CERTAIN TYPES OF GROUPS OF AUTOMORPHISMS OF A FACTOR

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In studying the crossed products of rings of operators, we have shown in [4] that an arbitrary countable group admits a faithful representation as a group of outer automorphisms of an approximately finite factor on a separable Hilbert space (the so-called outer automorphic representation). The object of the present paper is to discuss the algebraic properties of groups of outer automorphisms of the approximately finite factor, which are the outer automorphic representations of certain torsion free groups.

An automorphism (a group of automorphisms) of a factor is said to be *ergodic* if it leaves only the center elementwise invariant. Our first task is to ask whether there exists an ergodic group of automorphisms of the approximately finite factor. Indeed, we shall find the necessary and sufficient condition that the outer automorphic representation of a group be ergodic. The second one is to examine the outer automorphic representation of a free product of two arbitrary torsion free countable groups and show that the crossed products of the approximately finite factor by such groups of automorphisms are not approximately finite.

1. In the starting point, we shall recall the construction in [4] of the outer automorphic representation of a countably infinite group G. Let Δ be a set of all functions α on G taking only the values $0, 1$ such that $\alpha(g) = 0$ except for a finite number of g's. Then Δ is an additive group with the addition $[\alpha + \beta](g) = \alpha(g) + \beta(g) \pmod 2$. Further, define Δ' as a set of all functions φ on Δ taking only the values $0, 1$ such that $\varphi(\gamma) = 0$ except for a finite number of γ's, and make Δ' into a group by the addition $[\varphi + \psi](\gamma) = \varphi(\gamma) + \psi(\gamma) \pmod 2$. Now we make the pair $\mathcal{G} = (\Delta', \Delta)$ into a group by defining

$$(\varphi, \alpha)(\psi, \beta) = (\varphi^{\beta} + \psi, \alpha + \beta)$$

where $\varphi, \psi \in \Delta', \alpha, \beta \in \Delta$ and $\varphi^{\beta}(\gamma) = \varphi(\gamma + \beta)$. Let $\{V_{(\varphi,\alpha)}\}_{(\varphi,\alpha)\in\mathcal{G}}$ be unitary operators on $l_2(\mathcal{G})$ defined by $[V_{(\varphi,\alpha)}f]((\psi,\beta)) = f((\psi,\beta)(\varphi,\alpha))$ for all $f \in l_2(\mathcal{G})$, then a factor $\mathbf{M}$ generated by $\{V_{(\varphi,\alpha)}\}_{(\varphi,\alpha)\in\mathcal{G}}$ is an approximately finite factor on a separadle Hilbert space. Hereupon, for each $g \in G$, define an automorphism T_g on Δ by

$$[T_g\alpha]\,(g') = \alpha(gg') \text{ for all } g' \in G,$$

and successively an automorphism T'_g on Δ' by

$$[T'_g\,\varphi]\,(\gamma) = \varphi(T_{g^{-1}}\gamma) \text{ for all } \gamma \in \Delta.$$

Then the mappings $V_{(\varphi,\alpha)} \to V^{\theta_g} = V_{(T'_g\varphi\, T_g\alpha)}$ give a group of outer automorphisms of $\mathbf{M}$ and the mapping $g \to \theta_g$ gives the outer automorphic representation of G.

Following the previous paper [4], we shall state the necessary and sufficient condition that the outer automorphic representation of a group be ergodic.

THEOREM 1. *Let G be a countably infinite group with the unit e, then for each element $g \neq e$ in G, the corresponding outer automorphism θ_g is ergodic if and only if the order of g is infinite*

The proof is given successively by means of the following lemmas.

LEMMA 1. *If g is an element of finite order in G, then there exists $(\varphi, \alpha) \in \mathcal{G}$ such that $(\varphi, \alpha) = (T'_g\,\varphi, T_g\alpha)$.*

PROOF. Let us select an element $\alpha \in \Delta$ as follows: $\alpha(g') = 1$ on a finite set $F = \{e, g, g^2, \ldots\ldots, g^{n-1}(n: \text{the order of } g)\}$ and $= 0$ elsewhere, and choose an element $\varphi \in \Delta'$ such that $\varphi(\gamma) = 1$ on an $\alpha \in \Delta$ and $= 0$ elsewhere Then it is easily verified that $(\varphi, \alpha) = (T'_g\varphi, T_g\alpha)$, in fact, $[T_g\alpha](g') = 1$ on F and $= 0$ elsewhere, i.e. $T_g\alpha = \alpha$, and also $[T'_g\varphi](\gamma) = 1$ if $\gamma = T_g\alpha = \alpha$ and $= 0$ elsewhere, i.e. $T'_g\varphi = \varphi$.

LEMMA 2 *If g is an element of infinite order of G, then for any finite subset F of G, $Fg \neq F\,(gF \neq F)$.*

In fact, if $Fg = F$, for a fixed element $h \in F$, all of $hg^n\,(n = 0, 1, 2, \ldots)$ are distinct and belong to F and so it contradicts to the finiteness of F.

LEMMA 3. *If g is an element of infinite order of G, then $(\varphi, \alpha) \neq (T'_g\varphi, T_g\alpha)$ for all $(\varphi, \alpha) \neq (0, 0) \in \mathcal{G}$.*

PROOF. Let $(\varphi, \alpha) \neq (0, 0)$ be an element of $\mathcal{G}$ If $\alpha \neq 0$, i.e. $\alpha(g') = 1$ on a finite subset F of G and $= 0$ elsewhere, then $[T_g\alpha](g') = 1$ on a finite subset $g^{-1}F$ and $= 0$ elsewhere. However, since $F \neq g^{-1}F$ by Lemma 2, $T_g\alpha \neq \alpha$. If $\alpha = 0$, then φ may be taken as an element of Δ' such that $\varphi(\gamma) = 1$ on a finite subset Δ_0 of Δ and $= 0$ elsewhere. Putting $G_0 = \bigcup_{\gamma \in \Delta_0} \{g' : \gamma(g') = 1\}$, G_0 is a finite subset of G and each element $\alpha \in \Delta_0$ vanishes on $G - G_0$. But, since $G_0 \neq g^{-1}G_0$ by Lemma 2, there must be an element $\beta \in T_g\Delta_0$ which does not vanish on $G - G_0$, and so $\Delta_0 \neq T_g\Delta_0$, or $T'_g\varphi \neq \varphi$.

Therefore, for each $(\varphi, \alpha) \neq (0, 0) \in \mathcal{G}$, $(\varphi, \alpha) \neq (T'_g\varphi, T_g\alpha)$.

PROOF OF THE THEOREM. The necessity have been already proved by Lemma 1, and so it remains only to show the sufficiency. Let A be an element in $\mathbf{M}$ fixed under θ_g, then $A = A^{\theta_{g^n}}$ for all $n = 1, 2, \cdots\cdots$. According to Lemma 5.3.6 in [1], A is expressed in the form:

$$\sum\nolimits_{(\varphi,\alpha)\in\mathcal{G}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)}.$$

where Σ is taken in the sense of metric convergence in $\mathbf{M}$ and a family $\{\lambda_{(\varphi,\alpha)}\}$ of scalars is unique. Thus we have

$$\sum\nolimits_{(\varphi,\alpha)\in\mathcal{G}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)} = \sum\nolimits_{(\varphi,\alpha)\in\mathcal{G}} \lambda_{(\varphi,\alpha)} V_{(T'_{g^n}\varphi, T_{g^n}\alpha)}$$

for all n. Now suppose that $\lambda_{(\varphi,\alpha)} \neq 0$ for some $(\varphi, \alpha) \neq (0, 0)$, then since a sequence $(T'^{-1}_{g^n}\varphi, T^{-1}_{g^n}\alpha)$ consists of distinct elements of $\mathcal{G}$ from Lemma 3, by the uniqueness of the family $\{\lambda_{(\varphi,\alpha)}\}$ we obtain

$$\lambda_{(\varphi,\alpha)} = \lambda_{(T'^{-1}_g\varphi, T^{-1}_g\alpha)} = \lambda_{(T'^{-1}_{g^2}\varphi, T^{-1}_{g^2}\alpha)} = \cdots\cdots,$$

which contradicts $\sum_{(\varphi,\alpha)\in\mathcal{G}} |\lambda_{(\varphi,\alpha)}|^2 < \infty$. Therefore A is a scalar multiple of the identity.

2. Let G_i be two torsion free countable groups ($i = 1, 2$) and let $G = G_1 * G_2$ be the free product of G_i. From the preceding theorem we see that the outer automorphic representation of G is ergodic and in particular it consists of ergodic automorphisms of $\mathbf{M}$ except the unit. Furthermore we shall show that it possesses the distinguished algebraic property relative to the norm $[[\ \ ]]$ defined by the canonical trace Tr (Recall that for $A \in \mathbf{M}$, $[[A]] = Tr(A^*A)^{\frac{1}{2}}$).

LEMMA 4. *The outer automorphic representation of the group G with the unit e has the following property: There exists $g \neq e$ in G such that, for all $A \in M$ with $Tr(A) = 0$,*

$$k[[A]] \leqq [[A - A^{\theta_g}]] \leqq 2[[A]]$$

where k is a positive constant and does not depend on A.

PROOF. By the expression in [1: Lemma 5.3.6] used in the above section, an element A with $Tr(A) = 0$ is expressed in the following form:

$$\sum\nolimits_{\substack{(\varphi,\alpha)\in\mathcal{G} \\ (\varphi,\alpha)\neq(0,0)}} \lambda_{(\varphi,\alpha)} V_{(\varphi,\alpha)}$$

and hence $[[A - A^{\theta_{g^{-1}}}]] = \left(\sum_{\substack{(\varphi,\alpha)\in\mathcal{G} \\ (\varphi,\alpha)\neq(0,0)}} |\lambda_{(\varphi,\alpha)} - \lambda_{(T'_g\varphi, T_g\alpha)}|^2\right)^{\frac{1}{2}}$

for all g in G. Define the equivalence relation on $\mathcal{G} - (0, 0)$ as follows: $(\varphi, \alpha) \equiv (\psi, \beta) \pmod G$ if and only if there exists $g \in G$ such that $(T'_g\varphi, T_g\alpha) =$

(ψ, β), and decompose $\mathfrak{G}$ into equivalent classes $\{\mathfrak{G}_i\}$. Then the elements of $\mathfrak{G}_i$ can be written as $(T'_g\varphi, T_g\alpha)$ where (φ, α) is a fixed element in $\mathfrak{G}_i$ and g runs over G, and as all elements $(T'_g\varphi, T_g\alpha)$ are distinct by Theorem 1, their expressions are unique. Thus, putting $s_i = \left(\sum_{(\varphi,\beta)\in\mathfrak{G}_i}|\lambda_{(\psi,\beta)}|^2\right)^{\frac{1}{2}}$ and $\nu_i(E) = \sum_{h\in E}|\lambda_{(T'_h\varphi, T_h\alpha)}|^2$ for any subset E of G, then

$$s_i = \nu_i(G)^{\frac{1}{2}}.$$

Pick up $g_1 \neq e_1$ in G_1 and $g_2, g_3 \neq e_2$ in G_2 (e_1, e_2 being the unit of G_1, G_2 respectively) and put $P_{ij} = \left\{\sum_{(\psi,\beta)\in\mathfrak{G}_i}|\lambda_{(\psi,\beta)} - \lambda_{(T'_{g_j}\psi, T_{g_j}\beta)}|^2\right\}^{\frac{1}{2}}$ $(j = 1, 2, 3)$, then

$$P_{ij} = \left\{\sum_{h\in G}|\lambda_{(T'_h\varphi, T_h\alpha)} - \lambda_{(T'_{hg_j}\varphi, T_{hg_j}\alpha)}|^2\right\}^{\frac{1}{2}}$$

and

$$|\nu_i(E)^{\frac{1}{2}} - \nu_i(Eg_j)^{\frac{1}{2}}| \leqq P_{ij}\ (j = 1, 2, 3) \tag{1}$$

for any subset E of G. Now we define E_1 as the set of h's, $h \neq e$ having the normal form $h_1\ h_2 \ldots\ldots h_p$ in which h_p belongs to G_1 and $E_2 = G - E_1$. Then, by the inequality (1),

$$|\nu_i(E_1)^{\frac{1}{2}} - \nu_i(E_1g_2)^{\frac{1}{2}}| \leqq P_{i2}$$

$$|\nu_i(E_1g_2)^{\frac{1}{2}} - \nu_i(E_1g_2g_1)^{\frac{1}{2}}| \leqq P_{i1}$$

and hence we have

$$|\nu_i(E_1)^{\frac{1}{2}} - \nu_i(E_1g_2g_1)^{\frac{1}{2}}| \leqq P_{i1} + P_{i2}. \tag{2}$$

Since $E_1g_2g_1 \subsetneqq E_1$, $E_1 - E_1g_2g_1 \neq \phi$, and (2) gives

$$\nu_i(E_1 - E_1g_2g_1) = \nu_i(E_1) - \nu_i(E_1g_2g_1) = \left(\nu_i(E_1)^{\frac{1}{2}} - \nu_i(E_1g_2g_1)^{\frac{1}{2}}\right)$$

$$\cdot \left(\nu_i(E_1)^{\frac{1}{2}} + \nu_i(E_1g_2g_1)^{\frac{1}{2}}\right) \leqq 2s_i(P_{i1} + P_{i2}). \tag{3}$$

Furthermore, similarly we obtain

$$|\nu_i(E_1g_3g_1)^{\frac{1}{2}} - \nu_i(E_1)^{\frac{1}{2}}| \leqq P_{i3} + P_{i1}. \tag{4}$$

Therefore (2) and (4) yield that

$$|\nu_i(E_1g_2g_1)^{\frac{1}{2}} - \nu_i(E_1g_3g_1)^{\frac{1}{2}}| \leqq 2P_{i1} + P_{i2} + P_{i3},$$

and so $|\nu_i(E_1g_2g_1) - \nu_i(E_1g_3g_1)| \leqq 2\,s_i(2\,P_{i1} + P_{i2} + P_{i3})$. Since $E_1g_3g_1 \subset E_1 - E_1g_2g_1$, we have by (3)

$$\nu_i(E_1g_2g_1) \leqq 2s_i(2P_{i1} + P_{i2} + P_{i3}) + \nu_i(E_1g_3g_1)$$

$$\leqq 2s_i(3P_{i1} + 2P_{i2} + P_{i3}). \tag{5}$$

Thus by (3) and (5),

$$\nu_i(E_1) = \nu_i(E_1 - E_1 g_2 g_1) + \nu_i(E_1 g_2 g_1)$$
$$\leqq 2s_i(4P_{i1} + 3P_{i1} + P_{i3}). \quad (6)$$

Next, applying to E_2 the argument similar to that of the preceding paragraph, we obtain

$$|\ \nu_i(E_2 g_2)^{\frac{1}{2}} - \nu_i(E_2 g_3)^{\frac{1}{2}}\ | \leqq P_{i2} + P_{i3}$$

and also $|\nu_i(E_2 g_2) - \nu_i(E_2 g_3)| \leqq 2s_i(P_{i2} + P_{i3})$.
But, since $\nu_i(E_2 - E_2 g_3) \leqq 2s_i P_{i3}$ and $E_2 g_2 \subset E_2 - E_2 g_3$,

$$\nu_i(E_2 g_3) \leqq 2s_i(P_{i2} + P_{i3}) + 2s_i P_{i3} = 2s_i(P_{i2} + 2P_{i3}).$$

Thus $\quad \nu_i(E_2) = \nu_i(E_2 - E_2 g_3) + \nu_i(E_2 g_3) \leqq 2s_i(P_{i2} + 3P_{i3}). \quad (7)$

Therefore, (6) and (7) yield that

$$s_i^2 = \nu_i(G) = \nu_i(E_1) + \nu_i(E_2) \leqq 8s_i(P_{i1} + P_{i2} + P_{i3}),$$

and so $\quad s_i/8 \leqq P_{i1} + P_{i2} + P_{i3}$.

Now, let g be one of $g_j^{-1}(j = 1, 2, 3)$ such that $[[A - A^{\theta g}]] \geqq [[A - A^{\theta g_j^{-1}}]]$ $(j = 1, 2, 3)$. Then, noting that $[[A - A^{\theta g_j^{-1}}]]^2 = \sum_i P_{ij}^2\ (j = 1, 2, 3)$, we obtain

$$[[A - A^{\theta g}]]^2 \geqq \frac{1}{3}\sum_{j=1,2,3}[[A - A^{\theta g_j^{-1}}]]^2 = \frac{1}{3}\sum_i (P_{i1}^2 + P_{i2}^2 + P_{i3}^2)$$
$$\geqq \frac{1}{3^2}\sum_i (P_{i1} + P_{i2} + P_{i3})^2 \geqq \left(\sum_i s_i^2\right)/24^2 = ([[A]]/24)^2.$$

However since it holds clearly that $[[A - A^{\theta g}]] \leqq 2[[A]]$, we conclude that

$$[[A]]/24 \leqq [[A - A^{\theta g}]] \leqq 2[[A]].$$

Further, since it is evident from the above argument that $k = 1/24$ does not depend on A, the proof is completed.

For each $g \in G = G_1 * G_2$, let V_g be the unitary operator on $l_2(G)$ defined by $[V_g f](g') = f(g'g)\ (f \in l_2(G))$. Then it is directly deduced from Lemma 6.2.1 and Lemma 6.3.1 in [1] that the factor $\mathbf{M}$ generated by $\{V_g\}_{g \in G}$ is of type $\mathbf{II}_1$ and does not possess the following property (the so-called property (Γ)): Given any system $A_1, A_2, \ldots\ldots, A_m \in \mathbf{M}$ and any $\varepsilon > 0$ there exists a unitary $U = U(A_1, A_2, \ldots\ldots, A_m)$ with $Tr(U) = 0$ and $[[U^{-1}A_k U - A_k]] < \varepsilon$ for $k = 1, 2, \ldots\ldots, m$. Thus such a factor is not approximately finite. This fact was proved essentially by the following lemma (see [2: Lemma 10] or [1: Lemma 6.2.1]).

LEMMA 5. *Let $f(g)$ be a complex-valued function on $G = G_1 * G_2$ such that $\sum_{g \in G} |f(g)|^2 < \infty$ Then there exist $g_1, g_2 \in G$ such that for any $\varepsilon > 0$, $\left(\sum_{g \in G} |f(g_i^{-1} g g_i) - f(g)|^2\right)^{\frac{1}{2}} < \varepsilon\ (i = 1, 2)$ imply $\left(\sum_{g \in G, g \neq e} |f(g)|^2\right)^{\frac{1}{2}} < k\varepsilon$, where k*

does not depend on ε.

Using this lemma and Lemma 4, we shall show that the crossed product of the approximately finite factor by the outer automorphic representation of $G = G_1 * G_2$ is of type $\mathbf{II}_1$ and not approximatley finite. That is to say, we see that it is produced a factor of different algebraical type from an original one by the extension of a factor in this manner.

THEOREM 2. *Let* $\mathbf{M}$ *be the approximately finite factor and let* G *be a group of automorphisms of* $\mathbf{M}$ *which is the outer automorphic representation of the free product of two torsion free countable groups. Then the crossed product of* $\mathbf{M}$ *by* G *is of type* $\mathbf{II}_1$ *and not approximately finite.*

PROOF. We shall freely use the definition and the notation with respect to the crossed product in [3]. By [3 ; Theorem 4], $(\mathbf{M}, G)$ is a factor of type $\mathbf{II}_1$, and so it remains only to show that it is not approximately finite. Denote by g_1, g_2 the elements of G in Lemma 5 and g_3 the element of G stated in Lemma 4. Now, assume that the crossed product $(\mathbf{M}, G)$ has the property (Γ), that is, for any $\varepsilon > 0$, there exists a unitary $V = V(U_{g_1}, U_{g_2}, U_{g_3})$ in $(\mathbf{M}, G)$ with $Tr(V) = 0$ (where $U_{g_k}(k = 1, 2, 3)$ are unitary operators defined in [3 : Lemma 2]), and $[[U_{g_k} - V^{-1}U_{g_k}V]] < \varepsilon$ $(k = 1, 2, 3)$. By [3 : Theorem 1], V is uniquely expressed in the form $\sum_{g \in G} A_g U_g (A \in \mathbf{M} \otimes \mathbf{I})$ in the sense of metric convergence in $(\mathbf{M}, G)$, and $[[V]] = \left(\sum_{g \in G} [[A_g]]^2\right)^{\frac{1}{2}} = 1$.

Thus
$$\begin{aligned} \varepsilon > [[U_{g_k} - V^{-1}U_{g_k}V]] &= [[V - U_{g_k} V U_{g_k}^{-1}]] \\ &= \left[\left[\sum_{g \in G} A_g U_g - \sum_{g \in G} A_g^{g_k} U_{g_k g g_k^{-1}}\right]\right] \\ &= \left[\left[\sum_{g \in G} \left(A_g - A_{g_k^{-1} g g_k}^{g_k}\right) U_g\right]\right] \\ &= \left(\sum_{g \in G} \left[\left[A - A_{g_k^{-1} g g_k}^{g_k}\right]\right]^2\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, by Lemma 4, $[[A_e - A_e^{g_3}]] < \varepsilon$ (e ; the unit of G) implies $[[A_e]] \leqq k_1 \varepsilon$ where k_1 does not depend on ε. On the other hand, putting $f(g) = [[A_g]] = [[A_g^h]]$ $(h \in G)$, then

$$\left(\sum_{g \in G} |f(g_k^{-1} g g_k) - f(g)|^2\right)^{\frac{1}{2}} \leqq \left(\sum_{g \in G} \left[\left[A_g - A_{g_k^{-1} g g_k}^{g_k}\right]\right]^2\right)^{\frac{1}{2}} < \varepsilon.$$

Applying Lemma 5, we obtain $\left(\sum_{\substack{g \in G \\ g \neq e}} [[A_g]]^2\right)^{\frac{1}{2}} = \left(\sum_{\substack{g \in G \\ g \neq e}} |f(g)|^2\right)^{\frac{1}{2}} < k_2 \varepsilon$ where k_2 does not depend on ε. Therefore we can conclude that

$$1 = [[V]] = \left(\sum_{g \in G} [[A_g]]^2\right)^{\frac{1}{2}} \leqq [[A_e]] + \left(\sum_{\substack{g \in G \\ g \neq e}} [[A_g]]^2\right)^{\frac{1}{2}} < (k_1 + k_2)\varepsilon.$$

Hence, choosing $\varepsilon < 1/(k_1 + k_2)$ we have the contradiction, which completes the proof.

REMARK. It is readily seen from the preceding argument that the statements analogous to Lemma 4 and Theorem 2 are true for the free product of n arbitary torsion free countable groups ($n \geqq 2$).

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