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CERTAIN SCHUR-HADAMARD MULTIPLIERS IN THE SPACES C_p

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ABSTRACT. Let f be a continuously differentiable function on $[-1,1]$ satisfying $|f'(t)| \leq C|t|^\alpha$ for some $0 < C$, $\alpha < \infty$ and all $-1 \leq t \leq 1$, and let $\lambda = (\lambda_i) \in l_r$ satisfy $-1 \leq \lambda_i \leq 1$ for all i. Then

$$a_{f,\lambda} = \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \right)$$

is a Schur-Hadamard multiplier from C_{p_1} into C_{p_2} for all p_1, p_2 satisfying $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $p_2^{-1} \leq p_1^{-1} + \alpha/r$.

1. Introduction . Let C_∞ be the space of all compact operators on l_2 with the operator norm. For $1 \leq p < \infty$ let C_p be the Banach space of all $x \in C_\infty$ for which $||x||_p = (\text{trace}(x^*x)^{p/2})^{1/p} < \infty$. (See [**9**, Chapter III] for a detailed study of these spaces and related topics.) A matrix $a = (a(i,j))$ is said to be a *Schur-Hadamard multiplier* (or, briefly, *a multiplier*) *from* C_{p_1} *into* C_{p_2} if for every $b = (b(i,j))$ in C_{p_1} the Schur-Hadamard product of a and b, namely $a \circ b = (a(i,j)b(i,j))$, belongs to C_{p_2}. We denote by $M(C_{p_1}, C_{p_2})$ the space of all multipliers from C_{p_1} into C_{p_2} with the norm

$$||a||_{p_1,p_2} = \sup\{||a \circ b||_{p_2}; ||b||_{p_1} \leq 1\}.$$

In this note we study multipliers of the form

$$a_{f,\lambda} = (a_{f,\lambda}(i,j)),$$

where

$$a_{f,\lambda}(i,j) = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}, & \lambda_i \neq \lambda_j, \\ f'(\lambda_i), & \lambda_i = \lambda_j. \end{cases} \tag{1.1}$$

Here f is a continuously differentiable function on $[-1,1]$ and $\lambda = (\lambda_i)$ is a real sequence with $-1 \leq \lambda_i \leq 1$, $i = 1, 2, \ldots$. Such multipliers are of great importance in perturbation theory of linear operators (see [**1, 3, 5–8**]). The multiplier $a_{f,\lambda}$ plays the role of the Gâteaux derivative of the operator map $x \mapsto f(x)$, evaluated at the diagonal matrix $d(\lambda) = \text{diag}(\lambda_i)$, whenever the derivative exists.

In the papers [**1–5**] the authors study general multipliers from C_p into itself (and from $B(l_2)$ into itself) under the name "Stieltjes Double Operator Integrals". In their applications to the multipliers $a_{f,\lambda}$ there is no restriction on λ besides boundedness. As for f, boundedness of the derivative f' is clearly a necessary condition for the boundness of the multiplier $a_{f,\lambda}$ in C_p for all λ, but if $p \neq 2$ it is not sufficient and

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one has to impose more restrictive smoothness conditions on f. (The case $p = 2$ is trivial. C_2, the Hilbert-Schmidt class, is a Hilbert space by itself and thus its multipliers are precisely the coordinate-wise bounded matrices.) A typical result is the following

THEOREM 1.1 [**1**, THEOREM 5]. *Let f be so that $f' \in \text{Lip}_\alpha[-1,1]$ for some $\alpha > 0$, and let $\lambda = (\lambda_i)$ satisfies $-1 \leq \lambda_i \leq 1$ for all i. Then $a_{f,\lambda} \in M(B(l_2), B(l_2))$ and $a_{f,\lambda} \in M(C_p, C_p)$ for all $1 \leq p \leq \infty$.*

We are interested here in the little different problem of membership of $a_{f,\lambda}$ in $M(C_{p_1}, C_{p_2})$ for $p_1 \neq p_2$, mainly for $p_2 \leq 2 \leq p_1$. Roughly speaking, we require that $\lambda \in l_r$ for some $r < \infty$, and relax the Lip_α condition on f' by requiring only that $|f'(t)| \leq C|t|^\alpha$ for some $0 < C < \infty$ and all $-1 \leq t \leq 1$, where the exponent $\alpha > 0$ satisfies $1/p_2 \leq 1/p_1 + \alpha/r$. The condition $p_2 \leq 2 \leq p_1$ allows us to factor $a_{f,\lambda}$ through C_2, and thus to reduce the study of the multipliers $a_{f,\lambda}$ to that of the multipliers $b_\mu = (\mu_{\min\{i,j\}})$ where $\mu_i = |\lambda_i|^\beta$ for an appropriate β. Finally, using the triangular projection, we reduce the study of the multipliers b_μ to that of left and right multiplication by diagonal matrices. We also show that, in a sense, the result stated in the abstract is best possible.

2. The main result. We start with some known facts on multipliers which will be needed later.

PROPOSITION 2.1. $M(C_2, C_2) = 1_\infty(\mathbf{N} \times \mathbf{N})$.

PROOF. This follows immediately from $C_2 = l_2(\mathbf{N} \times \mathbf{N})$. □

PROPOSITION 2.2. *Let $1 \leq p_1 \leq q_1 \leq \infty$, $1 \leq q_2 \leq p_2 \leq \infty$. Then $M(C_{q_1}, C_{q_2}) \subseteq M(C_{p_1}, C_{p_2})$ and $||a||_{p_1,p_2} \leq ||a||_{q_1,q_2}$ for all $a \in M(C_{q_1}, C_{q_2})$.*

PROOF. This follows trivially from the fact that if $p \leq q$ then $C_p \subseteq C_q$ and $||x||_q \leq ||x||_p$ for all $x \in C_p$. □

COROLLARY 2.3. *Let $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$. Let $\lambda = (\lambda_i)$ be so that $-1 \leq \lambda_i \leq 1$ for all i, and let f be any continuously differentiable function on $[-1,1]$. Then $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$.*

PROOF. By Propositions 2.1 and 2.2, $a_{f,\lambda} \in M(C_2, C_2) \subseteq M(C_{p_1}, C_{p_2})$. □

Let D denotes the diagonal projection, i.e. $(Da)(i,j) = \delta_{i,j}a(i,i)$.

PROPOSITION 2.4.

$$D(M(C_{p_1}, C_{p_2})) = M(l_{p_1}, l_{p_2}) = l_{p_0}$$

where $1/p_2 = 1/p_1 + 1/p_0$.

PROOF. l_p is identified with $D(C_p)$, so

$$D(M(C_{p_1}, C_{p_2})) = M(D(C_{p_1}), D(C_{p_2})) = M(l_{p_1}, l_{p_2}).$$

The fact that $M(l_{p_1}, l_{p_2}) = l_{p_0}$ is well known. □

Next for $0 < \alpha < \infty$ let X_α be the space of all continuously differentiable functions on $[-1,1]$ satisfying $f(0) = 0$ and $|f'(t)| \leq C|t|^\alpha$ for some $0 < C < \infty$ and all $t \in [-1,1]$, with the norm

$$||f||_\alpha = \inf\{0 < C < \infty; |f'(t)| \leq C|t|^\alpha \text{ for all } -1 \leq t \leq 1\}.$$

Our main result is the following:

THEOREM 2.5. *Let* $0 < \alpha, r < \infty$. *Let* $f \in X_\alpha$ *and let* $\lambda = (\lambda_i) \in l_r$ *be so that* $-1 \le \lambda_i \le 1$ *for all* i. *Then*

$$a_{f,\lambda} = ((f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j))$$

is a multiplier from C_{p_1} *into* C_{p_2} *for all* p_1, p_2 *satisfying* $1 \le p_2 \le 2 \le p_1 \le \infty$ *and* $1/p_2 = 1/p_1 + \alpha/r$.

Moreover,

$$\|a_{f,\lambda}\|_{p_1,p_2} \le 2\|f\|_\alpha \cdot \|\lambda\|_r^\alpha.$$

We remark that using Proposition 2.2 one gets easily that, under the hypotheses of Theorem 2.5, $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$ for all $1 \le p_2 \le 2 \le p_1 \le \infty$ with $1/p_2 \le 1/p_1 + \alpha/r$.

The proof of Theorem 2.5 proceeds by a sequence of propositions.

PROPOSITION 2.6. *Let* $0 < \alpha < \infty$, *let* $f \in X_\alpha$ *and let* $\lambda = (\lambda_i)$ *be so that* $-1 \le \lambda_i \le 1$ *for all* i. *Let* $m_{f,\lambda,\alpha}$ *be defined by*

$$m_{f,\lambda,\alpha}(i,j) = \begin{cases} (f(\lambda_i) - f(\lambda_j))((\lambda_i - \lambda_j)\max\{|\lambda_i|^\alpha, |\lambda_j|^\alpha\})^{-1} \\ \qquad\qquad\qquad\qquad \text{if } \lambda_i \ne 0 \text{ or } \lambda_j \ne 0, \\ 0 \quad \text{if } \lambda_i = \lambda_j = 0. \end{cases} \tag{2.1}$$

Then $m_{f,\lambda,\alpha} \in M(C_2, C_2)$ *and* $\|m_{f,\lambda,\alpha}\|_{2,2} \le \|f\|_\alpha$.

PROOF. Using Proposition 2.1 we get

$$\|m_{f,\lambda,\alpha}\|_{2,2} = \sup_{i,j} |m_{f,\lambda,\alpha}(i,j)| \le \|f\|_\alpha. \quad \square$$

Next, let P_T denotes the (upper) triangular projection, that is

$$(P_T x)(i,j) = \begin{cases} x(i,j), & i \le j, \\ 0, & i > j. \end{cases}$$

It is known that P_T is bounded in C_p if and only if $1 < p < \infty$ (see [**10**, Chapter III and **11**]). In this case, let γ_p denotes the norm of P_T in C_p.

For any $\lambda = (\lambda_i) \in l_\infty$ let $d(\lambda) = \text{diag}(\lambda_i)$ be the diagonal matrix whose (i,i) entry is λ_i. We denote by L_λ and R_λ the operators of left and right multiplication by $d(\lambda)$, respectively. That is

$$(L_\lambda x)(i,j) = \lambda_i x(i,j), \qquad (R_\lambda x)(i,j) = x(i,j)\lambda_j.$$

Since L_λ, R_λ, P_T are (identified with) multipliers—they commute with each other.

Finally, for all sequences $\lambda = (\lambda_i)$ we define a matrix $b_\lambda = (b_\lambda(i,j))$ by

$$b_\lambda(i,j) = \lambda_{\min\{i,j\}}.$$

Notice that for all matrix x

$$b_\lambda \circ x = L_\lambda P_T x + R_\lambda (I - P_T)x = P_T L_\lambda x + (I - P_T) R_\lambda x. \tag{2.2}$$

PROPOSITION 2.7. *Let* $1 \le p_0, p_1, p_2 \le \infty$ *be so that* $1/p_2 = 1/p_0 + 1/p_1$. *Let* $\lambda \in l_{p_0}$. *Then* L_λ *and* R_λ *map* C_{p_1} *into* C_{p_2} *and* $\|L_\lambda\|_{p_1,p_2} = \|R_\lambda\|_{p_1,p_2} = \|\lambda\|_{p_0}$.

We omit the straightforward proof which depends on the "generalized Hölder inequality" in the spaces C_p (see [**9**, Chapter III]):

$$\|xy\|_{p_2} \leq \|x\|_{p_0}\|y\|_{p_1}, \qquad 1/p_2 = 1/p_0 + 1/p_1.$$

PROPOSITION 2.8. *Let $1 \leq p_0, p_1, p_2 \leq \infty$ be so that $1/p_2 = 1/p_0 + 1/p_1$. Let $\lambda \in l_{p_0}$. Then $b_\lambda \in M(C_{p_1}, C_{p_2})$ with norm $\|b_\lambda\|_{p_1,p_2} \leq A\|\lambda\|_{p_0}$ (the constant A depends only on p_1, p_2), except for the cases $p_1 = p_2 = \infty$ and $p_1 = p_2 = 1$.*

PROOF. If $1 < p_1 < \infty$ then P_T is bounded in C_{p_1}, and L_λ and R_λ map C_{p_1} into C_{p_2}. Using (2.2) and Proposition 2.7 we get $b_\lambda \in M(C_{p_1}, C_{p_2})$ and $\|b_\lambda\|_{p_1,p_2} \leq (1+2\gamma_{p_1})\|\lambda\|_{p_0}$. (If $1 < p_2 < \infty$ then, using the boundedness of P_T in C_{p_2}, we get in a similar way $b_\lambda \in M(C_{p_1}, C_{p_2})$ and $\|b_\lambda\|_{p_1,p_2} \leq (1 + 2\gamma_{p_2})\|\lambda\|_{p_0}$.) It remains to deal with the case $p_0 = p_2 = 1$, $p_1 = \infty$. For $x \in C_\infty$ define $x_n(i,j) = x(i,j)$ if $\min\{i,j\} = n$ and $x_n(i,j) = 0$ otherwise. Clearly, $x = \sum_{n=1}^\infty x_n$ and $\|x_n\|_1 \leq 2\|x\|_\infty$. So

$$\begin{aligned}\|b_\lambda \circ x\|_1 = \left\|\sum_{n=1}^\infty \lambda_n x_n\right\|_1 &\leq \sum_{n=1}^\infty |\lambda_n|\,\|x_n\|_1 \\ &\leq 2\|x\|_\infty \sum_{n=1}^\infty |\lambda_n| = 2\|x\|_\infty\|\lambda\|_1,\end{aligned}$$

i.e. $b_\lambda \in M(C_\infty, C_1)$ and $\|b_\lambda\|_{\infty,1} \leq 2\|\lambda\|_1$. □

REMARK. For general $\lambda \in l_\infty$, b_λ need not be defined on the whole of C_∞ (or C_1); this is the consequence of [**11**, Proposition 1.3] and the unboundedness of the triangular projection in C_∞ (respectively, in C_1). If $\lambda \in c_0$ and

$$\sum_{i=1}^\infty \lambda_i^*(2i-1)^{-1} < \infty,$$

where $\{\lambda_i^*\}_{i=1}^\infty$, is the nonincreasing rarrangement of $\{|\lambda_i|\}_{i=1}^\infty$, then

$$b_\lambda \in M(C_\infty, C_\infty) = M(C_1, C_1).$$

This follows by the above arguments from the fact that P_T acts continuously from the Macaev ideal

$$C_\omega = \left\{x \in C_\infty; \|x\|_\omega = \sum_{i=1}^\infty s_i(x)(2i-1)^{-1} < \infty\right\}$$

into C_∞ and from C_1 into $C_\Omega = C_\omega^*$. (Here $\{s_i(x)\}_{i=1}^\infty$ are the s-numbers of x; see [**10**, Chapter III].)

PROOF OF THEOREM 2.5. Since $\lambda_i \to 0$, we can assume without loss of generality that $\{|\lambda_i|\}_{i=1}^\infty$ is nonincreasing. Thus

$$\max\{|\lambda_i|^\alpha, |\lambda_j|^\alpha\} = |\lambda_{\min\{i,j\}}|^\alpha.$$

Let α_1 be defined by $1/2 = 1/p_1 + \alpha_1/r$ and let $\alpha_2 = \alpha - \alpha_1$. Clearly, $0 \leq \alpha_1, \alpha_2 \leq \alpha$ and $1/p_2 = 1/2 + \alpha_2/r$. We present the proof in the case $p_2 < 2 < p_1$, i.e. $0 < \alpha_1$, $\alpha_2 < \alpha$. The other cases are treated similarly, and are even easier.

Let $m_{f,\lambda,\alpha}$ be defined by (2.1), let $\mu^{(k)} = |\lambda|^{\alpha_k}$ $(k = 1, 2)$, i.e. $\mu_i^{(k)} = |\lambda_i|^{\alpha_k}$. Then $a_{f,\lambda}$ admits the following factorization:

$$a_{f,\lambda} = b_{\mu^{(2)}} \circ m_{f,\lambda,\alpha} \circ b_{\mu^{(1)}}.$$

Now, by Proposition 2.8 $b_{\mu^{(1)}} \in M(C_{p_1}, C_2)$, $b_{\mu^{(2)}} \in M(C_2, C_{p_2})$ and

$$||b_{\mu^{(1)}}||_{p_1,2} \leq \sqrt{2}||\mu^{(1)}||_{r/\alpha_1} = \sqrt{2}||\lambda||_r^{\alpha_1},$$
$$||b_{\mu^{(2)}}||_{2,p_2} \leq \sqrt{2}||\mu^{(2)}||_{r/\alpha_2} = \sqrt{2}||\lambda||_r^{\alpha_1}.$$

Since, by Proposition 2.6, $m_{f,\lambda,\alpha} \in M(C_2, C_2)$ and $||m_{f,\lambda,\alpha}||_{2,2} \leq ||f||_\alpha$ we finally get that $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$ and

$$\begin{aligned}||a_{f,\lambda}||_{p_1,p_2} &\leq ||b_{\mu^{(2)}}||_{2,p_2} \cdot ||m_{f,\lambda,\alpha}||_{2,2} \cdot ||b_{\mu^{(1)}}||_{p_1,2} \\ &\leq \sqrt{2}||\lambda||_r^{\alpha_2} \cdot ||f||_{\alpha} \cdot \sqrt{2}||\lambda||_r^{\alpha_1} = 2||f||_{\alpha}||\lambda||_r^{\alpha}. \quad \square\end{aligned}$$

3. Concluding remarks. Theorem 2.5 admits the following partial converse, which shows that our hypotheses on f and λ are optimal.

PROPOSITION 3.1. *Let $1 \leq p_2 < p_1 \leq \infty$, let $\alpha > 0$ and let $0 < r < \infty$ be so that $1/p_2 = 1/p_1 + \alpha/r$.*

(i) *Let $\lambda = (\lambda_i)$ be so that $-1 \leq \lambda_i \leq 1$ for all i, and assume that for all $f \in X_\alpha$, $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$. Then $\lambda \in l_r$.*

(ii) *Let f be a continuous differentiable function on $[-1, 1]$ so that for $\lambda = (\lambda_i) \in l_r$ with $-1 \leq \lambda_i \leq 1$ for all i, $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$. Then $f \in X_\alpha$.*

PROOF. (i) Consider $f(t) = t|t|^\alpha$. We have $f'(t) = (\alpha + 1)|t|^\alpha$ and so $f \in X_\alpha$. Since $a_{f,\lambda} \in M(C_{p_1}, C_{p_2})$, we get by Proposition 2.4 that

$$D(a_{f,\lambda}) = \text{diag}((\alpha + 1)(\lambda_i)^\alpha) \in M(l_{p_1}, l_{p_2}) = l_{r/\alpha}.$$

That is $\lambda \in l_r$.

(ii) Suppose that $f \notin X_\alpha$. Then there is a sequence $\{t_n\}_{n=1}^\infty$ in $[-1, 1]$ with $0 < |t_n| \leq 2^{-n(1+1/\alpha)}$ and $|f'(t_n)| \geq 2^n|t_n|^\alpha$ for all $n = 1, 2, \ldots$. Put

$$k_n = [2^{-nr/\alpha} \cdot |t_n|^{-r}], \qquad n = 1, 2, \ldots,$$

and let $\lambda = (\lambda_i)$ be the sequence in which t_n appears exactly k_n times. Then

$$\sum_{i=1}^\infty |\lambda_i|^r = \sum_{n=1}^\infty k_n|t_n|^r \leq \sum_{n=1}^\infty 2^{-nr/\alpha} < \infty,$$

but

$$\begin{aligned}\sum_{i=1}^\infty |f'(\lambda_i)|^{r/\alpha} &= \sum_{n=1}^\infty k_n|f'(t_n)|^{r/\alpha} \\ &\geq \sum_{n=1}^\infty k_n 2^{nr/\alpha}|t_n|^r \geq \sum_{n=1}^\infty (1 - 2^{-nr}) = \infty.\end{aligned}$$

It follows that $(f'(\lambda_i)) \notin M(l_{p_1}, l_{p_2})$, and so by Proposition 2.4,

$$a_{f,\lambda} \notin M(C_{p_1}, C_{p_2}).$$

A contradiction. $\square$

REMARK. Our methods and results can be extended by standard arguments to other symmetric norm ideals. For instance, if $a \in M(C_{p_1}, C_{p_2})$ then $a \in M(C_{p_2^*}, C_{p_1^*})$ where $1/p_j + 1/p_j^* = 1$, $j = 1, 2$. This is the consequence of the fact that the adjoint operator of the multiplier a is the multiplier $\bar{a} = (\overline{a(i,j)})$, the fact that $||\bar{a}||_{p,q} = ||a||_{p,q}$ and the well-known fact that $(C_p)^* = C_{p^*}$. It follows that if $\mathcal{F}$ is any interpolation functor and if $C_E = \mathcal{F}(C_{p_1}, C_{p_2^*}) = C_{\mathcal{F}(l_{p_1}, l_{p_2^*})}$, $C_F = \mathcal{F}(C_{p_2}, C_{p_1^*}) = C_{\mathcal{F}(l_{p_2}, l_{p_1^*})}$ then $a \in M(C_E, C_F)$. For instance, if $0 < \theta < 1$ and $p(\theta)$, $q(\theta)$ are defined by $1/p(\theta) = (1-\theta)/p_1 + \theta/p_2^*$ and $1/q(\theta) = (1-\theta)/p_2 + \theta/p_1^*$, then $a \in M(C_{p(\theta)}, C_{q(\theta)})$.

We conclude the paper with the following

CONJECTURE. *Theorem* 2.5 *remains true even in the case where* p_1, p_2 *are on the same side of* 2, *i.e.*, $1 \le p_2 \le p_1 \le 2$, *or* $2 \le p_2 \le p_1 \le \infty$.

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