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CERTAIN RESULTS ON KENMOTSU PSEUDO-METRIC MANIFOLDS

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Abstract. In this paper, a systematic study of Kenmotsu pseudo-metric manifolds are introduced. After studying the properties of this manifolds, we provide necessary and sufficient condition for Kenmotsu pseudo-metric manifold to have constant φ-sectional curvature, and prove the structure theorem for ξ-conformally flat and φ-conformally flat Kenmotsu pseudo-metric manifolds. Next, we consider Ricci solitons on this manifolds. In particular, we prove that an η-Einstein Kenmotsu pseudo-metric manifold of dimension higher than 3 admitting a Ricci soliton is Einstein, and a Kenmotsu pseudo-metric 3-manifold admitting a Ricci soliton is of constant curvature $-\varepsilon$.

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1. Introduction

The study of contact metric manifolds with associated pseudo-Riemannian metrics were first started by Takahashi [10] in 1969. Since then, such structures were studied by several authors mainly focusing on the special case of Sasakian manifolds. The case of contact Lorentzian structures (η, g), where η is a contact one-form and g is a Lorentzian metric associated to it, has a particular relevance for physics and was considered in [4] and [1]. A systematic study of almost contact semi-Riemannian manifolds was undertaken by Calvaruso and Perrone [3] in 2010, introducing all the technical apparatus which is needed for further investigations.

On the other hand, in 1972 Kenmotsu [9] investigated a class of contact metric manifolds satisfying some special conditions, and after onwards such manifolds are came to known as Kenmotsu manifolds. Recently, Wang and Liu [11] investigated almost Kenmotsu manifolds with associated pseudo-Riemannian metric. These are called almost Kenmotsu pseudo-metric manifolds. In this paper, we undertake the systematic study of Kenmotsu pseudo-metric manifolds.

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The present paper is organized as follows: In Section 2, we give the basics of Kenmotsu pseudo-metric manifold (M,g). Certain properties of Kenmotsu pseudo-metric manifolds are provided in Section 3. We devote Section 4 to the study of curvature properties of Kenmotsu pseudo-metric manifold (M,g) and gave necessary and sufficient condition for (M,g) to have constant φ-sectional curvature. In Section 5, we prove necessary and sufficient condition for Kenmotsu pseudo-metric manifold to be ξ-conformally flat (and φ-conformally flat). The last section is focused on the study of Kenmotsu pseudo-metric manifolds whose metric is a Ricci soliton. We show that if (M,g) is a Kenmotsu pseudo-metric manifold admitting a Ricci soliton, then the soliton constant $\lambda = 2n\varepsilon$, where $\varepsilon = \pm 1$. Moreover, we show that if M is an η-Einstein manifold of dimension higher than 3 admitting Ricci soliton, then M is Einstein. Further we show that, a Kenmotsu pseudo-metric manifold (M,g) of dimension 3 admitting Ricci soliton is of constant curvature $-\varepsilon$, where $\varepsilon = \pm 1$. Finally, an illustrative example is constructed which verifies our results.

2. PRELIMINARIES

Let M be a $(2n+1)$ dimensional smooth manifold. We say that M has an *almost contact structure* if there is a tensor field φ of type $(1,1)$, a vector field ξ (called the *characteristic vector field* or *Reeb vector field*), and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

If M with (φ, ξ, η)-structure is endowed with a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{2.2}$$

where $\varepsilon = \pm 1$, for all $X, Y \in TM$, then M is called an *almost contact pseudo-metric manifold*. The relation (2.2) is equivalent to

$$\eta(X) = \varepsilon g(X,\xi) \text{ along with } g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.3}$$

In particular, in an almost contact pseudo-metric manifold, it follows that $g(\xi,\xi) = \varepsilon$ and so, the characteristic vector field ξ is a unit vector field, which is either space-like or time-like, but cannot be light-like.

The *fundamental* 2*-form* of an almost contact pseudo-metric manifold is defined by

$$\Phi(X,Y) = g(X, \varphi Y),$$

which satisfies $\eta \wedge \Phi^n \neq 0$. An almost contact pseudo-metric manifold is said to be a *contact pseudo-metric manifold* if $d\eta = \Phi$. The Riemannian curvature tensor R is given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, which is opposite to the one used in [3]. The Ricci operator Q is determined by

$$Ric(X,Y) = g(QX,Y),$$

where Ric denotes the Ricci tensor. In an almost contact pseudo-metric manifold there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^n$, called a local pseudo φ-basis.

Consider the manifold $M \times \mathbb{R}$, where M is an almost contact pseudo-metric manifold. Denoting the vector field on $M \times \mathbb{R}$ by $(X, f\frac{d}{dt})$, where $X \in TM$, $t \in \mathbb{R}$, and f is a smooth function $M \times \mathbb{R}$, we define the structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

which defines an almost complex structure. If J is integrable, we say that the almost contact pseudo-metric structure (φ, ξ, η) is normal. Necessary and sufficient condition for integrability of J is [3]

$$[\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0.$$

The following can be easily obtained.

Proposition 1. *An almost contact pseudo-metric manifold is normal if and only if*

$$(\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_X\varphi)Y + (\nabla_X\eta)(Y)\xi = 0, \tag{2.4}$$

where ∇ is the Levi-Civita connection.

An *almost Kenmotsu pseudo-metric manifold* is an almost contact pseudo-metric manifold with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu pseudo-metric manifold is called a Kenmotsu pseudo-metric manifold [11]. Equivalently, from (2.4) we have the following:

Definition 1. Almost contact pseudo-metric manifold is said to be *Kenmotsu pseudo-metric manifold* if

$$(\nabla_X\varphi)Y = -\eta(Y)\varphi X - \varepsilon g(X, \varphi Y)\xi. \tag{2.5}$$

From (2.5), we see

$$\nabla\xi = I - \eta \otimes \xi. \tag{2.6}$$

A straight forward calculation gives the following:

Proposition 2. *On Kenmotsu pseudo-metric manifold (M, g), we have*

$$(\nabla_X\eta)Y = \varepsilon g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$\pounds_\xi g = 2g - \varepsilon\eta \otimes \eta, \tag{2.8}$$

$$\pounds_\xi\varphi = 0, \tag{2.9}$$

$$\pounds_\xi\eta = 0, \tag{2.10}$$

where $\pounds$ denotes the Lie derivative.

3. Some properties of Kenmotsu pseudo metric manifolds

For $X \in \operatorname{Ker}\eta$, either space-like or time-like, the *ξ-sectional curvature* $K(\xi, X)$ and *φ-sectional curvature* $K(X, \varphi X)$ are defined respectively as

$$K(\xi, X) = \frac{g(R(\xi, X)X, \xi)}{\varepsilon g(X, X)},$$
$$K(X, \varphi X) = \frac{g(R(\varphi X, X)X, \varphi X)}{g(X, X)^2}.$$

Now we prove:

Proposition 3. *If (M, g) is a Kenmotsu pseudo-metric manifold, then we have*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.1}$$
$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{3.2}$$
$$R(X, \xi)Y = \varepsilon g(X, Y)\xi - \eta(Y)X, \tag{3.3}$$
$$Ric(X, \xi) = -2n\eta(X) \quad (\Rightarrow Q\xi = -2n\varepsilon\xi), \tag{3.4}$$
$$K(\xi, \cdot) = -\varepsilon, \tag{3.5}$$
$$(\nabla_Z R)(X, Y, \xi) = \varepsilon\{g(X, Z)Y - g(Y, Z)X\} - R(X, Y)Z. \tag{3.6}$$

Proof. Equations (2.6) and (2.7) give (3.1). Equations (3.2), (3.3), (3.4) and (3.5) are consequences of (3.1). Equation (3.6) follows from (2.6), (2.7) and (3.1). □

Definition 2. An almost contact pseudo-metric manifold for which

$$\varphi^2(\nabla_W R)(X, Y, Z) = 0,$$

for all $X, Y, Z, W \in TM$ is said to be *globally φ-symmetric*.

Using (3.2) and (3.6), we have the following:

Corollary 1. *A globally φ-symmetric Kenmotsu pseudo-metric manifold is of constant curvature $-\varepsilon$.*

A Kenmotsu pseudo-metric manifold M is said to be η-Einstein if the Ricci tensor satisfies

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{3.7}$$

where a and b are certain smooth functions on M. If $b = 0$, then M is called an *Einstein* manifold.

From (3.4), we have

$$\varepsilon a + b = -2n. \tag{3.8}$$

Contracting (3.7) and using (3.8), we get

$$a = \left(\frac{r}{2n} + \varepsilon\right), \quad b = -\left(\frac{\varepsilon r}{2n} + 2n + 1\right), \tag{3.9}$$

where r is the scalar curvature. Thus, we have:

Proposition 4. *A Kenmotsu pseudo-metric manifold (M,g) is η-Einstein if and only if*

$$Ric(X,Y) = \left(\frac{r}{2n}+\varepsilon\right) g(X,Y) - \left(\frac{\varepsilon r}{2n}+2n+1\right)\eta(X)\eta(Y). \tag{3.10}$$

In particular, we have the following:

Corollary 2. *A Kenmotsu pseudo-metric manifold (M,g) is Einstein if and only if*

$$Ric(X,Y) = -2n\varepsilon g(X,Y). \tag{3.11}$$

Proposition 5. *If the Kenmotsu pseudo-metric manifold (M,g) is η-Einstein, then*

$$X(b)+2b\eta(X)=0, \tag{3.12}$$

for $n>1$, and for any vector field $X \in TM$.

Proof. Equation (3.10) is equivalent to

$$QY = aY + b\varepsilon\eta(Y)\xi, \tag{3.13}$$

where a and b are as in (3.9). It is well known that

$$\operatorname{div} Q = \frac{1}{2}Dr, \tag{3.14}$$

where D denotes the gradient. Equations (3.13) and (3.14) yields to

$$(n-1)Y(a) = \varepsilon\{\xi(b)\eta(Y)+2nb\eta(Y)\}.$$

For $Y=\xi$, it gives $\xi(b)=-2b$, and so we get (3.12) for $n>1$. □

Corollary 3. *If b (or a) is constant in an η-Einstein Kenmotsu pseudo-metric manifold, then it is Einstein.*

4. Curvature properties of Kenmotsu pseudo metric manifolds

First we prove the following Lemma which is very useful in subsequent sections.

Lemma 1. *On Kenmotsu pseudo-metric manifold (M,g), we have the following identities:*

$$\begin{aligned} R(X,Y)\varphi Z - \varphi R(X,Y)Z =& \varepsilon\{g(Y,Z)\varphi X - g(X,Z)\varphi Y \\ &+ g(X,\varphi Z)Y - g(Y,\varphi Z)X\}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} R(\varphi X,\varphi Y)Z = R(X,Y)Z &+ \varepsilon\{g(Y,Z)X - g(X,Z)Y \\ &+ g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y\}. \end{aligned} \tag{4.2}$$

Proof. The Ricci identity shows that

$$\nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi = R(X,Y)\varphi - \varphi R(X,Y).$$

Computing the left-hand side using (2.5) yields (4.1). The equation (4.2) is a result of (4.1). □

Note that the necessary and sufficient condition for a Sasakian pseudo-metric manifold to have constant φ-sectional curvature c is [10]

$$\begin{aligned}4R(X,Y)Z =&(c+3\varepsilon)\{g(Y,Z)X-g(X,Z)Y\}\\&+(\varepsilon c-1)\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\}\\&+(c-\varepsilon)\{\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(X,\varphi Z)\varphi Y\\&-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\}.\end{aligned}$$

Here we prove:

Theorem 1. *The necessary and sufficient condition for a Kenmotsu pseudo-metric manifold M to have constant φ-sectional curvature c is*

$$\begin{aligned}4R(X,Y)Z =&(c-3\varepsilon)\{g(Y,Z)X-g(X,Z)Y\}\\&+(c+\varepsilon)\{\varepsilon\eta(X)\eta(Z)Y-\varepsilon\eta(Y)\eta(Z)X\\&+\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(X,\varphi Z)\varphi Y\\&-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\}.\end{aligned}\tag{4.3}$$

Proof. Suppose that M has constant φ-sectional curvature c. Then for all vector fields $U,V\in \operatorname{Ker}\eta$, we have

$$R(U,\varphi U,U,\varphi U)=-cg(U,U)^2.\tag{4.4}$$

Using (4.1), we get

$$\begin{aligned}R(U,\varphi V,U,\varphi V)=&R(U,\varphi V,V,\varphi U)+\varepsilon\{g(U,U)g(V,V)\\&-g(U,V)^2-g(U,\varphi V)^2\},\end{aligned}\tag{4.5}$$

$$R(U,\varphi U,V,\varphi U)=R(U,\varphi U,U,\varphi V),\tag{4.6}$$

for all $U,V\in\operatorname{Ker}\eta$. Putting $U+V$ in (4.4), and using(4.2), (4.5), (4.6) and the first Bianchi identity, we obtain

$$\begin{aligned}&2R(U,\varphi U,U,\varphi V)+2R(V,\varphi V,V,\varphi U)+3R(U,\varphi V,V,\varphi U)+R(U,V,U,V)\\&=-c\{2g(U,V)^2+2g(U,U)g(U,V)+2g(U,V)g(V,V)+g(U,U)g(V,V)\}.\end{aligned}$$

Replacing V by $-V$ and then summing the resulting equation to the above equation gives

$$3R(U,\varphi V,V,\varphi U)+R(U,V,U,V)=-c\{2g(U,V)^2+g(U,U)g(V,V)\}.\tag{4.7}$$

Replacing V by φV in (4.7) and then using the identities (4.2) and (4.5), we get

$$4R(U,V,U,V)=(c-3\varepsilon)\{g(U,V)^2-g(U,U)g(V,V)\}-3(c+\varepsilon)g(U,\varphi V)^2.\tag{4.8}$$

For $U,V,Z,W\in\operatorname{Ker}\eta$, we determine $R(U+Z,V+W,U+Z,V+W)$ and then using (4.8) we obtain

$$4R(U,V,Z,W)+4R(U,W,Z,V)=(c-3\varepsilon)\{g(U,V)g(Z,W)$$

$$+g(U,W)g(V,Z)-2g(U,Z)g(V,W)\}-3(c+\varepsilon)\{g(U,\varphi V)g(Z,\varphi W)$$
$$+g(U,\varphi W)g(Z,\varphi V)\}. \tag{4.9}$$

Interchanging V and Z in (4.9), and then subtracting the resulting equation with (4.9) and by virtue of the first Bianchi identity we obtain

$$4R(U,W,Z,V)=(c-3\varepsilon)\{g(U,V)g(Z,W)-g(U,Z)g(V,W)\}$$
$$-(c+\varepsilon)\{g(U,\varphi V)g(Z,\varphi W)-g(U,\varphi Z)g(V,\varphi W)+2g(U,\varphi W)g(Z,\varphi V)\}. \tag{4.10}$$

Now if $X,Y,Z,W\in TM$, then replacing U,V,Z,W by $\varphi X,\varphi Y,\varphi Z,\varphi W$ in (4.10), and using (4.1), (4.2), and $\eta(R(X,Y)Z)=\eta(Y)g(X,Z)-\eta(X)g(Y,Z)$ we get (4.3). The converse is trivial. □

Theorem 2. *If a Kenmotsu pseudo-metric manifold has constant φ-sectional curvature c, then it is a space of constant curvature and $c=-\varepsilon$.*

Proof. From (4.3), it is easy to obtain (3.7), where $a=\frac{1}{2}(n(c-3\varepsilon)+(c+\varepsilon))$ and $b=\frac{-1}{2}\varepsilon(n+1)(c+\varepsilon)$. Since a and b are constants, from Corollary 3 it follows that $c=-\varepsilon$. □

5. Some structure theorems

The tangent space T_pM of an almost contact pseudo-metric manifold M can be decomposed as $T_pM=\varphi(T_pM)\oplus L(\xi_p)$, where $L(\xi_p)$ is the linear subspace of T_pM generated by ξ_p. Thus the conformal curvature tensor C is defined as a map

$$C:T_pM\times T_pM\times T_pM\to\varphi(T_pM)\oplus L(\xi_p),\qquad p\in M,$$

such that

$$C(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}\{Ric(Y,Z)X+g(Y,Z)QX-Ric(X,Z)Y$$
$$-g(X,Z)QY\}+\frac{r}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}. \tag{5.1}$$

Then there arise three cases:

- The projection of the image of C in $\varphi(T_pM)$ is zero, that is,
$$C(X,Y,Z,\varphi W)=0,\qquad \text{for any } X,Y,Z,W\in T_pM. \tag{5.2}$$
- Projection of the image of C in $L(\xi_p)$ is zero, that is,
$$C(X,Y)\xi=0,\qquad \text{for all } X,Y\in T_pM. \tag{5.3}$$
- Projection of the image of $C\mid_{\varphi(T_pM)\times\varphi(T_pM)\times\varphi(T_pM)}$ in $\varphi(T_pM)$ is zero, that is,
$$\varphi^2C(\varphi X,\varphi Y)\varphi Z=0,\qquad \text{for all } X,Y,Z\in T_pM. \tag{5.4}$$

An almost contact pseudo-metric manifold satisfying the cases (5.2), (5.3) and (5.4) are said to be conformally symmetric [14], ξ-conformally flat [13] and φ-conformally flat [2], respectively.

We begin with the following:

Theorem 3. *Let M be a ξ-conformally flat Kenmotsu pseudo-metric manifold of dimension higher than 3. Then the scalar curvature r of M satisfies*

$$Dr = \varepsilon\xi(r)\xi, \tag{5.5}$$

where D denotes gradient.

Proof. Since M is ξ-conformally flat, from (5.3) the equation (5.1) becomes

$$\begin{aligned} R(U,V)\xi =& \frac{1}{2n-1}\{Ric(V,\xi)U + \varepsilon\eta(V)QU - Ric(U,\xi)V - \varepsilon\eta(U)QV\} \\ & - \frac{\varepsilon r}{2n(2n-1)}\{\eta(V)U - \eta(U)V\}, \end{aligned} \tag{5.6}$$

for any $U, V \in TM$, and this further gives

$$\begin{aligned} R(U,\xi)V =& \frac{1}{2n-1}\{g(V,Q\xi)U + \varepsilon\eta(V)QU - g(QU,V)\xi - g(U,V)Q\xi\} \\ & - \frac{r}{2n(2n-1)}\{\varepsilon\eta(V)U - g(U,V)\xi\}. \end{aligned} \tag{5.7}$$

Putting $V = \xi$ in (5.6), then differentiating it covariently along W and using (5.7), we get:

$$\begin{aligned} (\nabla_W R)(U,\xi)\xi =& \frac{1}{2n-1}\{g((\nabla_W Q)\xi,\xi)U + \varepsilon(\nabla_W Q)U - g((\nabla_W Q)U,\xi)\xi \\ & - \varepsilon\eta(U)(\nabla_W Q)\xi\} - \frac{Wr}{2n(2n-1)}\{\varepsilon U - \varepsilon\eta(U)\xi\}. \end{aligned}$$

Taking the inner product of the above equation with Y and contracting with respect to U and W yield

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i,\xi)\xi, Y) =& \frac{1}{2n-1}\{g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi)\} \\ & + \frac{\varepsilon(2n-2)}{4n(2n-1)}\{Yr - \eta(Y)\xi(r)\}, \end{aligned} \tag{5.8}$$

where $\{e_i\}$ is a pseudo-orthonormal basis in M and $\varepsilon_i = g(e_i,e_i)$. From the second Bianchi identity we easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Y,\xi)\xi, e_i) = g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi). \tag{5.9}$$

Then from (5.8) and (5.9), noting that $n > 1$ we get

$$g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y, \xi) = \frac{\varepsilon}{4n}\{Yr - \eta(Y)\xi(r)\}.$$

Since ∇Q is symmetric, the above equation becomes

$$g((\nabla_Y Q)\xi, \xi) - g((\nabla_\xi Q)\xi, Y) = \frac{\varepsilon}{4n}\{Yr - \eta(Y)\xi(r)\}. \tag{5.10}$$

From (3.4), the left hand side of above equation vanishes. Then (5.10) leads to $Yr = \eta(Y)\xi(r)$ which gives (5.5). □

Theorem 4. *A Kenmotsu pseudo-metric manifold M is ξ-conformally flat if and only if it is an η-Einstein manifold.*

Proof. If M is ξ-conformally flat, then

$$\begin{aligned} R(X,\xi)\xi =& \frac{1}{2n-1}\{Ric(\xi,\xi)X + \varepsilon QX - Ric(X,\xi)\xi - \varepsilon\eta(X)Q\xi\} \\ & - \frac{\varepsilon r}{2n(2n-1)}\{X - \eta(X)\xi\}. \end{aligned}$$

Making use of equations (3.1) and (3.4) in above gives

$$Q = \left(\frac{r}{2n} + \varepsilon\right) I - \left(\frac{\varepsilon r}{2n} + 2n + 1\right)\varepsilon\eta \otimes \xi,$$

which is equivalent to (3.10).

Conversely, suppose that M is η-Einstein. Formula (5.1) gives

$$\begin{aligned} C(X,Y)\xi =& R(X,Y)\xi - \frac{1}{2n-1}\{Ric(Y,\xi)X + \varepsilon\eta(Y)QX - Ric(X,\xi)Y \\ & - \varepsilon\eta(X)QY\} + \frac{\varepsilon r}{2n(2n-1)}\{\eta(Y)X - \eta(X)Y\}. \end{aligned}$$

Now using identities (3.1), (3.4) and (3.13) results in

$$\begin{aligned} C(X,Y)\xi &= R(X,Y)\xi - \frac{1}{2n-1}\left\{(2n - \varepsilon a) + \frac{\varepsilon r}{2n}\right\}(\eta(X)Y - \eta(Y)X) \\ &= R(X,Y)\xi - (\eta(X)Y - \eta(Y)X) = 0, \end{aligned}$$

and this concludes the proof. □

Theorem 5. *A Kenmotsu pseudo-metric manifold of dimension higher than 3 is φ-conformally flat if and only if it is a space of constant cuvature $-\varepsilon$.*

Proof. Note that the φ-conformally flat condition $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ is equivalent to $C(\varphi X, \varphi Y, \varphi Z, \varphi W) = 0$, and so from (5.1) we get

$$\begin{aligned} & R(\varphi X, \varphi Y, \varphi Z, \varphi W) \\ =& \frac{1}{2n-1}\{Ric(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + g(\varphi Y, \varphi Z)Ric(\varphi X, \varphi W) \\ & - Ric(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - g(\varphi X, \varphi Z)Ric(\varphi Y, \varphi W)\} \end{aligned}$$

$$-\frac{r}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}. \tag{5.11}$$

Let $\{E_i=e_i, E_{n+i}=\varphi e_i, E_{2n+1}=\xi\}_{i=1}^{n}$ be a local pseudo-orthonormal φ-basis. Taking $X=W=E_i$ in (5.11) and summing, we get

$$\begin{aligned}
&\sum_{i=1}^{2n}\varepsilon_i R(\varphi E_i,\varphi Y,\varphi Z,\varphi E_i)\\
&\quad=\sum_{i=1}^{2n}\varepsilon_i\Big[\frac{1}{2n-1}\{Ric(\varphi Y,\varphi Z)g(\varphi E_i,\varphi E_i)+g(\varphi Y,\varphi Z)Ric(\varphi E_i,\varphi E_i)\\
&\qquad-Ric(\varphi E_i,\varphi Z)g(\varphi Y,\varphi E_i)-g(\varphi E_i,\varphi Z)Ric(\varphi Y,\varphi E_i)\}\\
&\qquad-\frac{r}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi E_i,\varphi E_i)-g(\varphi E_i,\varphi Z)g(\varphi Y,\varphi E_i)\}\Big]\\
&\quad=\left(\frac{2n-2}{2n-1}\right)Ric(\varphi Y,\varphi Z)+\frac{1}{2n-1}\left(\frac{r}{2n}+\varepsilon 2n\right)g(\varphi Y,\varphi Z),
\end{aligned} \tag{5.12}$$

where $\varepsilon_i=g(E_i,E_i)$. It can be easily verified that

$$\begin{aligned}
\sum_{i=1}^{2n}\varepsilon_i R(\varphi E_i,\varphi Y,\varphi Z,\varphi E_i)&=Ric(\varphi Y,\varphi Z)-\varepsilon R(\xi,\varphi Y,\varphi Z,\xi)\\
&=Ric(\varphi Y,\varphi Z)+\varepsilon g(\varphi Y,\varphi Z).
\end{aligned}$$

So that equation (5.12) becomes

$$Ric(\varphi Y,\varphi Z)=\left(\varepsilon+\frac{r}{2n}\right)g(\varphi Y,\varphi Z).$$

Substituting this in (5.11), one obtains

$$\begin{aligned}
&R(\varphi X,\varphi Y,\varphi Z,\varphi W)\\
&=\frac{r+4n\varepsilon}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}.
\end{aligned} \tag{5.13}$$

From (4.2), (4.1), (3.2) and (2.2), we get

$$\begin{aligned}
R(\varphi X,\varphi Y,\varphi Z,\varphi W)=&R(X,Y,Z,W)+\eta(Y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W)\\
&-\eta(Y)\eta(W)g(X,Z)+\eta(X)\eta(W)g(Y,Z).
\end{aligned} \tag{5.14}$$

Now (5.13) and (5.14) imply

$$\begin{aligned}
R(X,Y,Z,W)=&\frac{r+4n\varepsilon}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}\\
&-\eta(Y)\eta(Z)g(X,W)+\eta(X)\eta(Z)g(Y,W)\\
&+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z).
\end{aligned} \tag{5.15}$$

Now taking the scalar product of (4.1) with W and by virtue of (5.15) we get an equation and then contracting the resulting equation with respect to X and W gives

$$(2n-2)\left(\frac{r+4n\varepsilon}{2n(2n-1)}+\varepsilon\right)g(Y,\varphi Z)=0.$$

Since $n>1$, it follows that

$$r=-\varepsilon 2n(2n+1). \tag{5.16}$$

Using (5.16) and (2.2) in (5.15), we get

$$R(X,Y,Z,W)=-\varepsilon\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\},$$

and so that the manifold is of constant curvature $-\varepsilon$.

The converse is trivial. □

Corollary 4. *A conformally flat Kenmotsu pseudo-metric manifold of dimension higher than 3 is a space of constant cuvature $-\varepsilon$.*

The above corollary for Riemannian case has been proved in [9].

Now contracting (5.15), we obtain (3.10). Thus we can state the following:

Corollary 5. *A φ-conformally flat Kenmotsu pseudo-metric manifold is an η-Einstein manifold.*

In view of Theorem 4 and Corrollary 5, we have the following:

Corollary 6. *A φ-conformally flat Kenmotsu pseudo-metric manifold is always ξ-conformally flat.*

6. Ricci soliton on Kenmotsu pseudo-metric manifolds

A *Ricci soliton* on a pseudo-Riemannian manifold (M,g) is defined by

$$(\pounds_V g)(X,Y)+2Ric(X,Y)+2\lambda g(X,Y)=0, \tag{6.1}$$

where λ is a constant. Ricci soliton is a natural generalization of the Einstein metric (that is, $Ric(X,Y)=ag(X,Y)$, for some constant a), and is a special self similar solution of Hamilton's Ricci flow (see [8]) $\frac{\partial}{\partial t}g(t)=-2Ric(t)$ with initial condition $g(0)=g$. We say that the Ricci soliton is *steady* when $\lambda=0$, *expanding* when $\lambda>0$ and *shrinking* when $\lambda<0$.

Before producing the main results, we prove the following:

Lemma 2. *A Kenmotsu pseudo-metric manifold (M,g) satisfies*

$$(\nabla_X Q)\xi=-QX-2n\varepsilon X, \tag{6.2}$$

$$(\nabla_\xi Q)X=-2QX-4n\varepsilon X. \tag{6.3}$$

Proof. Differentiating $Q\xi = -2n\varepsilon\xi$, and recalling (2.6) provide (6.2).
Now differentiating (3.1) along W leads to

$$(\nabla_W R)(X,Y)\xi = -R(X,Y)W + \varepsilon g(X,W)Y - \varepsilon g(Y,W)X.$$

Contracting this with respect to X and W gives us

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i,Y)\xi, Z) = Ric(Y,Z) + 2ng(Y,Z). \tag{6.4}$$

From the second Bianchi identity, one can easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Z,\xi)Y, e_i) = g((\nabla_Z Q)\xi, Y) - g((\nabla_\xi Q)Z, Y). \tag{6.5}$$

Fetching (6.5) into (6.4) and with the aid of (6.2), we infer that

$$g((\nabla_\xi Q)Z,Y) = -2Ric(Y,Z) - 4ng(Y,Z),$$

which proves (6.3). □

Theorem 6. *Let (M,g) be a Kenmotsu pseudo-metric manifold. If (g,V) is a Ricci soliton, then the soliton constant $\lambda = 2n\varepsilon$, and so the soliton is either expanding or shrinking depending on the casual character of the Reeb vector field ξ.*

Proof. Differentiating (6.1) covariantly along Z gives

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z Ric)(X,Y). \tag{6.6}$$

From Yano [12], we know the following well known commutation formula:

$$\begin{aligned}(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z)\\ = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y),\end{aligned}$$

for all $X,Y,Z \in TM$. Since $\nabla g = 0$, the previous equation gives

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y), \tag{6.7}$$

for all $X,Y,Z \in TM$. As $\pounds_V \nabla$ is a symmetric, it follows from (6.7) that

$$\begin{aligned} &g((\pounds_V \nabla)(X,Y),Z)\\ &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(Z,X) - \frac{1}{2}(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{6.8}$$

Making use of (6.6) in (6.8) we have

$$g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z Ric)(X,Y) - (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(Z,X). \tag{6.9}$$

Putting $Y = \xi$ in (6.9) and using (6.2) and (6.3), we obtain

$$(\pounds_V \nabla)(X,\xi) = 2QX + 4n\varepsilon X.$$

Differentiating the preceding equation along Y and using (2.6), we obtain

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = -(\pounds_V \nabla)(X,Y) + 2\eta(Y)\{QX + 2n\varepsilon X\} + 2(\nabla_Y Q)X.$$

Feeding the above obtained expression into the following relation (see [12])

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z), \tag{6.10}$$

and using the symmetry of $\pounds_V \nabla$, we immediately obtain

$$\begin{aligned}(\pounds_V R)(X,Y)\xi =& 2\eta(X)\{QY + 2n\varepsilon Y\} - 2\eta(Y)\{QX + 2n\varepsilon X\}\\ &+ 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\}.\end{aligned} \tag{6.11}$$

Setting $Y = \xi$ in the foregoing equation, we get

$$(\pounds_V R)(X,\xi)\xi = 0. \tag{6.12}$$

Now taking the Lie-derivative of $R(X,\xi)\xi = -X + \eta(X)\xi$ along V gives

$$(\pounds_V R)(X,\xi)\xi - 2\eta(\pounds_V \xi)X + \varepsilon g(X, \pounds_V \xi)\xi = (\pounds_V \eta)(X)\xi,$$

which by virtue of (6.12) becomes

$$(\pounds_V \eta)(X)\xi = -2\eta(\pounds_V \xi)X + \varepsilon g(X, \pounds_V \xi)\xi. \tag{6.13}$$

With the help of (3.4), the equation (6.1) takes the form

$$(\pounds_V g)(X,\xi) = -2\lambda\varepsilon\eta(X) + 4n\eta(X). \tag{6.14}$$

Changing X to ξ in the aforementioned equation gives

$$\eta(\pounds_V \xi) = \lambda - 2n\varepsilon. \tag{6.15}$$

Now Lie-differentiating $\eta(X) = \varepsilon g(X,\xi)$ yields $(\pounds_V \eta)(X) = \varepsilon(\pounds_V g)(X,\xi) + \varepsilon g(X, \pounds_V \xi)$. Using this and (6.15) in (6.13) provides $(\lambda - 2n\varepsilon)(X - \eta(X)\xi) = 0$. Tracing the previous equation yield $\lambda = 2n\varepsilon$. □

Corollary 7. *A Kenmotsu manifold admitting the Ricci soliton is always expanding with $\lambda = 2n$.*

Lemma 3. *Let (M,g) be a Kenmotsu pseudo-metric manifold. If (g,V) is a Ricci soliton, then the Ricci tensor satisfies*

$$(\pounds_V Ric)(X,\xi) = -X(r) + \xi(r)\eta(X). \tag{6.16}$$

Proof. Contracting equation (6.11) with respect to X and recalling the well-known formulas

$$\operatorname{div} Q = \frac{1}{2} Dr \quad \text{and} \quad \operatorname{trace} \nabla Q = Dr,$$

we easily obtain

$$(\pounds_V Ric)(Y,\xi) = -Y(r) - 2\eta(Y)\{r + \varepsilon 2n(2n+1)\}. \tag{6.17}$$

Substituting $Y = \xi$, we have $(\pounds_V Ric)(\xi,\xi) = -\xi(r) - 2\{r + \varepsilon 2n(2n+1)\}$. On the other hand, contracting (6.12) gives $(\pounds_V Ric)(\xi,\xi) = 0$. Using this in the previous equation leads to

$$\xi(r) = -2(r + \varepsilon 2n(2n+1)). \tag{6.18}$$

Hence (6.18) and (6.17) give (6.16). □

Combining Theorem 3 and 4, we state the following:

Lemma 4. *An η-Einstein Kenmotsu pseudo-metric manifold M of dimension higher than 3 satisfies*

$$Dr = \varepsilon \xi(r)\xi. \tag{6.19}$$

Now we prove:

Theorem 7. *Let (M,g) be an η-Einstein Kenmotsu pseudo-metric manifold of dimension higher than 3. If (g,V) is a Ricci soliton, then M is Einstein.*

Proof. Making use of (6.19) in (6.16), we have $(\pounds_V Ric)(X,\xi) = 0$. Now, Lie-differentiating the first relation of (3.4) along V, using (3.10), (6.14), $\lambda = 2n\varepsilon$ and $\eta(\pounds_V \xi) = 0$, we obtain

$$(r + \varepsilon 2n(2n+1))\pounds_V \xi = 0.$$

If $r = -\varepsilon 2n(2n+1)$, then (3.10) shows that M is Einstein.

So we assume $r \neq -\varepsilon 2n(2n+1)$ in some open set $\mathcal{O}$ of M. Hence $\pounds_V \xi = 0$ on $\mathcal{O}$, and so it follows from (2.6) that

$$\nabla_\xi V = V - \eta(V)\xi. \tag{6.20}$$

Clearly, (6.14) shows that $(\pounds_V g)(X,\xi) = 0$. This together with (6.20), we have

$$g(\nabla_X V,\xi) = -g(\nabla_\xi V, X) = -g(X,V) + \eta(X)\eta(V). \tag{6.21}$$

From Duggal and Sharma [5], we know that

$$(\pounds_V \nabla)(X,Y) = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} V + R(V,X)Y.$$

Setting $Y = \xi$ in the above equation and by virtue of (2.6), (3.1), (6.20) and (6.21), we have $r = -\varepsilon 2n(2n+1)$. This leads to a contradiction as $r \neq -\varepsilon 2n(2n+1)$ on $\mathcal{O}$ and completes the proof. □

Now we consider Kenmotsu pseudo-metric 3-manifolds which admits Ricci solitons.

Theorem 8. *Let (M,g) be a Kenmotsu pseudo-metric 3-manifold. If (g,V) is a Ricci soliton, then M is of constant curvature $-\varepsilon$.*

Proof. The Riemannian curvature tenor of pseudo-Riemannian 3-manifold is given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{6.22}$$

Taking $Y = Z = \xi$ in (6.22) and using (3.1) and (3.4) gives

$$Q = \left(\frac{r}{2}+1\right) I - \left(\frac{r}{2}+3\right) \eta \otimes \xi. \tag{6.23}$$

Making use of this in (6.11) gives

$$\begin{aligned}(£_V R)(X,Y)\xi =& X(r)\{Y - \eta(Y)\xi\} + Y(r)\{-X + \eta(X)\xi\} \\ & - (r+6\varepsilon)\{\eta(Y)X - \eta(X)Y\}.\end{aligned} \tag{6.24}$$

Replacing Y by ξ in the above equation and comparing it with (6.12), we obtain

$$\{\xi(r) + (r+6\varepsilon)\}\{-X + \eta(X)\xi\} = 0.$$

Contracting the above equation gives $\xi(r) + (r+6\varepsilon) = 0$, and consequently it follows from (6.18) that $r = -6\varepsilon$. Then from (6.23) we have $QX = -2\varepsilon X$, and substituting this into (6.22) shows that M is of constant curvature $-\varepsilon$. □

Corollary 8. *There does not exist a Kenmotsu pseudo-metric manifold (M, g) admitting the Ricci soliton $(g, V = \xi)$.*

Proof. If $V = \xi$, then from (2.8) the Ricci soliton equation (6.1) would become

$$Ric = -(1+\lambda)g + \varepsilon \eta \otimes \eta, \tag{6.25}$$

which means M is η-Einstein. Then due to Theorem 7 and 8, M must be Einstein, and this will be a contradiction to equation (6.25) as $\varepsilon \neq 0$. □

Remark 1. Clearly, Theorem 7 and 8 are generalizations of the results of Ghosh proved in [6] and [7]. Note that our approach and technique to obtain the result is different to that of Ghosh.

Now we provide an example of a Kenmotsu pseudo-metric 3-manifold which admits a Ricci soliton and verify our results.

Example 1. Let $M = N \times I$, where N is an open connected subset of $\mathbb{R}^2$ and I is an open interval in $\mathbb{R}$. Let (x, y, z) be the Cartesian coordinates in M. Define the structure (φ, ξ, η, g) on M as follows:

$$\begin{aligned} &\varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \\ &\xi = \frac{\partial}{\partial z}, \quad \eta = dz, \\ &(g_{ij}) = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{2z} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}. \end{aligned}$$

Now from Koszul's formula, the Levi-Civita connection ∇ is given by

$$\begin{array}{lll} \nabla_{\partial_1}\partial_1 = -\varepsilon e^{2z}\partial_3, & \nabla_{\partial_1}\partial_2 = 0, & \nabla_{\partial_1}\partial_3 = \partial_1, \\ \nabla_{\partial_2}\partial_1 = 0, & \nabla_{\partial_2}\partial_2 = -\varepsilon e^{2z}\partial_3, & \nabla_{\partial_2}\partial_3 = \partial_2, \\ \nabla_{\partial_3}\partial_1 = \partial_1, & \nabla_{\partial_3}\partial_2 = \partial_2, & \nabla_{\partial_3}\partial_3 = 0, \end{array} \tag{6.26}$$

where $\partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. From (6.26), one can easily verify

$$(\nabla_{\partial_i}\varphi)\partial_j = -\eta(\partial_j)\varphi\partial_i - \varepsilon g(\partial_i, \varphi\partial_j)\xi, \tag{6.27}$$

for all $i, j = 1, 2, 3$, and so M is a Kenmotsu pseudo-metric manifold with the above (φ, ξ, η, g) structure.

With the help of (6.26), we find that:

$$\begin{gathered} R(\partial_1,\partial_2)\partial_3 = R(\partial_2,\partial_3)\partial_1 = R(\partial_1,\partial_3)\partial_2 = 0, \\ R(\partial_1,\partial_3)\partial_1 = R(\partial_2,\partial_3)\partial_2 = \varepsilon e^{2z}\partial_3, \\ R(\partial_1,\partial_2)\partial_1 = \varepsilon e^{2z}\partial_2, \quad R(\partial_2,\partial_3)\partial_3 = -\partial_2, \\ R(\partial_1,\partial_3)\partial_3 = -\partial_1, \quad R(\partial_1,\partial_2)\partial_2 = -\varepsilon e^{2z}\partial_1. \end{gathered} \tag{6.28}$$

Let $e_1 = e^{-z}\partial_1, e_2 = e^{-z}\partial_2$ and $e_3 = \xi = \partial_3$. Clearly, $\{e_1, e_2, e_3\}$ forms an orthonormal φ-basis of vector fields on M. Making use of (6.28) one can easily show that M is Einstein, that is, $Ric(Y,Z) = -2\varepsilon g(Y,Z)$, for any $Y, Z \in TM$.

Let us consider the vector field

$$V = \alpha\frac{\partial}{\partial y}, \tag{6.29}$$

where α is a non-zero constant. Making use of (6.26) one can easily show that V is Killing with respect to g, that is, we have

$$(\pounds_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0,$$

for any $X, Y \in TM$. Hence g is a Ricci soliton, that is, (6.1) holds true with V as in (6.29) and $\lambda = 2\varepsilon$. Further (6.28) shows that

$$R(X,Y)Z = -\varepsilon\{g(Y,Z)X - g(X,Z)Y\},$$

for any $X, Y \in TM$, which means M is of constant curvature $-\varepsilon$ and so Theorem 8 is verified.

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