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"CERTAIN RATIONAL FUNCTIONS WHOSE POWER SERIES HAVE POSITIVE COEFFICIENTS"

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Certain rational functions whose power series have positive coefficients

Richard Askey

In the late 1920's some very important work was done by Courant, Friedrichs, and Lewy on solving partial differential equations by approximating them with systems of algebraic equations and solving these algebraic equations. In trying to prove convergence of the solutions of the algebraic equations which approximate the wave equation in three variables the series

$$\frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)} = \sum_{n,m,k=0}^{\infty} A_{n,m,k}\, r^n s^m t^k$$

arose. The corresponding solution to the differential equation was positive and all of the coefficients they calculated also turned out to be positive, but they were unable to prove the positivity. In such circumstances the best course is to write to someone who is an expert on questions of this type. So in 1930 H. Lewy wrote to G. Szegö and in a very short while he received a solution. We will give a solution which is closely related to Szegö's solution. Szegö's idea was to use some of the special functions of mathematical physics. He solved the problem by using old results on Bessel functions and then he showed how to reduce the problem to an integral of products of Laguerre polynomials. We will follow this part of Szegö's paper and then estimate the integral of Laguerre polynomials. The Laguerre polynomial $L_n(x)$ comes from the generating function

$$\frac{e^{-xr/(1-r)}}{1-r} = \sum_{n=0}^{\infty} L_n(x)\, r^n.$$

It is easy to show that $L_n(x)$ is a polynomial of degree n. Also

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} L_n(x)\, L_m(x)\, e^{-x}dx\; r^n s^m = \int_0^{\infty} \frac{e^{-\frac{xr}{1-r}-\frac{xs}{1-s}}\, dx}{(1-r)(1-s)}$$

$$= \frac{1}{1-rs} = \sum_{n=0}^{\infty} (rs)^n$$

so
$$\int_0^\infty L_n(x)\, L_m(x)\, e^{-x}dx = \begin{cases} 0 & n \neq m, \\ 1 & n = m. \end{cases}$$

This is a standard argument and Szegö extended it as follows.
From the generating function we have

$$\frac{e^{-x/(1-r)}}{1-r} = \sum_{n=0}^{\infty} L_n(x)e^{-x}r^n.$$

Then
$$\frac{e^{-x\left(\frac{1}{1-r}+\frac{1}{1-s}+\frac{1}{1-t}\right)}}{(1-r)(1-s)(1-t)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} L_n(x)L_m(x)L_k(x)e^{-3x}\, r^n s^m t^k.$$

Integrating from 0 to ∞ gives

$$\frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)} = \sum \int_0^\infty L_n(x)L_m(x)L_k(x)e^{-3x}dx\ r^n s^m t^k$$

so
$$A_{n,m,k} = \int_0^\infty L_n(x)L_m(x)L_k(x)e^{-3x}dx.$$

When faced with a new integral which you can not find in any of the standard tables of integrals the first thing to look for is a similar integral. In Whittaker and Watson, Modern Analysis, the integral

(1) $$\int_{-1}^{1} P_n(x)\, P_m(x)\, P_k(x)dx$$

is evaluated. Here $P_n(x)$ is the Legendre polynomial defined by

$$\frac{1}{(1-2xr+r^2)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} P_n(x)r^n.$$

It satisfies the orthogonality condition

$$\int_{-1}^{1} P_n(x)\ P_m(x)dx = 0, \quad n \neq m.$$

The actual value of (1) is not important to us but the fact that it is nonnegative is essential. Now the problem that occurs is how to go from $P_n(x)$ to $L_n(x)$. Recall that $(1-\frac{x}{t})^t \to e^{-x}$ as $t \to \infty$. This suggests that we should consider polynomials orthogonal on $[-1,1]$ with respect to $(1-x)^{\alpha}$. Actually we will now define the polynomials which are orthogonal with respect to $(1-x)^{\alpha}(1+x)^{\beta}$. These polynomials are called $P_n^{(\alpha,\beta)}(x)$ and are normalize by $P_n^{(\alpha,\beta)}(1) =$
$= \dfrac{(\alpha+1)(\alpha+2)\ +\ (\alpha+n)}{1.2....n} = \binom{n+\alpha}{n}$.

The orthogonality relation is

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\ P_m^{(\alpha,\beta)}(x)\ (1-x)^{\alpha}(1+x)^{\beta}dx = 0, \quad m \neq n.$$

If we let $x = 1 - \frac{2y}{\beta}$ then this is

$$\int_0^{\beta} P_n^{(\alpha,\beta)}(1-\frac{2y}{\beta})P_m^{(\alpha,\beta)}(1-\frac{2y}{\beta})(\frac{2y}{\beta})^{\alpha}(1-\frac{y}{\beta})^{\beta}2^{\beta+1}\ \frac{dy}{\beta} = 0 \quad n \neq m.$$

If we multiply by $2^{-\alpha-\beta-1}\ \beta^{1+\alpha}$ this is

$$\int_0^{\beta} P_n^{(\alpha,\beta)}(1-\frac{2y}{\beta})P_m^{(\alpha,\beta)}(1-\frac{2y}{\beta})y^{\alpha}(1-\frac{y}{\beta})^{\beta}dy = 0 \quad m \neq n.$$

Letting $\beta \to \infty$ this should approach

$$\int_0^{\infty} L_n^{\alpha}(y)L_m^{\alpha}(y)y^{\alpha}\ e^{-y}dy = 0 \quad m \neq n$$

where $L_n^\alpha(x)$ are polynomials orthogonal with respect to $x^\alpha e^{-x}$ on $(0,\infty)$ which are normalized by $L_n^\alpha(0) = P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. This in fact is true and $L_n^0(x)$ is $L_n(x)$. The normalization is right for
$\frac{e^{-\frac{0r}{1-r}}}{1-r} = \sum_{n=0}^{\infty} L_n(x)r^n = \sum_{n=0}^{\infty} r^n$ so $L_n(0) = 1 = L_n^0(0)$. The polynomials $L_n^\alpha(x)$ come from the generating function

$$\frac{e^{-\frac{xr}{1-r}}}{(1-r)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^\alpha(x)r^n$$

and the orthogonality can be checked as before. We will return to $L_n^\alpha(x)$ later.

This shows us how to use $P_n^{(0,\beta)}(x)$ to get $L_n(x)$ but we still haven't solved the problem of how to go from $P_n(x)$ to $P_n^{(0,\beta)}(x)$. We will do it one step at a time. Let $w(x)$ be a nonnegative function on $[-1,1]$ and let $p_n(x)$ be the polynomials on $[-1,1]$ which are orthogonal with respect to $w(x)$. These polynomials are unique except for a multiplicative constant. We choose the polynomials to be orthonormal and choose the highest coefficient to be positive. Let $q_n(x)$ be the polynomials orthogonal with respect to $(1+x)\,w(x)$ and normalize them in the same way. Expand $(1+x)\,q_n(x)$ in terms of $p_k(x)$. This is

$$(1+x)\,q_n(x) = \sum_{k=0}^{n+1} a_{k,n}\,p_k(x),$$

where $a_{k,n}$ is given by

$$a_{k,n} = \int_{-1}^{1} (1+x)\,q_n(x)\,p_k(x)\,w(x)\,dx\,.$$

If $k < n$ then $a_{k,n} = 0$ since we have

$$\int_{-1}^{1} x^j\,q_n(x)(1+x)\,w(x)dx = 0, \quad j = 0,1,\ldots,n-1$$

and

$$p_k(x) = \sum_{j=0}^{k} \alpha_{j,k}\,x^j$$

Thus

$$(1+x)\ q_n(x) = A_n\ p_{n+1}(x) + B_n\ p_n(x).$$

Since the highest coefficients of $q_n(x)$ and $p_{n+1}(x)$ are positive so is A_n. Now we let $x = -1$. The left hand side is zero so we have

$$B_n\ p_n(-1) = -A_n\ p_{n+1}(-1).$$

All of the zeros of $p_n(x)$ are real and lie strictly between -1 and 1. Thus $p_n(-1)(-1)^n > 0$ and $p_{n+1}(-1)(-1)^{n+1} > 0$. This gives $B_n > 0$. Now we apply this to our problem of Jacobi polynomials.
Recall that

$$\int_{-1}^{1} P_n(x)\ P_m(x)\ P_k(x)dx \geq 0,$$

with $A_n > 0$, $B_n > 0$. Thus

$$\int_{-1}^{1} P_n^{(0,1)}(x)\ P_m^{(0,1)}(x)\ P_k^{(0,1)}(x)(1+x)^3dx$$

$$\int_{-1}^{1} [A_nP_{n+1}(x) + B_nP_n(x)]\,[A_mP_{m+1}(x) + B_mP_m(x)]\,[A_kP_{k+1}(x) + B_kP_k(x)]\,dx$$

and after multiplying these terms we see that

$$\int_{-1}^{1} P_n^{(0,1)}(x)\ P_m^{(0,1)}(x)\ P_k^{(0,1)}(x)\ (1+x)^3dx \geq 0.$$

If we continue in the same fashion we see that

$$\int_{-1}^{1} P_n^{(0,j)}(x)\ P_m^{(0,j)}(x)\ P_k^{(0,j)}(x)(1+x)^{3j}dx \geq 0.$$

Now let $x = 1 - \frac{2y}{j}$ and let $j \to \infty$. First multiply by $j2^{-3j-1}$ and this gives

$$2^{-3j-1}j \int_0^j P_n^{(0,j)}(1 - \tfrac{2y}{j})P_m^{(0,j)}(1 - \tfrac{2y}{j})P_k^{(0,j)}(1 - \tfrac{2y}{j})2^{3j+1}\ (1 - \tfrac{y}{j})^{3j}\ \frac{dy}{j} \to$$

$$\to \int_0^\infty L_n(y)\ L_m(y)\ L_k(y)^{-3y}dy$$

and the approximating integrals on the left are all nonnegative. Thus so is the limit on the right. To prove the strict positivity we must generalize the original problem. This generalization is also due to Szegö. The same argument which we gave before leads to

$$\frac{\Gamma(\alpha+1)}{[(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)]^{\alpha+1}} = \sum_{n,m,k=0}^{\infty} \int_0^\infty L_n^\alpha(x)L_m^\alpha(x)L_k^\alpha(x)x^\alpha e^{-3x}dx\ r^n s^m t^k$$

where $\Gamma(\alpha+1) = \int_0^\infty x^\alpha\ e^{-x}dx$.

Szegö showed that these integrals are nonnegative if $\alpha \geq -\frac{1}{2}$. They change sign for $\alpha < -\frac{1}{2}$ so let us consider the end case $\alpha = -\frac{1}{2}$. The proof we gave above can be repeated once we know that

$$\int_{-1}^{1} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)\ P_m^{(-\frac{1}{2},-\frac{1}{2})}(x)\ P_k^{(-\frac{1}{2},-\frac{1}{2})}(x)\ (1-x^2)^{-\frac{1}{2}}dx \geq 0.$$

If we let $x = \cos\theta$ this is the same as

$$\int_0^\pi P_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta)P_m^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta)P_k^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta)d\theta \geq 0.$$

But what are these functions $P_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta)$? $P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$ are orthogonal on $[-1,1]$ with respect to $(1-x^2)^{-\frac{1}{2}}$ and so $P_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta)$ are orthogonal on $[0,\pi]$ with respect to $d\theta$. They are also even functions of θ, since $\cos(-\theta) = \cos\theta$.

Thus they are constant multiples of $\cos n\theta$ and the nonnegativity of the above integral is just

$$\cos n\theta \cos m\theta = \tfrac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta]$$

since $\frac{1}{2} > 0$ is the only coefficient which does not vanish.

This gives $\int_0^\infty L_n^{-\frac{1}{2}}(x)\, L_m^{-\frac{1}{2}}(x)\, L_k^{-\frac{1}{2}}(x)\, x^{-\frac{1}{2}} e^{-3x} dx \geq 0$. However we can now use $x^2 = xx$ to again show $\int_0^\infty L_n(x)\, L_m(x)\, L_k(x) e^{-3x} dx \geq 0$ and this proof can be extended to give the strict positivity of this integral.

$$\frac{1}{[(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)]} = \frac{\Gamma(\frac{1}{2})}{[(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)]^{\frac{1}{2}}} \cdot$$

$$\cdot \frac{\Gamma(\frac{1}{2})}{[\qquad]^{\frac{1}{2}}} \cdot \frac{1}{[\Gamma(\frac{1}{2})]^2}$$

$$\sum_{n,m,k=0}^\infty \int_0^\infty L_n(x)\, L_m(x)\, L_k(x) e^{-3x} dx\, r^n\, s^m\, t^k =$$

$$= |\Gamma(\tfrac{1}{2})|^2 \sum \int_0^\infty L_{n_1}^{-\frac{1}{2}}(x)\, L_{m_1}^{-\frac{1}{2}}(x)\, L_{k_1}^{-\frac{1}{2}}(x)\, x^{-\frac{1}{2}} e^{-3x} dx \int_0^\infty L_{n_2}^{-\frac{1}{2}}(x)\, L_{m_2}^{-\frac{1}{2}}(x)\, L_{k_2}^{-\frac{1}{2}}(x) x^{-\frac{1}{2}} e^{-3x} dx$$

$$r^{n_1+n_2}\, s^{m_1+m_2}\, t^{k_1+k_2}$$

If we first sum over $n_1+n_2 = n$ and then over n we see that

$$\int_0^\infty L_n(x) L_m(x) L_k(x) e^{-3x} dx = \frac{1}{|\Gamma(\frac{1}{2})|^2} \sum_{\substack{n_1=0,1,\ldots,n \\ m_1=0,1,\ldots,m \\ k_1=0,1,\ldots,k}} \int_0^\infty L_{n-n_1}^{-\frac{1}{2}}(x) L_{m-m_1}^{-\frac{1}{2}}(x) L_{k-k_1}^{-\frac{1}{2}}(x) x^{-\frac{1}{2}} e^{-3x} dx$$

$$\int_0^\infty L_{n_1}^{-\frac{1}{2}}(x)\, L_{m_1}^{-\frac{1}{2}}(x)\, L_{k_1}^{-\frac{1}{2}}(x)\, x^{-\frac{1}{2}} e^{-3x} dx \;.$$

This gives the nonnegativity of the integral on the left without using $\int_{-1}^{1} P_n(x)\, P_m(x)\, P_k(x)dx \geq 0$ and if we can show that even one product on the right is positive the left hand side will also be positive. It is possible to show that

$$\int_0^\infty L_n^{-\frac{1}{2}}(x)\, L_m^{-\frac{1}{2}}(x)\, x^{-\frac{1}{2}}e^{-3x}dx > 0.$$

There are two ways to do this. One is to use the generating function and compute the integral. The other way is to use a very interesting theorem of Karlin and McGregor. If $p_n(x)$ denotes a set of polynomials orthogonal on $[0,\infty]$ with respect to a nonnegative measure $d\alpha(x)$ with $p_n(0) > 0$ then

$$\int_0^\infty p_n(x)\, p_m(x)e^{-\epsilon x}d\alpha(x) > 0, \quad \epsilon > 0.$$

The real interest in this result is the connection with probability theory. This positive number represents the probability of a birth and death process going from a population of size m to one of size n in time ϵ. Rather than give any details of this let us consider some related problems.

There are other interesting sets of orthogonal polynomials. For example, consider the measure with mass c^x at $x = 0,1,\ldots$. Let $M_n(x;c)$ be the polynomials orthogonal on $[0,\infty]$ with respect to this measure. That is

$$\sum_{x=0}^{\infty} c^x\, M_n(x;c)\, M_k(x;c) = 0, \quad n \neq k.$$

Normalize by $M_n(0;c) = 1$.

An analogue of Szegö's result would be

$$\sum_{x=0}^{\infty} M_n(x)\, M_k(x)\, M_l(x)c^{3x} > 0.$$

Unfortunately this result fails and I know no results of this typefor these polynomials. This type of result does not always hold for we can prove the following theorem.

$$\int_0^\infty L_n^\alpha(x)\, L_m^\alpha(x)\, L_k^\alpha(x) x^\alpha e^{-jx} dx \geq 0$$

does not hold for any j and all n,m,k, if $-1 < \alpha < -\frac{1}{2}$. The Karlin-McGregor result shows that if a result of this type holds for some j then it holds for all larger j. It is possible to extend the Szegö result to

$$\int_0^\infty L_n(x)\, L_m(x)\, L_k(x)\, e^{-2x} dx > 0, \text{ and so}$$

$$\frac{1}{(1-r)(1-s)(1+\frac{t}{2})+(1-r)(1+\frac{s}{2})(1-t)+(1+\frac{r}{2})(1-s)(1-t)} = \sum B_{n,m,k}\, r^n\, s^m\, t^k$$

has positive coefficients . As Szegö remarked there is another extension, this time to more variables. If $f(x) = (x-r)(x-s)(x-t)$ then $f'(1) = (1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)$. Similarly we can consider $f(x) = (x-x_1)\ldots(x-x_k)$. Then the same proof as above gives

$$\frac{1}{[f'(1)]^{\alpha+1}} = \sum A^\alpha_{n_1,\ldots,n_k}\, x_1^{n_1}\ldots\, x_k^{n_k}$$

with $A^\alpha_{n_1,\ldots,n_k} \geq 0$ if $\alpha \geq -\frac{1}{2}$, and $A^\alpha_{n_1,\ldots,n_k} > 0$.

Most good problems suggest other problems and have interesting applications. We have mentioned some of the further problems suggested by this. One other was suggested by H. Lewy. Show that the coefficients of

$$\left(\frac{1}{4-r-s-t-u}\right)\left(\frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-r)(1-u)+(1-s)(1-t)+(1-s)(1-u)+(1-t)(1-u)}\right)$$

are positive.

Unfortunately we have been unable to find a representations for these coefficients in terms of special functions which is simple enough to handle. The original problem has been done directly by Kaluza but his proof is very complicated and does not extend at all. Thus special functions seem to be the only good way of attacking such problems.

There are two applications besides the original one to difference equations. The first is to the construction of an interesting Banach algebra in which questions of analysis and probability theory can be asked. The other is probably no more than an interesting remark in classical algebra. From

$$\frac{1}{f'(1)} = \sum A_{n_1,\dots,n_k} x_1^{n_1} \dots x_k^{n_k}$$

we get

$$\frac{x^{k-1}}{f'(x)} = \sum_{n=0}^{\infty} \frac{1}{x^n} \sum_{n_1+\dots+n_k=n} A_{n_1,\dots,n_k} x_1^{n_1} \dots x_n^{n_k}$$

with $A_{n_1,\dots,n_k} > 0$.

Now we would like to show how functions like $\frac{x^{k-1}}{f'(x)}$ can be used.

$$\frac{1}{(1-x_1t)\dots(1-x_kt)} = (1+x_1t+x_2t^2+\dots)(1+x_2t+x_2^2t^2+\dots)\dots(1+x_kt+\dots)$$

$$= 1 + (x_1+\dots+x_k)t + (x_1^2+x_1x_2+\dots+x_1x_k+x_2^2+\dots+x_{k-1}x_k+x_k^2)t^2 + \dots$$

$$+ \sum_{1\leq i_1\leq i_2\leq\dots\leq i\leq k} x_i,\dots x_{ij}\; t^j + \dots = 1 + \sum_{j=1}^{\infty} \aleph_j\,(x_1,\dots,x_k)t^j.$$

$$= \frac{t^{-k}}{(t^{-1}-x_1)\dots(t^{-1}-x_k)} \quad .$$

Let $t^{-1} = z$. Then

$$\frac{z^k}{(z-x_1)\dots(z-x_k)} = \frac{z^k}{f(z)} = 1 + \sum_{j=1}^{\infty} \aleph_j\, z^{-j} \;.$$

But

$$\frac{1}{(z-x_1)\ldots(z-x_k)} = \frac{1}{f'(x_1)(z-x_1)} + \ldots + \frac{1}{f'(x_k)(z-x_k)}$$

or

$$1 + \sum_{j=1}^{\infty} \aleph_j \, z^{-j} = \sum_{i=1}^{k} \frac{z^k}{f'(x_i)(z-x_i)} = \sum_{i=1}^{k} \frac{z^{k-1}}{f'(x_i)(1 - \frac{x_i}{z})}$$

$$= \sum_{i=1}^{k} \sum_{l=0}^{\infty} \frac{x_i^{\,l}}{z^l} \frac{z^{k-1}}{f'(x_i)} = \sum_{l=0}^{\infty} z^{-l+k-1} \sum_{i=1}^{k} \frac{x_i^{\,l}}{f'(x_i)}$$

Thus

$$\aleph_{l-k+1} = \sum_{i=1}^{k} \frac{x_i^{\,l}}{f'(x_i)} .$$