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Research Article

Certain New Classes of Analytic Functions with Varying Arguments

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We introduce certain new classes $\kappa - V\mathrm{ST}(\alpha,\beta)$ and $\kappa - V\mathrm{UCV}(\alpha,\beta)$, which represent the κ uniformly starlike functions of order α and type β with varying arguments and the κ uniformly convex functions of order α and type β with varying arguments, respectively. Moreover, we give coefficients estimates, distortion theorems, and extreme points of these classes.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

that are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1 (see [1]). Let $\kappa - \mathrm{ST}(\alpha,\beta)$ denote the subclass of $\mathscr{A}$ consisting of functions $f(z)$ of the form (1) and satisfy the following inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \kappa\left|\frac{zf'(z)}{f(z)} - \beta\right|, \quad (0 \le \alpha < \beta \le 1;\ \kappa(1-\beta) < (1-\alpha);\ z \in U). \qquad (2)$$

Also let $\kappa - \mathrm{UCV}(\alpha,\beta)$ denote the subclass of $\mathscr{A}$ consisting of functions $f(z)$ of the form (1) and satisfy the following inequality:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \kappa\left|1 + \frac{zf''(z)}{f'(z)} - \beta\right|, \quad (0 \le \alpha < \beta \le 1;\ \kappa(1-\beta) < (1-\alpha);\ z \in U). \qquad (3)$$

It follows from (2) and (3) that

$$f \in \kappa - \mathrm{UCV}(\alpha,\beta) \Longleftrightarrow zf' \in \kappa - \mathrm{ST}(\alpha,\beta). \qquad (4)$$

The class $\kappa - \mathrm{ST}(\alpha,\beta)$ denote the class of κ uniformly starlike functions of order α and type β and the class $\kappa - \mathrm{UCV}(\alpha,\beta)$ denotes the class of κ uniformly convex functions of order α and type β.

Specializing parameters β, α, and κ, we obtain the following subclasses studied by various authors:

(i) $\kappa - \mathrm{ST}(\alpha,1) = \mathrm{SD}(\kappa,\alpha)$ and $\kappa - \mathrm{UCV}(\alpha,1) = \mathrm{KD}(\kappa,\alpha)$ (see [2, 3]);

(ii) $1 - \mathrm{ST}(\alpha,1) = S_p(\alpha)$ and $1 - \mathrm{UCV}(\alpha,1) = \mathrm{UCV}(\alpha)$ (see [4]);

(iii) $\kappa - \mathrm{ST}(0,1) = \kappa - \mathrm{ST}$ and $\kappa - \mathrm{UCV}(0,1) = \kappa - \mathrm{UCV}$ (see [5, 6]);

(iv) $1-\mathrm{ST}(0,1)=S_p$ and $1-\mathrm{UCV}(0,1)=\mathrm{UCV}$ (see [4, 7–10]).

Also we note that

$$\begin{aligned}
&1-\mathrm{ST}(\alpha,\beta)\\
&\quad=\mathrm{US}(\alpha,\beta)\\
&\quad=\left\{f(z)\in\mathscr{A}:\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\left|\frac{zf'(z)}{f(z)}-\beta\right|\right\},\\
&1-\mathrm{UCV}(\alpha,\beta)\\
&\quad=\mathrm{UC}(\alpha,\beta)\\
&\quad=\left\{f(z)\in\mathscr{A}:\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\left|1+\frac{zf''(z)}{f'(z)}-\beta\right|\right\},
\end{aligned}\tag{5}$$

which are the uniformly starlike functions of order α and type β and uniformly convex functions of order α and type β, respectively.

Definition 2 (see [11]). A function $f(z)$ of the form (1) is said to be in the class $V(\theta_n)$ if $f\in\mathscr{A}$ and $\arg(a_n)=\theta_n$ for all $n\geq 2$. If furthermore there exist a real number δ such that $\theta_n+(n-1)\delta\equiv\pi\,(\mathrm{mod}\,2\pi)$, then $f(z)$ is said to be in the class $V(\theta_n,\delta)$. The union of $V(\theta_n,\delta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers δ is denoted by V.

Let $\kappa-V\mathrm{ST}(\alpha,\beta)$ denote the subclass of V consisting of functions $f(z)\in\kappa-\mathrm{ST}(\alpha,\beta)$. Also Let $\kappa-V\mathrm{UCV}(\alpha,\beta)$ denote the subclass of V consisting of functions $f(z)\in\kappa-\mathrm{UCV}(\alpha,\beta)$.

In this paper we obtain coefficient bounds for functions in the classes $\kappa-V\mathrm{ST}(\alpha,\beta)$ and $\kappa-V\mathrm{UCV}(\alpha,\beta)$, respectively, further we obtain distortion bounds and the extreme points for functions in these classes.

2. Coefficient Estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0\leq\alpha<\beta\leq 1$, $\kappa(1-\beta)<(1-\alpha)$ and $z\in U$.

We shall need the following lemmas.

Lemma 3. *The sufficient condition for $f(z)\in\mathscr{A}$ given by* (1) *to be in the class $\kappa-\mathrm{ST}(\alpha,\beta)$ is that*

$$\sum_{n=2}^{\infty}\left[\kappa(n-\beta)+(n-\alpha)\right]|a_n|\leq 1-\alpha-\kappa(1-\beta).\tag{6}$$

Proof. It suffices to show that inequality (2) holds true. Upon using the fact that

$$\operatorname{Re}\{w\}>\kappa|w-\beta|+\alpha\Longleftrightarrow\operatorname{Re}\left\{\left(1+\kappa e^{i\theta}\right)w-\beta\kappa e^{i\theta}\right\}>\alpha,\tag{7}$$

then inequality (2) may be written as

$$\operatorname{Re}\left\{\left(1+\kappa e^{i\theta}\right)\frac{zf'(z)}{f(z)}-\beta\kappa e^{i\theta}\right\}>\alpha,\tag{8}$$

or

$$\operatorname{Re}\left\{\frac{A(z)}{B(z)}\right\}>\alpha,\tag{9}$$

where $A(z)=(1+\kappa e^{i\theta})zf'(z)-\beta\kappa e^{i\theta}f(z)$ and $B(z)=f(z)$, then condition (2) or (9) is equivalent to

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\geq 0.\tag{10}$$

We note that

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)|\\
&\quad=\left|\left[(1-\beta)\kappa e^{i\theta}+2-\alpha\right]z\right.\\
&\qquad\left.-\sum_{n=2}^{\infty}\left[(\beta-n)\kappa e^{i\theta}+\alpha-n-1\right]a_nz^n\right|
\end{aligned}\tag{11}$$

$$\begin{aligned}
&\quad\geq\left[-\kappa(1-\beta)+2-\alpha\right]|z|\\
&\qquad-\sum_{n=2}^{\infty}\left[(n-\beta)\kappa+n-\alpha+1\right]|a_n||z|^n,\\
&|A(z)-(1+\alpha)B(z)|\\
&\quad=\left|\left[(1-\beta)\kappa e^{i\theta}-\alpha\right]z\right.\\
&\qquad\left.+\sum_{n=2}^{\infty}\left[(n-\beta)\kappa e^{i\theta}+n-\alpha-1\right]a_nz^n\right|
\end{aligned}\tag{12}$$

$$\begin{aligned}
&\quad\leq\left[\kappa(1-\beta)+\alpha\right]|z|\\
&\qquad+\sum_{n=2}^{\infty}\left[(n-\beta)\kappa+n-\alpha-1\right]|a_n||z|^n.
\end{aligned}$$

Using (11) and (12), then we can obtain the following inequality:

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\\
&\quad\geq 2\left[(1-\alpha)-\kappa(1-\beta)\right]|z|\\
&\qquad-2\sum_{n=2}^{\infty}\left[(n-\beta)\kappa+n-\alpha\right]|a_n||z|^n.
\end{aligned}\tag{13}$$

The expression $|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$ is bounded below by 0 if

$$\begin{aligned}
&2\left[(1-\alpha)-(1-\beta)\kappa\right]|z|-2\sum_{n=2}^{\infty}\left[(n-\beta)\kappa+n-\alpha\right]|a_n||z|^n\\
&\quad>0,
\end{aligned}\tag{14}$$

or

$$\sum_{n=2}^{\infty}\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]\left|a_n\right|<1-\alpha-\kappa\left(1-\beta\right). \quad (15)$$

Hence the proof of Lemma 3 is completed. □

By using (4) and (6) we can obtain the following lemma.

Lemma 4. *A function $f(z)$ of the form* (1) *is in the class $\kappa-UCV(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}n\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]\left|a_n\right|<1-\alpha-\kappa\left(1-\beta\right). \quad (16)$$

Remark 5. Putting $\beta=1$ in Lemmas 3 and 4, we obtain the results obtained by Shams et al. [3, Theorems 2.1, 2.2, resp.].

In the following theorems, we show that the conditions (6) and (16) are also necessary for functions $f(z)\in\kappa-VST(\alpha,\beta)$ and $\kappa-VUCV(\alpha,\beta)$, respectively.

Theorem 6. *Let $f(z)$ be of the form* (1)*, then $f(z)\in\kappa-VST(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]\left|a_n\right|<1-\alpha-\kappa\left(1-\beta\right). \quad (17)$$

Proof. In view of Lemma 3, we need only to show that function $f(z)\in\kappa-VST(\alpha,\beta)$ satisfies the coefficient inequality (17). If $f(z)\in\kappa-VST(\alpha,\beta)$, then from (2), we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\kappa\left|\frac{zf'(z)}{f(z)}-\beta\right|, \quad (18)$$

thus we have

$$\operatorname{Re}\left\{\frac{(1-\alpha)+\sum_{n=2}^{\infty}(n-\alpha)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}\right\}>\kappa\left|\frac{(1-\beta)+\sum_{n=2}^{\infty}(n-\beta)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}\right|. \quad (19)$$

Since $f(z)\in V$, $f(z)$ lies in the class $V(\theta_n,\delta)$ for some sequence $\{\theta_n\}$ and a real number δ such that

$$\theta_n+(n-1)\delta\equiv\pi\,(\operatorname{mod}2\pi). \quad (20)$$

Set $z=re^{i\delta}$ in (19), then we obtain

$$\frac{(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\left|a_n\right|r^{n-1}}{1-\sum_{n=2}^{\infty}a_nr^{n-1}}>\kappa\left[\frac{(1-\beta)+\sum_{n=2}^{\infty}(n-\beta)\left|a_n\right|r^{n-1}}{1-\sum_{n=2}^{\infty}\left|a_n\right|r^{n-1}}\right]. \quad (21)$$

Letting $r\rightarrow1$, then we have

$$\sum_{n=2}^{\infty}\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]\left|a_n\right|<1-\alpha-\kappa\left(1-\beta\right). \quad (22)$$

Hence the proof of Theorem 6 is completed. □

Corollary 7. *If $f(z)\in\kappa-VST(\alpha,\beta)$, then*

$$\left|a_n\right|\leq\frac{1-\alpha-\kappa\left(1-\beta\right)}{\kappa\left(n-\beta\right)+\left(n-\alpha\right)}\quad(n\geq2). \quad (23)$$

The equality holds for the function

$$f(z)=z+\frac{1-\alpha-\kappa\left(1-\beta\right)}{\kappa\left(n-\beta\right)+\left(n-\alpha\right)}e^{i\theta_n}z^n\quad(n\geq2;\ z\in U). \quad (24)$$

Using the same technique used in Theorem 6 we get the following theorem

Theorem 8. *Let $f(z)$ of the form* (1)*, then $f(z)\in\kappa-VUCV(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}n\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]\left|a_n\right|<1-\alpha-\kappa\left(1-\beta\right). \quad (25)$$

Corollary 9. *If $f(z)\in\kappa-VUCV(\alpha,\beta)$, then*

$$\left|a_n\right|\leq\frac{1-\alpha-\kappa\left(1-\beta\right)}{n\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]}\quad(n\geq2). \quad (26)$$

The equality holds for the function

$$f(z)=z+\frac{1-\alpha-\kappa\left(1-\beta\right)}{n\left[\kappa\left(n-\beta\right)+\left(n-\alpha\right)\right]}e^{i\theta_n}z^n\quad(n\geq2;\ z\in U). \quad (27)$$

3. Distortion Theorems

Theorem 10. *Let the function $f(z)$ defined by* (1) *be in the class $\kappa-VST(\alpha,\beta)$. Then*

$$|z|-\frac{1-\alpha-\kappa\left(1-\beta\right)}{\kappa\left(2-\beta\right)+\left(2-\alpha\right)}|z|^2\leq\left|f(z)\right|\leq|z|+\frac{1-\alpha-\kappa\left(1-\beta\right)}{\kappa\left(2-\beta\right)+\left(2-\alpha\right)}|z|^2. \quad (28)$$

The result is sharp.

Proof. We employ the same technique as used by Silverman [11]. In view of Theorem 6, since

$$\Phi(n)=\kappa\left(n-\beta\right)+\left(n-\alpha\right) \quad (29)$$

is an increasing function of $n\ (n\geq2)$, we have

$$\Phi(2)\sum_{n=2}^{\infty}\left|a_n\right|\leq\sum_{n=2}^{\infty}\Phi(n)\left|a_n\right|\leq1-\alpha-\kappa\left(1-\beta\right), \quad (30)$$

that is,

$$\sum_{n=2}^{\infty}\left|a_n\right|\leq\frac{1-\alpha-\kappa\left(1-\beta\right)}{\Phi(2)}. \quad (31)$$

Thus we have

$$|f(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} |a_n|,$$
$$|f(z)| \le |z| + \frac{1-\alpha-\kappa(1-\beta)}{\kappa(2-\beta)+(2-\alpha)}|z|^2. \tag{32}$$

Similarly, we get

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} |a_n||z|^n \ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$
$$\ge |z| - \frac{1-\alpha-\kappa(1-\beta)}{\kappa(2-\beta)+(2-\alpha)}|z|^2. \tag{33}$$

This completes the proof of Theorem 10. Finally the result is sharp for the following function:

$$f(z) = z + \frac{1-\alpha-\kappa(1-\beta)}{\kappa(2-\beta)+(2-\alpha)} e^{i\theta_2} z^2, \tag{34}$$

at $z = \pm|z|e^{-i\theta_2}$. □

Corollary 11. *Under the hypotheses of Theorem 8, $f(z)$ is included in a disc with center at the origin and radius r_1 given by*

$$r_1 = 1 + \frac{1-\alpha-\kappa(1-\beta)}{\kappa(2-\beta)+(2-\alpha)}. \tag{35}$$

Theorem 12. *Let the function $f(z)$ defined by (1) be in the class $\kappa - V\mathrm{ST}(\alpha,\beta)$. Then*

$$1 - \frac{2[1-\alpha-\kappa(1-\beta)]}{\kappa(2-\beta)+(2-\alpha)}|z|$$
$$\le |f'(z)|$$
$$\le 1 + \frac{2[1-\alpha-\kappa(1-\beta)]}{\kappa(2-\beta)+(2-\alpha)}|z|. \tag{36}$$

The result is sharp.

Proof. Similarly $\Phi(n)/n$ is an increasing function of $n\ (n \ge 2)$, in view of Theorem 6, we have

$$\frac{\Phi(2)}{2}\sum_{n=2}^{\infty} n|a_n| \le \sum_{n=2}^{\infty}\Phi(n)|a_n| \le 1-\alpha-\kappa(1-\beta), \tag{37}$$

that is,

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{2[1-\alpha-\kappa(1-\beta)]}{\Phi(2)}. \tag{38}$$

Thus we have

$$|f'(z)| \le 1 + |z|\sum_{n=2}^{\infty} n|a_n|$$
$$\le 1 + \frac{2[1-\alpha-\kappa(1-\beta)]}{\kappa(2-\beta)+(2-\alpha)}|z|. \tag{39}$$

Similarly

$$|f'(z)| \ge 1 - |z|\sum_{n=2}^{\infty} n|a_n|$$
$$\ge 1 - \frac{2[1-\alpha-\kappa(1-\beta)]}{\kappa(2-\beta)+(2-\alpha)}|z|. \tag{40}$$

Finally, we can see that the assertions of Theorem 12 are sharp for the function $f(z)$ defined by (34). This completes the proof of Theorem 12. □

Corollary 13. *Under the hypotheses of Theorem 12, $f'(z)$ is included in a disc with center at the origin and radius r_2 given by*

$$r_2 = 1 + \frac{2[1-\alpha-\kappa(1-\beta)]}{\kappa(2-\beta)+(2-\alpha)}. \tag{41}$$

Using the same technique used in Theorems 10 and 12, we get the following theorems.

Theorem 14. *Let the function $f(z)$ defined by (1) be in the class $\kappa - V\mathrm{UCV}(\alpha,\beta)$. Then*

$$|z| - \frac{1-\alpha-\kappa(1-\beta)}{2[\kappa(2-\beta)+(2-\alpha)]}|z|^2$$
$$\le |f(z)|$$
$$\le |z| + \frac{1-\alpha-\kappa(1-\beta)}{2[\kappa(2-\beta)+(2-\alpha)]}|z|^2. \tag{42}$$

The result is sharp for the following function:

$$f(z) = z + \frac{1-\alpha-\kappa(1-\beta)}{2[\kappa(2-\beta)+(2-\alpha)]} e^{i\theta_2} z^2. \tag{43}$$

Theorem 15. *Let the function $f(z)$ defined by (1) be in the class $\kappa - V\mathrm{UCV}(\alpha,\beta)$. Then*

$$1 - \frac{1-\alpha-\kappa(1-\beta)}{\kappa(2-\beta)+(2-\alpha)}|z|$$
$$\le |f'(z)|$$
$$\le 1 + \frac{1-\alpha-\kappa(1-\beta)}{\kappa(2-\beta)+(2-\alpha)}|z|. \tag{44}$$

The result is sharp for the function given by (43).

4. Extreme Points

Theorem 16. *Let the function $f(z)$ defined by (1) be in the class $\kappa - V\mathrm{ST}(\alpha,\beta)$, with $\arg a_n = \theta_n$, where $\theta_n + (n-1)\delta \equiv \pi\ (\mathrm{mod}\ 2\pi)$. Define*

$$f_1(z) = z,$$
$$f_n(z) = z + \frac{1-\alpha-\kappa(1-\beta)}{\kappa(n-\beta)+(n-\alpha)} e^{i\theta_n} z^n \quad (n \ge 2;\ z \in U). \tag{45}$$

Then $f(z) \in \kappa - V\mathrm{ST}(\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, where $\mu_n \geq 0$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. If $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$ with $\mu_n \geq 0$ and $\sum_{n=1}^{\infty} \mu_n = 1$, then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n f_n(z) \\ &= \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z) \\ &= \left(1 - \sum_{n=2}^{\infty} \mu_n\right) z \\ &\quad + \sum_{n=2}^{\infty} \left[\mu_n \left(z + \frac{1-\alpha-\kappa(1-\beta)}{\kappa(n-\beta)+(n-\alpha)} e^{i\theta_n} z^n\right)\right] \\ &= z + \sum_{n=2}^{\infty} \frac{1-\alpha-\kappa(1-\beta)}{\kappa(n-\beta)+(n-\alpha)} \mu_n e^{i\theta_n} z^n. \end{aligned} \tag{46}$$

But according to (17), we can see that

$$\begin{aligned} &\sum_{n=2}^{\infty} [\kappa(n-\beta)+(n-\alpha)] \left| \frac{1-\alpha-\kappa(1-\beta)}{\kappa(n-\beta)+(n-\alpha)} \mu_n e^{i\theta_n} \right| \\ &\quad = \sum_{n=2}^{\infty} [\kappa(n-\beta)+(n-\alpha)] \frac{1-\alpha-\kappa(1-\beta)}{\kappa(n-\beta)+(n-\alpha)} \mu_n \\ &\quad = \sum_{n=2}^{\infty} [1-\alpha-\kappa(1-\beta)] \mu_n \\ &\quad = (1-\mu_1)[1-\alpha-\kappa(1-\beta)] \leq [1-\alpha-\kappa(1-\beta)]. \end{aligned} \tag{47}$$

Then $f(z)$ satisfies (17), hence $f(z) \in \kappa - V\mathrm{ST}(\alpha,\beta)$.

Conversely, let the function $f(z)$ defined by (1) be in the class $\kappa - V\mathrm{ST}(\alpha,\beta)$, and define

$$\begin{aligned} \mu_n &= \frac{[\kappa(n-\beta)+(n-\alpha)]}{1-\alpha-\kappa(1-\beta)} |a_n|, \quad n \geq 2, \\ \mu_1 &= 1 - \sum_{n=2}^{\infty} \mu_n. \end{aligned} \tag{48}$$

From Theorem 6, $\sum_{n=2}^{\infty} \mu_n \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_n f_n(z) = \mu_n z + a_n z^n$, then

$$\sum_{n=1}^{\infty} \mu_n f_n(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z). \tag{49}$$

This completes the proof of Theorem 16. □

Finally using the same technique used in Theorem 16 we get the following theorem.

Theorem 17. *Let the function $f(z)$ defined by (1) be in the class $\kappa - V\mathrm{UCV}(\alpha,\beta)$, with $\arg a_n = \theta_n$, where $\theta_n + (n-1)\delta \equiv \pi \pmod{2\pi}$. Define*

$$\begin{aligned} f_1(z) &= z, \\ f_n(z) &= z + \frac{1-\alpha-\kappa(1-\beta)}{n[\kappa(n-\beta)+(n-\alpha)]} e^{i\theta_n} z^n \\ &\qquad (n \geq 2; z \in U). \end{aligned} \tag{50}$$

Then $f(z) \in \kappa - V\mathrm{UCV}(\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, where $\mu_n \geq 0$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

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