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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING COMPLEX ORDER

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Abstract

In the present investigation, we consider an unified class of functions of complex order. Necessary and sufficient condition for functions to be in this class is obtained. The results obtained in this paper generalizes the results obtained by Srivastava and Lashin [10], and Ravichandran *et al.* [4].

1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^2 + \cdots \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ .

A function $f \in \mathcal{A}$ is subordinate to an univalent function $g \in \mathcal{A}$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

Let Ω be the family of analytic functions $\omega(z)$ in the unit disc Δ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(\omega(z))$.

Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Ma and Minda [2] introduced and studied the class $S^*(\phi)$, consists of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta).$$

Recently, Ravichandran et al. [4] defined classes related to the class of starlike functions of complex order defined as

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Definition 1.1. *Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [2], Shanmugam and Sivasubramanian [7] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\frac{\alpha z^2 f''(z) + zf'(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \phi(z),$$

where $\phi(z)$ satisfies the condition mentioned in Definition 1.1. Kamali and Akbulut [1] introduced and studied a new class of functions $f \in T$ for which

$$\Re\left(\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2\ f''(z) + zf'(z)}\right) > \beta,\ 0 \leq \alpha < 1,\ 0 \leq \beta < 1.$$

We remark here that the class of functions T is the familiar class of functions introduced and studied by Silverman [9]. In a later investigation, this particular class introduced by Kamali and Akbulut was generalized by Shanmugam et al. [8].

In this paper, we introduce a more general class of complex order $M[b,\alpha](\phi)$ which we define below.

Definition 1.2. *Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M[b,\alpha](\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying*

$$1 + \frac{1}{b}\left(\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2\ f''(z) + zf'(z)} - 1\right) \prec \phi(z), \quad (0 \leq \alpha < 1)$$

Clearly,

$$M[b,0] \equiv C_b(\phi).$$

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $M[b,\alpha](\phi)$ which we have defined. The Motivation of this paper is to generalize the results obtained by Ravichandran et al. [4] and also by Srivastava and Lashin [10].

In order to prove our main results, we need the following lemmas.

The following result follows from a result of Ruscheweyh [5] for functions in the class $S^*(\phi)$ (see Ruscheweyh [6, Theorem 2.37, pages 86–88]).

Lemma 1.3. *[4] Let ϕ be a convex function defined on Δ, $\phi(0) = 1$. Define $F(z)$ by*

$$F(z) = z \exp\left(\int_0^z \frac{\phi(x) - 1}{x} dx\right). \tag{1.2}$$

Let $q(z) = 1 + c_1 z + \cdots$ be analytic in Δ. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z) \tag{1.3}$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{1.4}$$

Lemma 1.4. *[3, Corollary 3.4h.1, p.135] Let $q(z)$ be univalent in Δ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $\dfrac{zq'(z)}{\varphi(q(z))}$ is starlike, then*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)),$$

implies that $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2 Subordination Results

By making use of Lemma 1.3, we have the following:

Theorem 2.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.3. The function $f \in M[b, \alpha](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s\left[\alpha z^2 \ f''(tz) + zf'(tz)\right]}{t\left[\alpha z^2 \ f''(sz) + zf'(sz)\right]}\right)^{1/b} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.1}$$

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{\alpha z^2 \ f''(z) + zf'(z)}{z}\right)^{1/b}. \tag{2.2}$$

By taking logarithmic derivative of $p(z)$ given by (2.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left\{\frac{\alpha z^3 f'''(z) + (1 + 2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 \ f''(z) + zf'(z)} - 1\right\}. \tag{2.3}$$

The result now follows from Lemma 1.3. $\square$

For $\alpha = 0$ we get the following .

Corollary 2.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.3. The function $f \in C_b(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{sf'(tz)}{tf'(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}.$$

Theorem 2.3. *Let ϕ starlike with respect to 1 and $F(z)$ is given by (1.2) be starlike. If $f \in M[b, \alpha](\phi)$, then we have*

$$\frac{\alpha z^2\ f''(z) + zf'(z)}{z} \prec \left(\frac{F(z)}{z}\right)^b. \tag{2.4}$$

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \left(\frac{\alpha z^2\ f''(z) + zf'(z)}{z}\right)^{1/b}, \quad q(z) := \left(\frac{F(z)}{z}\right).$$

Then a computation yields

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left\{\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2\ f''(z) + zf'(z)} - 1\right\}.$$

and

$$\frac{zq'(z)}{q(z)} = \left(\frac{zF'(z)}{F(z)} - 1\right) = \phi(z) - 1.$$

Since $f \in M[b, \alpha](\phi)$, we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left\{\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2\ f''(z) + \alpha zf'(z)} - 1\right\} \prec \phi(z) - 1 = \frac{zq'(z)}{q(z)}.$$

The result now follows by an application of Lemma 1.4. $\square$

By taking $\phi(z) = \dfrac{1+z}{1-z}$, and $\alpha = 0$ in Theorem 2.3, we get another result of Srivastava and Lashin [10]:

Corollary 2.4. *If $f \in C_b$, then*

$$f'(z) \prec \frac{1}{(1-z)^{2b}}.$$

By taking $\phi(z) = \dfrac{1+z}{1-z}$, and $\alpha = 1$ in Theorem 2.3, we get another interesting new result.

Corollary 2.5. *If $f \in C_b$, then*

$$f'(z) + zf''(z) \prec \frac{1}{(1-z)^{2b}}.$$

3 Coefficients Estimates

In this section, we introduce the β-convex functions involving complex order defined as follows.

Definition 3.1. *Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi'(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmmetric with respect to the real axis. Then the class $M_{\beta,b}(\phi)$ consists of all functions $f \in$ satisfying*

$$1 + \frac{1}{b}\left[(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] \prec \phi(z) \quad (0 \leq \beta \leq 1).$$

We note that,

$$M_{0,1}(\phi) \equiv \mathcal{S}^*(\phi)$$

and

$$M_{1,1}(\phi) \equiv C(\phi),$$

the classes introduced by Ma and Minda [2]. Also, we note that,

$$M_{0,b}(\phi) \equiv \mathcal{S}_b^*(\phi)$$

and

$$M_{1,b}(\phi) \equiv C_b(\phi),$$

the classes studied by Ravichandran et al. [4]. To prove our main result, we need the following :

Lemma 3.2. *[4] If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then*

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}$$

Our main result is the following :

Theorem 3.3. *Let $0 \leq \beta \leq 1$. Further let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, $z \in \Delta$, where B_n' are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $M_{\beta,b}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+2\beta)} \max\left\{1, \left|\frac{B_2}{B_1} + (1 - 2\mu + \beta(3-4\mu))\frac{bB_1}{(1+\beta)^2}\right|\right\}.$$

The result is sharp.

Proof. If $f(z) \in M_{\beta,b}(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0)=0$ and $|w(z)|<1$ in Δ such that

$$1+\frac{1}{b}\left[(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]=\phi(w(z)). \tag{3.1}$$

Define $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots. \tag{3.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z))>0$ and $p_1(0)=1$. Define the function $p(z)$ by

$$p(z)=1+\frac{1}{b}\left[(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right]=1+b_1z+b_2z^2+\cdots. \tag{3.3}$$

In view of equations (3.1), (3.2), (3.3), we have

$$p(z)=\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{3.4}$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\left(c_3-\frac{c_1^3}{4}-c_1c_2\right)z^3+\cdots\right]$$

and therefore

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]z^2+\cdots.$$

from this equation (3.4), we obtain

$$b_1=\frac{B_1c_1}{2}, \tag{3.5}$$

$$b_2=\frac{1}{2}\left(B_1\left(c_2-\frac{1}{2}c_1^2\right)\right)+\frac{1}{4}B_2c_1^2. \tag{3.6}$$

from the equation (3.3), we obtain

$$a_2=\frac{bb_1}{(1+\beta)}, \tag{3.7}$$

$$a_3=\frac{bb_2+(1+3\beta)a_2^2}{2(1+2\beta)}. \tag{3.8}$$

By applying (3.5), (3.6) in (3.7) and (3.8) we have

$$\begin{aligned} a_2 &= \frac{bB_1c_1}{2(1+\beta)}, \\ a_3 &= \frac{bB_1c_2}{4(1+2\beta)}+\frac{c_1^2}{8(1+2\beta)}\left[\frac{(1+3\beta)}{(1+\beta)^2}b^2B_1^2-b(B_1-B_2)\right]. \end{aligned}$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4(1+2\beta)} \left[c_2 - vc_1^2\right] \tag{3.9}$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{bB_1}{(1+\beta)^2}\left(1 + 3\beta - 2\mu(1+2\beta)\right)\right].$$

Our result now follows by the application of Lemma 3.2. The result is sharp for the function f defined by

$$1 + \frac{1}{b}\left[(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] = \phi(z^2)$$

and

$$1 + \frac{1}{b}\left[(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] = \phi(z).$$

□

For $\beta = 0$, in Theorem 3.3 we get the result obtained by Ravichandran et al. [4].

Corollary 3.4. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $S_b^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1|b|}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu)bB_1\right|\right\}.$$

The result is sharp.

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