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# Certain subclasses of Spiral-like univalent functions related with Pascal distribution series

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ABSTRACT. The purpose of the present paper is to find the sufficient conditions for the subclasses of analytic functions associated with Pascal distribution to be in subclasses of spiral-like univalent functions and inclusion relations for such subclasses in the open unit disk $\mathbb{D}$. Further, we consider the properties of integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

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## 1. Introduction

Denote by $\mathcal{A}$ the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be subclass of $\mathcal{A}$ whose members are given by (1.1)

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Dedicated to Prof. H. M. Srivasatava, on his $80^{th}$ birth Anniversary and to my father Prof. P. M. Gangadharan.

and are univalent in $\mathbb{D}$. For functions $f \in \mathcal{S}$ be given by (1.1) and $g \in \mathcal{S}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

The two well known subclass of $\mathcal{S}$, are namely the class of starlike and convex functions (for details see Robertson [19]). A function $f \in \mathcal{S}$ given by (1.1) is said to be starlike of order $\gamma$ $(0 \leq \gamma < 1)$, if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \quad (z \in \mathbb{D}).$$

This function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{D})$ is starlike with respect to the origin.

A function $f \in \mathcal{S}$ is said to be convex of order $\gamma$ $(0 \leq \gamma < 1)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \quad (z \in \mathbb{D}).$$

This class is denoted by $\mathcal{K}(\gamma)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. By Alexander's relation(see [3]), it is a known fact that

$$f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

A function $f \in \mathcal{S}$ is said to be spiral-like if

$$\Re\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0$$

for some $\alpha$ with $\mid \alpha \mid < \frac{\pi}{2}$ and for all $z \in \mathbb{D}$. This class of spiral-like function was introduced in[27]. Also $f(z)$ is convex spiral-like if $zf'(z)$ is spiral-like. For instance, in 1974,a subclass of spiral-like functions was familiarized by Silvia[21],who gave some amazing properties of this function class. Consequently, Umarani [29] dened and deliberate another function class of spiral-like functions.Lately, certain properties of spiral-like close-to-convex functions associated with conic domains has been studied extensively by Srivastava et al.,[25] (see also [23, 26] and the references cited therein). Due to Murugusundramoorthy [9] (see also [10]), we consider subclasses of spiral-like functions as below:

**Definition 1.1.** For $0 \leq \rho < 1$, $0 \leq \gamma < 1$ then

$$\mathcal{S}(\xi,\gamma,\rho) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(e^{i\xi}\frac{zf'(z)}{(1-\rho)f(z) + \rho z f'(z)}\right) > \gamma\cos\xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}.$$

By virtue of Alexander's relation, we define the following subclass:

**Definition 1.2.** For $0 \leq \rho < 1$, $0 \leq \gamma < 1$ then

$$\mathcal{K}(\xi,\gamma,\rho) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(e^{i\xi}\frac{zf''(z) + f'(z)}{f'(z) + \rho z f''(z)}\right) > \gamma\cos\xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}.$$

By specialising the parameter $\rho = 0$ we remark the following :

**Definition 1.3.** For $0 \leq \gamma < 1$ then

$$\mathcal{S}(\xi,\gamma) := \left\{ f \in \mathcal{S} : \text{Re } \left( e^{i\xi} \frac{zf'(z)}{f(z)} \right) > \gamma \cos \xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}$$

and

$$\mathcal{K}(\xi,\gamma) := \left\{ f \in \mathcal{S} : \text{Re } \left( e^{i\xi} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right) > \gamma \cos \xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}.$$

The above function classes $\mathcal{S}(\xi,\gamma), \mathcal{K}(\xi,\gamma)$ and $\mathcal{S}(\xi,\gamma,\rho)$(see [9, 10]) have been studied and generalized by different view points and perspectives. Now we state the necessary sufficient conditions for $f$ in the above classes relevant for current study.

**Lemma 1.1** ([9, 10]). *A function $f(z)$ given by (1.1) is a member of $\mathcal{S}(\xi,\gamma,\rho)$ if*

$$\sum_{n=2}^{\infty} [(1-\rho)(n-1)\sec \xi + (1-\gamma)(1+n\rho-\rho)]|a_n| \leq 1-\gamma, \tag{1.2}$$

*where $|\xi| < \frac{\pi}{2}, 0 \leq \rho < 1, 0 \leq \gamma < 1$.*

**Lemma 1.2.** *A function $f(z)$ given by (1.1) is a member of $\mathcal{S}(\xi,\gamma,\rho)$ if*

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec \xi + (1-\gamma)(1+n\rho-\rho)]|a_n| \leq 1-\gamma, \tag{1.3}$$

*where $|\xi| < \frac{\pi}{2}, 0 \leq \rho < 1, 0 \leq \gamma < 1$.*

*Proof.* By Alexander type Theorem (see [3]) ,we have $f \in \mathcal{K}(\xi,\gamma,\rho)$ if and only if $zf' \in \mathcal{S}(\xi,\gamma,\rho)$, Thus $z + \sum\limits_{n=2}^{\infty}(na_n)z^n$ is in $\mathcal{S}(\xi,\gamma,\rho)$. Hence by wringing $a_n$ by $na_n$in Lemma 1.1 we get the desired result.

**Lemma 1.3.** *Let$f(z)$ be given by (1.1). Then $f \in \mathcal{S}(\xi,\gamma)$ if*

$$\sum_{n=2}^{\infty} [(n-1)\sec \xi + (1-\gamma)]|a_n| \leq 1-\gamma, \tag{1.4}$$

*where $|\xi| < \frac{\pi}{2}, 0 \leq \gamma < 1$.*

**Lemma 1.4.** *A function $f(z)$ given by (1.1) is a member of $\mathcal{K}(\xi,\gamma)$ if*

$$\sum_{n=2}^{\infty} n[(n-1)\sec \xi + (1-\gamma)]|a_n| \leq 1-\gamma, \tag{1.5}$$

*where $|\xi| < \frac{\pi}{2}, 0 \leq \gamma < 1$.*

**Definition 1.4.** *A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{R}^{\tau}(\vartheta,\delta)$, $(\tau \in \mathbb{C}\backslash\{0\}, \ 0 < \vartheta \leq 1; \delta < 1)$, if it satisfies the inequality*

$$\left| \frac{(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1}{2\tau(1-\delta) + (1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1} \right| < 1 \quad (z \in \mathbb{D}).$$

*The class $\mathcal{R}^{\tau}(\vartheta,\delta)$ was introduced earlier by Swaminathan* [28]*(for special cases see the references cited there in).*

**Lemma 1.5** ([28]). *If $f \in \mathcal{R}^{\tau}(\vartheta,\delta)$ is of form (1.1), then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{1.6}$$

*The bounds given in (1.6) is sharp.*

A variable $x$ is said to be *Pascal distribution* if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$$(1-q)^m, \frac{qm(1-q)^m}{1!}, \frac{q^2m(m+1)(1-q)^m}{2!}, \frac{q^3m(m+1)(m+2)(1-q)^m}{3!} \cdots$$

, respectively, where $q$ and $m$ are called the parameters, and thus

$$P(x=k) = \binom{k+m-1}{m-1} q^k(1-q)^m, k = 0, 1, 2, 3, \ldots.$$

Lately, El-Deeb et al.[5](also see [1]) introduced a power series whose coefficients are probabilities of Pascal distribution

$$\Theta_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{D}$$

where $m \geq 1; 0 \leq q \leq 1$ and one can easily verify that the radius of convergence of above series is infinity by ratio test. Now, we define the linear operator

$$\Lambda_q^m(z) : \mathcal{A} \to \mathcal{A}$$

defined by the convolution or Hadamard product

$$\Lambda_q^m f(z) = \Theta_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m a_n z^n, \qquad z \in \mathbb{D}.$$

In recent years,several interesting subclasses of analytic functions were introduced and investigated from different view points.Stimulated by prior results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example,[2, 7, 8, 20, 22, 15, 24, 28]) and by the recent investigations related with distribution series (see for example, [1, 4, 5, 6, 12, 11, 17, 16, 18], we obtain sufficient condition for the function $\Phi_q^m$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions belonging in $\mathcal{R}^{\tau}(\vartheta,\delta)$ by smearing convolution operator. Finally, we afford conditions for the integral operator $\mathcal{G}_q^m(z) = \int_0^z \frac{\Theta_q^m(t)}{t} dt$ belonging to the above classes.

## 2. Inclusion Results

In order to substantiate our main results, we will use the following symbolizations, for $m \geq 1$ and $0 \leq q < 1$:

$$\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n = \frac{1}{(1-q)^m}; \quad \sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{1}{(1-q)^{m+1}}$$

$$\text{and} \quad \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n = \frac{1}{(1-q)^{m+2}}. \tag{2.1}$$

By modest computation we get the subsequent relations:

$$\sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 = \frac{1}{(1-q)^m} - 1 \tag{2.2}$$

$$\sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1} = qm \sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{qm}{(1-q)^{m+1}} \tag{2.3}$$

and

$$\begin{aligned} \sum_{n=2}^{\infty} (n-1)(n-2) \binom{n+m-2}{m-1} q^{n-1} &= q^2\, m(m+1) \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n \\ &= \frac{q^2\, m(m+1)}{(1-q)^{m+2}}. \end{aligned} \tag{2.4}$$

**Theorem 2.1.** *Let* $m > 0$. *Then* $\Theta_q^m(z) \in \mathcal{S}(\xi, \gamma, \rho)$ *if*

$$[(1-\rho)\sec\xi + \rho(1-\gamma)] \frac{qm}{(1-q)^{m+1}} \leq 1 - \gamma. \tag{2.5}$$

*Proof.* Since

$$\Theta_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m z^n.$$

Using the Lemma 1.1, it suffices to show that

$$\sum_{n=2}^{\infty} [(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)] \leq 1 - \gamma. \tag{2.6}$$

From (2.6) we let

$$\begin{aligned} M_1(\xi,\gamma,\rho) &= \sum_{n=2}^{\infty} [(1-\rho)\sec\xi(n-1) + (1-\gamma)(1+n\rho-\rho)] \\ &\quad \times \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &= [(1-\rho)\sec\xi + \rho(1-\gamma)](1-q)^m \sum_{n=2}^{\infty} (n-1) \\ &\quad \times \binom{n+m-2}{m-1} q^{n-1} + (1-\gamma)(1-q)^m \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} \\ &= [(1-\rho)\sec\xi + \rho(1-\gamma)](1-q)^m qm \sum_{n=0}^{\infty} \binom{n+m}{m} q^n \\ &\quad + (1-\gamma)(1-q)^m \left( \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 \right) \end{aligned}$$

$$
\begin{aligned}
&= [(1-\rho)\sec\xi+\rho(1-\gamma)](1-q)^m\frac{qm}{(1-q)^{m+1}}\\
&+ (1-\gamma)(1-q)^m\left(\frac{1}{(1-q)^m}-1\right)\\
&= [(1-\rho)\sec\xi+\rho(1-\gamma)]\frac{qm}{(1-q)}+(1-\gamma)\left(1-(1-q)^m\right).
\end{aligned}
$$

But $M_1(\xi,\gamma,\rho)$ is constrained above by $1-\gamma$ if and only if (2.5) holds.

**Theorem 2.2.** *Let $m>0$. Then $\Theta_q^m(z)\in\mathcal{K}(\xi,\gamma,\rho)$ if*

$$
\begin{aligned}
&[(1-\rho)\sec\xi+\rho(1-\gamma)]\frac{m(m+1)q^2}{(1-q)^2}+[2(1-\rho)\sec\xi+(1-\gamma)(4-\rho)]\frac{mq}{1-q}\\
+\ &[(1-\gamma)(2-\rho)]\left(1-(1-q)^m\right)\le 1-\gamma. \qquad (2.7)
\end{aligned}
$$

*Proof.* In view of Lemma 1.2, we have to show that

$$
\sum_{n=2}^{\infty}n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\le 1-\gamma. \qquad (2.8)
$$

Writing $n=(n-1)+1$ and $n^2=(n-1)(n-2)+3(n-1)+1$.
From (2.8), consider the expression

$$
\begin{aligned}
M_2(\xi,\gamma,\rho) &= \sum_{n=2}^{\infty}n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\\
&\times \binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&= [(1-\rho)\sec\xi+\rho(1-\gamma)](1-q)^m\sum_{n=2}^{\infty}n^2\binom{n+m-2}{m-1}q^{n-1}\\
&- (1-\rho)[\sec\xi-(1-\gamma)(1-q)^m\sum_{n=2}^{\infty}n\binom{n+m-2}{m-1}q^{n-1}\\
&= [(1-\rho)\sec\xi+\rho(1-\gamma)](1-q)^m\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}\\
&+ [2(1-\rho)\sec\xi+(1-\gamma)(4-\rho)](1-q)^m\sum_{n=2}^{\infty}(n-1)\\
&\times \binom{n+m-2}{m-1}q^{n-1}\\
&+ [(1-\gamma)(2-\rho)](1-q)^m\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}\\
&= [(1-\rho)\sec\xi+\rho(1-\gamma)](1-q)^m q^2 m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n
\end{aligned}
$$

$$+ \quad [2(1-\rho)\sec\xi + (1-\gamma)(4-\rho)]\,(1-q)^m mq \sum_{n=0}^{\infty}\binom{n+m}{m}q^n$$

$$+ \quad [(1-\gamma)(2-\rho)]\,(1-q)^m\left[\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\right].$$

Now by using (2.2)-(2.4), we get

$$\begin{aligned} M_2(\xi,\gamma,\rho) \quad &= \quad [(1-\rho)\sec\xi + \rho(1-\gamma)]\frac{m(m+1)q^2}{(1-q)^2} \\ &+ \quad [2(1-\rho)\sec\xi + (1-\gamma)(4-\rho)]\,\frac{mq}{1-q} \\ &+ \quad [(1-\gamma)(2-\rho)]\,(1-(1-q)^m)\,. \end{aligned}$$

Hence, $M_2(\xi,\gamma,\rho)$ is bounded above by $1-\gamma$ if (2.7) is satisfied.

## 3. Image Properties of $\Lambda_q^m$ operator

Making use of the Lemma 1.5, we will focus the influence of the Pascal distribution series on the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$.

**Theorem 3.1.** *Let $m>0$, and $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$. Then $\Lambda_q^m f(z)$ is in $\mathcal{S}(\xi,\gamma,\rho)$ if*

$$\begin{aligned} &\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\Big\{\ [(1-\rho)\sec\xi + \rho(1-\gamma)]\,[1-(1-q)^m] \\ + \quad &\frac{(1-\rho)(1-\gamma-\sec\xi)}{q(m-1)}\,[(1-q)-(1-q)^m - q(m-1)(1-q)^m]\Big\} \\ \leq \quad & 1-\gamma. \end{aligned} \tag{3.1}$$

*Proof.* In view of Lemma 1.1, it is required to show that

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n| \leq 1-\gamma.$$

Let

$$\begin{aligned} M_3(\xi,\gamma,\rho) \quad &= \quad \sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)] \\ &\times \quad \binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n|. \end{aligned}$$

Since $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$, then by Lemma 1.5, we have

$$|a_n| \leq \frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)}, \quad n\in\mathbb{N}\setminus\{1\}$$

and $1+\vartheta(n-1)\geq \vartheta n$. Thus,we have

$$\begin{aligned}
M_3(\xi,\gamma,\rho) &\leq \frac{2|\tau|(1-\delta)}{\vartheta}\left[\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\right. \\
&\quad\times \left.\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right] \\
&= \frac{2|\tau|(1-\delta)}{\vartheta}(1-q)^m\left[\sum_{n=2}^{\infty}[(1-\rho)\sec\xi+\rho(1-\gamma)]\right. \\
&\quad+ \left.(1-\rho)(1-\gamma-\sec\xi)\frac{1}{n}\right]\binom{n+m-2}{m-1}q^{n-1}\Bigg].
\end{aligned}$$

Using (2.2), we get

$$\begin{aligned}
M_3(\xi,\gamma,\rho) &= \frac{2|\tau|(1-\delta)}{\vartheta}(1-q)^m\{[(1-\rho)\sec\xi+\rho(1-\gamma)] \\
&\quad\times\left[\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n-1\right] \\
&\quad+\frac{(1-\rho)(1-\gamma)}{q(m-1)}\left[\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n-1-(m-1)q\right]\Bigg\} \\
&= \frac{2|\tau|(1-\delta)}{\vartheta}\Big\{[(1-\rho)\sec\xi+\rho(1-\gamma)][1-(1-q)^m] \\
&\quad+\frac{(1-\rho)(1-\gamma-\sec\xi)}{q(m-1)} \\
&\quad\times[(1-q)-(1-q)^m-q(m-1)(1-q)^m]\Big\}.
\end{aligned}$$

But $M_3(\xi,\gamma,\rho)$ is bounded by $1-\gamma$, if (3.1) holds. This completes the proof of Theorem 3.1.

Applying Lemma 1.2 and using the same procedure as in the proof of Theorem 2.2, we have the subsequent result.

**Theorem 3.2.** *Let $m>0$, and $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$. Then $\Lambda_q^m f(z)$ is in $\mathcal{K}(\xi,\gamma,\rho)$ if*

$$\begin{aligned}
&\frac{2|\tau|(1-\delta)}{\vartheta}\left[[(1-\rho)\sec\xi+(1-\gamma)]\frac{m(m+1)q^2}{(1-q)^2}\right. \\
&+\left.[2(1-\rho)\sec\xi+(1-\gamma)(4-\rho)]\frac{mq}{1-q}+[(1-\gamma)(2-\rho)](1-(1-q)^m)\right] \\
&\leq 1-\gamma.
\end{aligned}$$

## 4. An integral operator

**Theorem 4.1.** *If the function $\mathcal{G}_q^m(z)$ is given by*

$$\mathcal{G}_q^m(z) = \int_0^z \frac{\Theta_q^m(t)}{t} dt, \;\; z \in \mathbb{D} \tag{4.1}$$

*then $\mathcal{G}_q^m(z) \in \mathcal{K}(\xi, \gamma, \rho)$ if*

$$[(1-\rho)\sec\xi + \rho(1-\gamma)] \frac{qm}{(1-q)^{m+1}} \le 1-\gamma.$$

*Proof.* Since

$$\mathcal{G}_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n}$$

then by Lemma 1.2, we requisite to prove that

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)] \times \frac{1}{n} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \le 1-\gamma,$$

or, consistently

$$\sum_{n=2}^{\infty} [(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \le 1-\gamma.$$

The enduring part of the proof of Theorem 4.1 is parallel to that of Theorem 2.1, and so we omit the details.

**Theorem 4.2.** *Let $m > 0$, and the integral operator $\mathcal{G}_q^m$ as assumed by (4.1). Then $\mathcal{G}_q^m$ is in $\mathcal{S}(\xi, \gamma, \rho)$ if*

$$[(1-\rho)\sec\xi + \rho(1-\gamma)]\,[1-(1-q)^m] + \frac{(1-\rho)(1-\gamma-\sec\xi)}{q(m-1)}[(1-q)-(1-q)^m - q(m-1)(1-q)^m] \le 1-\gamma.$$

*Proof.* Since

$$\mathcal{G}_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n}$$

then by Lemma 1.1, we requisite to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \le 1-\gamma.$$

Thus, we have

$$M_4(\xi,\gamma,\rho) = \sum_{n=2}^{\infty} \frac{1}{n}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)]$$

$$\begin{aligned} &\times \quad . \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\ &= \quad (1-q)^m \left[ \sum_{n=2}^{\infty} [(1-\rho)\sec\xi + \rho(1-\gamma)] \right. \\ &+ \quad \left. (1-\rho)(1-\gamma-\sec\xi)\frac{1}{n} \right] \binom{n+m-2}{m-1} q^{n-1} \Bigg]. \end{aligned}$$

Using (2.2), we get

$$\begin{aligned} M_4(\xi,\gamma,\rho) &= (1-q)^m \left\{ [(1-\rho)\sec\xi + \rho(1-\gamma)] \left[ \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 \right] \right. \\ &+ \left. \frac{(1-\rho)(1-\gamma-\sec\xi)}{q(m-1)} \left[ \sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n - 1 - (m-1)q \right] \right\} \\ &= \Big\{ [(1-\rho)\sec\xi + \rho(1-\gamma)] \left[1-(1-q)^m\right] \\ &+ \frac{(1-\rho)(1-\gamma-\sec\xi)}{q(m-1)} \left[(1-q)-(1-q)^m - q(m-1)(1-q)^m\right] \Big\}. \end{aligned}$$

But $M_4(\xi,\gamma,\rho)$ is confined by $1-\gamma$, if (3.1) holds. This concludes the proof of Theorem 4.2.

## 5. Corollaries and consequences

By taking $\rho = 0$ in Theorems 2.1-4.2, we attain the sufficient condition for Pascal distribution series be in the function classes $\mathcal{S}(\xi,\gamma)$ and $\mathcal{K}(\xi,\gamma)$ as identified in following corollaries.

**Corollary 5.1.** *Let $m > 0$, then $\Theta_q^m$ is in $\mathcal{S}(\xi,\gamma)$ if*

$$\frac{qm\sec\xi}{(1-q)^{m+1}} \le 1-\gamma.$$

**Corollary 5.2.** *Let $m > 0$, then $\Theta_q^m$is in $\mathcal{K}(\xi,\gamma)$ if*

$$\begin{aligned} &[\sec\xi + (1-\gamma)]\frac{m(m+1)q^2}{(1-q)^2} + [2\sec\xi + 4(1-\gamma)]\frac{mq}{1-q} \\ +\ &[2(1-\gamma)]\left(1-(1-q)^m\right) \le 1-\gamma. \end{aligned}$$

**Corollary 5.3.** *Let $f \in \mathcal{R}^\tau(\vartheta,\delta)$ then $\Lambda_q^m$ is in $\mathcal{S}(\xi,\gamma)$ if*

$$\begin{aligned} &\frac{2\,|\tau|\,(1-\delta)}{\vartheta} \Big\{ \sec\xi\left[1-(1-q)^m\right] \\ +\ &\frac{(1-\gamma-\sec\xi)}{q(m-1)}\left[(1-q)-(1-q)^m - q(m-1)(1-q)^m\right] \Big\} \le 1-\gamma. \end{aligned}$$

**Corollary 5.4.** *Let* $f \in \mathcal{R}^{\tau}(\vartheta, \delta)$, *then* $\Lambda_q^m$ *is in* $\mathcal{K}(\xi, \gamma)$ *if*

$$\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\left[[\sec\xi + (1-\gamma)]\frac{m(m+1)q^2}{(1-q)^2} + [2\sec\xi + 4(1-\gamma)]\,\frac{mq}{1-q}\right.$$
$$\left. + \quad [2(1-\gamma)]\,(1-(1-q)^m)\ \right] \leq 1-\gamma.$$

**Corollary 5.5.** *Let* $m > 0$, *then* $\mathcal{G}_q^m(z)$,*as assumed by (4.1) is in* $\mathcal{K}(\xi, \gamma)$ *if*

$$\frac{qm\sec\xi}{(1-q)^{m+1}} \leq 1-\gamma.$$

**Corollary 5.6.** *Let* $m > 0$, *then* $\mathcal{G}_q^m(z)$,*as assumed by (4.1) is in* $\mathcal{S}(\xi, \gamma)$ *if*

$$\sec\xi\,[1-(1-q)^m]$$
$$+ \quad \frac{(1-\gamma-\sec\xi)}{q(m-1)}\,[(1-q)-(1-q)^m - q(m-1)(1-q)^m] \leq 1-\gamma.$$

**Conclusions** In this investigation, we obtain sufficient conditions and inclusion results for functions $f \in \mathcal{A}$ to be in the classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$, and information regarding the images of functions by applying convolution operator with Pascal distribution series. Also, certain special cases are also discussed. Further certain analytic Spiral-like functions of complex order can be defined and inclusion properties based on general distribution series be discussed based on this study.

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