizer $G^{\tilde{Y}}$ of the corresponding vector $\tilde{Y} \in \tilde{V}$ as explained in 2.0.4. We assume the matrix $iR$ is positive definite and set $\dim_{\boldsymbol{C}} V = m$.

5.1.1. The group of real points $G_\infty$ is a real reductive Lie group whose associated symmetric space is

$$\mathfrak{D} = \left\{ \sigma = \begin{bmatrix} b_\sigma \\ \mathsf{a}_\sigma \\ 1 \end{bmatrix} \in \tilde{V}_\infty \mid i\tilde{R}[\sigma] = iR[\mathsf{a}_\sigma] - 2\mathrm{Im}(b_\sigma) < 0 \right\}.$$

The transform of a point $\sigma \in \mathfrak{D}$ by an element $g \in G_\infty$ is denoted by $g\langle\sigma\rangle \in \mathfrak{D}$, which is defined to be the point of $\mathfrak{D}$ such that $g\sigma = c_{g,\sigma}\, g\langle\sigma\rangle$ with a scalar $c_{g,\sigma} \in \boldsymbol{C}^\times$.

Fix a base point $\sigma_0 = \begin{bmatrix} (1+\sqrt{D})/2 \\ 0_m \\ 1 \end{bmatrix} \in \mathfrak{D}$. Then $K_\infty$, the stabilizer in $G_\infty$ of the point $\sigma_0$, is a maximal compact subgroup of $G_\infty$. Since the signature of $i\tilde{R}$ is $((m+1)+, 1-)$, $G_\infty$ is a realization of the real-rank-one unitary group $\mathrm{U}(m+1, 1)$, and $K_\infty \cong \mathrm{U}(m+1) \times \mathrm{U}(1)$. Since $iR$ is positive definite, $G_{0,\infty}$ is compact.

5.1.2. The group $G_{0,\mathrm{f}}$ acts on the set of all the $\mathcal{O}$-lattices in $V$. Fix a maximal $\mathcal{O}$-integral lattice $\mathcal{M}$ in $(R, V)$ and let $K_{0,\mathrm{f}}$ be the stabilizer of $\mathcal{M}$ in $G_{0,\mathrm{f}}$; then $K_{0,\mathrm{f}}$ is a maximal compact subgroup of $G_{0,\mathrm{f}}$. Similarly $K_\mathrm{f}$ denotes the maximal compact subgroup of $G_\mathrm{f}$, which stabilizes the maximal $\mathcal{O}$-integral lattice $\tilde{\mathcal{M}} = \mathcal{O} \oplus \mathcal{M} \oplus \mathcal{O}$ in $(\tilde{R}, \tilde{V})$.

5.1.3. The symmetric space associated with the Lie group $G_\infty^{\tilde{Y}}$ is

$$\mathfrak{D}^{\tilde{Y}} = \{\sigma \in \mathfrak{D} \mid \tilde{R}(\tilde{Y}, \sigma) = 0\} = \left\{ \begin{bmatrix} b_\sigma \\ \mathsf{a}_\sigma \\ 1 \end{bmatrix} \in \mathfrak{D} \,\middle|\, R(Y, \mathsf{a}_\sigma) = 0 \right\},$$

which is a divisor of the $(m+1)$-dimensional complex manifold $\mathfrak{D}$. Since $\sigma_0 \in \mathfrak{D}^{\tilde{Y}}$, the intersection $K_\infty^{\tilde{Y}} = G_\infty^{\tilde{Y}} \cap K_\infty$ is a maximal compact subgroup of $G_\infty^{\tilde{Y}}$. We have isomorphisms:

$$G_\infty^{\tilde{Y}} \cong \mathrm{U}(m, 1), \quad K_\infty^{\tilde{Y}} \cong \mathrm{U}(m) \times \mathrm{U}(1).$$

5.2. Assumptions. In the remaining part of this paper, we hold the following two assumptions on $R$ and $Y$.

(A1): $Y \in \mathcal{M}^*_{\mathrm{prim}}$, $R[Y]^{-1}Y \in \mathcal{M}_{\mathrm{prim}}$,

(A2): for each prime $p$, the localization $R_p$ of $R$ at $p$ is isotropic.

From (A1), we have

LEMMA 25. (1) *The direct sum decomposition of $\mathcal{O}$-lattice $\mathcal{M} = R[Y]^{-1}Y\mathcal{O} \oplus (Y^\perp \cap \mathcal{M})$ holds. The lattice $Y^\perp \cap \mathcal{M}$ is maximal $\mathcal{O}$-integral in $(R \mid Y^\perp, Y^\perp)$.*

(2) *For any prime $p$, we have $R[Y]^{-1} \in \mathcal{O}_p^\times \cup \pi\mathcal{O}_p^\times$.*

PROOF. The assertion (1) is proved directly. Since $Y_0 = R[Y]^{-1}Y$ belongs to $\mathcal{M}$, we obtain $R[Y_0] \in \mathcal{O}$, which yields $R[Y]^{-1} \in \mathcal{O}$. Let $p$ be a prime. Suppose $R[Y]^{-1} \in \pi^a\mathcal{O}_p$ with $a \geqslant 2$. Since $Y \in \mathcal{M}^*$ and $R[\pi^{a-1}Y] \in \pi^{a-2}\mathcal{O}_p \subset \mathcal{O}_p$, the lattice $\mathcal{M}_p + \pi^{-1}R[Y]^{-1}Y\mathcal{O}_p$ is an $\mathcal{O}_p$-integral lattice containing $\mathcal{M}_p$. By the maximality of $\mathcal{M}_p$, $\mathcal{M}_p + \pi^{-1}R[Y]^{-1}Y\mathcal{O}_p$ has to coincide with $\mathcal{M}_p$, or equivalently $\pi^{-1}R[Y]^{-1}Y \in \mathcal{M}_p$. This contradicts the primitivity of $R[Y]^{-1}Y$ in $\mathcal{M}_p$. Hence $R[Y]^{-1} \in \mathcal{O}_p - \pi^2\mathcal{O}_p$. □

Let $K_{\mathrm{f}}^{\tilde{Y}}$ (resp. $K_{0,\mathrm{f}}^{Y}$) be the stabilizer of $\tilde{\mathcal{M}} \cap \tilde{Y}^{\perp}$ (resp. $\mathcal{M} \cap Y^{\perp}$) in $G_{\mathrm{f}}^{\tilde{Y}}$ (resp. $G_{0,\mathrm{f}}^{Y}$). Then $K_{\mathrm{f}}^{\tilde{Y}}$ and $K_{0,\mathrm{f}}^{Y}$ yield maximal compact subgroups of $G_{\mathrm{f}}^{\tilde{Y}}$ and $G_{0,\mathrm{f}}^{Y}$, respectively, and $K_{0,\mathrm{f}}^{Y} = G_{0,\mathrm{f}}^{Y} \cap K_{0,\mathrm{f}}$, $K_{\mathrm{f}}^{\tilde{Y}} = G_{\mathrm{f}}^{\tilde{Y}} \cap K_{\mathrm{f}}$.

Set $K_{\boldsymbol{A}}^{\tilde{Y}} = K_{\infty}^{\tilde{Y}} K_{\mathrm{f}}^{\tilde{Y}}$. Then $K_{\boldsymbol{A}}^{\tilde{Y}}$ is a maximal compact subgroup of $G_{\boldsymbol{A}}^{\tilde{Y}}$ and the decomposition $G_{\boldsymbol{A}}^{\tilde{Y}} = P_{\boldsymbol{A}}^{\tilde{Y}} K_{\boldsymbol{A}}^{\tilde{Y}}$ holds.

REMARK. The first assumption (A1) forces that the prime 2 is unramified in $E/\boldsymbol{Q}$ if $m$ is odd. To confirm this, suppose $m$ is odd and $2|D$. Then Lemma 11 yields $\mathrm{ord}_2(R[Y]/\sqrt{D})$ $= -\mathrm{ord}_2(D)$. Combining this with Lemma 25 (2), we obtain $\mathrm{ord}_2(D) \in \{0, 1\}$. which is absurd since $\mathrm{ord}_2(D)$ should be 2 or 3.

The second assumption (A2) necessarily implies $m > 1$.

5.3. Normalizations of Haar measures. Let $\mathrm{d}\zeta_\infty$ be the standard Lebesgue measure of $\boldsymbol{R}$. For each prime $p$, let $\mathrm{d}\zeta_p$ be the Haar measure of $\boldsymbol{Q}_p$ such that $\mathrm{vol}(\boldsymbol{Z}_p) = 1$. Then the product of $\mathrm{d}\zeta_v$'s affords $\boldsymbol{A}$ a unique Haar measure $\mathrm{d}\zeta$ such that $\mathrm{vol}(\boldsymbol{Q}\backslash\boldsymbol{A}) = 1$; $\mathrm{d}\zeta$ is self dual with respect to the basic character $\psi : \boldsymbol{Q}\backslash\boldsymbol{A} \to \boldsymbol{C}^{\times}$ such that $\psi_\infty(x_\infty) = \exp(2\pi\sqrt{-1}x_\infty)$ for all $x_\infty \in \boldsymbol{R}$. Here, for any place $p \leqslant \infty$ of $\boldsymbol{Q}$, $\psi_p$ denotes the $p$-component of $\psi$.

For a finite dimensional $E$-vector space $U$, we put the adele space $U_{\boldsymbol{A}}$ the Haar measure such that $\mathrm{vol}(U_{\boldsymbol{A}}/U) = 1$. Then we normalize the Haar measure $\mathrm{d}n$ (resp. $\mathrm{d}n'$) of the unipotent group $N_{\boldsymbol{A}}$ (resp. $N_{\boldsymbol{A}}^{\tilde{Y}}$) so that $\mathrm{d}n = \mathrm{d}X\mathrm{d}\xi$ (resp. $\mathrm{d}n' = \mathrm{d}Z\mathrm{d}\zeta$) if $n = \mathsf{n}(X;\xi)$ (resp. $n' = \mathsf{n}(Z;\zeta)$). Let $\mathrm{d}l$ be the Haar measure of the compact group $K_{\boldsymbol{A}}^{\tilde{Y}}$ such that $\mathrm{vol}(K_{\boldsymbol{A}}^{\tilde{Y}}) = 1$. Let $\mathrm{d}^{\times}t = \otimes'\mathrm{d}^{\times}t_p$ be the Haar measure of the multiplicative group $E_{\boldsymbol{A}}^{\times}$ which is a product of Haar measures $\mathrm{d}^{\times}t_p$ on $E_p^{\times}$ such that $\mathrm{vol}(\mathcal{O}_p^{\times}) = 1$ if $p < \infty$ and $\mathrm{d}^{\times}t_\infty = (2\pi)^{-1}r^{-1}\mathrm{d}r\mathrm{d}\theta$ with $(r,\theta)$ the polar coordinates of $E_\infty^{\times} \cong \boldsymbol{C}^{\times}$. Fix a Haar measure $\mathrm{d}g_0$ of $G_{0,\boldsymbol{A}}^{Y}$ such that $\mathrm{vol}(G_{0,\boldsymbol{Q}}^{Y}\backslash G_{0,\boldsymbol{A}}^{Y}) = 1$. By the Iwasawa decomposition $G_{\boldsymbol{A}}^{\tilde{Y}} = P_{\boldsymbol{A}}^{\tilde{Y}} K_{\boldsymbol{A}}^{\tilde{Y}}$, we take the Haar measure $\mathrm{d}h$ of $G_{\boldsymbol{A}}^{\tilde{Y}}$ so that the formula

$$
\begin{aligned}
\int_{P_{\boldsymbol{Q}}^{\tilde{Y}}\backslash G_{\boldsymbol{A}}^{\tilde{Y}}} f(h)\mathrm{d}\dot{h} = & \int_{E^{\times}\backslash E_{\boldsymbol{A}}^{\times}} |\mathrm{N}(t)|_{\boldsymbol{A}}^{-m}\mathrm{d}^{\times}t \int_{G_{0,\boldsymbol{Q}}^{Y}\backslash G_{0,\boldsymbol{A}}^{Y}} \mathrm{d}\dot{g}_0 \int_{N_{\boldsymbol{Q}}^{\tilde{Y}}\backslash N_{\boldsymbol{A}}^{\tilde{Y}}} \mathrm{d}\dot{n}' \\
& \times \int_{K_{\boldsymbol{A}}^{\tilde{Y}}} f(n'\mathsf{m}(t;g_0)l)\mathrm{d}l\,, \quad (f \in L^1(P_{\boldsymbol{Q}}^{\tilde{Y}}\backslash G_{\boldsymbol{A}}^{\tilde{Y}}))
\end{aligned}
\tag{5.1}
$$

holds.

5.4. Eisenstein series. Since $G_0^Y$ is $\boldsymbol{R}$-isotropic, the space $G_{0,\boldsymbol{Q}}^{Y}\backslash G_{0,\boldsymbol{A}}^{Y}/K_{0,\mathrm{f}}^{Y}G_{0,\infty}^{Y}$ is a finite set. For a function $f$ on $G_{0,\boldsymbol{Q}}^{Y}\backslash G_{0,\boldsymbol{A}}^{Y}/K_{0,\mathrm{f}}^{Y}G_{0,\infty}^{Y}$, define a $\boldsymbol{C}$-valued function $f(s;h)$ in $(s,h) \in \boldsymbol{C} \times G_{\boldsymbol{A}}^{\tilde{Y}}$ by the formula

$$
f(s;\mathsf{m}(t;g_0)nl) = |\mathrm{N}(t)|_{\boldsymbol{A}}^{s+m/2} f(g_0)\,, \quad (t \in E_{\boldsymbol{A}}^{\times}, g_0 \in G_{0,\boldsymbol{A}}^{Y}, n \in N_{\boldsymbol{A}}^{\tilde{Y}}, l \in K_{\mathrm{f}}^{\tilde{Y}}K_{\infty}^{\tilde{Y}})\,.
$$

The Eisenstein series relevant to our purpose is a right $K_{\mathrm{f}}^{\tilde{Y}} K_{\infty}^{\tilde{Y}}$-invariant and left $G_{\boldsymbol{Q}}$-invariant smooth function on $G_{\boldsymbol{A}}^{\tilde{Y}}$ which is originally given by the absolutely convergent series

$$E(f; s; g) = \sum_{\gamma \in P_{\boldsymbol{Q}}^{\tilde{Y}} \backslash G_{\boldsymbol{Q}}^{\tilde{Y}}} f(s; \gamma g), \quad g \in G_{\boldsymbol{A}}^{\tilde{Y}} \tag{5.2}$$

for $\mathrm{Re}(s) > m/2$; it has a meromorphic continuation to the whole $s$-plane ([10, IV], [6]).

5.5. Rankin-Selberg integrals. For the notion of automorphic forms and cusp forms on an adele group, we refer to [10, I.2.17, I.2.18].

Let $(\tau, W)$ be an irreducible unitary representation of $K_\infty$ containing a non-zero $K_{\infty}^{\tilde{Y}}$-fixed vector $v_0$. Let $F : G_{\boldsymbol{Q}} \backslash G_{\boldsymbol{A}} \to W$ be a cusp form such that

$$F(g k_{\mathrm{f}} k_\infty) = \tau(k_\infty)^{-1} F(g), \quad k_{\mathrm{f}} k_\infty \in K_{\mathrm{f}} K_\infty . \tag{5.3}$$

Consider the integral

$$Z_{f,Y}^{F}(s) := \int_{G_{\boldsymbol{Q}}^{\tilde{Y}} \backslash G_{\boldsymbol{A}}^{\tilde{Y}}} E(f; s - 1/2; h) \langle v_0 | F(h) \rangle \mathrm{d}\dot{h}, \quad s \in \boldsymbol{C}, \tag{5.4}$$

where $\langle x \mid y \rangle$ is the inner-product of $W$, which is antilinear with respect to the first variable $x$. Since $E(f; s-1/2)$ is an automorphic form on $G_{\boldsymbol{A}}^{\tilde{Y}}$ and $F$ is a cusp form on $G_{\boldsymbol{A}}$, the integrand is a rapidly decreasing function on $G_{\boldsymbol{A}}^{\tilde{Y}}$ ([10, I.2.12]), which guarantees the convergence of the integral (5.4) for all $s \in \boldsymbol{C}$ where $E(f; s - 1/2)$ is regular. Moreover, $Z_{f,Y}^{F}(s)$ yields a meromorphic function on $\boldsymbol{C}$, which is holomorphic outside the poles of the Eisenstein series $E(f; s - 1/2; h)$.

5.6. Whittaker integrals. For $X \in V$, let $\psi_X$ be the character of $N_{\boldsymbol{A}}$ defined by

$$\psi_X(\mathsf{n}(Z; \zeta)) = \psi(\tau R(X, Z)), \quad \mathsf{n}(Z; \zeta) \in N_{\boldsymbol{A}} . \tag{5.5}$$

Note $\psi_X$ is trivial on the subgroup $N_{\boldsymbol{Q}}$.

Our aim in this section is to show that the integral $Z_{f,Y}^{F}(s)$ is expressed as a Mellin transform of the integral

$$\varphi_{f,X}^{F}(g) := \int_{G_{0,\boldsymbol{Q}}^{X} \backslash G_{0,\boldsymbol{A}}^{X}} f(g_0) \mathrm{d}\dot{g}_0 \int_{N_{\boldsymbol{Q}} \backslash N_{\boldsymbol{A}}} F(n \mathsf{m}(1; g_0) g) \psi_X(n)^{-1} \mathrm{d}\dot{n}, \qquad X \in V, \quad g \in G_{\boldsymbol{A}}, \tag{5.6}$$

which we call the *Whittaker integral* of $F$ along $(f, X)$. The function $\varphi_{f,Y}^{F} : G_{\boldsymbol{A}} \to W$ is bounded, since $F$ is bounded on $G_{\boldsymbol{A}}$ and $G_{0,\boldsymbol{Q}} \backslash G_{0,\boldsymbol{A}} \times N_{\boldsymbol{Q}} \backslash N_{\boldsymbol{A}}$ is compact.

When $X \in EY - \{0\}$, it is easy to see that $\varphi_{f,X}^{F}$ has the equivariance:

$$\varphi_{f,X}^{F}(n \mathsf{m}(1; k_{0,\mathrm{f}} g_{0,\infty}) g k_{\mathrm{f}} k_\infty) = \psi_X(n) \tau(k_\infty)^{-1} \varphi_{f,X}^{F}(g), \qquad (n \in N_{\boldsymbol{A}},\ k_{0,\mathrm{f}} g_{0,\infty} \in K_{0,\mathrm{f}}^{Y} G_{0,\infty}^{Y},\ k_{\mathrm{f}} k_\infty \in K_{\mathrm{f}} K_\infty). \tag{5.7}$$

5.7. A basic identity. Here is the main theorem of this section.

THEOREM 26. *The integral*

$$\zeta(\varphi^F_{f,Y}; s) := \int_{E^\times_{\mathbf{A}}} \langle v_0 | \varphi^F_{f,Y}(\mathsf{m}(t; 1_m))\rangle |\mathrm{N}(t)|_{\mathbf{A}}^{s-(m+1)/2} \mathrm{d}^\times t$$

*converges absolutely in* $\mathrm{Re}(s) > (m+1)/2$ *and*

$$Z^F_{f,Y}(s) = \zeta(\varphi^F_{f,Y}; s), \quad \mathrm{Re}(s) > (m+1)/2 .$$

PROOF. Let $\mathrm{Re}(s) > (m+1)/2$. From (5.2) and (5.4), by using the integration formula (5.1), we obtain

$$Z^F_{f,Y}(s) = \int_{E^\times \backslash E^\times_{\mathbf{A}}} \mathrm{d}^\times t \int_{G^Y_{0,\mathbf{Q}} \backslash G^Y_{0,\mathbf{A}}} \mathrm{d}\dot{g}_0 \times \int_{N^{\tilde{Y}}_{\mathbf{Q}} \backslash N^{\tilde{Y}}_{\mathbf{A}}} \mathrm{d}\dot{n}' |\mathrm{N}(t)|_{\mathbf{A}}^{s-(m+1)/2} f(g_0) \langle v_0 | F(n' \mathsf{m}(t; g_0))\rangle \tag{5.8}$$

after a standard argument. Note the integral over the compact group $K^{\tilde{Y}}_{\mathbf{A}}$ yields the factor 1 since $F$ has the $K^{\tilde{Y}}_{\mathbf{A}}$-equivariance (5.3) and $v_0$ is fixed by $K^{\tilde{Y}}_{\infty}$.

LEMMA 27. *For any* $g \in G_{\mathbf{A}}$, *we have*

$$\int_{G^Y_{0,\mathbf{Q}} \backslash G^Y_{0,\mathbf{A}}} f(g_0) \mathrm{d}g_0 \int_{N^{\tilde{Y}}_{\mathbf{Q}} \backslash N^{\tilde{Y}}_{\mathbf{A}}} \langle v_0 | F(n' \mathsf{m}(1; g_0) g)\rangle \mathrm{d}\dot{n}' = \sum_{\alpha \in E^\times} \langle v_0 | \varphi^F_{f,Y}(\mathsf{m}(\alpha; 1_m) g)\rangle . \tag{5.9}$$

PROOF. Fix $g \in G_{\mathbf{A}}$. Since the smooth function on $E_{\mathbf{A}}$

$$\Phi_g(\alpha) := \int_{Y^\perp_{\mathbf{A}}/Y^\perp_{\mathbf{Q}}} \mathrm{d}Z \int_{\mathbf{A}/\mathbf{Q}} \langle v_0 | F(\mathsf{n}(\alpha Y + Z; \zeta) g)\rangle \mathrm{d}\zeta , \quad \alpha \in E_{\mathbf{A}}$$

is $E$-periodic, the Fourier inversion formula yields the identity

$$\sum_{\alpha_0 \in E} \hat{\Phi}_g(\alpha_0) = \Phi_g(0) \tag{5.10}$$

with $\hat{\Phi}_g(\alpha_0) = \int_{E_{\mathbf{A}}/E} \Phi_g(\alpha) \psi((R[Y]/\sqrt{D}) \mathrm{tr}_{E/\mathbf{Q}}(\bar{\alpha}_0 \alpha))^{-1} \mathrm{d}\alpha$ for $\alpha_0 \in E$. By the normalization of the Haar measure of $N_{\mathbf{A}}$ and that of $N^{\tilde{Y}}_{\mathbf{A}}$ (see 5.3), we have

$$\hat{\Phi}_g(\alpha_0) = \int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \langle v_0 | F(n \mathsf{m}(\alpha_0; 1_m) g)\rangle \psi_Y(n)^{-1} \mathrm{d}n , \quad (\alpha_0 \neq 0) ,$$

$$\Phi_g(0) = \int_{N^{\tilde{Y}}_{\mathbf{Q}} \backslash N^{\tilde{Y}}_{\mathbf{A}}} \langle v_0 | F(n' g)\rangle \mathrm{d}n' .$$

Hence the identity (5.10) takes the form

$$\hat{\Phi}_g(0) + \sum_{\alpha_0 \in E^\times} \int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \langle v_0 | F(n \mathsf{m}(\alpha_0; 1_m) g)\rangle \psi_Y(n)^{-1} \mathrm{d}n = \int_{N^{\tilde{Y}}_{\mathbf{Q}} \backslash N^{\tilde{Y}}_{\mathbf{A}}} \langle v_0 | F(n' g)\rangle \mathrm{d}n' .$$

By the cuspidality of $F$, the first term $\hat{\Phi}_g(0)$ of the left-hand side equals zero. To obtain (5.9), we first replace $g$ with $\mathsf{m}(1; g_0)g$, multiply the both sides of the identity by $f(g_0)$ and then integrate with respect to $g_0 \in G^Y_{0,\boldsymbol{Q}} \backslash G^Y_{0,\boldsymbol{A}}$. □

By (5.8) and (5.9), we obtain

$$Z^F_{f,Y}(s) = \int_{E^\times \backslash E^\times_{\boldsymbol{A}}} |\mathrm{N}(t)|_{\boldsymbol{A}}^{s-(m+1)/2} \Big( \sum_{\alpha \in E^\times} \langle v_0 | \varphi^F_{f,Y}(\mathsf{m}(\alpha t; 1_m)) \rangle \Big) \mathrm{d}^\times t = \zeta(\varphi^F_{f,Y}; s).$$

This completes the proof. □

**6. Computation of non-Archimedean zeta-integrals.** We retain the notations and the assumptions made in Section 5. In this section, we fix a prime number $p$ and let $E_p$ denote the quadratic $\boldsymbol{Q}_p$-algebra $E \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$ with the maximal order $\mathcal{O}_p = \mathcal{O} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$. The $p$-components of $K_{\mathrm{f}}$, $K_{0,\mathrm{f}}$, $K^{\tilde{Y}}_{\mathrm{f}}$ and $K^Y_{0,\mathrm{f}}$ are denoted by $K_p$, $K_{0,p}$, $K^{\tilde{Y}}_p$ and $K^Y_{0,p}$, respectively.

6.1. Local zeta-integrals. Let $\mathcal{W}^Y_p$ be the space of all the locally constant functions $\varphi : G_p \to \boldsymbol{C}$ such that

$$\varphi(n\mathsf{m}(1; k_0)gk) = \psi_{Y,p}(n)\varphi(g), \quad n \in N_p, \quad k_0 \in K^Y_{0,p}, \quad k \in K_p \tag{6.1}$$

(cf. (5.7)). Here $\psi_{Y,p}$ is the $p$-component of the character $\psi_Y : N_{\boldsymbol{A}} \to \boldsymbol{C}^{(1)}$ defined by (5.5).

Let $\mathcal{H}_p$ (resp. $\mathcal{H}^Y_p$) be the Hecke algebra for $(G_p, K_p)$ (resp. $(G^Y_{0,p}, K^Y_{0,p})$). The space $\mathcal{W}^Y_p$ becomes a double $\mathcal{H}^Y_p \times \mathcal{H}_p$-module by the action

$$(\phi_0 * \varphi * \phi)(x) = \int_{G^Y_{0,p}} \int_{G_p} \phi_0(g_0)\varphi(g_0^{-1}xg)\phi(g)\mathrm{d}g_0\mathrm{d}g, \quad (\phi_0, \phi) \in \mathcal{H}^Y_p \times \mathcal{H}_p,$$

where $\mathrm{d}g$ (resp. $\mathrm{d}g_0$) is the Haar measure of $G_p$ (resp. $G^Y_{0,p}$) such that $\mathrm{vol}(K_p) = 1$ (resp. $\mathrm{vol}(K^Y_{0,p}) = 1$). Our aim in this section is to evaluate the *local zeta-integral*

$$\zeta_p(\varphi; s) := \int_{E_p^\times} \varphi(\mathsf{m}(t; 1_m))|\mathrm{N}(t)|_p^{s-(m+1)/2}\mathrm{d}^\times t \tag{6.2}$$

for an $\mathcal{H}^Y_p \times \mathcal{H}_p$-eigenfunction $\varphi \in \mathcal{W}^Y_p$. Here is the result.

THEOREM 28. *Let $\varphi \in \mathcal{W}^Y_p$ be an $\mathcal{H}^Y_p \times \mathcal{H}_p$-eigenfunction corresponding to the character $(\Lambda_0, \Lambda)$, i.e., $\phi_0 * \varphi * \phi = \Lambda_0(\phi_0)\Lambda(\phi)\varphi$ for all $(\phi_0, \phi) \in \mathcal{H}^Y_p \times \mathcal{H}_p$. Suppose $\varphi$ is bounded on $G_p$. Then the integral* (6.2) *converges on* $\mathrm{Re}(s) > (m+1)/2$, *and*

$$\zeta_p(\varphi; s) = \frac{L(s, \Lambda)}{L(s+1/2, \Lambda_0)} \frac{1}{\zeta_{m,p}(2s)} \varphi(1), \quad \mathrm{Re}(s) > (m+1)/2$$

*with*

$$\zeta_{m,p}(s) = \begin{cases} (1-p^{-s})^{-1} & (m \equiv 1 \pmod 2), \\ (1-\omega_p(p)p^{-s})^{-1} & (m \equiv 0 \pmod 2,\ p \notin \mathrm{R}(E)), \\ 1 & (m \equiv 0 \pmod 2,\ p \in \mathrm{R}(E)). \end{cases}$$

6.2. Computation at non-split primes. We assume $E_p = \boldsymbol{Q}_p(\sqrt{D})$ is a field and use the notations in Section 3 and Subsection 4.1. By the assumption (A2) in 5.2, we may set $(R, \mathcal{M}_p) = (S_{\nu+1}, L_{\nu+1})$ and $(\tilde{R}, \tilde{\mathcal{M}_p}) = (S_{\nu+2}, L_{\nu+2})$ for a $\nu \in \boldsymbol{N}$ with a Witt tower $\{(S_\nu, V_\nu)\}_{\nu\in\boldsymbol{N}}$. Let $n_0$ denote the size of $S_0$. Then $m = 2\nu+n_0+2$ and we have identifications $(G_{0,p}, K_{0,p}) = (G_{\nu+1}, K_{\nu+1})$ and $(G_p, K_p) = (G_{\nu+2}, K_{\nu+2})$. Put $\partial = \partial_R(\mathcal{M}_p) = \partial_{S_0}(L_0)$. Fix $\varphi \in \mathcal{W}_p^Y$ and let $\Lambda_0$ and $\Lambda$ be as in Theorem 28.

Note that the vector $Y$ is reduced for $(R, \mathcal{M}_p)$ by Lemma 25.

LEMMA 29. (1) *If* $l \in \boldsymbol{Z}$ *and* $l < 0$, *then* $\varphi(\mathsf{m}(\pi^l; 1_m)) = 0$.
(2) *If* $g_0 \in G_{0,p}$ *is such that* $g_0^{-1}Y \notin L^*_{\nu+1}$, *then* $\varphi(\mathsf{m}(1; g_0)) = 0$.

PROOF. Let $l \in \boldsymbol{Z}$ and $g_0 \in G_{0,p}$. Suppose $\bar{\pi}^l g_0^{-1}Y \notin L^*_{\nu+1}$. Then $\psi_p(\tau R(Y, \pi^l g_0 Z)) \neq 1$ for some $Z \in L_{\nu+1}$. Since $R[Z] \in \sqrt{D}\tau(\mathcal{O}_p)$, we can write $R[Z] = a - \bar{a}$ with an $a \in \mathcal{O}_p$. Then $\zeta = \bar{a} + 2^{-1}R[Z] \in \boldsymbol{Q}_p$ and $\mathsf{n}(Z; \zeta) \in N_p \cap K_p$. The equivariance (6.1) of $\varphi$ yields the formula

$$\varphi(\mathsf{m}(\pi^l; g_0)) = \varphi(\mathsf{m}(\pi^l; g_0)\mathsf{n}(Z; \zeta)) = \psi_p(\tau R(Y, \pi^l g_0 Z))\varphi(\mathsf{m}(\pi^l; g_0)),$$

which in turn gives $\varphi(\mathsf{m}(\pi^l; g_0)) = 0$. This proves (1) and (2). Note $\bar{\pi}^l Y \notin L^*_{\nu+1}$ for all $l < 0$, since $Y$ is $\mathcal{O}_p$-primitive in $L^*_{\nu+1} = \mathcal{M}^*_p$. □

LEMMA 30. *Let* $F_\varphi(T) \in \boldsymbol{C}[[T]]$ *be the formal power series*

$$F_\varphi(T) := \sum_{l=0}^{\infty} \varphi(\mathsf{m}(\pi^l; 1_m))T^l .$$

*If* $\varphi$ *is bounded on* $G_p$, *then* $\zeta_p(\varphi; s) = F_\varphi(q^{-s+(m+1)/2})$ *for* $\mathrm{Re}(s) > (m+1)/2$.

PROOF. This follows from the definition (6.2) by $E_p^\times = \bigcup_{l\in\boldsymbol{Z}} \pi^l \mathcal{O}_p^\times$ and Lemma 29 (1). Note the assumption that $\varphi$ is bounded, combined with Lemma 29 (1), yields a majoration of the integral $\zeta(|\varphi|; \mathrm{Re}(s))$ by the geometric series $\sum_{l=0}^{\infty} q^{(-\mathrm{Re}(s)+(m+1)/2)l}$, which is convergent on $\mathrm{Re}(s) > (m+1)/2$. □

LEMMA 31. *For each* $l \in \boldsymbol{N}$, $0 \leqslant r \leqslant \nu+2$,

$$\begin{aligned}(\varphi * \tilde{c}^{(r)}_{\nu+2})&(\mathsf{m}(\pi^l; 1_m))\\ &= q^{2\nu+n_0+3}\varphi(r-1, l+1) + \varphi(r-1, l-1) + q^r\varphi(r, l)\\ &\quad + \begin{cases} C^{(r-2)}_{\nu+1}\varphi(r-2, l) + D^{(r-1)}\varphi(r-1, l) & (l > 0)\,, \\ \varphi'(r-2, 0) - q^{r-2}\varphi''(r-2, 0) + q^{r-1}\varphi''(r-1, 0) - q^{r-e/2}\varphi(r-1, 0) & \\ & (l = 0)\,, \end{cases}\end{aligned}$$

*with*

$$\varphi(r,l)=\sum_{h\in\tilde{c}^{(r)}_{\nu+1}/K_{\nu+1}}\varphi(\mathrm{m}(\pi^l;h))\,,$$

$$\varphi'(r,0)=\sum_{\substack{h\in\tilde{c}^{(r)}_{\nu+1}/K_{\nu+1},X\in\pi^{-1}L_{\nu+1}/L_{\nu+1},\\ \sqrt{D}^{-1}S_{\nu+1}[X]\in\tau(\pi^{-1}\mathcal{O}_p),hX\in\pi^{-1}L_{\nu+1}\,,\\ \zeta\in(2^{-1}S_{\nu+1}[X]+\pi^{-1}\mathcal{O}_p)\cap\boldsymbol{Q}_p)/\boldsymbol{Z}_p}}\psi_p(\tau S_{\nu+1}(Y,hX))\varphi(\mathrm{m}(1;h))\,,$$

$$\varphi''(r,0)=\sum_{\substack{h\in\tilde{c}^{(r)}_{\nu+1}/K_{\nu+1},\\ z\in L'_0/L_0,\\ \zeta\in(2^{-1}S_0[z]+\pi^{-1}\mathcal{O}_p)\cap\boldsymbol{Q}_p)/\boldsymbol{Z}_p}}\psi_p\left(\tau S_{\nu+1}\left(Y,h\begin{bmatrix}0_{\nu+1}\\ z\\ 0_{\nu+1}\end{bmatrix}\right)\right)\varphi(\mathrm{m}(1;h))\,,$$

*and* $\varphi(r,l)=0$ *if* $r<0$ *or* $l<0$. *Here* $C^{(r-2)}_{\nu+1}$ *and* $D^{(r-1)}$ *are the numbers defined by* (4.2), $\psi_p:\boldsymbol{Q}_p\to\boldsymbol{C}^{(1)}$ *is the p-component of the basic character* $\psi$.

PROOF. This follows from Lemma 13. □

PROPOSITION 32. *Let* $\mathbf{s}\in(\boldsymbol{C}^\times)^{\nu+2}/W_{\nu+2}$ *be the Satake parameter of* $\Lambda$. *Then*

$$F_\varphi(T)P_{\nu+2}(q^{-(\nu+1+(n_0+1)/2)}T;\mathbf{s})=\sum_{k=0}^{2\nu+4}(-1)^k(q^{-(\nu+1+(n_0+1)/2)}T)^k\sum_{r=0}^{\nu+1}B_{\varphi,k}(r)\tag{6.3}$$

*with*

$$\begin{aligned}B_{\varphi,k}(r)=\big(&a_{\nu+1,k}(r)-q^{-(\nu+1+(n_0+1)/2)}(D^{(r)}+q^r)a_{\nu+1,k-1}(r)\\ &-q^{-(\nu+1+(n_0+1)/2)}C^{(r)}_{\nu+1}a_{\nu+1,k-1}(r+1)\big)\varphi(r,0)\\ &+q^{-(\nu+1+(n_0+1)/2)}a_{\nu+1,k-1}(r+1)\varphi'(r,0)\\ &+q^{-(\nu+1+(n_0+1)/2)+r}(a_{\nu+1,k-1}(r)-a_{\nu+1,k-1}(r+1))\varphi''(r,0)\,.\end{aligned}\tag{6.4}$$

PROOF. Similar to the proof of [14, Proposition 1 (p. 349)]. □

PROPOSITION 33. *Set* $\tilde{c}^{(r)}_Y=\{h\in G^Y_{\nu+1}\,|\,\mathrm{rank}_{\mathcal{O}_p/\pi\mathcal{O}_p}(\pi h\pmod{\pi\mathcal{O}_p})=r\}=G^Y_{\nu+1}\cap\tilde{c}^{(r)}_{\nu+1}$. *Then* $\varphi(r,0)=\varphi'(r,0)=\varphi''(r,0)=0$ *if* $r>\nu'=\nu(S_{\nu+1}|Y^\perp)$. *If* $0\leqslant r\leqslant\nu'$, *then*

$$\varphi(r,0)=(\tilde{c}^{(r)}_Y*\varphi)(1)\,,\quad \varphi'(r,0)=C'_r\varphi(r,0)\,,\quad \varphi''(r,0)=C''_r\varphi(r,0)\,,$$

*where*

$$C'_r=q^{1-e/2}\sum_{\substack{X\in\mathcal{U}_{\nu+1}\\ \tilde{c}^{(r)}_Y X\in\pi^{-1}L_{\nu+1}}}\psi_p(\tau S_{\nu+1}(Y,X))\,,$$

$$C''_r=q^{1-e/2}\sum_{z\in\mathcal{U}_0}\psi_p\left(\tau S_{\nu+1}\left(Y,\begin{bmatrix}0\\ z\\ 0\end{bmatrix}\right)\right)\,.$$

PROOF. If $r > \nu'$, then $\tilde{c}_Y^{(r)} = \emptyset$ by Lemma 12. Hence the first assertion follows. In order to show the second statement, first note that for each $X$ the number of $\zeta \in (2^{-1}S_{\nu+1}[X] + \pi^{-1}\mathcal{O}_p) \cap \boldsymbol{Q}_p)/\boldsymbol{Z}_p$ is $q^{1-e/2}$. By this remark, combined with Lemma 29, we write $\varphi'(r,0)$ as a sum of $q^{1-e/2}\psi_p(\tau S_{\nu+1}(Y, hX))\varphi(\mathsf{m}(1;h))$ over all $(h, X) \in (\tilde{c}_{\nu+1}^{(r)}/K_{\nu+1}) \times (\pi^{-1}L_{\nu+1}/L_{\nu+1})$ such that

$$h^{-1}X \in L_{\nu+1}^*\,, \tag{6.5}$$

$$hX \in \pi^{-1}L_{\nu+1}\,, \quad S_{\nu+1}[X]/\sqrt{D} \in \tau(\pi^{-1}\mathcal{O}_p)\,. \tag{6.6}$$

Since $Y$ is reduced for $(S_{\nu+1}, L_{\nu+1})$, the condition (6.5) implies $h \in G_{\nu+1}^Y K_{\nu+1}$ by Lemma 21. Hence we can write the set of cosets $h \in \tilde{c}_{\nu+1}^{(r)}/K_{\nu+1}$ satisfying (6.5) as $(\tilde{c}_{\nu+1}^{(r)} \cap G_{\nu+1}^Y K_{\nu+1})/K_{\nu+1} \cong \tilde{c}_Y^{(r)}/K_{\nu+1}^Y$. Thus, in the summation defining $\varphi'(r,0)$, changing the range of $h$ from $\tilde{c}_{\nu+1}^{(r)}/K_{\nu+1}$ to $\tilde{c}_Y^{(r)}/K_{\nu+1}^Y$ does not affect $\varphi'(r,0)$. Let $c_Y^{(r)} \in G_{\nu+1}^Y$ be a representative of $\tilde{c}_Y^{(r)}/K_{\nu+1}^Y$. Then for those $h \in \tilde{c}_Y^{(r)}/K_{\nu+1}^Y$, the first condition in (6.6) is equivalent to $c_Y^{(r)}X \in \pi^{-1}L_{\nu+1}$, independent of individual $h$. Hence $\varphi'(r,0)$ is factored into the product of $C'_r$ and $\sum_{h \in \tilde{c}_Y^{(r)}/K_{\nu+1}^Y} \varphi(\mathsf{m}(1;h)) = (\tilde{c}_Y^{(r)} * \varphi)(1)$. This proves the formula for $\varphi'(r,0)$. Similar arguments yield formulas of $\varphi(r,0)$ and $\varphi''(r,0)$. □

The numbers $C'_r$ and $C''_r$ are evaluated in terms of $\beta_Y$ (Lemma 20) and $\rho_Y$ (Lemma 18).

LEMMA 34. *For* $0 \leqslant r \leqslant \nu'$,

$$C'_r = q^{r+1-e/2}(-q^{\nu+n_0-r+e/2} + q^{\nu+1-r}(q^{\partial} + \beta_Y))\,,$$
$$C''_r = q^{\partial+1-e/2}(1 - \delta(Y \notin {L'_{\nu+1}}^*)) = q^{\partial+1-e/2} - q\rho_Y\,,$$

*and*

$$\begin{aligned} B_{\varphi,k}(r) = \big\{ & a_{\nu+1,k}(r) - q^{-(\nu+1+(n_0+1)/2)+r+1}\rho_Y a_{\nu+1,k-1}(r) \\ & + q^{-(\nu+1+(n_0+1)/2)+r+1}(-q^{2\nu+n_0-2r+1} + q^{\nu-r+1-e/2}\beta_Y + \rho_Y) \\ & \times a_{\nu+1,k-1}(r+1)\big\} \Lambda_0(\tilde{c}_Y^{(r)})\varphi(1)\,. \end{aligned} \tag{6.7}$$

PROOF. Let us compute $C'_r$. By Lemma 6, choosing a Witt basis of $\mathcal{M}_p$ properly, we may assume that the identification $(R, V_p) = (S_{\nu+1}, L_{\nu+1})$ is made so that $Y = \begin{bmatrix} 0_r \\ Y' \\ 0 \end{bmatrix}$ with $Y' = \begin{bmatrix} 0_{\nu-r} \\ a \\ \mathsf{a} \\ 1 \\ 0_{\nu-r} \end{bmatrix}$, $(a \in \mathcal{O}_p, \mathsf{a} \in L_0^*)$. Then the element $c_{\nu+1}^{(r)}$ fixes the vector $Y$ if $0 \leqslant r \leqslant \nu$, namely $c_{\nu+1}^{(r)} \in G_{\nu+1}^Y$ $(0 \leqslant r \leqslant \nu)$. The condition $c_Y^{(r)}X \in \pi^{-1}L_{\nu+1}$, $X \in \mathcal{U}_{\nu+1}$ for a vector $X = \begin{bmatrix} x_1 \\ X' \\ y_1 \end{bmatrix}$, $(x_1, y_1 \in E_p^r, X' \in V_{\nu-r+1})$ is equivalent to

$$x_1 \in (\pi^{-1}\mathcal{O}_p/\mathcal{O}_p)^r\,, \quad y_1 = 0\,, \quad X' \in \mathcal{U}_{\nu-r+1}\,.$$

Hence $C'_r = q^{r+1-e/2}\theta_{\nu-r+1}(Y')$ with $\theta_n$ the exponential sum studied in 3.7. Using Lemma 20 (2), Lemma 18 and Lemma 19, we have

$$C'_r = q^{r+1-e/2}(-q^{\nu+n_0-r+e/2} + q^{\nu-r+1}(q^{\partial} + \beta_{Y'}))\,.$$

Note $\beta_{Y'} = \beta_Y$, since $\mathcal{V}_{0,Y} = \mathcal{V}_{0,Y'}$.

The evaluation of $C''_r$ is simpler. Since $\mathcal{U}_0 = L'_0/L_0$, we have $C''_r = q^{1-e/2}\theta'_0(\mathsf{a})$. Use Lemma 20 (1) to obtain $C''_r = q^{\partial+1-e/2}\delta(\mathsf{a} \in L'_0{}^*)$. By $\delta(Y \in L'_{\nu+1}{}^*) = \delta(\mathsf{a} \in L'_0{}^*)$, the conclusion follows.

Using Proposition 33 and the values of $C'_r$, $C''_r$, from (6.4), we obtain the formula (6.7) by a computation. □

Set $\nu' = \nu(S_{\nu+1}|Y^{\perp})$, $n'_0 = n_0(S_{\nu+1}|Y^{\perp})$ and $\partial' = \partial_{S_{\nu+1}|Y^{\perp}}(L_{\nu+1} \cap Y^{\perp})$. Since $Y$ is reduced for $(S_{\nu+1}, L_{\nu+1})$, by Lemma 5, there exists an anisotropic skew-hermitian matrix $S'_0$ (among the ones listed in Lemma 8) such that $(S_{\nu+1}|Y^{\perp}, Y^{\perp}) \cong (S'_{\nu'}, \mathcal{O}_p^{2\nu'+n'_0})$. Then the Witt tower $\{(S'_n, V_n)\}_{n\in N}$ determines the coefficients $\{b_{n,k}(r)\}$ of Hecke polynomials in the same way as the Witt tower $\{(S_n, V_n)\}_{n\in N}$ determines the coefficients $\{a_{n,k}(r)\}$. Lemma 25 (2), combined with Lemma 11, implies that possible values of $(n'_0, \partial')$ are $(n_0 - 1, \partial - 1)$, $(n_0 - 1, \partial)$ and $(n_0 + 1, \partial)$.

LEMMA 35. (1) *Suppose* $(n'_0, \partial') = (n_0 - 1, \partial - 1)$. *Set* $\tilde{b}_{n,k}(r) = b_{n,k}(r) + Ab_{n,k-1}(r)$ *with* $A = -q^{\partial-n_0/2+1-e/2}$. *Then*

$$\begin{aligned} a_{n,k}(r) &- q^{-(n+(n_0+1)/2)+\partial+r+1-e/2}a_{n,k-1}(r) \\ &- q^{-(n+(n_0+1)/2)+r+1-e/2}(q^{n-r}-1)(q^{n+n_0-r+e/2-1}+q^{\partial})a_{n,k-1}(r+1) \\ &= q^{-k/2}\tilde{b}_{n,k}(r) \end{aligned}$$

*for* $0 \leqslant k \leqslant 2n+1, 0 \leqslant r \leqslant n$.

(2) *Suppose* $(n'_0, \partial') = (n_0 - 1, \partial)$. *Then*

$$a_{n+1,k}(r) + (q^{(n_0-1)/2} - q^{n-r+(n_0+1)/2})a_{n+1,k-1}(r+1) = q^{-k/2}b_{n+1,k}(r)$$

*for* $0 \leqslant k \leqslant 2(n+1), 0 \leqslant r \leqslant n+1$.

(3) *Suppose* $(n'_0, \partial') = (n_0 + 1, \partial)$. *Set* $\tilde{b}_{n,k}(r) = b_{n,k}(r) - (A+B)b_{n,k-1}(r) + ABb_{n,k-2}(r)$ *with* $A = q^{-n_0/2}$, $B = -q^{\partial-n_0+1/2}$. *Then*

$$a_{n,k}(r) - (q^{n-r-1+(n_0+1)/2} + q^{\partial-n_0/2})a_{n,k-1}(r+1) = q^{-k/2}\tilde{b}_{n-1,k}(r) \tag{6.8}$$

*for* $0 \leqslant k \leqslant 2n, 0 \leqslant r \leqslant n-1$.

PROOF. Consider the case $(n'_0, \partial') = (n_0 + 1, \partial)$; we then have $\nu' = \nu$. The formula (6.8) for $(n, k, r)$ such that $k \in \{2n, 2n-1\}$ and $0 \leqslant r \leqslant n-1$ is proved by a direct calculation with the aid of Lemma 24. Note this in particular cares the case of $n = 1$. Let us prove (6.8) by induction on $n$. Suppose $n > 1$, $0 \leqslant k \leqslant 2n$ and $0 \leqslant r \leqslant n$. Let us consider the case $r = 0$ first. Use Lemma 23 (1) to write $a_{n+1,k}(0) - (q^{n+(n_0+1)/2} + q^{\partial-n_0/2})a_{n+1,k-1}(1) - q^{-k/2}\tilde{b}_{n,k}(0)$ in terms of $a_{n,k'}(i)$, $\tilde{b}_{n-1,k'}(i)$; then by induction assumption we can write $\tilde{b}_{n-1,k'}(i)$ in terms of $a_{n,k''}(j)$. After a straightforward but tedious computation, we obtain

$$\begin{aligned}&a_{n+1,k}(0)-(q^{n+(n_0+1)/2}+q^{\partial-n_0/2})a_{n+1,k-1}(1)-q^{-k/2}\tilde{b}_{n,k}(0)\\&=q^{-(1+n_0/2)}(q^{n+1+n_0-1/2}+q^{\partial})\{a_{n,k-1}(1)+a_{n,k-3}(1)\\&\quad -q^{-(n+(n_0+1)/2)}(qa_{n,k-2}(0)+C_n^{(1)}a_{n,k-2}(2)+D^{(1)}a_{n,k-2}(1))\}\,.\end{aligned}$$

The formula inside the curly bracket on the right-hand side is zero by Lemma 23 (2).

Consider the case $r>0$. Since the formula is obvious when $k=0$, we assume $k>0$. Then using Lemma 23 (1) (i), we have

$$\begin{aligned}&a_{n+1,k}(r)-(q^{n-r+(n_0+1)/2}+q^{\partial-n_0/2})a_{n+1,k-1}(r+1)-q^{-k/2}\tilde{b}_{n,k}(r)\\&=q^{-(n+(n_0+1)/2)}\{a_{n,k-1}(r-1)-(q^{n-r+(n_0+1)/2}+q^{\partial-n_0/2})a_{n,k-2}(r)\\&\quad -q^{-(k-1)/2}\tilde{b}_{n-1,k-1}(r-1)\}\end{aligned}$$

after a computation. By the induction assumption, the right-hand side is zero. This proves (6.8) completely. □

PROPOSITION 36. *Let* $\mathbf{s}\in(\boldsymbol{C}^{\times})^{\nu+2}/W_{\nu+2}$ *and* $\mathbf{s}_0\in(\boldsymbol{C}^{\times})^{\nu'}/W_{\nu'}$ *be the Satake parameters of* $\Lambda$ *and* $\Lambda_0$, *respectively. Then we have*

$$F_{\varphi}(T)=\frac{P_{\nu'}(q^{-(\nu+1+(n_0+1)/2)-1/2}T;\mathbf{s}_0)}{P_{\nu+2}(q^{-(\nu+1+(n_0+1)/2)}T;\mathbf{s})}B_Y(q^{-(\nu+1+(n_0+1)/2)-1/2}T)\varphi(1)$$

*with*

$$B_Y(T)=\begin{cases}1+q^{\partial-n_0/2+1-e/2}T\,, & (n_0',\partial')=(n_0-1,\partial-1)\,,\\ 1\,, & (n_0',\partial')=(n_0-1,\partial)\,,\\ (1-q^{-n_0/2}T)(1+q^{\partial-(n_0-1)/2}T)\,, & (n_0',\partial')=(n_0+1,\partial)\,.\end{cases}$$

PROOF. Consider the case $(n_0',\partial')=(n_0+1,\partial)$. In this case, $\nu'=\nu$. From the Table 1 in Lemma 11, we have $\rho_Y=0$, $e=1$ and $\beta_Y=-q^{\partial}$. The formula (6.7) is simplified as

$$B_{\varphi,k}(r)=(a_{\nu+1,k}(r)-q^{(n_0+1)/2}(q^{\nu-r}+q^{\partial-n_0-1/2})a_{\nu+1,k-1}(r+1))\Lambda_0(\tilde{c}_Y^{(r)})\varphi(1)$$

and this equals $q^{-k/2}\tilde{b}_{n-1,k}(r)\Lambda_0(\tilde{c}_Y^{(r)})\varphi(1)$ by Lemma 35. By definition (see Lemma 23), $P_{\nu'}(q^{-1/2}T_0;\mathbf{s}_0)=\sum_{k=0}^{2\nu}(-1)^kq^{-k/2}T_0^k\sum_{r=0}^{\nu}b_{\nu,k}(r)\Lambda_0(\tilde{c}_Y^{(r)})$ with $T_0=q^{-(\nu+1+(n_0+1)/2)}T$. By (6.3) and (6.8), we have

$$\begin{aligned}&F_{\varphi}(T)P_{\nu+2}(T_0;\mathbf{s})\\&=\sum_{k=0}^{2(\nu+1)}(-1)^kT_0^k\sum_{r=0}^{\nu}q^{-k/2}(b_{\nu,k}(r)-(A+B)b_{\nu,k-1}(r)+ABb_{\nu,k-2}(r))\Lambda_0(\tilde{c}_Y^{(r)})\varphi(1)\\&=P_{\nu}(q^{-1/2}T_0;\mathbf{s}_0)(1+(A+B)q^{-1/2}T_0+AB(q^{-1/2}T_0)^2)\varphi(1)\\&=P_{\nu}(q^{-1/2}T_0;\mathbf{s}_0)(1-q^{-(n_0+1)/2}T_0)(1+q^{\partial-n_0/2}T_0)\varphi(1).\end{aligned}$$

This proves the desired formula. The remaining cases are similar. □

Now Theorem 28 follows from Proposition 36 combined with the following lemma which is a direct consequence of the definition of local $L$-factors recalled in 4.1.

LEMMA 37. *If* $T = q^{-s+(m+1)/2}$, *then*

$$\frac{P_{\nu'}(q^{-(\nu+1+(n_0+1)/2)-1/2}T;\mathbf{s}_0)}{P_{\nu+2}(q^{-(\nu+1+(n_0+1)/2)}T;\mathbf{s})}B_Y(q^{-(\nu+1+(n_0+1)/2)-1/2}T) = \frac{L(s,\Lambda_p)}{L(s+1/2,\Lambda_{0,p})}\frac{1}{\zeta_{m,p}(2s)}.$$

6.3. Computation at split primes. In this subsection, we use the settings and the notations in 4.2. Recall that $R = (T, -{}^tT)$ with some $T \in \mathrm{GL}_m(\boldsymbol{Z}_p)$ and hence $\tilde{R} = (\tilde{T}, -{}^t\tilde{T})$ with $\tilde{T} = \begin{bmatrix} & & -1 \\ & T & \\ 1 & & \end{bmatrix} \in \mathrm{GL}_{m+2}(\boldsymbol{Z}_p)$. Then $G_p = \{(g_1, g_2) \in \mathrm{GL}_{m+2}(\boldsymbol{Q}_p)^2 \mid {}^tg_2\tilde{T}g_1 = \tilde{T}\}$ is identified with $\mathrm{GL}_{m+2}(\boldsymbol{Q}_p)$ by the first projection. Similarly $G_{0,p} \cong \mathrm{GL}_m(\boldsymbol{Q}_p)$. Put

$$\gamma(X_1, X_2; z) = \begin{bmatrix} 1 & {}^tX_1 & z \\ & 1_m & X_2 \\ & & 1 \end{bmatrix}, \quad (X_1, X_2, z) \in \boldsymbol{Q}_p^m \times \boldsymbol{Q}_p^m \times \boldsymbol{Q}_p .$$

Then for $X = (X_1, X_2) \in E_p^m$ and $\zeta \in \boldsymbol{Q}_p$, we have $\mathsf{n}(X;\zeta) = \gamma(-{}^tTX_2, X_1; \zeta - 2^{-1}{}^tX_2TX_1)$ by the identification $G_p = \mathrm{GL}_{m+2}(\boldsymbol{Q}_p)$ made above.

Let us write $Y = (Y', Y'')$, and $D_0 \in \boldsymbol{Z}_p^\times$ a solution of the equation $t^2 = D$, i.e.,$\sqrt{D} = (D_0, -D_0)$.

LEMMA 38. *Let* $\varphi \in \mathcal{W}_p^Y$.

(1) *If* $t_1, t_2 \in \boldsymbol{Q}_p^\times$, $X_1, X_2 \in \boldsymbol{Q}_p^m$ *and* $h \in \mathrm{GL}_m(\boldsymbol{Q}_p)$ *satisfy* $t_1{}^th^{-1}X_1 \in \boldsymbol{Z}_p^m$ *and* $t_2hX_2 \in \boldsymbol{Z}_p^m$, *then*

$$\varphi(\mathrm{diag}(t_1, h, t_2^{-1})\gamma(X_1, X_2; \zeta)) = \varphi(\mathrm{diag}(t_1, h, t_2^{-1})).$$

(2) *Let* $t_1, t_2 \in \boldsymbol{Q}_p^\times$ *and* $h \in \mathrm{GL}_m(\boldsymbol{Q}_p)$. *Then* $\varphi(t_1, h, t_2^{-1}) = 0$ *unless*

$$t_1h^{-1}Y' \in \boldsymbol{Z}_p^m, \quad t_2{}^th{}^tTY'' \in \boldsymbol{Z}_p^m .$$

PROOF. By (6.1), we have

$$\begin{aligned}&\varphi(\mathrm{diag}(t_1, h, t_2^{-1})\gamma(X_1, X_2; \zeta)) \\ &\quad = \psi_p((-t_2/D_0){}^tY''ThX_2)\psi_p((t_1/D_0){}^tY'{}^th^{-1}X_1)\varphi(\mathrm{diag}(t_1, h, t_2^{-1})).\end{aligned}$$

Noting $D_0 \in \boldsymbol{Z}_p^\times$, $T \in \mathrm{GL}_m(\boldsymbol{Z}_p)$ and $\psi_p|\boldsymbol{Z}_p^\times = 1$, we have the first part of the lemma. To obtain the second part, it suffices to note that $\varphi(\mathrm{diag}(t_1, h, t_2^{-1})\gamma(X_1, X_2; 0)) = \varphi(\mathrm{diag}(t_1, h, t_2^{-1}))$ for $(X_1, X_2) \in \boldsymbol{Z}_p^m \oplus \boldsymbol{Z}_p^m$. □

LEMMA 39. *Let* $F_\varphi(T_1, T_2) \in \boldsymbol{C}[[T_1, T_2]]$ *be the formal power series*

$$F_\varphi(T_1, T_2) = \sum_{l_1, l_2 \geqslant 0} \varphi(\mathrm{diag}(p^{l_1}, 1_m, p^{-l_2}))T_1^{l_1}T_2^{l_2}.$$

*If* $\varphi$ *is bounded on* $G_p$, *then* $\zeta_p(\varphi; s) = F_\varphi(p^{-s+(m+1)/2}, p^{-s+(m+1)/2})$ *for* $\mathrm{Re}(s) > (m + 1)/2$.

PROOF. This follows from the definition (6.2) by the decomposition

$$E_p^\times = \bigcup_{l_1,l_2\in \boldsymbol{Z}} (p^{l_1}\boldsymbol{Z}_p^\times \times p^{l_2}\boldsymbol{Z}_p^\times)$$

and Lemma 38 (2). Note that $p^{l_1}Y' \notin Z_p^m$ if $l_1 < 0$ and $p^{l_2}Y'' \notin \boldsymbol{Z}_p^m$ if $l_2 < 0$, since $Y = (Y', Y'')$ is assumed to be $\mathcal{O}_p$-primitive in $\mathcal{M} = \boldsymbol{Z}_p^m \oplus \boldsymbol{Z}_p^m$. Since $\varphi$ is bounded, by Lemma 38 (2), the integral $\zeta(|\varphi|; \mathrm{Re}(s))$ is majorized by the geometric series

$$\sum_{l_1,l_2\geqslant 0} q^{(-\mathrm{Re}(s)+(m+1)/2)l_1} q^{(-\mathrm{Re}(s)+(m+1)/2)l_2}\,,$$

which is convergent in $\mathrm{Re}(s) > (m+1)/2$. □

For $i, j \geqslant 0$ such that $i + j \leqslant m$, put $c_m^{(i,j)} = p^{(1,\dots,1,0,\dots,0,-1,\dots,-1)}$ (1 appears $i$ times and $-1$ appears $j$ times in the exponent of $p$) and set $\tilde{c}_m^{(i,j)} = K_m c_m^{(i,j)} K_m$. We use the same notation $\tilde{c}_m^{(i,j)}$ to denote its characteristic function. Fix a complete set of representatives $R_m^{(i,j)}$ of $K_m/K_m \cap c_m^{(i,j)} K_m (c_m^{(i,j)})^{-1}$.

LEMMA 40. (1) *For* $0 \leqslant i \leqslant m+2$, *the double coset* $\tilde{c}_{m+2}^{(i,0)}$ *is a disjoint union of the following left* $K_{m+2}$ *cosets.*

- $\mathrm{diag}(1, \alpha c_m^{(i,0)}, 1)\gamma(0, Y_1; 0)K_{m+2}$ *with*

$$\alpha \in R_m^{(i,0)}\,, \quad Y_1 = \begin{bmatrix} y_1 \\ 0_{m-i} \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m\,.$$

- $\mathrm{diag}(1, \alpha c_m^{(i-1,0)}, p)K_{m+2}$ *with* $\alpha \in R_m^{(i-1,0)}$.
- $\mathrm{diag}(p, \alpha c_m^{(i-1,0)}, 1)\gamma(X_2, Y_2; z_2)K_{m+2}$ *with*

$$\alpha \in R_m^{(i-1,0)}\,, \quad z_2 \in p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p\,,$$

$$X_2 = \begin{bmatrix} 0_{i-1} \\ x_2 \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m\,, \quad Y_2 = \begin{bmatrix} y_1 \\ 0_{m-i+1} \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m\,.$$

- $\mathrm{diag}(p, \alpha c_m^{(i-2,0)}, p)\gamma(X_3, 0; 0)K_{m+2}$ *with*

$$\alpha \in R_m^{(i-2,0)}\,, \quad X_3 = \begin{bmatrix} 0_{i-2} \\ x_2 \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m\,.$$

(2) *For* $0 \leqslant j \leqslant m+2$, *the double coset* $\tilde{c}_{m+2}^{(0,j)}$ *is a disjoint union of the following left* $K_{m+2}$ *cosets.*

- $\mathrm{diag}(1, \alpha c_m^{(0,j)}, 1)\gamma(X_1, 0; 0)K_{m+2}$ *with*

$$\alpha \in R_m^{(0,j)}\,, \quad X_1 = \begin{bmatrix} 0_{m-j} \\ x_2 \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m\,.$$

- $\mathrm{diag}(p^{-1}, \alpha c_m^{(0,j-1)}, 1)K_{m+2}$ *with* $\alpha \in R_m^{(0,j-1)}$.

- $\mathrm{diag}(1, \alpha c_m^{(0,j-1)}, p^{-1})\gamma(X_2', Y_2'; z_2')K_{m+2}$ *with*

$$\alpha \in R_m^{(0,j-1)}, \quad z_2' \in p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p,$$

$$X_2' = \begin{bmatrix} 0_{m-j+1} \\ x_2' \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m, \quad Y_2' = \begin{bmatrix} y_1 \\ 0_{j-1} \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m.$$

- $\mathrm{diag}(p^{-1}, \alpha c_m^{(0,j-2)}, p^{-1})\gamma(0, Y_3; 0)K_{m+2}$ *with*

$$\alpha \in R_m^{(0,j-2)}, \quad Y_3 = \begin{bmatrix} y_1 \\ 0_{j-2} \end{bmatrix} \in p^{-1}\boldsymbol{Z}_p^m/\boldsymbol{Z}_p^m.$$

PROOF. This is proved by the elementary divisor theory. □

LEMMA 41. *For* $0 \leqslant i \leqslant m+2$, $l_1, l_2 \in \boldsymbol{N}$,

$$\begin{aligned}(\varphi * \tilde{c}_{m+2}^{(i,0)})(\mathrm{diag}(p^{l_1}, 1_m, p^{-l_2})) &= p^i\varphi(i; l_1, l_2) + \varphi(i-1; l_1, l_2-1) \\ &\quad + p^{m+1}\varphi(i-1; l_1+1, l_2) + p^{m-i+2}\varphi(i-2; l_1+1, l_2-1)\end{aligned}$$

*with*

$$\varphi(i; l_1, l_2) = \sum_{\alpha \in R_m^{(i,0)}} \varphi(\mathrm{diag}(p^{l_1}, \alpha c_m^{(i,0)}, p^{-l_2})), \quad (0 \leqslant i \leqslant m)$$

*and* $\varphi(i; l_1, l_2) = 0$ *if* $i < 0$ *or* $i > m$.

PROOF. By the Iwasawa decomposition of the double coset $\tilde{c}_{m+2}^{(i,0)}$ given in Lemma 40, the integral

$$(\varphi * \tilde{c}_{m+2}^{(i,0)})(\mathrm{diag}(p^{l_1}, 1_m, p^{-l_2})) = \sum_{g \in \tilde{c}_{m+2}^{(i,0)}/K_{m+2}} \varphi(\mathrm{diag}(p^{l_1}, 1_m, p^{-l_2})g)$$

is a sum of the following four terms.

$$I_1 = \sum_{\substack{\alpha \in R_m^{(i,0)} \\ y_1 \in (p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p)^i}} \varphi\big(\mathrm{diag}(p^{l_1}, \alpha c_m^{(i,0)}, p^{-l_2})\gamma\big(0, \left[\begin{smallmatrix} y_1 \\ 0_{m-i} \end{smallmatrix}\right]; 0\big)\big),$$

$$I_2 = \sum_{\alpha \in R_m^{(i-1,0)}} \varphi(\mathrm{diag}(p^{l_1}, \alpha c_m^{(i-1,0)}, p^{-l_2+1})),$$

$$\begin{aligned} I_3 = \sum_{\substack{\alpha \in R_m^{(i-1,0)}, z_2 \in p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p \\ x_2 \in (p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p)^{m-i+1}, y_1 \in (p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p)^{i-1}}} & \varphi\big(\mathrm{diag}(p^{l_1+1}, \alpha c_m^{(i-1,0)}, p^{-l_2}) \\ & \times \gamma\big(\left[\begin{smallmatrix} 0_{i-1} \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_1 \\ 0_{m-i+1} \end{smallmatrix}\right]; z_2\big)\big), \end{aligned}$$

$$I_4 = \sum_{\substack{\alpha \in R_m^{(i-2,0)} \\ x_2 \in (p^{-1}\boldsymbol{Z}_p/\boldsymbol{Z}_p)^{m-i+2}}} \varphi\big(\mathrm{diag}(p^{l_1+1}, \alpha c_m^{(i-2,0)}, p^{-l_2+1})\gamma\big(\left[\begin{smallmatrix} 0_{i-2} \\ x_2 \end{smallmatrix}\right]; 0\big)\big).$$

Now apply Lemma 38 to see that $I_1$ equals

$$\sum_{\substack{\alpha\in R_m^{(i,0)}\\ y_1\in(p^{-1}\mathbf{Z}_p/\mathbf{Z}_p)^i}} \varphi(\mathrm{diag}(p^{l_1},\alpha c_m^{(i,0)},p^{-l_2})) = \sharp(p^{-1}\mathbf{Z}_p/\mathbf{Z}_p)^i \sum_{\alpha\in R_m^{(i,0)}} \varphi(\mathrm{diag}(p^{l_1},\alpha c_m^{(i,0)},p^{-l_2}))$$
$$= p^i\varphi(i;l_1,l_2)\,.$$

Similarly we have $I_2=\varphi(i-1;l_1,l_2-1)$, $I_3=p^{m+1}\varphi(i-1;l_1+1,l_2)$ and $I_4=p^{m-i+2}\varphi(i-2;l_1+1,l_2-1)$. □

LEMMA 42. *Let* $\mathbf{s}\in(\boldsymbol{C}^\times)^{m+2}/S_{m+2}$ *be the Satake parameter of* $\Lambda$. *We have*

$$F_\varphi(T_1,T_2)P^{(1)}_{m+2}(p^{-(m+1)/2}T_1;\mathbf{s})$$
$$=\sum_{i=0}^{m+1}(-1)^i p^{-i(m+1)+i(i-1)/2}\sum_{l_2=0}^{\infty}(p^i\varphi(i;0,l_2)+\varphi(i-1;0,l_2)T_2)T_1^iT_2^{l_2}\,.$$

PROOF. Since

$$P^{(1)}_{m+2}(T_1;\mathbf{s})=\sum_{i=0}^{m+2}(-1)^ip^{-i(m+2-i)/2}\Lambda_p(\tilde{c}^{(i,0)}_{m+2})T_1^i \tag{6.9}$$

([12, p. 269]), we have

$$F_\varphi(T_1,T_2)P^{(1)}_{m+2}(p^{-(m+1)/2}T_1;\mathbf{s})$$
$$=\sum_{l_1,l_2\geqslant0}\varphi(\mathrm{diag}(p^{l_1},1,p^{-l_2}))T_1^{l_1}T_2^{l_2}\sum_{i=0}^{m+2}(-1)^ip^{-i(m+2-i)/2-i(m+1)/2}\Lambda_p(\tilde{c}^{(i,0)}_{m+2})T_1^i$$
$$=\sum_{l_2\geqslant0}T_2^{l_2}\sum_{l_1\geqslant0}\sum_{i=0}^{m+2}(-1)^ip^{-i(m+1)+i(i-1)/2}(\varphi*\tilde{c}^{(i,0)}_{m+2})(\mathrm{diag}(p^{l_1},1,p^{-l_2}))T_1^{i+l_1}$$
$$=\sum_{l_2\geqslant0}T_2^{l_2}\sum_{l_1\geqslant0}\sum_{i=0}^{m+2}(-1)^ip^{-i(m+1)+i(i-1)/2}\big\{p^i\varphi(i;l_1,l_2)+\varphi(i-1;l_1,l_2-1)$$
$$+p^{m+1}\varphi(i-1;l_1+1,l_2)+p^{m-i+2}\varphi(i-2;l_1+1,l_2-1)\big\}T_1^{i+l_1}$$
$$=\sum_{l_2\geqslant0}T_2^{l_2}\sum_{\substack{0\leqslant i\leqslant m+2\\ k\geqslant i}}(-1)^ip^{-i(m+1)+i(i-1)/2}\big\{p^i\varphi(i;k-i,l_2)+\varphi(i-1;k-i,l_2-1)$$
$$+p^{m+1}\varphi(i-1;k-i+1,l_2)+p^{m-i+2}\varphi(i-2;k-i+1,l_2-1)\big\}T_1^k$$
$$=\sum_{l_2\geqslant0}\sum_{k\geqslant0}T_1^kT_2^{l_2}\Bigg\{\sum_{\substack{0\leqslant i\leqslant m+2\\ k\geqslant i}}(-1)^ip^{-i(m+1)+i(i-1)/2}\cdot p^i\varphi(i;k-i,l_2)$$
$$+\sum_{\substack{0\leqslant i\leqslant m+1\\ k>i}}(-1)^{i+1}p^{-i(m+1)+i(i-1)/2}\cdot p^i\varphi(i;k-i,l_2)$$

$$+\sum_{\substack{0\leqslant i\leqslant m+2\\ k\geqslant i}}(-1)^i p^{-i(m+1)+i(i-1)/2}\varphi(i-1;k-i,l_2-1)$$

$$\left.+\sum_{\substack{0\leqslant i\leqslant m+2\\ k>i}}(-1)^{i+1} p^{-i(m+1)+i(i-1)/2}\varphi(i-1;k-i,l_2-1)\right\}$$

$$=\sum_{0\leqslant i\leqslant m+1}(-1)^i p^{-i(m+1)+i(i-1)/2}\sum_{l_2\geqslant 0}(p^i\varphi(i;0,l_2)+\varphi(i-1;0,l_2-1))T_1^iT_2^{l_2}\,.$$

□

LEMMA 43. *For $i\geqslant 0, l_2\geqslant 0$, we have*

$$\begin{aligned}&\varphi(i;0,l_2)P^{(2)}_{m+2}(p^{-(m+1)/2}T_2;\mathbf{s})\\ &\quad=\sum_{j=0}^{m+2}(-1)^j p^{-j(m+1)+j(j-1)/2}\big(p^j\tilde{\varphi}(i,j;l_2)+\tilde{\varphi}'(i,j-1;l_2)\\ &\qquad+p^{m+1}\tilde{\varphi}(i,j-1;l_2+1)+p^{m-j+2}\tilde{\varphi}'(i,j-2;l_2+1)\big)T_2^j\end{aligned}$$

*with*

$$\tilde{\varphi}(i,j;l_2)=\sum_{\substack{h_1\in\tilde{c}_m^{(i,0)}/K_m,\\ h_1^{-1}Y',p^{l_2}{}^th_1{}^tTY''\in\boldsymbol{Z}_p^m}}\ \sum_{h_2\in\tilde{c}_m^{(0,j)}/K_m}\varphi(\mathrm{diag}(1,h_1h_2,p^{-l_2}))\,,\tag{6.10}$$

$$\tilde{\varphi}'(i,j;l_2)=\sum_{\substack{h_1\in\tilde{c}_m^{(i,0)}/K_m,\\ h_1^{-1}Y',p^{l_2}{}^th_1{}^tTY''\in\boldsymbol{Z}_p^m}}\ \sum_{h_2\in\tilde{c}_m^{(0,j)}/K_m}\varphi(\mathrm{diag}(p^{-1},h_1h_2,p^{-l_2}))\,.\tag{6.11}$$

PROOF. By Lemma 38 (2), we can write $\varphi(i;0,l_2)$ as a sum of $\varphi(\mathrm{diag}(1,h,p^{-l_2}))$ over all $h\in\tilde{c}_m^{(i,0)}/K_m$ such that $h^{-1}Y'\in\boldsymbol{Z}_p^m$ and $p^{l_2}{}^th{}^tTY''\in\boldsymbol{Z}_p^m$. Since

$$P^{(2)}_{m+2}(T_2;\mathbf{s})=\sum_{j=0}^{m+2}(-1)^j p^{-j(m+2-j)/2}\Lambda(\tilde{c}^{(0,j)}_{m+2})T_2^j\,,\tag{6.12}$$

we can calculate $\varphi(i;0,l_2)P^{(2)}_{m+2}(p^{-(m+1)/2}T_2;\mathbf{s})$ using Lemma 41 by a similar way to Lemma 42. □

LEMMA 44. *We have $\tilde{\varphi}'(i,j;0)=0$ for $0\leqslant i,j\leqslant m$.*

PROOF. By Lemma 38, we have $\varphi(\mathrm{diag}(p^{-1},h,1))=0$ unless $p^{-1}h^{-1}Y''\in\boldsymbol{Z}_p^m$, ${}^th{}^tTY''\in\boldsymbol{Z}_p^m$, a fortiori ${}^tY''TY'\in p\boldsymbol{Z}_p$. The assumption that $Y$ should be reduced for $(R,\mathcal{M}_p)$ means $R[Y]\in\mathcal{O}_p^\times$, or equivalently ${}^tY''TY'\in\boldsymbol{Z}_p^\times$. Hence $\varphi(\mathrm{diag}(p^{-1},h,1))=0$ for any $h\in\mathrm{GL}_m(\boldsymbol{Q}_p)$. □

LEMMA 45. *For* $0 \leqslant i, j \leqslant m$, *put*

$$S_m^{(i,j)} = \{(h_1, h_2) \in (\tilde{c}_m^{(i,0)}/K_m) \times (\tilde{c}_m^{(0,j)}/K_m) \mid h_1^{-1}Y', {}^th_1{}^tTY'', (h_1h_2)^{-1}Y', {}^t(h_1h_2){}^tTY'' \in \mathbf{Z}_p^m\}.$$

*Then*

$$\tilde{\varphi}(i, j; 0) = \sum_{(h_1,h_2)\in S_m^{(i,j)}} \varphi(\mathrm{diag}(1, h_1h_2, 1)).$$

*In particular, we have* $\tilde{\varphi}(i, j; 0) = 0$ *if* $i = m$ *or* $j = m$.

PROOF. The first assertion is a consequence of Lemma 38 and the definition (6.10). Assume $i = m$. Then the condition $h_1 \in \tilde{c}_m^{(i,0)}$ yields $h_1 = pk_1$ with some $k_1 \in K_m$. Combining this with the condition $h_1^{-1}Y' \in \mathbf{Z}_p^m$, we obtain $Y' \in p\mathbf{Z}_p^m$, contradictory to $Y' \in \mathbf{Z}_p^m - p\mathbf{Z}_p^m$. Hence $S_m^{(i,j)} = \emptyset$ and $\tilde{\varphi}(i, j; 0) = 0$ if $i = m$.

Suppose $(h_1, h_2) \in S_m^{(i,m)}$. Then the condition $h_2 \in \tilde{c}_m^{(0,m)}$ yields $h_2 = p^{-1}k_2$ with some $k_2 \in K_m$; this, together with ${}^t(h_1h_2){}^tTY'' \in \mathbf{Z}_p^m$, implies ${}^th_1{}^tTY'' \in p\mathbf{Z}_p^m$. Since $h_1^{-1}Y' \in \mathbf{Z}_p^m$, we obtain ${}^tY''TY' \in p\mathbf{Z}_p$, contradictory to $R[Y] \in \mathcal{O}_p^\times$. Hence $S_m^{(i,j)} = \emptyset$ and $\tilde{\varphi}(i, j; 0) = 0$ if $j = m$. □

LEMMA 46.

$$F_\varphi(T_1, T_2)P_{m+2}^{(1)}(p^{-(m+1)/2}T_1; \mathbf{s})P_{m+2}^{(2)}(p^{-(m+1)/2}T_2; \mathbf{s}) = (1 - p^{-(m+1)}T_1T_2)\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}(-1)^{i+j}p^{-(i+j)m+i(i-1)/2+j(j-1)/2}\tilde{\varphi}(i, j; 0)T_1^iT_2^j.$$

PROOF. From Lemmas 42 and 43,

$$\begin{aligned}(6.13)\quad & F_\varphi(T_1,T_2)P_{m+2}^{(1)}(p^{-(m+1)/2}T_1; \mathbf{s})P_{m+2}^{(2)}(p^{-(m+1)/22}T_2; \mathbf{s}) \\ &= \sum_{i=0}^{m+1}(-1)^i p^{-i(m+1)+i(i-1)/2}T_1^i \\ &\quad \times \sum_{l_2\geqslant 0}\{p^i\varphi(i; 0, l_2) + \varphi(i-1; 0, l_2)T_2\}P_{m+2}^{(2)}(p^{-(m+1)/2}T_2; \mathbf{s})T_2^{l_2} \\ &= \sum_{i=0}^{m+1}(-1)^i p^{-i(m+1)+i(i-1)/2}T_1^i \sum_{l_2\geqslant 0}\sum_{j=0}^{m+2}(-1)^j p^{-j(m+1)+j(j-1)/2}T_2^{j+l_2}\end{aligned}$$

$$\times \big\{p^{i+j}\tilde{\varphi}(i,j;l_2) + p^i\tilde{\varphi}'(i,j-1;l_2) + p^{i+m+1}\tilde{\varphi}(i,j-1;l_2+1) + p^{i-j+m+2}\tilde{\varphi}'(i,j-2;l_2+1) + \big(p^j\varphi(i-1,j;l_2) + \tilde{\varphi}'(i-1,j-1;l_2) + p^{m+1}\tilde{\varphi}(i-1,j-1;l_2+1) + p^{m-j+2}\tilde{\varphi}'(i-1,j-2;l_2+1)\big)T_2\big\}$$

$$= \sum_{i=0}^{m+1}(-1)^i p^{-i(m+1)+i(i-1)/2}T_1^i \times (p^i\Phi(i;T_2) + \Phi(i-1;T_2)T_2 + p^i\Phi'(i;T_2) + \Phi'(i-1;T_2)T_2)\,,$$

where, for each $i$, we set

$$\Phi(i;T_2) = \sum_{l_2\geqslant 0}\sum_{j=0}^{m+1}(-1)^j p^{-j(m+1)+j(j-1)/2}T_2^{j+l_2} \times (p^j\tilde{\varphi}(i,j;l_2) + p^{m+1}\tilde{\varphi}(i,j-1;l_2+1))\,,$$

$$\Phi'(i;T_2) = \sum_{l_2\geqslant 0}\sum_{j=0}^{m+1}(-1)^j p^{-j(m+1)+j(j-1)/2}T_2^{j+l_2} \times (\tilde{\varphi}'(i,j-1;l_2) + p^{m-j+2}\tilde{\varphi}'(i,j-2;l_2+1))\,.$$

By making a change of variables $j + l_2 = k$ in the summation with respect to $l_2$, we easily obtain

$$\Phi(i;T_2) = \sum_{j=0}^{m+1}(-1)^j p^{-j(m+1)+j(j-1)/2}p^j\tilde{\varphi}(i,j;0)T_2^j\,,$$

$$\Phi'(i;T_2) = \sum_{j=0}^{m+1}(-1)^j p^{-j(m+1)+j(j-1)/2}\tilde{\varphi}'(i,j-1;0)T_2^j\,.$$

By these expressions of $\Phi(i;T_2)$ and $\Phi'(i;T_2)$, from the last formula of (6.13), we have

$$F_\varphi(T_1,T_2)P^{(1)}_{m+2}(p^{-(m+1)/2}T_1;\mathbf{s})P^{(2)}_{m+2}(p^{-(m+1)2}T_2;\mathbf{s})$$
$$= \sum_{i=0}^{m+1}(-1)^i p^{-i(m+1)+i(i-1)/2}T_1^i\sum_{j=0}^{m+1}(-1)^j p^{-j(m+1)+j(j-1)/2}T_2^j \times \big\{p^{i+j}\tilde{\varphi}(i,j;0) + p^i\tilde{\varphi}'(i,j-1;0) + p^j\tilde{\varphi}(i-1,j;0)T_2 + \tilde{\varphi}'(i-1,j-1;0)T_2\big\}$$
$$= (1 - p^{-(m+1)}T_1T_2)\sum_{i=0}^{m-1}(-1)^i p^{-im+i(i-1)/2}T_1^i \times \sum_{j=0}^{m-1}(-1)^j p^{-jm+j(j-1)/2}T_2^j\tilde{\varphi}(i,j;0)$$

using Lemmas 44 and 45 to prove the last equality. □

6.3.1. Since $Y = (Y', Y'')$ is primitive in $\mathcal{M}_p^* (= \mathcal{M}_p)$, $Y'$ and $Y''$ belong to $\boldsymbol{Z}_p^m - p\boldsymbol{Z}_p^m$. Since $Y$ is reduced for $(R, \mathcal{M}_p)$, we have ${}^tY''{}^tTY'' \in \boldsymbol{Z}_p^\times$. Hence we may assume

$$Y' = \begin{bmatrix} 1 \\ 0_{m-1} \end{bmatrix}, \quad {}^tTY'' = \begin{bmatrix} u_1 \\ \mathsf{u}_2 \end{bmatrix} \quad (u_1 \in \boldsymbol{Z}_p^\times,\ \mathsf{u}_2 \in \boldsymbol{Z}_p^{m-1}).$$

By the identification $G_{0,p} = \mathrm{GL}_m(\boldsymbol{Q}_p)$, the subgroup $G_{0,p}^Y = \{(h_1, h_2) \in G_{0,p} \mid h_1 Y' = Y', h_2 Y'' = Y''\}$ (resp. $K_{0,p}^Y$) is identified with

$$\begin{gathered} {}^0\mathrm{GL}_{m-1}(\boldsymbol{Q}_p) = \left\{ \left[ \begin{smallmatrix} 1 & u_1^{-1}{}^t\mathsf{u}_2(1_{m-1}-h) \\ 0_{m-1,1} & h \end{smallmatrix} \right] \middle| h \in \mathrm{GL}_{m-1}(\boldsymbol{Q}_p) \right\}, \\ (\text{resp. } {}^0K_{m-1} = {}^0\mathrm{GL}_{m-1}(\boldsymbol{Q}_p) \cap \mathrm{GL}_m(\boldsymbol{Z}_p)). \end{gathered}$$

For $0 \leqslant i, j \leqslant m-1$, let ${}^0c_{m-1}^{(i,j)}$ and ${}^0\tilde{c}_{m-1}^{(i,j)}$ be the image of $c_{m-1}^{(i,j)}$ and $\tilde{c}_{m-1}^{(i,j)}$ by the obvious isomorphism ${}^0\mathrm{GL}_{m-1}(\boldsymbol{Q}_p) \cong \mathrm{GL}_{m-1}(\boldsymbol{Q}_p)$.

LEMMA 47. *Let $0 \leqslant i, j \leqslant m-1$. The natural inclusion from ${}^0\mathrm{GL}_{m-1}(\boldsymbol{Q}_p)$ into $\mathrm{GL}_m(\boldsymbol{Q}_p)$ induces bijections*

$$\begin{aligned} {}^0\tilde{c}_{m-1}^{(i,0)}/{}^0K_{m-1} &\cong \{h_1 \in \tilde{c}_m^{(i,0)}/K_m \mid h_1^{-1}Y', {}^th_1{}^tTY'' \in \boldsymbol{Z}_p^m\}, \\ {}^0\tilde{c}_{m-1}^{(0,j)}/{}^0K_{m-1} &\cong \{h_1 \in \tilde{c}_m^{(0,j)}/K_m \mid h_1^{-1}Y', {}^th_1{}^tTY'' \in \boldsymbol{Z}_p^m\}. \end{aligned}$$

PROOF. By the Iwasawa decomposition of $\mathrm{GL}_m(\boldsymbol{Q}_p)$, we may assume that a coset $h_1 \in \tilde{c}_m^{(i,0)}/K_m$ is represented by a matrix of the form

$$\begin{bmatrix} a & X \\ 0 & h \end{bmatrix}, \quad (a \in \boldsymbol{Q}_p^\times, X \in M_{1,m-1}(\boldsymbol{Q}_p), h \in \mathrm{GL}_{m-1}(\boldsymbol{Q}_p)).$$

From the condition $h_1^{-1}Y' \in \boldsymbol{Z}_p^m$ we have $a^{-1} \in \boldsymbol{Z}_p$. Another condition ${}^th_1{}^tTY'' \in \boldsymbol{Z}_p^m$ is equivalent to $au_1 \in \boldsymbol{Z}_p$, ${}^tXu_1 + {}^th\mathsf{u}_2 \in \boldsymbol{Z}_p^{m-1}$. Since $u_1 \in \boldsymbol{Z}_p^\times$, we have $a \in \boldsymbol{Z}_p$. Thus $a \in \boldsymbol{Z}_p^\times$. This means we may assume $a = 1$. Then the formula

$$h_1 \begin{bmatrix} 1 & -u_1^{-1}({}^t\mathsf{c} - {}^t\mathsf{u}_2) \\ 0 & 1_{m-1} \end{bmatrix} = \begin{bmatrix} 1 & u_1^{-1}{}^t\mathsf{u}_2(1_{m-1} - h) \\ 0 & h \end{bmatrix}$$

with $\mathsf{c} = {}^tXu_1 + {}^th\mathsf{u}_2 \in \boldsymbol{Z}_p^{m-1}$ shows that $h_1$ lies in the image of the map ${}^0\mathrm{GL}_{m-1}(\boldsymbol{Q}_p) \to \mathrm{GL}_m(\boldsymbol{Q}_p)$ modulo $K_m$. □

PROPOSITION 48. *Let $\mathbf{s}_0 \in (\boldsymbol{C}^\times)^{m-1}/S_{m-1}$ be the Satake parameter of $\Lambda_0$. Then*

$$\begin{aligned} &F_\varphi(T_1, T_2) P_{m+2}^{(1)}(p^{-(m+1)/2}T_1; \mathbf{s}) P_{m+2}^{(2)}(p^{-(m+1)/2}T_2; \mathbf{s}) \\ &\quad = (1 - p^{-(m+1)}T_1T_2) P_{m-1}^{(1)}(p^{-(m+2)/2}T_1; \mathbf{s}_0) P_{m-1}^{(2)}(p^{-(m+2)/2}T_2; \mathbf{s}_0). \end{aligned}$$

PROOF. By Lemmas 45 and 47, we have

$$\tilde{\varphi}(i,j;0)=\sum_{\substack{h_1\in {}^0\tilde{c}^{(i,0)}_{m-1}/{}^0K_{m-1}\\ h_2\in {}^0\tilde{c}^{(0,j)}_{m-1}/{}^0K_{m-1}}}\varphi(\mathrm{diag}(1,h_1h_2,1))=\Lambda_0({}^0\tilde{c}^{(i,0)}_{m-1})\Lambda_0({}^0\tilde{c}^{(0,j)}_{m-1})\varphi(1)\,.$$

By Lemma 46, (6.9) and (6.12), we have the conclusion. □

**7. Archimedean Whittaker functions.** We retain the notations in Section 5.

Let $\mathcal{W}^Y_\infty$ be the space of right $K_\infty$-finite $C^\infty$-functions $\varphi : G_\infty \to \boldsymbol{C}$ which satisfies the two conditions:

(a) $\varphi(n\mathsf{m}(1;k_0)g)=\psi_{Y,\infty}(n)\varphi(g)$ for any $n\in N_\infty$ and any $k_0\in G^Y_{0,\infty}$. (cf. (5.7).) Here $\psi_{Y,\infty}: N_\infty\to \boldsymbol{C}^{(1)}$ is the archimedean component of the character $\psi_Y$ defined by (5.5).

(b) $\varphi$ is uniformly of moderate growth, i.e., there exists a constant $r\in\boldsymbol{R}$ such that for each $D\in U(\mathfrak{g})$ the estimation

$$|R_D\varphi(g_\infty)|\leqslant C|\mathrm{Tr}({}^t\bar{g}_\infty g_\infty)|^r\,,\quad g_\infty\in G_\infty \tag{7.1}$$

holds with a constant $C>0$. Here $\mathfrak{g}$ is the Lie algebra of $G_\infty$, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $R_D$ the right-action by $D$.
By the right translation, $\mathcal{W}^Y_\infty$ becomes a $(\mathfrak{g},K_\infty)$-module. For an irreducible $(\mathfrak{g},K_\infty)$-module $(\pi,H_\pi)$, the $\pi$-isotypic part of $\mathcal{W}^Y_\infty$, which we denote by $\mathcal{W}^Y_\infty(\pi)$, is defined to be the image of the natural map $H_\pi\otimes\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(H_\pi,\mathcal{W}^Y_\infty)\to\mathcal{W}^Y_\infty$.

We study the functions $\varphi\in\mathcal{W}^Y_\infty(\pi)$ for two special cases:

- (Case 1). $\pi$ is a class one principal series representation.
- (Case 2). $\pi$ is a unitarizable non-trivial representation such that $\mathrm{H}^{1,1}(\mathfrak{g},K_\infty;\pi)\neq 0$.

In practice, we take an irreducible unitary representation $(\tau,W)$ of $K_\infty$ and consider the space $\mathcal{W}^Y_\tau(\pi)=(\mathcal{W}^Y_\infty(\pi)\otimes W)^{K_\infty}$ consisting of $W$-valued functions.

Let $\Omega$ be the Casimir element of $\mathrm{U}(m+1,1)$ corresponding to the $\mathrm{U}(m+1,1)$-invariant $\boldsymbol{R}$-bilinear form $(X_1,X_2)\mapsto 2^{-1}\mathrm{tr}(X_1X_2)$ on $\mathfrak{u}(m+1,1)$.

7.1. Case 1. For $\nu\in\boldsymbol{C}$, let $\pi(\nu)$ be the representation $\pi(\nu)$ of $G_\infty\cong\mathrm{U}(m+1,1)$ induced from the one dimensional representation $(P_\infty\ni)\ \mathsf{m}(t;g_0)n\mapsto|\mathrm{N}(t)|^{(\nu+m+1)/2}$ of $P_\infty$. Take $\tau_0$ to be the one dimensional trivial representation of $K_\infty$, and consider a function $\varphi\in\mathcal{W}^Y_{\tau_0}(\pi(\nu))$. Since the Casimir operator $\Omega$ acts on $\pi(\nu)$ by the scalar $\nu^2-(m+1)^2$ (see [19, Proposition 6.2.2 (1)]), the function $\phi(t)=\varphi(\mathsf{m}(t;1_m))$ $(t>0)$ satisfies

$$\partial^2\phi-2(m+1)\partial\phi-16\pi^2|R[Y]/\sqrt{D}|t^2\phi=\{\nu^2-(m+1)^2\}\phi\,,$$

with $\partial=t(\partial/\partial t)$ the Euler operator. By examining the differential equation, it is easy to see that there exists, up to a constant multiple, a unique function $\varphi_0^{\pi(\nu)}\in\mathcal{W}^Y_{\tau_0}(\pi(\nu))$ such that

$$\varphi_0^{\pi(\nu)}(\mathsf{m}(t;1_m))=t^{m+1}K_\nu\left(4\pi t\left|\frac{R[Y]}{\sqrt{D}}\right|^{1/2}\right),\quad t>0\,. \tag{7.2}$$

Here $K_\nu(z)$ is the modified Bessel function.

7.2. Case 2.

7.2.1. Invariant tensors. Let $\sigma_0$ be the base point of $\mathfrak{D}$ defined in the paragraph 5.1.1. Set

$$\mathsf{v}_0^- = |\tilde{R}[\sigma_0]|^{-1/2}\sigma_0 = |D|^{-1/4}\begin{bmatrix}(1+\sqrt{D})/2\\ 0_m\\ 1\end{bmatrix}, \quad \mathsf{v}_{\tilde{Y}}^+ = |\tilde{R}[\tilde{Y}]|^{-1/2}\tilde{Y} = |\varDelta|^{-1/2}\begin{bmatrix}0\\ Y\\ 0\end{bmatrix}.$$

The orthogonal complement $\sigma_0^\perp$ of $\sigma_0$ in $\tilde{V}_\infty = \boldsymbol{C}^{m+2}$ is a positive definite $K_\infty$-irreducible subspace with the induced inner product $\langle \mathsf{v}, \mathsf{v}'\rangle = i\tilde{R}(\mathsf{v}, \mathsf{v}')$. For $\mathsf{f} \in \mathrm{End}_{\boldsymbol{C}}(\sigma_0^\perp)$, let $\mathsf{f}^* \in \mathrm{End}_{\boldsymbol{C}}(\sigma_0^\perp)$ be its adjoint, i.e., $\langle \mathsf{f}(\mathsf{v}), \mathsf{v}'\rangle = \langle \mathsf{v}, \mathsf{f}^*(\mathsf{v}')\rangle$ for $\mathsf{v}, \mathsf{v}' \in \sigma_0^\perp$. Then $\langle \mathsf{f}_1|\mathsf{f}_2\rangle = \mathrm{tr}_{\sigma_0^\perp}(\mathsf{f}_1\mathsf{f}_2^*)$ yields a $K_\infty$-invariant Hermitian inner product on the $\boldsymbol{C}$-vector space $\mathrm{End}_{\boldsymbol{C}}(\sigma_0^\perp)$. Set

$$\mathsf{E} = \mathrm{End}_{\boldsymbol{C}}(\sigma_0^\perp), \quad \mathsf{E}^\circ = \{\mathsf{f} \in \mathsf{E} \,|\, \langle \mathsf{f}|1_{\sigma_0^\perp}\rangle = 0\}.$$

Then $\mathsf{E} = \mathsf{E}^\circ \oplus \langle 1_{\sigma_0^\perp}\rangle_{\boldsymbol{C}}$ is a $K_\infty$-irreducible decomposition. We denote the action of $K_\infty$ on $\mathsf{E}$ by $\tau_{1,1}$, i.e., $\tau_{1,1}(k)\mathsf{f} = k\mathsf{f}k^{-1}$ for $k \in K_\sigma$ and $\mathsf{f} \in \mathsf{E}$. The subrepresentation on $\mathsf{E}^\circ$ is denoted by $\tau_{1,1}^\circ$.

The $K_\infty^{\tilde{Y}}$-module $\sigma_0^\perp$ has two irreducible components; the one dimensional space $\langle \tilde{Y}\rangle_{\boldsymbol{C}}$ and its orthogonal complement $\tilde{Y}^\perp \cap \sigma_0^\perp$. For two vectors $\mathsf{v}_1, \mathsf{v}_2 \in \sigma_0^\perp$, let us define $\mathsf{X}(\mathsf{v}_1|\mathsf{v}_2) \in \mathsf{E}$ by

$$\mathsf{X}(\mathsf{v}_1|\mathsf{v}_2)(\mathsf{v}) = \langle \mathsf{v}, \mathsf{v}_2\rangle \mathsf{v}_1 \quad (\mathsf{v} \in \sigma_0^\perp).$$

The formula $\mathsf{X}(\mathsf{v}_1|\mathsf{v}_2)^* = \mathsf{X}(\mathsf{v}_2|\mathsf{v}_1)$ is easily proved. For any $\mathsf{f} \in \mathsf{E}$ let $\mathsf{f}^\circ$ be its orthogonal projection to $\mathsf{E}^\circ$, or explicitly $\mathsf{f}^\circ = \mathsf{f} - (1/(m+1))\langle \mathsf{f}|1_{\sigma_0^\perp}\rangle 1_{\sigma_0^\perp}$.

LEMMA 49. *The $K_\infty^{\tilde{Y}}$-fixed part of $\mathsf{E}$ is two dimensional space generated by $\mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+)$ and $1_{\sigma_0^\perp}$, and the vector $\mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+)^\circ$ spans the $K_\infty^{\tilde{Y}}$-fixed part of $\mathsf{E}^\circ$:*

$$\mathsf{E}^{K_\infty^{\tilde{Y}}} = \langle \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+), 1_{\sigma_0^\perp}\rangle_{\boldsymbol{C}}, \quad (\mathsf{E}^\circ)^{K_\infty^{\tilde{Y}}} = \langle \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+)^\circ\rangle_{\boldsymbol{C}}.$$

PROOF. First note $K_\infty \cong \mathrm{U}(m+1) \times \mathrm{U}(1)$ and $K_\infty^{\tilde{Y}} \cong \mathrm{diag}(\mathrm{U}(m), 1) \times \mathrm{U}(1)$. Since any irreducible representation of $\mathrm{U}(m+1)$ contains the trivial representation of $\mathrm{U}(m)$ at most once, we have $\dim((\mathsf{E}^\circ)^{K_\infty^{\tilde{Y}}}) \leqslant 1$ and $\dim(\mathsf{E}^{K_\infty^{\tilde{Y}}}) \leqslant 2$. It is obvious that $\mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+)$ and $1_{\sigma_0^\perp}$ are $K_\infty^{\tilde{Y}}$-fixed and are linearly independent. □

The group $G_{0,\infty}$ coincides with the stabilizer in $P_\infty$ of the vector $\sigma_0$. The group $P_\infty$ acts on the unitary character group of $N_\infty$ naturally. The compact group $G_{0,\infty}^Y$ coincided with the group of elements of $G_{0,\infty}$ which fix the character $\psi_{\infty,Y}$. Consider the unit vector

$$\mathsf{v}_0^+ = |D|^{-1/4}\begin{bmatrix}(1-\sqrt{D})/2\\ 0_m\\ 1\end{bmatrix}.$$

Then $(\sigma_0^\perp)^{G_{0,\infty}} = \langle \mathsf{v}_0^+\rangle_{\boldsymbol{C}}$ and $(\sigma_0^\perp)^{G^Y_{0,\infty}} = \langle \mathsf{v}_0^+, \mathsf{v}_{\tilde{Y}}^+\rangle_{\boldsymbol{C}}$. Set

$$\text{(7.3)}\qquad \mathsf{y}^{00} = \left(\frac{m+1}{m}\right)^{1/2} \mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_0^+)^\circ, \quad \mathsf{y}^{01} = -\mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_{\tilde{Y}}^+)^\circ, \quad \mathsf{y}^{10} = \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_0^+)^\circ,$$

$$\text{(7.4)}\qquad \mathsf{y}^{11} = -\left(\frac{1}{m(m-1)}\right)^{1/2} (m\mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+)^\circ + \mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_0^+)^\circ).$$

LEMMA 50. *The 4 vectors $\mathsf{y}^{ij}$ $(i, j = 0, 1)$ form an orthonormal basis of the space of $G^Y_{0,\infty}$-fixed part of $\mathsf{E}^\circ$. Set $X_m = \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_0^+)$. Then the operators $\tau_{1,1}(X_m)$ and $\tau_{1,1}(X_m^*)$ keep the space $(\mathsf{E}^\circ)^{G^Y_{0,\infty}} = \langle \mathsf{y}^{ij} \mid i, j = 0, 1\rangle_{\boldsymbol{C}}$ invariant; their action is explicitly given by*

$$\text{(7.5)}\qquad \begin{aligned} \tau_{1,1}(X_m)\begin{bmatrix}\mathsf{y}^{00}\\ \mathsf{y}^{01}\\ \mathsf{y}^{10}\\ \mathsf{y}^{11}\end{bmatrix} &= \begin{bmatrix}0 & 0 & A_0 & 0\\ A_0 & 0 & 0 & A_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & A_1 & 0\end{bmatrix}\begin{bmatrix}\mathsf{y}^{00}\\ \mathsf{y}^{01}\\ \mathsf{y}^{10}\\ \mathsf{y}^{11}\end{bmatrix},\\ \tau_{1,1}(X_m^*)\begin{bmatrix}\mathsf{y}^{00}\\ \mathsf{y}^{01}\\ \mathsf{y}^{10}\\ \mathsf{y}^{11}\end{bmatrix} &= \begin{bmatrix}0 & A_0 & 0 & 0\\ 0 & 0 & 0 & 0\\ A_0 & 0 & 0 & A_1\\ 0 & A_1 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathsf{y}^{00}\\ \mathsf{y}^{01}\\ \mathsf{y}^{10}\\ \mathsf{y}^{11}\end{bmatrix}\end{aligned}$$

*where $A_0 = ((m+1)/m)^{1/2}$ and $A_1 = ((m-1)/m)^{1/2}$.*

PROOF. For simplicity set $W = \langle \mathsf{v}_0^-, \mathsf{v}_0^+\rangle_{\boldsymbol{C}}^\perp$. Since $\mathsf{v}_0^+$ is $G_{0,\infty}$-fixed, the $G_{0,\infty}$-irreducible decomposition $\sigma_0^\perp = W \oplus \langle \mathsf{v}_0^+\rangle_{\boldsymbol{C}}$ yields the decomposition

$$\mathsf{E} \cong \mathrm{End}(W) \oplus W \oplus W^* \oplus \langle \mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_0^+)\rangle_{\boldsymbol{C}}$$

of $G_{0,\infty}$-modules. Noting that $W = \langle \mathsf{v}_0^-, \mathsf{v}_0^+, \mathsf{v}_{\tilde{Y}}^+\rangle_{\boldsymbol{C}}^\perp \oplus \langle \mathsf{v}_{\tilde{Y}}^+\rangle_{\boldsymbol{C}}$ is an irreducible decomposition of $G^Y_{0,\infty}$-module, the subspaces $\boldsymbol{C}\mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_0^+)$, $\boldsymbol{C}\mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_{\tilde{Y}}^+)$ and $\langle \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+), \mathrm{pr}_W\rangle_{\boldsymbol{C}}$ of $\mathsf{E}$ correspond to $W^{G^Y_{0,\infty}}$, $(W^*)^{G^Y_{0,\infty}}$ and $\mathrm{End}(W)^{G^Y_{0,\infty}}$ on the right-hand side, respectively. Here $\mathrm{pr}_W \in \mathsf{E}$ is the orthogonal projector to $W$. Thus

$$\mathsf{E}^{G^Y_{0,\infty}} = \langle \mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_0^+), \mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_{\tilde{Y}}^+), \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_0^+), \mathsf{X}(\mathsf{v}_{\tilde{Y}}^+|\mathsf{v}_{\tilde{Y}}^+), \mathrm{pr}_W\rangle_{\boldsymbol{C}}.$$

Taking projection to $\mathsf{E}^\circ$, we obtain $(\mathsf{E}^\circ)^{G^Y_{0,\infty}} = \langle \mathsf{y}^{ij} \mid i, j = 0, 1\rangle_{\boldsymbol{C}}$ because $\mathsf{X}(\mathsf{v}_0^+|\mathsf{v}_0^+)^\circ = -\mathrm{pr}_W^\circ$. By direct computation, we can check that $\{\mathsf{y}^{ij}\}$ is an orthonormal system in $\mathsf{E}^\circ$. The table (7.5) can also be checked by a direct computation. Note the action of $\mathrm{Lie}(K_\infty)_{\boldsymbol{C}} \cong \mathsf{E}$ on $\mathsf{E}$ is given by the bracket: $\tau_{1,1}(X)Z = [X, Z] = XZ - ZX$. □

7.2.2. Certain cohomological representations. Choose an orthonormal basis $\{\mathsf{v}_j\}_{j=1}^m$ of $\sigma_0^\perp$ such that $\mathsf{v}_m = \mathsf{v}_{\tilde{Y}}^+$ and set $\mathsf{v}_{m+1} = \mathsf{v}_0^-$. Then we have an isomorphism $\mathrm{c} : G_\infty \to \mathrm{U}(m+1, 1)$ such that $d\mathrm{c}_{\boldsymbol{C}}(\mathsf{X}(\mathsf{v}_j|\mathsf{v}_i)) = E_{ij}$ $(1 \leqslant i, j \leqslant m+1)$, where $d\mathrm{c}_{\boldsymbol{C}} : \mathfrak{g}_{\boldsymbol{C}} \to \mathfrak{gl}_{m+1}(\boldsymbol{C})$ is the complexification of the tangent map $d\mathrm{c}$ and $E_{ij}$ are the matrix units of $\mathfrak{gl}_{m+1}(\boldsymbol{C})$. Let $T$ be the compact Cartan subgroup of $\mathrm{U}(m+1, 1)$ formed by all the diagonal matrices in $\mathrm{U}(m+1, 1)$. Let $\{\varepsilon_j\}_{1\leqslant j\leqslant m+1}$ be the basis of $\mathfrak{t}_{\boldsymbol{C}}^*$ dual to the basis $E_{jj}$ $(1 \leqslant j \leqslant m+1)$ of $\mathfrak{t}_{\boldsymbol{C}}$. Here $\mathfrak{t}_{\boldsymbol{C}}$

is the complexified Lie algebra of $T$. For a $\mathfrak{t}_C$-root $\beta$, let $\mathfrak{g}_C(\beta)$ denote the $\beta$-root space in $\mathfrak{g}_C$. Let $\mathfrak{q}$ be the sum of those $\mathfrak{t}_C$-root spaces $\mathfrak{g}_C(\beta)$ such that $\beta(E_{11} - E_{mm}) \geqslant 0$. Then $\mathfrak{q}$ is a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$ in the sense of [22]. Here $\theta$ is the Cartan involution of $\mathfrak{g}$ corresponding to $K_\infty$.

The construction in [22] yields an irreducible unitarizable $(\mathfrak{g}, K_\infty)$-module $A_\mathfrak{q}$ such that $\mathrm{H}^{1,1}(\mathfrak{g}, K_\infty; A_\mathfrak{q}) \neq 0$, which we denote by $\pi_{11}$. By [22, Proposition 6.1], the representation $\pi_{11}$ is characterized by the two properties: (1) $\pi_{11}$ contains the $K_\infty$-type $\tau^\circ_{1,1}$ and (2) the Casimir element $\Omega$ acts on $\pi_{11}$ by 0.

7.2.3. An explicit formula of Whittaker functions.

PROPOSITION 51. *Let $\varphi \in \mathcal{W}^Y_{\tau^\circ_{1,1}}(\pi_{11})$. There exists a constant $C_\varphi$ such that $\varphi = C_\varphi \varphi_0^{\pi_{11}}$, where $\varphi_0^{\pi_{11}} \in \mathcal{W}^Y_{\tau^\circ_{1,1}}(\pi_{11})$ is given by*

$$\varphi_0^{\pi_{11}}(\mathsf{m}(t; 1_m)) = \left(4\pi \left|\frac{R[Y]}{\sqrt{D}}\right|^{1/2}\right)^{-(m+1)} \sum_{i,j=0,1} \phi_{ij}\left(4\pi t \left|\frac{R[Y]}{\sqrt{D}}\right|^{1/2}\right)\mathsf{y}^{ij}, \quad t > 0 \tag{7.6}$$

*with*

$$\phi_{00}(t) = \left(\frac{m}{m+1}\right)^{1/2} t^{m+3} K_{m-1}(t), \tag{7.7}$$

$$\phi_{01}(t) = \phi_{10}(t) = \left(\frac{m}{m+1}\right)^{1/2}\left(\frac{d}{dt} - \frac{2(m+1)}{t}\right)\phi_{00}(t),$$

$$\phi_{11}(t) = \left(\frac{m-1}{m+1}\right)^{1/2}\phi_{00}(t) - \frac{2m^{1/2}(m-1)^{1/2}}{t}\phi_{10}(t). \tag{7.8}$$

PROOF. Note the highest $\mathfrak{t}_C$-weight of $\tau^\circ_{1,1}$ is $\varepsilon_1 - \varepsilon_m$. It is known that the highest $\mathfrak{t}_C$-weight of a $K_\infty$-type of $\pi_{11}$ is contained in the cone $\{(a+1)\varepsilon_1 - (b+1)\varepsilon_m + (b-a)\varepsilon_{m+1} \mid a, b \in N\}$. In particular, the $\mathfrak{t}_C$-weights $-\varepsilon_m + \varepsilon_{m+1}$ and $\varepsilon_1 - \varepsilon_{m+1}$ are not the highest weights of $K_\infty$-types of $\pi_{11}$. Hence, $\nabla^{-1}\varphi = 0$, $\nabla^{+(m+1)}\varphi = 0$ holds, where $\nabla^i$ is the Schmid operator ([19], [16]). Since the function $t \mapsto \varphi(\mathsf{m}(t; 1_m))$ takes its values in $(\mathsf{E}^\circ)^{G^Y_{0,\infty}}$, it can be written as $\sum_{i,j=0,1}\phi_{ij}(t)\mathsf{y}^{ij}$ with some functions $\phi_{ij}(t)$. By the same way as [16], using Lemma 50, one can deduce the equations among $\phi_{ij}$'s.

Here is the result. Let $\partial = t(d/dt)$, the Euler operator.

- *The equation $\Omega w = 0$:*

$$\partial^2\phi - 2(m+1)\partial\phi + \mathsf{A}(t)\phi = 0, \quad \phi = \begin{bmatrix}\phi_{00}\\ \phi_{10}\\ \phi_{01}\\ \phi_{11}\end{bmatrix} \tag{7.9}$$

with

$$\mathsf{A}(t) = -N^2t^2 1_4 - 2Nt \begin{bmatrix} 0 & A_0 & A_0 & 0 \\ A_0 & 0 & 0 & A_1 \\ A_0 & 0 & 0 & A_1 \\ 0 & A_1 & A_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2m+1 & 0 & 0 \\ 0 & 0 & 2m+1 & 0 \\ 0 & 0 & 0 & 4m \end{bmatrix}$$

and $A_0 = ((m+1)/m)^{1/2}$, $A_1 = ((m-1)/m)^{1/2}$, $N = 4\pi|R[Y]/\sqrt{D}\,|^{1/2}$.

- *The equation* $\nabla^{-1}w = 0$:

$$\partial\phi_{00} - 2(m+1)\phi_{00} - NtA_0\phi_{10} = 0\,, \tag{7.10}$$

$$\partial\phi_{01} - (2m+1)\phi_{01} - Nt\frac{A_0}{m+1}\phi_{00} - NtA_1\phi_{11} = 0\,. \tag{7.11}$$

- *The equation* $\nabla^{+(m+1)}w = 0$:

$$\partial\phi_{00} - 2(m+1)\phi_{00} - NtA_0\phi_{01} = 0\,, \tag{7.12}$$

$$\partial\phi_{10} - (2m+1)\phi_{10} - Nt\frac{A_0}{m+1}\phi_{00} - NtA_1\phi_{11} = 0\,. \tag{7.13}$$

From (7.9), (7.10) and (7.12), we obtain

$$\partial^2\phi_{00}(t) - 2(m+3)\partial\phi_{00}(t) + (-N^2t^2 + 8(m+1))\phi_{00}(t) = 0\,,$$

which, by putting $\phi_{00}(t) = t^{m+5/2}u(t)$, is transformed to the classical Whittaker's differential equation

$$\frac{d^2u}{dz^2} + \left(\frac{-1}{4} + \frac{1/4 - (m-1)^2}{z^2}\right)u = 0$$

with respect to the new variable $z = 2Nt$. Hence $u(t)$ has to be proportional to $W_{0,m-1}(2Nt)$ since $\varphi(\mathsf{m}(t;\, 1_m))$ should be of polynomial growth as $t \to +\infty$. □

**8. Computation of Archimedean local-zeta integrals.** We retain the notations in Sections 5 and 7.

The aim in this section is to evaluate the *local-zeta integral*

$$\zeta_\infty(\varphi; s) = \int_{\boldsymbol{C}^\times} \langle v_0|\varphi(\mathsf{m}(t;\, 1_m))\rangle |t|_{\boldsymbol{C}}^{s-(m+1)/2} \mathrm{d}^\times t\,, \quad \varphi \in \mathcal{W}_\tau^Y(\pi)\,. \tag{8.1}$$

Here $(\tau, W)$ is an irreducible unitary representation of $K_\infty$ with a $K_\infty^{\tilde{Y}}$-fixed unit vector $v_0 \in W$ and $\langle\,|\,\rangle$ is the inner-product of $W$. (Note $|t|_{\boldsymbol{C}} = t\bar{t}$ for $t \in \boldsymbol{C}$.)

LEMMA 52. *We have*

$$\zeta_\infty(\varphi; s) = \int_0^\infty \langle v_0|\varphi(\mathsf{m}(t;\, 1_m))\rangle t^{2s-m-2} \mathrm{d}t\,. \tag{8.2}$$

PROOF. Write the integral (8.1) by the polar coordinates on $\boldsymbol{C}^\times$. Then use the $K_\infty^{\tilde{Y}}$-invariance of the vector $v_0$ to compute the integral on the unit circle. □

We compute the zeta-integral (8.1) more concretely for (Case 1) and (Case 2) discussed in 7.1 and 7.2.

Let $\varepsilon \in \{0, 1\}$ be the parity of $m$. Set $\Gamma_{\boldsymbol{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\boldsymbol{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ with $\Gamma(s)$ the gamma function.

8.1. Case 1. We consider the case when $\pi$ is the spherical principal series representation $\pi(\nu)$ and $(\tau_0, W_0)$ the trivial representation with $v_0 = 1 \in W_0 = \boldsymbol{C}$.

PROPOSITION 53. *Let $\varphi_0^{\pi(\nu)} \in \mathcal{W}_{\tau_0}^{Y}(\pi(\nu))$ be the function whose restriction to the split torus $\mathsf{m}(t; 1_m)$ $(t > 0)$ is given by* (7.2). *Then $\zeta_\infty(\varphi_0^{\pi(\nu)}; s)$ is convergent on* $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|/2$, *and*

$$\zeta_\infty(\varphi_0^{\pi(\nu)}; s) = 2^{-(\varepsilon+9)/2}|D|^{(m+\varepsilon-2)/4}|\mathrm{N}(\mathfrak{d}_R(\mathcal{M}))|^{1/4}|R[Y]|^{1/2} \times (2|D|^{-1/2})^s \frac{L_\infty(s, \pi(\nu))}{L_\infty(s+1/2, \mathcal{M}\cap Y^\perp)} \frac{1}{\zeta_{m,\infty}(2s)} \tag{8.3}$$

*with*

$$L_\infty(s, \pi(\nu)) = |\mathrm{N}(\mathfrak{d}_R(\mathcal{M}))|^{s/2}|D|^{[(m+2)/2]s}\Gamma_{\boldsymbol{C}}(s+\nu/2)\,\Gamma_{\boldsymbol{C}}(s-\nu/2) \times \prod_{j=1}^{[m/2]} \Gamma_{\boldsymbol{C}}(s+(m+1)/2-j)^2\Gamma_{\boldsymbol{C}}(s)^{\varepsilon}\,, \tag{8.4}$$

$$L_\infty(s, \mathcal{M}\cap Y^\perp) = |\mathrm{N}(\mathfrak{d}_{R|Y^\perp}(\mathcal{M}\cap Y^\perp))|^{s/2}|D|^{[(m-1)/2]s} \times \prod_{j=1}^{[(m-1)/2]} \Gamma_{\boldsymbol{C}}(s+m/2-j)^2\,\Gamma_{\boldsymbol{C}}(s)^{1-\varepsilon}. \tag{8.5}$$

*We also set*

$$\zeta_{m,\infty}(s) = |D|^{(1-\varepsilon)s/2}\Gamma_{\boldsymbol{R}}(s-\varepsilon+1)\,. \tag{8.6}$$

PROOF. Set $N = 4\pi t|R[Y]/\sqrt{D}|^{1/2}$. By the formula (7.2) and the definition (8.2),

$$\begin{aligned}\zeta_\infty(\varphi_0^{\pi(\nu)}; s) &= \int_0^\infty t^{m+1}K_\nu(Nt)t^{2s-m-2}\mathrm{d}t\\ &= N^{-2s}\int_0^\infty K_\nu(t)t^{2s-1}\mathrm{d}t\\ &= 2^{2s-2}N^{-2s}\Gamma(s+\nu/2)\Gamma(s-\nu/2)\end{aligned}$$

for $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|/2$. Here we use [2, 6.561, 16 (p. 668)] to prove the third equality. The remaining part of the proof is a direct computation. We use the relation $\mathrm{N}(\mathfrak{d}_R(\mathcal{M})) = \mathrm{N}(\mathfrak{d}_{R|Y^\perp}(\mathcal{M}\cap Y^\perp))|R[Y]|^{-2}$, which is a consequence of Lemma 25. □

8.2. Case 2. Let $\pi_{11}$ and $(\tau, W) = (\tau_{1,1}^{\circ}, \mathsf{E}^{\circ})$ be as in the paragraph 7.2.2. Then $v_0 = \mathsf{X}(\mathsf{v}_{\tilde{Y}}^{+}|\mathsf{v}_{\tilde{Y}}^{+})^{\circ}$ is a $K_\infty^{\tilde{Y}}$-fixed unit vector of $\mathsf{E}^{\circ}$.

PROPOSITION 54. *Let $\varphi_0^{\pi_{11}} \in \mathcal{W}^{Y}_{\tau^\circ_{1,1}}(\pi_{11})$ be the function whose restriction to the split torus $\mathsf{m}(t;1_m)$ $(t>0)$ is given by* (7.6). *Then $\zeta(\varphi_0^{\pi_{11}};s)$ is convergent on* $\mathrm{Re}(s) > (m-1)/2$, *and*

$$\zeta_\infty(\varphi_0^{\pi_{11}};s) = \frac{-m\pi^{m+1}}{m+1} 2^{m-(\varepsilon+3)/2} |D|^{(m+\varepsilon-2)/4} |\mathrm{N}(\mathfrak{d}_R(\mathcal{M}))|^{1/4} |R[Y]|^{1/2} \times (2|D|^{-1/2})^s \frac{L_\infty(s,\pi_{11})}{L_\infty(s+1/2, \mathcal{M}\cap Y^\perp)} \frac{1}{\zeta_{m,\infty}(2s)} \prod_{j=2}^{m} (s+(m+1)/2-j)^{-1}$$

*with*

$$L_\infty(s,\pi_{11}) = |\mathrm{N}(\mathfrak{d}_R(\mathcal{M}))|^{s/2} |D|^{[(m+2)/2]s} \Gamma_{\mathbf{C}}(s+(m+1)/2)^2 \times \prod_{j=1}^{[m/2]} \Gamma_{\mathbf{C}}\,(s+(m+1)/2-j)^2\, \Gamma_{\mathbf{C}}(s)^{\varepsilon}\,. \tag{8.7}$$

PROOF. By (7.3), we have

$$v_0 = \frac{-1}{\sqrt{m(m+1)}} (\mathsf{y}^{00} + (m^2-1)^{1/2}\mathsf{y}^{11})\,.$$

Substitute this and the formula (7.6) to the integral (8.2); then $\zeta_\infty(\varphi_0^{\pi_{11}},s)$ equals $(-1/\sqrt{m(m+1)})N^{-2s}$ times

$$\begin{aligned}
&\int_0^\infty (\phi_{00}(t) + (m^2-1)^{1/2}\phi_{11}(t)) t^{2s-m-2}\mathrm{d}t \\
&= \int_0^\infty \left\{ \left( m + \frac{4m(m^2-1)}{t^2}\right)\phi_{00}(t) - \frac{2m(m-1)}{t}\phi'_{00}(t) \right\} t^{2s-m-2}\mathrm{d}t \\
&= 2m(m-1)(2s+m-1)\int_0^\infty \phi_{00}(t)t^{2s-m-4}\mathrm{d}t + m\int_0^\infty \phi_{00}(t)t^{2s-m-2}\mathrm{d}t
\end{aligned} \tag{8.8}$$

if $\mathrm{Re}(s) > (m-1)/2$. Here, to prove the second equality we apply the integration-by-part and eliminate $\phi'_{00}$, noting that $\phi_{00}(t)$ is of exponential decay as $t\to\infty$ and $K_{m-1}(t) = O(t^{-(m-1)})$ as $t\to+0$. By (7.7) and the formula [2, 6.561, 16 (p. 668)], we have

$$\int_0^\infty \phi_{00}(t)t^{2s-m-2}\mathrm{d}t = (m/(m+1))^{1/2}\, 2^{2s}\Gamma(s+(m+1)/2)\Gamma(s-(m-3)/2)\,.$$

Use this formula to compute the integrals in the last form of (8.8); then we obtain

$$\begin{aligned}
\zeta_\infty(\varphi_0^{\pi_{11}},s) &= \frac{-m}{m+1}N^{-2s}2^{2s}\left\{ \frac{(m-1)(2s+m-1)}{2}\Gamma(s+(m-1)/2)\Gamma(s-(m-1)/2) \right. \\
&\qquad \left. + \Gamma(s+(m+1)/2)\Gamma(s-(m-3)/2) \right\} \\
&= \frac{-m}{m+1}N^{-2s}2^{2s}\Gamma(s+(m+1)/2)^2 \prod_{j=2}^{m}(s+(m+1)/2-j)^{-1}
\end{aligned}$$

by using the equation $\Gamma(x+1) = x\Gamma(x)$ several times. The remaining part of the proof is a direct computation. □

**9. Global results.** We retain the notations and the assumptions made in Section 5. Let $(\tau, W)$ be an irreducible unitary representation of $K_\infty$ with a non-zero $K_\infty^{\tilde{Y}}$-fixed vector $v_0 \in W$. Let $F : G_{\boldsymbol{Q}} \backslash G_{\boldsymbol{A}} \to W$ be a cusp form with the $K_f K_\infty$-equivariance (5.3). Suppose $F$ is a Hecke eigenfunction, i.e., there exists a $\boldsymbol{C}$-algebra homomorphism $\Lambda_p : \mathcal{H}_p \to \boldsymbol{C}$ for each prime $p$ such that

$$F * \phi = \Lambda_p(\phi) F \,, \quad \phi \in \mathcal{H}_p \,.$$

Then the $L$-function of $F$ is defined to be the Euler product

$$L(s, F) = \prod_p L(s, \Lambda_p) \,,$$

over all the prime numbers $p$, where $L(s, \Lambda_p)$ is the local $L$-factor attached to the character $\Lambda_p$ of $\mathcal{H}_p$ for each $p$ (see Section 4). It is known that the infinite product $L(s, F)$ converges absolutely for $\mathrm{Re}(s) > c$ with a sufficiently large $c > 0$.

Our aim in this section is to study the automorphic $L$-function $L(s, F)$ of $F$ by the integral (5.4), relying on the results of Murase and Sugano which we shall recall below.

9.1. Murase-Sugano's results on global $L$-functions. Let us assume that the function $f : G_{0,\boldsymbol{Q}}^Y \backslash G_{0,\boldsymbol{A}}^Y / K_{0,f}^Y G_{0,\infty}^Y \to \boldsymbol{C}$ used to form the Eisenstein series (see 5.4) is also a Hecke eigenfunction, i.e., there exists a $\boldsymbol{C}$-algebra homomorphism $\Lambda_{0,p} : \mathcal{H}_p^Y \to \boldsymbol{C}$ for each prime $p$ such that $\phi_0 * f = \Lambda_{0,p}(\phi_0) f$ for all $\phi_0 \in \mathcal{H}_p^Y$.

THEOREM 55 (Murase and Sugano [8]). *Suppose the class number of $E$ is one. Define the completed $L$-function $\hat{L}(s, f) := L(s, f) L_\infty(s, \mathcal{M} \cap Y^\perp)$ with the gamma factor $L_\infty(s, \mathcal{M} \cap Y^\perp)$ given by* (8.5). *Then,*

(1) *The holomorphic function $\hat{L}(s, f)$ originally defined on some right-half plane is meromorphically continued to the whole complex plane with the functional equation $\hat{L}(s, f) = \hat{L}(1-s, f)$.*

(2) *The meromorphic function $\hat{L}(s, f)$ on $\boldsymbol{C}$ is holomorphic except possible simple poles at $s = m/2 - j$ $(0 \leqslant j \leqslant m-1)$.*

(3) *The function $\hat{L}(s, f)$ has a pole at $s = m/2$ if and only if $f$ is a constant function.*

The normalized Eisenstein series associated to $f$ is defined by

$$E^*(f; s; g) = (2|D|^{-1/2})^{-s} \hat{\zeta}_m(2s+1) \hat{L}(s+1, f) E(f; s; g) \,.$$

Here $\zeta_m(s)$ is the completed Riemann zeta function $\hat{\zeta}(s)$ for an odd $m$, and is the completed Dirichlet $L$-function $\hat{L}(s, \omega)$ for an even $m$. We need the following result.

THEOREM 56 (Murase and Sugano [8]). *Suppose the class number of $E$ is one. Then the function $E^*(f; s; g)$ is meromorphic on the whole $s$-plane $\boldsymbol{C}$ and invariant by the substitution of the variable $s \to -s$. It is holomorphic except possible simple poles at $s =$*

*$m/2-k$ $(0 \leqslant k \leqslant m)$. The residue at its right most possible pole $s=m/2$ is the constant*

$$\mathrm{Res}_{s=m/2}E^*(s;\, f;\, g) = f(1)\zeta_m(m)\mathrm{Res}_{s=m/2}\hat{L}(s,\, f)\,.$$

9.2. An estimation of Whittaker integrals. Recall the Whittaker integral of $F$ defined by (5.6).

LEMMA 57. *The function $\varphi^F_{f,Y}|G_\infty$ belongs to the space $\mathcal{W}^Y_\infty \otimes W$.*

PROOF. By the definition of the automorphic forms [10, I.2.17], there exists a constant $r \in \boldsymbol{R}$ such that for each $D \in U(\mathfrak{g})$ the estimation $\|R_D F(g)\| \leqslant C_0\|g\|^r_{G_{\boldsymbol{A}}}$ holds for all $g \in G_{\boldsymbol{A}}$ with a constant $C_0 > 0$. Here $\|\cdot\|_{G_{\boldsymbol{A}}}$ is a height function of $G_{\boldsymbol{A}}$ ([10, I.2.2]). Since $G^Y_{0,\boldsymbol{Q}}\backslash G^Y_{0,\boldsymbol{A}} \times N_{\boldsymbol{Q}}\backslash N_{\boldsymbol{A}}$ is compact, by the properties of the height function [10, (ii),(iii) (p. 20)], we obtain the estimation

$$\|R_D F(n\mathsf{m}(1;\, g_0)g_\infty)\| \leqslant C_1|\mathrm{Tr}({}^t\bar{g}_\infty g_\infty)|^r\,, \quad g_0 \in G^Y_{0,\boldsymbol{A}},\ \ n \in N_{\boldsymbol{A}},\ \ g_\infty \in G_\infty$$

with a constant $C_1 > 0$. From this, the estimation for $\varphi^F_{f,Y}|G_\infty$ follows by integration (see (5.6)). □

9.3. Automorphic $L$-functions for wave-forms. Let $(\tau, W) = (\tau_0, W_0)$ be the trivial representation of $K_\infty$. A cusp form $F$ is called a *wave-form* if it is an eigenfunction of the Casimir operator $\Omega$. Let $\nu^2-(m+1)^2$ with $\nu \in \boldsymbol{C}$ be the eigenvalue, i.e., $\Omega F = \{\nu^2-(m+1)^2\}F$. Let $\varphi^F_{f,Y}$ be the Whittaker integral of $F$ along $(f, Y)$ defined by (5.6). Since the restriction $\varphi^F_{f,Y}|G_\infty$ belongs to $\mathcal{W}^Y_{\tau_0}(\pi(\nu))$, the result of 7.1 yields the unique constant $c_{f,Y}(F) \in \boldsymbol{C}$ such that

$$\varphi^F_{f,Y}(\mathsf{m}(t;\, 1_m)) = c_{f,Y}(F)\varphi_0^{\pi(\nu)}(\mathsf{m}(t;\, 1_m))\,, \quad t > 0\,.$$

We call the number $c_{f,Y}(F)$ the $(f, Y)$-*Whittaker coefficient* of $F$.

THEOREM 58. *Let $\hat{L}(s, F) = L(s, F)L_\infty(s, \pi(\nu))$ be the completed $L$-function of $F$ with the gamma factor defined by* (8.4). *Then for $s \in \boldsymbol{C}$ such that* $\mathrm{Re}(s) > (m+1)/2$,

$$\int_{G^{\tilde{Y}}_{\boldsymbol{Q}}\backslash G^{\tilde{Y}}_{\boldsymbol{A}}} E^*(f;\, s-1/2;\, h)F(h)\mathrm{d}h = B_0 c_{f,Y}(F)\hat{L}(s, F)$$

*with $B_0 = 2^{-(\varepsilon+8)/2}|D|^{(m+\varepsilon-3)/4}|\mathrm{N}(\mathfrak{d}_R(\mathcal{M}))|^{1/4}|R[Y]|^{1/2}$. Here $\varepsilon \in \{0, 1\}$ is the parity of $m$.*

PROOF. By the property of $Z^F_{f,Y}(s)$ noted in 4.1, this follows from Theorem 26, Theorem 28 and Proposition 53. Note $|\mathrm{Re}(\nu)| \leqslant m+1$, since $F \in L^2(G_{\boldsymbol{Q}}\backslash G_{\boldsymbol{A}})$ implies $\pi(\nu)$ is unitarizable. □

THEOREM 59. *Assume the class number of $E$ is one. Suppose $c_{f,Y}(F) \neq 0$ for some $Y$ and $f$ as above.*

(1) *The completed $L$-function $\hat{L}(s, F)$ is continued to a meromorphic function on the whole complex plane with the functional equation $\hat{L}(1-s, F) = \hat{L}(s, F)$.*

(2) *The meromorphic function* $\hat{L}(s, F)$ *is holomorphic on* $\boldsymbol{C}$ *except at possible simple poles* $s = (m+1)/2 - j$ $(0 \leqslant j \leqslant m)$.

(3) *If* $f$ *is not constant, then* $\hat{L}(s, F)$ *is holomorphic at* $s = (m+1)/2$. *If* $f$ *is the constant function* 1, *then*

$$\mathrm{Res}_{s=(m+1)/2}\hat{L}(s, F) = B_0^{-1} c_{1,Y}(F)^{-1} \hat{\zeta}_m(m)\{\mathrm{Res}_{s=m/2}\hat{L}(s, 1)\} \int_{G_{\boldsymbol{Q}}^{\tilde{Y}} \backslash G_{\boldsymbol{A}}^{\tilde{Y}}} F(h)\mathrm{d}h\,.$$

PROOF. This follows from Theorems 56 and 58. □

COROLLARY 60. *The following two conditions on* $F$ *are equivalent.*

(1) *The integral* $\int_{G_{\boldsymbol{Q}}^{\tilde{Y}} \backslash G_{\boldsymbol{A}}^{\tilde{Y}}} F(h)\mathrm{d}h$ *is not zero.*

(2) $c_{1,Y}(F) \neq 0$ *and the* $L$*-function* $L(s, F)$ *has a pole at* $s = (m+1)/2$.

9.4. Automorphic $L$-functions for certain harmonic forms. Let $(\tau, W) = (\tau_{1,1}^{\circ}, \mathsf{E}^{\circ})$ and $\pi_{11}$ be as in 8.2. Assume $F$ belongs to the space $\{L^2(G_{\boldsymbol{Q}}\backslash G_{\boldsymbol{A}})^{\infty} \otimes W\}^{K_{\mathrm{f}}K_{\infty}}$ and satisfies $\Omega F = 0$. Here $L^2(G_{\boldsymbol{Q}}\backslash G_{\boldsymbol{A}})^{\infty}$ denotes the space of smooth vectors in $L^2(G_{\boldsymbol{Q}}\backslash G_{\boldsymbol{A}})$. By the characterizing property of $\pi_{11}$ recalled in the paragraph 8.2.2, the functions $g \mapsto \langle w|F(g)\rangle$ $(w \in \mathsf{E}^{\circ})$ generate a $\pi_{11}$-isotypic $(\mathfrak{g}, K_{\infty})$-submodule of finite length in $L^2(G_{\boldsymbol{Q}}\backslash G_{\boldsymbol{A}})^{\infty}$. Let $\varphi_{f,Y}^F$ be the Whittaker integral of $F$ along $(f, Y)$. Since the restriction $\varphi_{f,Y}^F|G_{\infty}$ belongs to the space $\mathcal{W}_{\tau_{1,1}^{\circ}}^{Y}(\pi_{11})$, Proposition 51 yields the unique constant $c_{f,Y}(F) \in \boldsymbol{C}$ such that

$$\varphi_{f,Y}^F(\mathsf{m}(t; 1_m)) = c_{f,Y}(F)\varphi_0^{\pi_{11}}(\mathsf{m}(t; 1_m))\,, \quad t > 0$$

where $\varphi_0^{\pi_{11}}$ is the function constructed in Proposition 51. We call the number $c_{f,Y}(F)$ the $(f, Y)$*-Whittaker coefficient* of $F$.

THEOREM 61. *Let* $\hat{L}(s, F) = L(s, F)L_{\infty}(s, \pi_{11})$ *be the completed* $L$*-function with the gamma factor defined by* (8.7)*. Let* $v_{11} = \mathsf{X}(\mathsf{v}_{\tilde{Y}}^{+}|\mathsf{v}_{\tilde{Y}}^{+})^{\circ}$. *Then for* $s \in \boldsymbol{C}$ *such that* $\mathrm{Re}(s) > (m+1)/2$,

$$\int_{G_{\boldsymbol{Q}}^{\tilde{Y}} \backslash G_{\boldsymbol{A}}^{\tilde{Y}}} E^*(f; s-1/2; h)\langle v_{11}|F(h)\rangle \mathrm{d}h = B_1 c_{f,Y}(F) \prod_{j=2}^{m} (s + (m+1)/2 - j)^{-1} \hat{L}(s, F)\,,$$

*where* $B_1 = -2^{m+3}\pi^{m+1}B_0(m/(m+1))$ *with* $B_0$ *the same constant as in Theorem* 58.

PROOF. By the same reasoning as Theorem 58, this follows from Theorems 26 and 28 and Proposition 54. □

THEOREM 62. *Assume the class number of* $E$ *is one. Suppose* $c_{f,Y}(F) \neq 0$ *for some* $(f, Y)$ *as above.*

(1) *The completed* $L$*-function* $\hat{L}(s, F)$ *is continued to a meromorphic function on the whole complex plane with the functional equation* $\hat{L}(1-s, F) = (-1)^{m-1}\hat{L}(s, F)$.

(2) *The meromorphic function* $\hat{L}(s, F)$ *is holomorphic on* $\boldsymbol{C}$ *except at possible simple poles* $s = (m+1)/2, (-m+1)/2$.

(3) *If* $f$ *is not constant, then* $\hat{L}(s, F)$ *is holomorphic at* $s = (m+1)/2$. *If* $f$ *is the constant function* 1, *then*

$$\mathrm{Res}_{s=(m+1)/2}\hat{L}(s, F)$$
$$= B_1^{-1}\,(m-1)!\,c_{1,Y}(F)^{-1}\hat{\zeta}_m(m)\{\mathrm{Res}_{s=m/2}\hat{L}(s, 1)\}\int_{G_{\boldsymbol{Q}}^{\tilde{Y}}\backslash G_{\boldsymbol{A}}^{\tilde{Y}}}\langle v_{11}\mid F(h)\rangle \mathrm{d}h\,.$$

PROOF. This follows from Theorems 56 and 61. □

COROLLARY 63. *The following two conditions on* $F$ *are equivalent.*

(1) *The integral* $\int_{G_{\boldsymbol{Q}}^{\tilde{Y}}\backslash G_{\boldsymbol{A}}^{\tilde{Y}}}\langle v_{11}|F(h)\rangle \mathrm{d}h$ *is not zero.*

(2) $c_{1,Y}(F) \neq 0$ *and the* $L$*-function* $L(s, F)$ *has a pole at* $s = (m+1)/2$.

**10. Examples.** Let us give examples of $(R, \mathcal{M}, Y)$ which satisfies the assumptions in 5.2.

LEMMA 64. *Let* $R = -\sqrt{D}T$ *with* $T$ *a positive definite symmetric matrix belonging to* $\mathrm{GL}_m(\boldsymbol{Z})$. *Suppose* $m \not\equiv 2 \pmod 4$. *Then there exists a maximal* $\mathcal{O}$*-integral lattice* $\mathcal{M}$ *in* $(R, E^m)$ *containing* $\mathcal{O}^m$ *such that* $\mathfrak{d}_R(\mathcal{M}) = \sqrt{D}^{\varepsilon}\mathcal{O}$ *with* $\varepsilon \in \{0, 1\}$ *the parity of* $m$.

PROOF. Let $\Lambda$ be the set of all the $\mathcal{O}$-integral lattices in $(R, E^m)$ containing $\mathcal{O}^m$; the set $\Lambda$ is not empty since $\mathcal{O}^m \in \Lambda$. Since $\mathcal{L} \in \Lambda$ is $\mathcal{O}$-integral, the inclusion $\mathcal{O}^m \subset \mathcal{L}$ yields $\mathcal{L} \subset R^{-1}\mathcal{O}^m$. Any maximal element $\mathcal{M}$ of $\Lambda$, whose existence is ensured by the fact that $R^{-1}\mathcal{O}^m$ is Noetherian, is a maximal $\mathcal{O}$-integral lattice in $(R, E^m)$. Since $\mathcal{O}^m \subset \mathcal{M} \subset \mathcal{M}^* \subset R^{-1}\mathcal{O}^m$, $\sharp(\mathcal{M}^*/\mathcal{M})$ divides $\sharp(R^{-1}\mathcal{O}^m/\mathcal{O}^m) = |D|^m$, which means $\mathfrak{d}_R(\mathcal{M}_p) = \mathcal{O}_p$ for all $p \in \mathrm{I}(E) \cup \mathrm{S}(E)$. Let $p \in \mathrm{R}(E)$. If $m$ is odd, then, by Lemma 8, we have necessarily $\mathfrak{d}_R(\mathcal{M}_p) = \sqrt{D}\mathcal{O}_p$. This proves the assertion. Let us consider the case when $m$ is a multiple of 4. Then $\det R = D^{m/2} = \mathrm{N}(\sqrt{D})^{m/2} \in \mathrm{N}(E_p^\times)$. By Lemma 8 and Lemma 5, this implies that $\mathcal{M}$ is split, i.e., $\mathcal{M}_0 = \{0\}$ in the decomposition (3.1). Thus $\mathfrak{d}_R(\mathcal{M}_p) = \mathcal{O}_p$. This proves the assertion. □

EXAMPLE 1. Let $m = 4k+1$ and $T = {}^tT \in \mathrm{GL}_{4k}(\boldsymbol{Z})$ be positive definite. Suppose $D \equiv 1 \pmod 4$. Choose a maximal $\mathcal{O}$-integral lattice $\mathcal{L}$ in $(-\sqrt{D}T, E^{4k})$ such that $\mathfrak{d}_{-\sqrt{D}T}(\mathcal{L}) = \mathcal{O}$ by Lemma 64. Set $V = E \oplus E^{4k}$, $R = \mathrm{diag}(-\sqrt{D}, -\sqrt{D}T)$, $\mathcal{M} = \mathcal{O} \oplus \mathcal{L}$. Then since $\mathfrak{d}_R(\mathcal{M}) = \sqrt{D}\mathcal{O}$, $\mathcal{M}$ is a maximal $\mathcal{O}$-integral lattice in $(R, V)$ by Proposition 9.

EXAMPLE 2. Let $m = 4k+2$ and $T = {}^tT \in \mathrm{GL}_{4k+1}(\boldsymbol{Z})$ be positive definite. Choose a maximal $\mathcal{O}$-integral lattice $\mathcal{L}$ in $(-\sqrt{D}T, E^{4k+1})$ such that $\mathfrak{d}_{-\sqrt{D}T}(\mathcal{L}) = \sqrt{D}\mathcal{O}$ by Lemma 64. Set $V = E \oplus E^{4k+1}$ and define $R$, $\mathcal{M}$ by the same formula in Example 1. Then $\mathfrak{d}_R(\mathcal{M}) = D\mathcal{O}$. Suppose $|D|$ is a product of primes of the form $4l+3$ ($l \in \boldsymbol{N}$). Since $-\det(R) = \mathrm{N}(\sqrt{D}^{2k+1}) \in \mathrm{N}(E^\times)$, $\det(R) \notin \mathrm{N}(E_p^\times)$ for any $p \in \mathrm{R}(E)$. Hence $\mathcal{M}$ is a maximal $\mathcal{O}$-integral lattice in $(R, V)$ by Proposition 9.

In both of these examples, the vector $Y = (1/\sqrt{D}, 0) \in V$ satisfies the assumption in the paragraph 5.2.1.

REMARK. Let $(R, \mathcal{M}, Y)$ be as in Examples 1 and 2 above. In [21], we show that there exist infinitely many linearly independent Hecke eigen wave-cusp-forms $F : G_{\boldsymbol{Q}}\backslash G_{\boldsymbol{A}}/K_{\mathrm{f}}K_\infty \to \boldsymbol{C}$ such that $c_{1,Y}(F) \neq 0$ and $\int_{G_{\boldsymbol{Q}}^{\tilde{Y}}\backslash G_{\boldsymbol{A}}^{\tilde{Y}}} F(h)\mathrm{d}h \neq 0$.

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